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# NILPOTENT SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH FUCHSIAN SINGULARITIES

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Let $K$ be a field of characteristic $p \neq 2$ say algebraically closed. Let $L$ be a linear differential operator

$$L = D^2 - aD - b \in K(x)[D] \tag{0.0}$$

with $D = d/dx$ . Let $\{\gamma_1 , \ldots , \gamma_m , \gamma_\infty = \infty\} = T$ be the set of singularities of $L$ and let

$$f(x) = \prod_{i=1}^{m} (x - \gamma_i) . \tag{0.1}$$

We assume

(0.2) All the singularities of $L$ are fuchsian.

(0.3) The exponents of $L$ at each singularity lie in $\underset{\sim}{F}_p$ .

(0.4) $L$ is nilpotent but does not have two solutions in $K(x)$ linearly independent over $K(x^p)$ .

By "nilpotent", we mean that $L$ has a non-trivial solution in $K(x)$ , and that the equation for the wronskian,

$$Dw = wa , \tag{0.4.1}$$

has a non-trivial solution in $K(x)$ . We may assume that the zeros and poles of $w$ lie in $T$ .

We use the word "exponent" to refer to a root of the indicial polynomial.

For $i = 1 , \ldots , m , \infty$ , let $e_i$ , $e'_i$ be the exponents at $\gamma_i$ .

We choose a solution $u$ of $L$ in $K[x]$ , unique up to factor in $K$ , by the condition that no zero of $u$ is of order greater than $p - 1$ .

We write

$$u = g(x) \prod_{i=1}^{m} (x - \gamma_i)^{\tilde{e}_i} , \tag{0.5}$$

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where

$$g \in K[x] \ , \ (g , f) = 1 \ , \ \tilde{e}_i \in [0 , p-1] \ .$$

We define $\tilde{e}_\infty$ by the condition that $\tilde{e}_\infty \in [0 , p-1)$ ,

(0.6) $$\tilde{e}_\infty \equiv - \deg u \mod p \ .$$

Clearly the $\tilde{e}_i$ represent exponents of $L$ . For all $s \in \underset{\sim}{N}$ , we write

(0.7) $$D^s = a_s D + b_s \mod K(x)[D] L \ ,$$

with $a_s$ , $b_s \in K(x)$ . It is known that

$$a_s = 0 = b_s \ , \quad \forall \ s \geq 2p \ .$$

An ad hoc proof is given in § 4.5 below :

Having defined $\tilde{e}_i (i = 1 , \ldots , m , \infty)$ , we define $e_i \ (\in \underset{\sim}{F}_p)$ to be the class of $\tilde{e}_i$ , and we define $e_i'$ to be the other exponent at $\gamma_i$ (of course we may have $e_i = e_i'$ ). Thus we have uniquely defined the difference, $e_i - e_i'$ , of exponents at $\gamma_i$ . We define $t_i \in [0 , p-1)$

(0.8) $$t_i \mod p = e_i - e_i' \quad (i = 1 , \ldots , m , \infty) \ .$$

The object of this section is to prove the following lemma.

**1. LEMMA.**

(1.1) $$(p-1)(m-1) = 2 \deg g + (t_1 + \ldots + t_m + t_\infty) + pt$$

<u>where</u> $t \in \underset{\sim}{N}$ , $t \geq 0$ .

**2. LEMMA.**

(2.1) $$f(x)^{p-1} \, a_p = g(x)^2 \prod_{i=1}^m (x - \gamma_i)^{t_i} \, \theta(x^p) \ ,$$

<u>where</u>

$\theta \in K[x]$

$g \in K[x]$

$g$ <u>is prime to</u> $f$

$g$ <u>has only simple zeros.</u>

We commence our treatment with an elementary proposition.

3. PROPOSITION. - For each $s \in \mathbb{N}$, $a_s f(x)^{s-1} \in K[x]$,

(3.1) $$\deg a_s f(x)^{s-1} \leq (s-1)(m-1) .$$

Proof. - By differentiating (0.7) and using $L$ to reduce the $D^2$ on the right hand side, we obtain the recursion formula

(3.2) $$\begin{pmatrix} a_{s+1} \\ b_{s+1} \end{pmatrix} = \begin{pmatrix} a'_s \\ b'_s \end{pmatrix} + \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} a_s \\ b_s \end{pmatrix}$$

On the other hand,

(3.3) $$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} .$$

By hypothesis for $1 \leq i \leq m$, $a_2 = a$ (resp. $b_2 = b$) has a pole at $\gamma_i$ of order not greater than one (resp. two). By induction on $s$ and the recursion formula, we show that, for $s \geq 1$,

(3.4) $a_s$ (resp. $b_s$) has a pole at $\gamma_i$ of order not greater than $s - 1$ (resp. $s$).

This shows that $a_s f(x)^{s-1}$ is a polynomial.

The condition that $L$ is fuchsian everywhere implies that we may write $L$ in the form

(3.5) $$L = D^2 + \sum_{i=1}^m \frac{A_i}{x - \gamma_i} D + \sum_{i=1}^m \left(\frac{B_i}{x-\gamma_i} + \frac{C_i}{(x-\gamma_i)^2}\right) ,$$

where $A_i$, $B_i$, $C_i \in K$ for $i = 1, 2, \ldots, m$.

The condition at infinity implies that

(3.6) $$\sum_{i=1}^m B_i = 0 .$$

Thus $\deg a_2$ (resp. $\deg b_2$) $\leq -1$ (resp. $-2$) and by (3.2) and induction we show that

(3.7) $$\deg a_s \text{ (resp. } \deg b_s) \leq -(s-1) \text{ (resp. } -s) .$$

This completes the proof of the proposition.

4. - We now commence the proofs of the lemmas. By hypothesis, the $K(x^p)$-space of solutions of $L$ in $K(x)$ has dimension one but in a suitable differential extension field $F$, the $F^p$ space of solutions of $L$ has dimension two.

More explicitly, we choose $F$ so as to contain, $\tau$, a solution of

(4.1) $$\tau' = w/u^2$$

and then, by a well known calculation using (0.4.1),

$$L(u\tau) = \tau L(u) + \frac{w}{u}(\tau' \frac{u^2}{w}) = 0 ,$$

while the wronskian

(4.2) $$\begin{vmatrix} u\tau & (u\tau)' \\ u & u' \end{vmatrix} = - w$$

which shows that $u$, $u\tau$ are linearly independent over the kernel of $D$ in $F$.

We now apply (0.7) and conclude that

$$(\tau u)^{(s)} = a_s(\tau u)' + b_s(\tau u)$$

(4.3)

$$u^{(s)} = a_s u' + b_s u .$$

Eliminating $b_s$, we obtain

(4.4) $$a_s = \sum_{\substack{i+j=s \\ i\geq 1}} \frac{\tau'^{(i-1)}}{\tau'} \frac{u^{(j)}}{u} \binom{s}{i} ,$$

a formula involving $u$ and $\tau'$ but not $\tau$. We observe that this formula is independent of the characteristic.

**(4.5) Remark.** - Since $u$ and $\tau'$ lie in $K(x)$ they are annihilated by $D^p$. For $s \geq 2p$ either $i - 1$ or $j$ on the right side of (4.4) exceeds $p - 1$ which shows again that $a_s = 0$ for $s \geq 2p$.

In particular, for $s = p$, the above formula gives

(4.6) $$a_p = (u^2/w)\ D^{p-1}\ (w/u^2) .$$

Now $a_p \neq 0$ as otherwise $\frac{w}{u^2}$ would lie in the kernel of $D^{p-1}$ in $K(x)$ , i. e.,

$$\frac{w}{u^2} \in K(x^p) + K(x^p)\ x + \dots + K(x^p)\ x^{p-2}$$

which would show that (4.1) has a solution $\tau$ in $K(x)$ contrary to hypothesis concerning the dimensionality of the kernel of $L$ in $K(x)$ (as $K(x^p)$ space).

By the same argument since $1 , x , \dots , x^{p-1}$ is basis of $K(x)$ as $K(x^p)$ space, we conclude that $D^{p-1}$ maps $K(x)$ into $K(x^p)$ . Hence

(4.7) $$a_p \in \frac{u^2}{w}\ K(x^p)\ .$$

Putting

$$Q_p = a_p\ f(x)^{p-1}\ ,$$

we have

(4.3) $$Q_p \in \frac{u^2}{w}\ \frac{1}{f(x)}\ K(x^p)\ .$$

We have defined $g$ as the factor of $u$ prime to $f(x)$ . If $x_0$ is a zero of $g$ then the indicial polynomial of $L$ at $x_0$ has $0 , 1 \pmod p$ as zeros and by definition $u$ has no zero of order greater than $p - 1$ . This shows that the zeros of $g$ are simple.

5. - We continue our proof of the lemmas. We will show

(5.1) $$\frac{u^2}{w}\ \frac{1}{f(x)} \in g(x)^2 \prod_{i=1}^m (x - \gamma_i)^{t_i}\ K(x^p)\ .$$

With this in mind, we use (3.5) to deduce

(5.2) $$\begin{aligned} e_i + e_i' &= 1 - A_i\ , \quad i = 1 , \dots , m \\ e_\infty + e_\infty' &= \textstyle\sum_{i=1}^m A_i - 1 \end{aligned}$$

while

(5.3) $$w \in \prod_{i=1}^m (x - \gamma_i)^{-\bar{A}_i}\ K(x^p)\ ,$$

where $\bar{A}_i$ is a representative in $\underset{\sim}{N}$ of $A_i$ $(1 \leqslant i \leqslant m)$ . Thus the order of $\gamma_i$ as zero of the left side of (5.1) is congruent mod p to $2e_i + A_i - 1 = e_i - e_i' \equiv t_i$ .

This together with our discussion of $g$ , the factor of $u$ prime to $f$ , concludes the demonstration of (5.1).

We now estimate the degree of the left side of (5.1). By hypothesis $\deg u = -\tilde{e}_\infty \equiv -e_\infty$ and so

$$\deg \frac{u^2}{wf(x)} \equiv -2e_\infty - m + \sum_{i=1}^m A_i .$$

By (5.2) this is the same as $-m + 1 - t_\infty$ . Thus from (5.1),

$$(5.4) \qquad m + 2\deg g + t_\infty + \sum_{i=1}^m t_i \equiv 1 \mod p .$$

By (4.8), (5.1), we obtain (2.1) with $\theta \in K(x)$ . We assert that $\theta$ is a polynomial. Indeed $Q_p$ is a polynomial and $Q_p/\theta(x^p)$ is, by (2.1), a polynomial with zeros of order bounded by $p - 1$ . Thus $\theta$ must be a polynomial. This completes the proof of Lemma 2. We continue with the proof of Lemma 1. By proposition 3,

$$(5.5) \qquad (p-1)(m-1) \geq \text{degree } Q_p = 2 \text{ degree } g + \sum_{i=1}^m t_i + p \deg \theta .$$

Let

$$(5.6) \qquad \rho = (p-1)(m-1) - 2 \text{ degree } g - \sum_{i=1}^m t_i .$$

Then

$$(5.7) \qquad \rho \geq p \text{ degree } \theta \geq 0 .$$

On the other hand, by (5.4) and (5.6),

$$\rho \equiv t_\infty \mod p .$$

And hence, by (5.7),

$$\rho = t_\infty + pt$$

for some $t \geq 0$ . Substitution in (5.6) completes the proof of Lemma 1.

6. Remark. - We view the sum of the $t_i$ as the analogue of the sum of the angles of the image of the upper half plane under a ratio of solutions of $L$ if $K$ were say the reals and the $\gamma_i$ were all real.

7. - In general we are given $L$ but not $u$ and so there are two choices of $t_i$ for each $i$ . Thus in applying Lemma 1 there are $2^{m+1}$ choices for $(t_1, \ldots, t_m, t_\infty)$ and $t$ is not known.

COROLLARY. - *If* $m = 2$ *then under hypotheses* (0.2)-(0.4), *we have* $t = 0$ , *and* there is just one possible choice for $t_0$ , $t_1$ , $t_\infty$ . Equation (1.1) takes the form

$$p - 1 = 2 \deg g + t_0 + t_1 + t_\infty .$$

Proof. - It is clear from (1.1) that $t = 0$ . Since $p \neq 2$ , it follows that

$$t_0 + t_1 + t_\infty \equiv 0 \mod 2$$

(7.1)

$$p - 1 \geq t_0 + t_1 + t_\infty .$$

Now each $t_i$ is fixed by $L$ up to the transformation

$$t_i \longrightarrow p - t_i .$$

The condition of parity shows that such a transformation, if applied at all, must be applied to two of the $t_i$ , say to $t_0$ , $t_1$ and we would then have

(7.2) $$p - 1 \geq p - t_0 + p - t_1 + t_\infty .$$

This is inconsistent with (7.1) as the sum would give

$$p - 1 \geq 2p + 2t_\infty \geq 2p .$$

Remark. - The degree of $g$ is at most $\frac{p-1}{2}$ and this occurs precisely, when $t_0 = t_1 = t_\infty = 0$ , for example in the case of the differential operator associated to the hypergeometric function $F(1/2 , 1/2 , 1 ; x)$ .