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ON ORDINARY LINEAR p-ADIC DIFFERENTIAL EQUATIONS WITH ALGEBRAIC FUNCTION COEFFICIENTS

by Bernard DWORK

(based on joint work with P. ROBBA [3])

In previous articles ([3], [4]), we considered differential equations whose coefficients are analytic elements. The central idea was to consider the behavior of the equation on the generic disk. The purpose of this article is to extend this procedure to the case in which the coefficients are algebraic functions.

Indeed, an early example concerning such differential equations was given by TATE ([2], § 5) who (in effect) discussed the question : Let $y^2 = x(x - 1)(x - \lambda)$, λ constant. Does there exist a constant C (depending upon λ) such that the equation

(1) $$C \frac{dz}{dx} = z/y$$

has a bounded solution on the disk at infinity, $\complement D(0 , 1^+)$? This formulation is not quite correct as the x-plane does not give a good parametrization of the elliptic curve at infinity but it follows from the general theorems of this article that the disk at infinity plays no special role and the question is the existence of a solution of (1) which is an analytic function on a residue class (<u>Remark</u> : The answer is yes if the elliptic curve has non-supersingular reduction and in that case, C is any unit determination at λ of $\sqrt{-1}\, F(\frac{1}{2} , \frac{1}{2} , 1 ; \lambda)$ viewed as algebraic element in the sense of CHRISTOL ([1]).

We consider a linear differential equation whose coefficients are both analytic functions on an annulus Δ (of center zero and outer radius 1) and are algebraic over the ring $H(\Delta)$ of analytic elements on Δ . The ring $H(\Delta)$ has a natural imbedding in E , the field of analytic elements on the generic disk. We demonstrate the existence of a non-canonical isomorphism mapping such algebraic-analytic functions on Δ into bounded functions on $D(t , 1^-)$ algebraic over E in such a way that the boundary norm goes over into the sup norm on $D(t , 1^-)$ (cf. corollary 1.10 below).

Using this isomorphism, we show that there is no difficulty in translating information concerning solutions lying in $\mathfrak{A}_\Delta$ into information concerning solutions in the generic disk and conversely. In particular, we get one usual comparison theorem (cf. lemma 3.4, theorem 4.4 below) and the rules of growth of solutions.

It seems likely that this work may be extended to differential equations whose coefficients (as function of x) are algebraic elements in the sense referred to above.

0. Notation.

0.1 : Let K be a field of characteristic zero complete under a non-archimedean valuation with residue class field of characteristic $p \neq 0$. Let Ω be an algebraically closed, maximally complete field under a valuation extending that of K and linearly disjoint from $E_0 = K(X)$ over K_0. Let Ω have a valuation ring containing an element, t, whose image in the residue class field of Ω is transcendental over the residue class field of K.

0.2 : For each $a \in \Omega$ and each positive real number r, let

$$D(a , r^-) = \{x \in \Omega ; \ |x - a| < r\}$$

$$D(a , r^+) = \{x \in \Omega ; \ |x - a| \leq r\} .$$

For $f \in \Omega[[x - a]]$, $f = \sum_{\nu=0}^{\infty} b_\nu (x - a)^\nu$, analytic on $D(a , r^-)$, let for $\rho < r$,

$$|f|_a (\rho) = \sup_\nu |b_\nu| \rho^\nu .$$

This is extended to functions f, meromorphic in $D(0 , r^-)$ by writting

$$|f|_a (\rho) = |g|_a (\rho)/|h|_a (\rho)$$

if $f = g/h$, both g and h being analytic on $D(a , r^-)$.

0.3 : Let E be the completion of E_0 under the Gauss norm

$$f \to |f|_0 (1) .$$

We shall write $|f|_E$ for this norm on E.

0.4 : For each bounded subset A of Ω such that $d(A , CA) > 0$ (or the union of such a bounded set with the complement in $\Omega \cup \{\infty\}$ of $CD(0 , r)$ for some $r > 0$. Let $H(A)$ be the completion under the topology of uniform convergence of the subspace of E_0 consisting of rational functions with coefficients in K having no poles in A. Under the sup norm, $\| \|_A$, $H(A)$ is a Banach space. Let $M(A)$ denote the quotient field of $H(A)$.

0.5 : Let $\pi = (\pi_\nu)_{\nu \in \mathbb{N}}$ be a non-increasing sequence of positive real numbers with

$$\pi_0 = 1$$

(0.5.1)

$$\pi_\nu / \pi_{\nu+1} \text{ monotonically increasing} .$$

Let W_a^π be the Banach space of germs of analytic functions at a

$$u = \sum_{\nu=0}^{\infty} b_\nu (x - a)^\nu$$

such that

$$\|u\|_\pi = \sup_\nu \pi_\nu |b_\nu| < +\infty$$

with norm $u \to \|u\|_\pi$.

0.6 : We use $\mathcal{a}_a^\rho$ to denote the space of functions analytic in $D(a , \rho^-)$

with topology of uniform convergence on disks $D(a, r^-)$ with $r < \rho$. We write $\mathfrak{a}_a$ instead of $\mathfrak{a}_a^1$ (if $a \in D(0, 1^+)$).

0.7 : In particular in 0.5, if $\pi_\nu = 1$ for all $\nu \in \underset{\sim}{N}$, we denote π by $\pi^{1,0}$ and write $\| \|_{1,0}$ for the norm of W_t^π (and for the operator norm on that space) and write $W_a^{1,0}$ for W_a^π.

1. Algebraic functions analytic on an annulus.

For each annulus,

(1.1) $$\Delta = \Delta_b = D(0, 1^-) - D(0, b^-), \qquad b \in [0, 1)$$

let $H(\Delta)$ be the field of analytic elements on Δ, $\mathfrak{M}(\Delta)$ the quotient field of $H(\Delta)$, $\mathfrak{a}_\Delta$ the ring of functions analytic on Δ and $\mathfrak{M}^a(\Delta)$ the field of those elements in the quotient field of $\mathfrak{a}_\Delta$ which are algebraic over $\mathfrak{M}(\Delta)$.

Let $\mathfrak{M}$ (resp : $\mathfrak{M}^a$) be the inductive limit of the fields $\mathfrak{M}(\Delta_b)$ (resp $\mathfrak{M}^a(\Delta_b)$) as $b \to 1$. If $u \in \mathfrak{M}^a$, then u is a ratio of elements of $\mathfrak{a}_\Delta$ for suitable Δ, but also satisfies a polynomial equation with coefficients in $\mathfrak{M}(\Delta)$. It follows that u has only a finite set of zeros and poles in Δ. This shows that $u \in \mathfrak{a}_{\Delta'}$, for some $\Delta' = \Delta_{b'}$, for some $b' \in [b, 1)$. Indeed it is clear that $\mathfrak{M}$ is the inductive limit of the rings $H(\Delta_b)$ and that $\mathfrak{M}^a$ is the inductive limit of the ring of elements of $\mathfrak{a}_{\Delta_b}$ which are algebraic over $H(\Delta_b)$.

The boundary norm

(1.2) $$\|u\|_\sigma = \lim\sup_{r\uparrow 1} |u|_0(r)$$

for elements of the quotient field of $\mathfrak{a}_\Delta$ may be written more simply as

(1.3) $$\|u\|_\sigma = \lim_{r\to 1} |u|_0(r) = \lim_{b\to 1} \|u\|_{\Delta_b} \quad \text{for } u \in \mathfrak{M}^a .$$

There is a natural map of $\mathfrak{M}$ into E. Our first object is to show that each element of $\mathfrak{M}^a$ has a branch in $W_t^{1,0}$.

1.4 : THEOREM. - <u>An irreducible polynomial in one variable with coefficients in $\mathfrak{M}$ remains irreducible over</u> F.

<u>Proof</u>. - Let f be an irreducible monic polynomial in $\mathfrak{M}[Y]$. Hence f has coefficients in $H(\Delta)$ for suitable annulus Δ. Suppose

(1.4.1) $$f = gh ,$$

where g (resp. h) is monic element of $E[Y]$ of degree n (resp. m). Viewing (1.4.1) on a non-linear equation satisfied by $n + m$ unspecified coefficients of g and h, we are led to consider the tangential mapping

$$(\mathfrak{a}_t[Y])_{n-1} \times (\mathfrak{a}_t[Y])_{m-1} \to (\mathfrak{a}_t[Y])_{n+m-1}$$
$$(G, H) \to Gh + gH .$$

This map is trivially injective as otherwise if $Gh + gH = 0$ with $H \neq 0$ then

gH is divisible by h while $\deg H < m = \deg h$ which shows that $(g, h) \neq 1$ and thus $(f, f') \neq 1$ which contradicts the irreducibility of f over $\mathfrak{M}$. It now follows from [3] (theorem 3.16) that the coefficients of f and h lie in $H[\Delta']$ for some

$$\Delta' = \Delta_{b'} \subset \Delta .$$

This contradicts the irreducibility of f over $\mathfrak{M}$. This completes the proof of the theorem.

For $u \in E$, we may think of u as analytic element on $D(t, 1^-)$ and the Gauss norm, the sup norm on $D(t, 1^-)$ and the specialization norm

(1.5) $$u \to |u(t)|$$

all coincide. Of course $u(t) = 0$ if and only if u is the zero element of E. Thus each element v of the algebraic closure E^{alg} of E may be imbedded in the ring, $\mathcal{O}_t$, of germs of analytic functions at t. Again for such v, $v(t) = 0$ only if $v = 0$. Since E is complete, the Gauss valuation has unique extension to E^{alg}.

1.6. PROPOSITION. - <u>The unique extension of Gauss norm to</u> E^{alg} <u>(viewed as imbedded in</u> $\mathcal{O}_t$) <u>coindides with the specialization norm</u> (1.5).

<u>Proof</u>. - Evident.

1.7 : We observe that if $u \in \mathcal{O}_t$ is algebraic over E then u is bounded on its disk of convergence with center t. However the example $x^{1/p} = \exp(p^{-1} \log(x/t))$ shows that u need not converge in $D(t, 1^-)$, and hence need not be in $W_t^{1,0}$. However $E^{alg} \cap W_t^{1,0}$ is a field.

1.8. LEMMA. - <u>The boundary norm valuation of</u> $\mathfrak{M}$ <u>has unique extension,</u> $\| \|_\sigma$, <u>to</u> $\mathfrak{M}^{alg}$, <u>its algebraic closure. If</u> $v \in \mathfrak{M}^{alg}$ <u>let</u> u <u>be any solution in</u> $\mathcal{O}_t$ <u>of the irreducible polynomial over</u> $\mathfrak{M}$ <u>satisfied by</u> v. <u>Then</u>

(1.8.1) $$\|v\|_\sigma = |u(t)| .$$

<u>Proof</u>. - The field E is the completion of $\mathfrak{M}$ under the boundary norm valuation. The uniqueness of the extension of boundary norm valuation thus follows from theorem 1.4, equation (1.8.1) merely states that the valuation of $\mathfrak{M}(v)$ is given by the isomorphism of that field into $E(u)$ together with the restriction to that field of the valuation of E^{alg}.

1.8.2. COROLLARY. - <u>The boundary norm is the unique valuation of</u> $\mathfrak{M}^a$ <u>which extends the boundary norm valuation of</u> $\mathfrak{M}$. <u>Thus by</u> (1.8.1) <u>the boundary norm valuation on</u> $\mathfrak{M}^a$ <u>coincides with the norm obtained by specializing a branch at</u> t.

1.9. THEOREM. - <u>Each element</u> u <u>of</u> $\mathfrak{M}^a$ <u>has a branch in</u> $W_t^{1,0}$. <u>More explicitly, the irreducible polynomial in</u> $\mathfrak{M}[V]$ <u>satisfied by</u> u <u>splits in</u> $W_t^{1,0}$.

Proof. - Let f be the irreducible polynomial for u in $\mathfrak{M}[V]$. Let v be a root of f in $\mathcal{O}_t$. Then $u \to v$ defines an isomorphism of $\mathfrak{M}(u)$ into $E(v)$. Each derivative $u^{(m)}$ of u lies in $\mathfrak{M}(u)$ and its image is $v^{(m)}$, the m^{th} derivative of v. This shows that

(1.9.1) $$\|u^{(m)}\|_\sigma = |v^{(m)}(t)| .$$

We assert that for each $m \geq 0$,

(1.9.2) $$\|u^{(m)}/m!\|_\sigma \leq \|u\|_\sigma .$$

Indeed by hypothesis, there exists Δ such that $u \in \mathfrak{A}_\Delta$. If $z \in \Delta$ then

$$D(z , |z|^-) \subset \Delta$$

and hence

$$\left|\frac{u^{(m)}(z)}{m!}\right| \; |z|^m \leq \|u\|_\Delta$$

and taking the supremum on the left side over all z on the circumference of center 0 and radius r, we obtain

$$r^m |u^{(m)}/m!|_0 (r) \leq \|u\|_\Delta$$

for all r close enough to 1. Taking the limit as r goes to 1 gives,

$$\|u^{(m)}/m!\|_\sigma \leq \|u\|_\Delta .$$

But Δ is given by 1.1 and taking limits on $b \to 1$ gives (1.9.2) as asserted. It follows from (1.9.1), (1.9.2) and the Taylor series for v at t that v lies in $W_t^{1,0}$ and then holds for each solution of f at t. This completes the proof of the theorem.

1.10. COROLLARY. - <u>The natural mapping of</u> $\mathfrak{M}$ <u>into</u> E <u>has a (non-canonical) extension</u> τ <u>of</u> $\mathfrak{M}^a$ <u>into</u> $E^{alg} \cap W_t^{1,0}$ <u>which commutes with differentiation,</u> d/dx. <u>For</u> $u \in \mathfrak{M}^a$ <u>we have</u>

(1.11) $$\|u\|_\sigma = \|\tau u\|_{1,0} .$$

2. Differential equations on the generic disk.

Let E' be a field of elements of $\mathfrak{A}_t$ (hence without zeros in $D(t , 1^-)$) which is complete under the sup norm and closed under differentiation. Let $\mathcal{R}_{E'} = E'[D]$, the ring of linear differential operators with coefficients in E'. Then $\mathcal{R}_{E'}$ operates on W_t^π with the operator norm

$$\|\Sigma\, C_m D^m\|_\pi = \sup_m |m!\; C_m|_{E'}/\pi_m .$$

2.1. THEOREM. - <u>Let</u> $L \in \mathcal{R}_{E'}$, <u>let</u> R <u>be the monic generator of the</u> π<u>-closure in</u> $\mathcal{R}_{E'}$ <u>of the ideal</u> $\mathcal{R}_{E'} L$.

<u>Then</u>

$$\mathrm{Ker}_t\, R = W_t^\pi \cap \mathrm{Ker}_t\, L .$$

2.2. COROLLARY. - If L is of order n , then

$$\mathcal{A}_t \cap \mathrm{Ker}_t\, L \subset W_t^{n-1} .$$

The proofs are the same as given by ROBBA ([4], § 2) the point being that at each step E may be replaced by E' .

3. Differential equations over an annulus.

Let τ be as in corollary 1.10 an imbedding of $\mathcal{M}^a$ into $W_t^{1,0}$. Our object is to compare differential equations having coefficients in $\mathcal{M}^a$ with equations having coefficients in $\tau\mathcal{M}^a$.

We set

$$\mathcal{M}_t^a = \tau\mathcal{M}^a \subset W_t^{1,0}$$

$$E' = \text{completion of } \mathcal{M}_t^a \text{ in } W_t^{1,0} \quad (\text{so } E' \supset E)$$

$$\mathcal{R}_0 = \mathcal{M}^a[D]$$

$$\mathcal{R}' = E'[D]$$

$$\mathcal{R}_t = \mathcal{M}_t^a[D] .$$

Thus $\mathcal{R}_t \subset \mathcal{R}'$ and the isomorphism τ extends trivially to an isomorphism of $\mathcal{R}_0$ with $\mathcal{R}_t$ which we again denote by τ .

Let $L \in \mathcal{R}_0$ and let τL be its image in $\mathcal{R}_t$.

The π-closure in $\mathcal{R}'$ of the ideal $\mathcal{R}'\tau L$ has monic generator R as left ideal of $\mathcal{R}'$.

Let $\{R_s\}_{s\in\mathbb{N}}$ be a sequence of monic elements of $\mathcal{R}_t$ such that

(3.1) $$\|R_s - R\|_\pi < 1/s \quad \forall\, s \in \mathbb{N}$$

(such a sequence exists as $\mathcal{R}_t$ is dense $\mathcal{R}'$).

LEMMA 3.2. - There exists a sequence $\{P_s\}_{s\in\mathbb{N}}$ of elements of $\mathcal{R}_t$ such that

(3.2.1) $$R_s = P_s \mod \mathcal{R}_t L_t .$$

(3.2.2) $$\|R_s\|_\pi < 1/s .$$

Proof. - The proof is based on the fact that $\mathcal{M}_t^a$ is dense in E' . To complete the proof, see [4], lemme 3.1.

3.3. LEMMA. - Let $L \in \mathcal{R}_0$, let Δ be an annulus such as 1.1. If u is quotient of elements of $\mathcal{A}_\Delta$, then

$$\|u^{-1} Lu\|_\sigma \leq \|\tau L\|_{1,0} .$$

If u is ratio of bounded elements of $\mathcal{A}_\Delta$, then

$$\|Lu\| \leq \|\tau L\|_{1,0} \|u\|_{1,0} .$$

Proof. - See [4], lemme 3.3.

3.4. LEMMA. - Let $L \in \mathcal{R}_0$. Clearly L is stable on $\mathfrak{a}_\Delta$ and on its quotient field, if b (in equation 1.1) is close enough to 1. We assert that

$$\dim \text{bdd}\ \mathrm{Ker}_t\ \tau L \geq \dim \text{bdd}\ \mathrm{Ker}_\Delta\ L$$

where the right side refers to the kernel of L in the space of all ratio g/h where g and h are bounded elements of $\mathfrak{a}_\Delta$.

Proof. - See [4] theoreme 3.4, [3] theorem 2.4.

THEOREM 3.5. - If $L \in \mathcal{R}_0$ and there exists $u\ (\neq 0)$ in the quotient field of $\mathfrak{a}_\Delta$ such that $Lu = 0$ then the bdd Ker τL is not trivial.

Proof. - See [4] theorem 3.5.

THEOREM 3.6. - Let L be a monic element of $\mathcal{R}_0$ such that

(i) The coefficients of L lie in $\mathfrak{a}_\Delta$,

(ii) The solutions at t of τL lie in $\mathfrak{a}_t$,

then L has a full set of solutions in $W_t^{1,0}$.

Proof. - Let n be the order of L. We write for all $n \in \underset{\sim}{N}$,

(3.6.1) $$D^m/m! \equiv \sum_{j=0}^{n-1} B_{m,j}\ D^j \mod \mathcal{R}_0 L$$

where each $B_{m,j} \in \mathfrak{M}^a$. Then

(3.6.2) $$D^m/m! \equiv \sum_{j=0}^{n-1} (\tau\ B_{m,j})\ D^j \mod \mathcal{R}_t(\tau L)\ .$$

It follows from hypothesis (ii) and corollary 2.2 that

(3.6.3) $$\|\tau\ B_{m,j}\|_{1,0} = 0(m^{n-1})\ .$$

Hence by 1.11,

(3.6.4) $$\|B_{m,j}\|_\sigma = 0(m^{n-1})$$

but by hypothesis (i) each $B_{m,j} \in \mathfrak{a}_0$ and hence the sup norm coincides with the boundary norm. Then in particular

(3.6.5) $$|B_{m,j}(0)| = 0(m^{n-1})\ .$$

This completes the proof of the theorem.

4. Comparison of the radii of convergence.

Let $\mathcal{R}_0$, $\mathcal{R}'$, $\mathcal{R}_t$, τ be as in § 3. Let L be a monic element of $\mathcal{R}_0$, let $r \in (0, 1)$, and let Γ be the monic divisor of τL in $\mathcal{R}'$ defined by

(4.1) $$\mathrm{Ker}_t\ \Gamma = \mathfrak{a}_c^r \cap \mathrm{Ker}_t\ \tau L\ .$$

Then

(4.2) $$_{\tau}L = \Lambda \cdot \Gamma \ ,$$

a factorization in $\mathcal{R}'$. Let Λ be of order m, and Γ of order n.

4.3. THEOREM. - The factors Λ and Γ lie in $\mathcal{R}_t$.

Proof. - The field E' is the completion of M_t^a in $W_t^{1,0}$ and hence is of infinite dimension as E-space. However the field generated over E by the coefficients of $_{\tau}L$ is a finite extension of E and hence complete. It follows from § 2 above that the coefficients of Γ lie in this field. Thus letting θ be an element of $\mathfrak{M}^a$ such that the coefficients of L lie in $\mathfrak{M}(\theta)$ and letting θ_t denote $_{\tau}\theta$, we may write

(4.3.1) $$\Lambda = \sum_{i=0}^{\nu-1} \theta_t^i \Lambda_i + D^m$$ $$\Gamma = \sum_{i=0}^{\nu-1} \theta_t^i \Gamma_i + D^n$$

where each Γ_i, Λ_i lies in $\mathcal{R} = E[D]$, order $\Gamma_i < n$, order $\Lambda_i < m$, and ν is the degree of θ over $\mathfrak{M}$. We choose an annulus Δ (as in equation 1.1) such that the coefficients of L lie in $H(\Delta)[\theta]$, such that the degree of θ over $\mathfrak{M}(\Delta)$ is again ν and such that $d\theta/dx$ is a linear combination of powers of θ with coefficients in $H(\Delta)$. This equation (4.2) represents a system of non-linear differential equations defined over $H(\Delta) \subset E$ and satisfied by the coefficients (in E) of the operator Λ_i, Γ_i. If

(4.3.2) $$\lambda \in (\mathfrak{A}_t[D])_{m-1} , \qquad \lambda \neq 0$$ $$\gamma \in (\mathfrak{A}_t[D])_{n-1}$$

and

$$\lambda\Gamma + \Lambda\gamma = 0$$

then precisely as in the proof of [4] (theorem 4), Λ has a non-trivial kernel in $\mathfrak{A}_t^r$ and this contradicts the definition of Γ. Thus

(4.3.4) $$(\lambda , \gamma) \to \lambda\Gamma + \Lambda\gamma$$

is an injective mapping of

$$(\mathfrak{A}_t[D])_{m-1} \times (\mathfrak{A}_t[D])_{n-1} \text{ into } (\mathfrak{A}_t[D])_{n+m-1} .$$

However (4.3.4) is not the tangential mapping to be associated with the "variety" (4.2). The tangential mapping is the map

$$((\mathfrak{A}_t[D])_{m-1})^{\nu} \times ((\mathfrak{A}_t[D]_{n-1})^{\nu} \to (\mathfrak{A}_t[D])_{n+m-1}$$

given by

(4.3.5) $$(\lambda_0,\lambda_1,\dots,\lambda_{\nu-1} ; \gamma_0,\dots,\gamma_{n-1}) \to (\sum_{i=0}^{\nu-1} \theta_t^i \lambda_i) \Gamma + \Lambda \sum_{i=0}^{\nu-1} \theta_t^i \gamma_i .$$

This mapping has a kernel for a trivial reason (if $\nu \neq 0$),

$$\sum_{i=0}^{\nu-1} \theta_t^i \gamma_i \quad 0 , \qquad \gamma_0,\gamma_1,\dots,\gamma_{\nu-1} \in \mathfrak{A}_t[D]$$

does not imply that all the γ_i are zero. To overcome this difficulty we reinterpret (4.2) by writting

(4.3.6) $$\tau L = D^{n+m} + \sum_{i=0}^{\nu-1} \theta_t^i L_i$$

where each L_i has coefficients in $H(\Delta)$. Let $\theta_1, \dots, \theta_\nu$ be the distinct conjugates of θ_t over E. Then (4.2) is equivalent to the system

(4.3.7) $D^{n+m} + \sum_{i=0}^{\nu-1} \theta_j^i L_i = (D^m + \sum_{i=0}^{\nu-1} \theta_j^i \Lambda_i)(D^n + \sum_{i=0}^{\nu-1} \theta_j^i \Gamma_i)$, $j = 1, 2, \dots, \nu$
which we view as a system of non-linear equations for the coefficients of the Λ_i and the Γ_i. The tangent mapping is now given by

(4.3.8) $$(\lambda_0, \dots, \lambda_{\nu-1} ; \gamma_0, \dots, \gamma_{\nu-1}) \to (\tau_j \lambda)(\tau_j \Gamma) + (\tau_j \Lambda)(\tau_j \gamma)$$

where the λ_i, γ_i are as in equation (4.3.5) and

(4.3.9) $$\begin{aligned} \tau_j \lambda &= \sum_{i=0}^{\nu-1} \theta_j^i \lambda_i \\ \tau_j \gamma &= \sum_{i=0}^{\nu-1} \theta_j^i \gamma_i \\ \tau_j \Gamma &= \sum_{i=0}^{\nu-1} \theta_j^i \Gamma_i + D^n \\ \tau_j \Lambda &= \sum_{i=0}^{\nu-1} \theta_j^i \Lambda_i + D^m . \end{aligned}$$

For an element in the kernel of this mapping we have (by our analysis of (4.3.3))

(4.3.10) $$\tau_j \gamma = 0 , \qquad j = 1, 2, \dots, \nu$$

and hence each $\gamma_i = 0$. Likewise for the λ_i. We conclude from [3] (theorem 3.1.b) that each Λ_i, Γ_i lies in $\mathfrak{M}(\Delta')[D]$ for some $\Delta' = \Delta_{b'} \subset \Delta_b = \Delta$; i. e. each Λ_i, Γ_i lies in $\mathfrak{M}[D]$ and hence Γ, Λ lie in $\mathfrak{M}_t^a[D]$ as asserted.

4.4. THEOREM. - Let $L \in \mathcal{R}_0$. Then

$$\dim(\mathrm{Ker}_t\, \tau L) \cap \mathfrak{a}_t \geq \dim \mathrm{Ker}\, L \cap \mathfrak{a}_\Delta .$$

Proof. - This follows from the preceeding theorem (cf. [3], theorem 4.2.2).

4.5. COROLLARY. - Let L be monic element of $\mathcal{R}_0$ of order n such that

(i) The coefficients of L lie in $\mathfrak{a}_0$,

(ii) The equation $Ly = 0$ has a full set of independent solutions in $\mathfrak{a}_0$.

Then L has a full set of solutions in $W_0^{1,n-1}$.

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