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Foundations for stepwise refinement of Program Specifications via Cylindric Algebra Theory

I. Németi

1. Introduction.

To investigate connections between different theories formulated in completely different languages is a problem in many branches of computer science , e. g. in structured programming, in structuring program specifications, Burstall-Goguen (77),(79), Goguen-Burstall (78),(80),(79), Dömölki (79), Mosses (79), in data types and semantics, Blum-Estes (77), in A. I. , Andréka-Gergely-Németi (72), McCarthy-Hayes (69), Andréka-Németi (79a) etc... The connections between the different theories and languages can be called interpretations or trnslations but translations would suggest something much simpler than the thing we have in mind. Most often they are called Theory Morphisms. One point to be stressed is that between two theories there are usually many theory morphisms. The subject of investigation here is actually a category consisting of theories and their morphisms. The notion of a theory morphism from the theory T into the theory T' was defined for example in Winkowski (78) §1 (p. 277) and there it was called a modelling μ of the theory T in the other one T' . In that paper theory morphisms are used to study Computer Simulation.

There is a related branch of "standard" Universal Algebra called Lattice of Varieties, see Grätzer (79) p. 389. That lattice is a special subcategory of the Category of Theories considered here. For the purposes reported here, it is too special for two reasons:

1. The main point in the presently reviewed field is that the theories are of different similarity types.
2. Between two theories there are many morphisms.

(In some Computer Science papers the Category of Theories was called "Hierarchy of Languages" to emphasize that the underlying languages, similarity types, or even logics are usually different, see Andréka-Gergely-Német (72), Rattray-Rus (77), Andréka-Németi (79a), Sain (79b).)

Here we quote three approaches which are complementary (and are most useful when applied together).

(1.1) Let the theories Th_1 and Th_2 be equational but possibly heterogenous (many sorted). Let $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ be the countably generated free algebras of the varieties $Mod(Th_1)$ and $Mod(Th_2)$ respectively. Then a suitable homomorphism $h: \underset{\sim}{F}_1 \longrightarrow \underset{\sim}{F}_2$ could establish a connection between the two theories i. e. between the two varieties. The problem is that $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ are usually of completely different similarity types ! Hence the notion of a homomorphism between them is just meaningless. To alleviate this problem, Blum-Estes (77) generalized the usual notion of homomorphism to be defined between algebras of different similarity types. One definition could be to say that $f: \underset{\sim}{A} \longrightarrow \underset{\sim}{B}$ is a generalized homomorphism if f takes every term function of $\underset{\sim}{A}$ into some term function of $\underset{\sim}{B}$. That is, the image of an operation of $\underset{\sim}{A}$ is required to be a function definable in $\underset{\sim}{B}$. One can then make restrictions (or generalizations) on the notion of definability used e. g. term-definable, first-order definable, implicitely definable by first order formulas etc... Then the Theory of Definability which is a branch of Model Theory, see for example the Chang-Keisler monograph, can be used as a guide for choosing the notion with the desired properties from a fairly broad spectrum of existing and well understood ones. For some varying choices see Blum-Estes (77), Blum-Lynch (79a),(79b).

(1.2) One of the main aims of Algebraïc Theories of Lawvere is to deal with the present problem. I. e. the aim is to investigate the "structure" or "system" consisting of several theories of several similarity types and several possible "interpretations" (i. e. connections) between these theories and languages. Hence this is not a mere translation of Universal Algebra into category theoretical language, but instead this is an approach to a problem inherent both in Universal Algebra and in Model Theory: to a problem which has not been attacked in "standard" Univeral Algebra or Model Theory yet. Though it should be mentioned her that: the Lattice of varieties, Reducts, Clone Algebras, see Grätzer (79), are branches in "standard" Universal Algebra which might be applicable to the present problem. Our reason for pointing this out is that there have been misinterpretations saying Lawvere's Algebraïc Theories were merely "new bottles for old wine".

Algebraïc Theories have been widely applied to the quoted Computer Science problems and from these applications the theory itself benefited considerably. Some of the references are: Elgot (71),(75), Tiuryn (79a), Wand (75a),(77a), Goguen-Burstall (78), Wagner et al (77),(76), and other works of the ADJ team. The field is too extensive to make proper references here: our references are samples chosen in a random manner.

Burstall and Goguen have made a distinction here which is worth of emphasizing. Namely: Stepwise refinement of programs and stepwise refinement of specifications are two different matters which need different tools. The former needs Rational Algebraïc Theories, ADJ (76), or iterative ones, Elgot (75), while the second needs only plain Algebraïc Theories. In program specifications, in their stepwise refinement, etc..., no algorithms are involved. Specifications are only declarative statements. Hence in the theory (or foundations) of Specifications no algorithmic or iterative notions are needed.

Specifications are important in themselves. Specifications are answers to "what" while programs are answers to "how". Sometimes it is more important to understand clearly what we are trying to do than to understand how we are trying to do it. It is valuable to be aware of what one really does intend to do and what one does not (and what one just happens to do without really intending).

An autonomous theory of specifications is badly needed. An approach to that is Burstall-Goguen (77),(79), Goguen-Burstall (78),(80),(79), Mosses (79) etc...

(1.3) The category of all first-order theories and their morphisms can be treated naturally by using Cylindric Algebras, Németi-Sain (78). Here a theory morphism correspond to an homomorphism between two cylindric algebras. This correspondence works both ways. A representation theorem to this effect was formulated in Németi-Sain (78), and Sain (79b). See Thm. 5 in the present paper. The category of theories obtained this way is complete and cocomplete, thence "Theory Procedures" of Burstall-Goguen (77) do work in this setting too, see Thm. 4 in this paper. Here, the so-called "Regular Cylindric Algebras" are the main tool for handling all first order theories see Prop. 1 in Németi-Sain (78). For regular cylindric algebras see Henkin-Monk-Tarski (79), Andréka-Gergely-Németi (77), Németi-Sain (78). In the latter two references they were called "i-finite" instead of "regular". "Base homomorphisms" of cylindric set algebras do represent theory morphisms see Sain (79b), Andréka-Németi (79b).

By this kind of Algebraïc Logic one can go beyond classical first order logic and languages as was shown in Andréka-Gergely-Németi (77) and Németi-Sain (78).

In this paper we shall not explain in more detail than it was done in sec. 1 , 2 , how and why the category of all first order theories and theory morphisms is considered to be the basic device for stepwise refinement of program specifications. For detailed explanations the reader is kindly referred to e. g. Burstall-Goguen (77),(79) and Goguen-Burstall (78)-(80). However, if we keep in mind that algebraic theories (and their morphisms) are special cases of first order theories (and their morphisms) then reading sec. 1 , 2 again might be sufficient to see the point, i. e. to see that an appropriate study of the category of all first order theories and their morphisms is somehow a mathematical foundation for an autonomous theory of specifications (in the sense outlined in sec. 1 , 2).

2. The category of all first order theories.

Definition 1 (theories, concepts).

(i) By a similarity type t we understand a pair $t = \langle H, t' \rangle$ such that t' is a function $t' : Dom(t') \longrightarrow N$ where N is the set of natural numbers and $0 \notin N$, and $H \subseteq Dom(t')$. Let $\langle r,n \rangle \in t'$ (i. e. let $t'(r) = n$). If $r \in H$ then r is said to be an $(n-1)$-ary function symbol, if $r \notin H$ then r is said to be an n -ary relation symbol.

(ii) Throughout the paper $x : Ord \rightarrowtail Sets$ is a fixed one-one function defined on the class Ord of all ordinals. I. e. if $\alpha < \beta$ are two ordinals then $x_\alpha \neq x_\beta$. Let $\alpha \in Ord$ and let t be a similarity type. F_t^α denotes the set of all first order formulas with equality and with variables in $\{x_i : i < \alpha\}$, see e. g. Chang-Keisler (73), Monk (76).

(iii) Let α be an ordinal, $Ax \subseteq F_t^\alpha$ and $\varphi \in F_t^\alpha$.
The semantic consequence $Ax \models \varphi$ is defined the usual way, see e. g. the references in (ii).

(iv) By a <u>theory</u> T in α variables we understand a pair $T = \langle Ax, F_t^\alpha \rangle$ such that t is a similarity type and $Ax \subseteq F_t^\alpha$. See Def. 11.29. of Monk (76).

(v) Let T be a theory. Then the set C_T of the <u>concepts</u> of T is defined as follows. Let $T = \langle Ax, F_t^\alpha \rangle$. Then $\equiv_T \subseteq F_t \times F_t$ is defined by

$$(\forall \varphi, \psi \in F_t^\alpha)\left[\varphi \equiv_T \psi \iff A_x \models (\varphi \longleftrightarrow \psi)\right].$$

Now $C_T \overset{df}{\equiv} F_t^\alpha / \equiv_T$.

<u>End</u> of Definition 1.

<u>Convention.</u>

Throughout, α denotes an arbitrary ordinal.
Let i be a number.
Then $t_i = \langle H_i, t_i' \rangle$, $Ax_i \subseteq F_{t_i}^\alpha$, $T_i = \langle Ax_i, F_{t_i}^\alpha \rangle$.

<u>Definition</u> 2 (interpretations - cf. Monk (65), Def. 11.43, theory morphisms, concept-interpretations).

Let T_1 and T_2 be theories in α variables.

(i) Let $m : F_{t_1}^\alpha \longrightarrow F_{t_2}^\alpha$.
We define $\langle T_1, m, T_2 \rangle$ to be an <u>interpretation</u> going from T_1 into T_2 (or an interpretation of T_1 in T_2) iff conditions (a)-(c) below hold:

(a) $m(x_i = x_j) = (x_i = x_j)$ for every $i,j < \alpha$,

(b) $m(\varphi \wedge \psi) = m(\varphi) \wedge m(\psi)$,
$m(\neg \varphi) = \neg\, m(\varphi)$,
$m(\exists x_i \varphi) = \exists x_i\, m(\varphi)$,
for all $\varphi, \psi \in F_{t_1}^\alpha$ and $i < \alpha$,

(c) $Ax_2 \models m(\varphi)$ for all $\varphi \in F^{\alpha}_{t_1}$ such that $Ax_1 \models \varphi$.

We shall often say that m is an interpretation but in these cases we actually mean $< T_1, m, T_2 >$.

(ii) Let m, n be two interpretations of T_1 in T_2. The interpretations $< T_1, m, T_2 >$, $< T_1, n, T_2 >$ are defined to be <u>semantically equivalent</u>, in symbols $m \equiv n$, iff condition (a) below holds.

(a) $Ax_2 \models \left[m(\varphi) \longleftrightarrow n(\varphi) \right]$ for all $\varphi \in F^{\alpha}_{t_1}$.

(iii) Let $< T_1, m, T_2 >$ be an interpretation. We define the equivalence class $m/\equiv$ of m (or more precisely $< T_1, m, T_2 > / \equiv$) to be:

$m/\equiv \ \stackrel{df}{=} \ \{ < T_1, n, T_2 > \ : \ n \equiv m \}$.

(iv) By a <u>theory morphism</u> $\mu : T_1 \longrightarrow T_2$ going from T_1 into T_2 we understand an equivalence class of interpretations of T_1 in T_2, i. e. μ is a theory morphism $\mu : T_1 \longrightarrow T_2$ iff $\mu = m/\equiv$ for some interpretation $< T_1, m, T_2 >$.

<u>End</u> of Definition 2.

<u>Definition</u> 3 (presentations of theory morphisms).

Let T_1 and T_2 be two theories in α variables.

(i) By a <u>presentation</u> of interpretations from T_1 to T_2 we understand a mapping $p : t_1' \longrightarrow F^{\alpha}_{t_2}$.

(ii) The interpretations $< T_1, m, T_2 >$ satisfies the presentation $p : t_1' \longrightarrow F^{\alpha}_{t_2}$ iff for every $< r, n > \in t_1'$ conditions (a) and (b) below hold.

(a) If $r \in H_1$ then $m(r(x_0, \ldots, x_{n-2}) = x_{n-1}) = p(r, n)$.

(b) If $r \notin H_1$ then $m(r(x_0,\dots,x_{n-1})) \equiv p(r,n)$.

We define the theory morphism μ to satisfy the presentation p if $< T_1,m,T_2 >$ satisfies p for some $< T_1,m,T_2 > \in \mu$.

End of Definition 3.

Proposition 1 (presentations determine morphisms uniquely).

Let T_1 and T_2 be two theories in α variables.
Let $p: t_1' \longrightarrow F^{\alpha}_{t_2}$ be a presentation of interpretations from T_1 to T_2 .
Then there is at most one theory morphism which satisfies p .

QED

Definition 4.

(i) TH^{α} is defined to be the pair $TH^{\alpha} \overset{df}{=} < ObTH^{\alpha}, MorTH^{\alpha} >$ of classes where

$ObTH^{\alpha} \overset{df}{=} \{ T : T$ is a theory in α variables $\}$ (i. e.

$ObTH^{\alpha} = \{ < Ax, F^{\alpha}_t > :$ t is an arbitrary similarity type and $Ax \subseteq F^{\alpha}_t \}$)

$MorTH^{\alpha} \overset{df}{=} \{ < T_1,\mu,T_2 > : \mu$ is a theory morphism $\mu: T_1 \longrightarrow T_2$ and $T_1, T_2 \in ObTH^{\alpha} \}$.

(ii) Let $\mu: T_1 \longrightarrow T_2$ and $\nu: T_2 \longrightarrow T_3$ be two theory morphisms. We define the composition $\nu.\mu: T_1 \longrightarrow T_3$ to be the (unique) theory morphism for which

$(\exists m \in \mu, \exists n \in \nu)\ \nu.\mu = (n.m)/\equiv$

where the function $(n.m): F^{\alpha}_{t_1} \longrightarrow F^{\alpha}_{t_3}$ is defined by

$(n.m)(\varphi) = n(m(\varphi))$ for all $\varphi \in F^{\alpha}_{t_1}$.

(iii) Let $T = < Ax, F^{\alpha}_t >$ be a theory. The identity function $Id_{F^{\alpha}_t}$ is defined to be $Id_{F^{\alpha}_t} \overset{df}{=} \{ <\varphi,\varphi> : \varphi \in F^{\alpha}_t \}$.

The identity morphism Id_T on T is defined to be

$$\mathrm{Id}_T \overset{\mathrm{df}}{=} (\mathrm{Id}_{F_t^\alpha})/\equiv \ .$$

<u>End</u> of Definition 4.

From now on we shall use the basic notions of category theory, see Herrlich-Strecker (73).

<u>Proposition</u> 2.

TH^α <u>is a category with objects</u> $ObTH^\alpha$, <u>morphisms</u> $MorTH^\alpha$, <u>composition</u> $\nu . \mu$ <u>for any</u> $\mu, \nu \in MorTH^\alpha$ <u>and identity morphisms</u> Id_T <u>for all</u> $T \in ObTH^\alpha$.

<u>Proof</u>.

<u>Fact</u> 2.1. Let $\mu : T_1 \longrightarrow T_2$ be a theory morphism. Then there is a unique mapping $\bar{\mu} : C_{T_1} \longrightarrow C_{T_2}$ between the sets of concepts (of T_1 and T_2 resp.) such that

$(\forall m \in \mu)(\forall \varphi \in F_{t_1}^\alpha) \ \bar{\mu}(\varphi/\equiv_{T_1}) = m(\varphi)/\equiv_{T_2}$.

<u>Fact</u> 2.2. $< ObTH^\alpha$, Interpretations $>$ is a category with usual set theoretic composition $< T_1, (n.m), T_3 >$ of interpretations $< T_1, m, T_2 >$ and $< T_2, n, T_3 >$ and with identity $< T, \mathrm{Id}_{F_t^\alpha}, T >$ for every theory $T = < Ax, F_t^\alpha >$.

<u>QED</u>

Now we shall use the category theoretic notion of <u>isomorhism</u> between theories. Two theories T_1 and T_2 are defined to be <u>isomorphic</u> iff in the category TH^α the objects T_1 and T_2 are isomorphic in the category theoretical sense, i. e. iff there are two morphisms $\mu : T_1 \longrightarrow T_2$ and $\nu : T_2 \longrightarrow T_1$ such that $\mu . \nu = \mathrm{Id}_{T_2}$ and $\nu . \mu = \mathrm{Id}_{T_1}$.

Proposition 3.

Let $T = < Ax, F_t^{\alpha} >$ be an arbitrary theory with $t = < H, t' >$. Then there exists a theory $T^+ = < Ax^+, F_{t^+}^{\alpha} >$ such that (i)-(iv) below hold.

(i) $t^+ = < 0, t' >$ (i. e. t^+ is the same as t' except that there is no function symbol in t^+) where 0 is the empty set.

(ii) T and T^+ are isomorphic, i. e. $T \cong T^+$.

(iii) Let the presentation $p: t' \longrightarrow F_{t^+}^{\alpha}$ be defined as $p(r,n) \overset{df}{=} r(x_0, \ldots, x_{n-1})$ for every $< r,n > \in t'$. Then there is an isomorphism $\mu: T \longrightarrow T^+$ which satisfies p , i. e.

$$(\exists m \in \mu)(\forall < r,n > \in t')\; r(x_0, \ldots, x_{n-1}) = \begin{cases} m(r(x_0, \ldots, x_{n-2}) = x_{n-1} & \text{if } r \in H \\ m(r(x_0, \ldots, x_{n-1})) & \text{if } r \notin H \end{cases} .$$

(iv) The above interpretation $m: F_t^{\alpha} \longrightarrow F_{t^+}^{\alpha}$ is effective (is a computable function).

Proof.

The proof is based on Thm. 10.5 of Bell-Machover (77).

QED

To formulate Theorem 4 below we use the notion of completeness of categories in the sense of Herrlich-Strecker (73) or equivalently MacLane (71). For those unfamiliar with these notions we shall recall them after having formulated Thm. 4 .

Theorem 4.

(i) The category TH^{α} of all theories is complete and cocomplete.

(ii) There is an effective procedure to construct the limits and colimits of the effectively given diagrams in TH^{α} .

Before proving Thm. 4 we shall recall the basic notions used in its formulation, and shall try to illustrate its meaning.

By a small category we understand a category $\underline{C} = \langle Ob\underline{C}, Mor\underline{C} \rangle$ such that $Mor\underline{C}$ is a set.

Examples.

(1) Every partially ordered set is a small category: let $\langle P, \leq \rangle$ be a partial order, then $\langle P, \{\langle a,b \rangle \in P \times P : a \leq b\} \rangle$ is a small category.

(2) $\underline{P} \overset{df}{=}$
$$\begin{tikzcd} & 1 \arrow[loop right] \\ 0 \arrow[loop left] \arrow[ur, "a"] \arrow[dr, "b"'] & \\ & 2 \arrow[loop right] \end{tikzcd}$$
is a small category .

(3)
$$\begin{tikzcd} 0 \arrow[loop right] & 1 \arrow[loop right] \end{tikzcd}$$
is a small category.

(4) $\underline{E} \overset{df}{=}$
$$\begin{tikzcd} 0 \arrow[loop left] \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & 1 \arrow[loop right] \end{tikzcd}$$
is a small category.

End of examples.

By a diagram in the category TH^{α} we understand a functor $D: \underline{C} \longrightarrow TH^{\alpha}$ where $\underline{C}$ is a small category, and a functor is a pair $D = \langle D_0, D_1 \rangle$ of functions $D_0: Ob\underline{C} \longrightarrow ObTH^{\alpha}$, $D_1: Mor\underline{C} \longrightarrow MorTH^{\alpha}$ such that D preserves the basic structure of $\underline{C}$ i. e. (i)-(iii) below hold.

(i) If $f: A \longrightarrow B$ in $\underline{C}$, then $D_1(f): D_0(A) \longrightarrow D_0(B)$ in TH^{α} .

(ii) $D_1(f.g) = D_1(f).D_1(g)$ for all $f,g \in Mor\underline{C}$.

(iii) $D_1(Id_A) = Id_{D_0(A)}$ for all $A \in Ob\underline{C}$.

The category $\underline{C}$ is called the index category of the diagram D .

Examples.

(5) Let $\underline{P}$ be the category in Example (2). Let $D: \underline{P} \longrightarrow TH^{\alpha}$ be any functor. Let $T_i = D_0(i)$ for $i < 3$ and let $\mu \overset{df}{=} D_1(a)$, $\nu \overset{df}{=} D_1(b)$. By definition, T_0 , T_1 , T_2 are theories and μ, ν are theory morphisms. Then the diagram D is usually illustrated as

$$\begin{tikzcd} & T_1 \\ T_0 \arrow[ur, "\mu"] \arrow[dr, "\nu"'] & \\ & T_2 \end{tikzcd}$$

We do not indicate the identity morphisms $D_1(i,i)$, $i < 3$, if not needed.

The historical reason for calling these functors diagrams is the possibility of illustrating them as we have just done.

(6) In the above diagram we may choose $D_0(1) = D_0(2)$. Then we obtain

$$\begin{tikzcd} & T_1 \\ T_0 \arrow[ur, "\mu"] \arrow[dr, "\nu"'] & \\ & T_1 \end{tikzcd}$$

(7) If we replace the index category of the above diagram by $\underline{E}$ given in Example (4) then we obtain

$$\begin{tikzcd} T_0 \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\nu"'] & T_1 \end{tikzcd}$$

It is important to keep in mind that the diagrams (6) and (7) above are different !

End of Examples.

Definition 5 (cone, colimit, limit).

Let $D: \underline{I} \longrightarrow TH^{\alpha}$ be a diagram. Let $\underline{I} = \langle I, M \rangle$.

(i) A cone over D is a system $\langle \langle h_i : i \in I \rangle , T \rangle$ such that $T \in \mathrm{ObTH}^{\alpha}$, $(\forall i \in I)\ h_i \in \mathrm{MorTH}^{\alpha}$ and for every $f \in M$ if $f: i \longrightarrow j$ in $\underline{I}$ then $h_i = h_j . D_1(f)$ in TH^{α} .

(ii) The <u>colimit</u> of D in TH^{α} is a cone $<< g_i : i \in I > , G >$ over D such that for every cone $<< h_i : i \in I > , T >$ over D there is a unique morphism $\mu: G \longrightarrow T$ such that $(\forall i \in I)\ h_i = \mu . g_i$.

(iii) The <u>limit</u> of D is defined exactly as above but all the arrows are reversed.

<u>End</u> of Definition 5.

<u>Examples.</u>

(8) The colimit of the diagram
$$\begin{tikzcd} & T_1 \\ T_0 \arrow[ur, "\mu"] \arrow[dr, "\nu"'] & \\ & T_2 \end{tikzcd}$$
given in Example (5) is a theory T_3 and two theory morphisms γ , ρ such that

$$\begin{tikzcd} & T_1 \arrow[dr, "\gamma"] & \\ T_0 \arrow[ur, "\mu"] \arrow[dr, "\nu"'] & & T_3 \\ & T_2 \arrow[ur, "\rho"'] & \end{tikzcd}$$

and $\gamma . \mu = \rho . \nu$, and for any other T_4 and β , ∂ if $\beta . \mu = \partial . \nu$ then there is a unique $\pi: T_3 \longrightarrow T_4$ such that $\pi . \gamma = \pi . \rho$

$$\begin{tikzcd} & T_1 \arrow[dr, "\gamma"] \arrow[drr, bend left, "\beta"] & & \\ T_0 \arrow[ur, "\mu"] \arrow[dr, "\nu"'] & & T_3 \arrow[r, "\pi"] & T_4 \\ & T_2 \arrow[ur, "\rho"'] \arrow[urr, bend right, "\partial"'] & & \end{tikzcd}$$

(9) The colimit of the diagram in Example (7) is a <u>single</u> theory morphism γ and a theory T_3 such that

$$\begin{tikzcd} T_0 \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\nu"'] & T_1 \arrow[r] & T_3 \end{tikzcd}$$

and $\gamma . \mu = \gamma . \nu$ etc...

Note that the difference between diagrams (6) and (7) shows itself in that in the colimit of (6) there are two morphisms γ, ρ from T_1 into T_3 while in the colimit of (7) there is only one.

(10) Let the index category $\underline{I}$ of the diagram D be the one in Example (3). Then our diagram D has the shape

$$T_0 \qquad\qquad T_1$$

I. e. no morphisms except the identity ones.
The limit of this diagram is a theory T_2 and two morphisms π_0 , π_1 such that

$$\begin{tikzcd}
 & T_2 \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
T_0 & & T_1
\end{tikzcd}$$

anf for every similar cocone $< T_3, \beta, \partial >$ there is a unique morphism $\mu: T_3 \longrightarrow T_2$ such that $\beta = \pi_0 \cdot \mu$ and $\partial = \pi_1 \cdot \mu$.

$$\begin{tikzcd}
 & T_3 \arrow[d, "\mu"] \arrow[ddl, bend right, "\beta"'] \arrow[ddr, bend left, "\partial"] & \\
 & T_2 \arrow[dl, "\pi_0"] \arrow[dr, "\pi_1"'] & \\
T_0 & & T_1
\end{tikzcd}$$

End of Examples.

Limits of the kind of Example (10) are called products and are denoted by $T_0 \times T_1$. If T_0 and T_1 are two theories then

$$\begin{tikzcd}
 & T_0 \times T_1 \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
T_0 & & T_1
\end{tikzcd}$$

is defined to be the limit in Example (10) and is called the product of T_0 and T_1. The morphisms π_0 and π_1 are called the projections of the product.

Definition 6.

A category $\underline{K}$ is said to be complete and cocomplete if for every small category $\underline{I}$ and for every diagram (i. e. functor) $D: \underline{I} \longrightarrow \underline{K}$ both the limit and the colimit of D exixt in $\underline{K}$.

End of Definition 6.

By the aboves we see that Theorem 4 says that every diagram $D: \underline{I} \longrightarrow TH^{\alpha}$ has both limit and colimit in TH^{α}. I. e. in the category TH^{α} of all theories all possible limits and colimits exist (and can be constructed).

Examples.

(14) Let $T_0 \stackrel{df}{=} \langle \emptyset, F^{\omega}_{t_0} \rangle$, $T_1 \stackrel{df}{=} \langle Ax_1, F^{\omega}_{t_1} \rangle$, where $t_0 = \langle \emptyset, \{ \langle R,2 \rangle \} \rangle$, $t_1 = \langle \{+\}, \{ \langle +, 3 \rangle \} \rangle$ and $Ax_1 \stackrel{df}{=} \{ (x_0+x_0 = x_0), ((x_0+x_1)+x_2=x_0+(x_1+x_2)), x_0+x_1=x_1+x_0 \}$. Let $\mu : T_0 \longrightarrow T_1$ and $\nu : T_0 \longrightarrow T_1$ be two theory morphisms such that for some $m \in \mu$ and $n \in \nu$ we have

$m(R(x_0,x_1)) = x_0+x_1=x_1$ and $n(R(x_0,x_1) = x_0+x_1=x_0$.

(a) Consider the diagram

$$\begin{tikzcd} & T_0 \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ T_1 & & T_1 \end{tikzcd}$$

The colimit of this diagram is

$$\begin{tikzcd} & T_0 \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ T_1 \arrow[dr, "\rho"'] & & T_1 \arrow[dl, "\partial"] \\ & T_2 & \end{tikzcd}$$

where T_2 = "Lattice theory" , i. e.

$T_2 = < Ax_2, F^{\omega}_{t_2} >$, where $t_2 = < \{+, .\}, \{< +,3 >, < .,3 >\} >$

and

$$Ax_2 = \{(x_0+(x_0.x_1)=x_0), (x_0.(x_0+x_1)=x_0)\}$$
$$\cup \{(x_0.x_0=x_0), ((x_0.x_1).x_2=x_0.(x_1.x_2)), (x_0.x_1=x_1.x_0)\}$$
$$\cup Ax_1 \quad .$$

ρ and ∂ are such that $r(x_0+x_1=x_2) = x_0+x_1=x_2$ and $d(x_0+x_1=x_2) = x_0.x_1=x_2$ for some $r \in \rho$ and $d \in \partial$.

(b) Consider the diagram

$$T_0 \underset{\nu}{\overset{\mu}{\rightrightarrows}} T_1$$

The colimit of this diagram is

$$T_0 \underset{\nu}{\overset{\mu}{\rightrightarrows}} T_1 \xrightarrow{\rho} T_2$$

where $T_2 = < Ax_2, F^{\omega}_{t_1} >$ and

$$Ax_2 = \{x_0+x_1=x_1+x_0, (x_0+x_1)+x_2=x_0+(x_1+x_2), x_0+x_0=x_0,$$
$$x_0+x_1=x_0 \longrightarrow x_1=x_0 \}$$

<u>Proof.</u>

<u>Proof of</u> (a).

1). Proof of $\rho.\mu = \partial.\nu$:

$r(m(R(x_0,x_1))) = r(x_0+x_1=x_1) = x_0+x_1=x_1$,

$d(n(R(x_0,x_1))) = d(x_0+x_1=x_0) = x_0 . x_1=x_0$.

We have to prove $r.m = d.n$, i. e. we have to show $(x_0+x_1=x_1)/\equiv_{T_2} = (x_0 . x_1=x_0)/\equiv_{T_2}$, i. e. that $Ax_2 \models (x_0+x_1=x_1 \longleftrightarrow x_0 . x_1=x_0)$.

Suppose $x_0+x_1=x_1$. Then $x_0 . x_1=x_0 . (x_0+x_1)=x_0$ by $(x_0 . (x_0+x_1)=x_0) \in Ax_2$.

We obtain $Ax_2 \models (x_0 . x_1=x_0 \longrightarrow x_0+x_1=x_1)$ similarly.

2). Suppose $\rho' . \mu = \partial' . \nu$. We have to show $\pi . \rho = \rho'$ and $\pi . \partial = \partial'$ for some theory morphism π .

$$\begin{tikzcd}
 & T_0 \arrow[dl, "\nu"'] \arrow[dr, "\mu"] & \\
T_1 \arrow[dr, "\partial"] \arrow[ddr, "\partial'"'] & & T_1 \arrow[dl, "\rho"'] \arrow[ddl, "\rho'"] \\
 & T_2 \arrow[d, "\pi"] & \\
 & T_2' = \langle Ax_2', T^{\omega}_{t_2'} \rangle &
\end{tikzcd}$$

Let $r' \in \rho'$ and $d' \in \partial'$.

$Ax_2' \models (r'(x_0+x_1=x_1) \longleftrightarrow d'(x_0+x_1=x_0))$ by $\rho' . \mu = \partial' . \nu$.

Let $p(x_0+x_1=x_2) \overset{df}{=} r'(x_0+x_1=x_2)$ and

$p(x_0 . x_1=x_2) \overset{df}{=} d'(x_0+x_1=x_2)$.

We have to show that p determines a theory morphism $\pi : T_2 \longrightarrow T_2'$. I. e. we have to show that $(\forall \varphi \in Ax_2)\ Ax_2' \models p(\varphi)$.

Notation: $r'(+) = \oplus$, $d'(+) = \otimes$.

We know that

$$Ax_2' \models \{x_0 \oplus x_0=x_0, (x_0 \oplus x_1) \oplus x_2=x_0 \oplus (x_1 \oplus x_2), \\ x_0 \oplus x_1=x_1 \longleftrightarrow x_0 \otimes x_1=x_0 \} .$$

Now $p(x_0+(x_0 . x_1)=x_0) = x_0 \oplus (x_0 \otimes x_1)=x_0$.

We have to show $Ax_2^! \models x_0 \oplus (x_0 \otimes x_1) = x_0$.
$x_0 \oplus (x_0 \oplus x_1) = (x_0 \oplus x_0) \oplus x_1 = x_0 \oplus x_1$ and therefore
$x_0 \otimes (x_0 \oplus x_1) = x_0$.
Similarly for the other elements of Ax_2 .

Proof of (b).

The proof is based on the fact that
$Th(Ax_2) = Th(Ax_1 \cup \{x_0+x_1=x_0 \longleftrightarrow x_0+x_1=x_1\})$.

QED

End of Examples.

Definition 7.

Let α be any ordinal. The class Lf_α of Cylindric Algebras was defined in the monograph Henkin-Monk-Tarski (71). Here Lf_α is considered a category the usual way:

The objects of Lf_α are the cylindric algebras Lf_α and the morphisms of Lf_α are the homomorphisms between algebras in Lf_α .

In more detail:

Lf_α is defined the following way.
First a similarity type $\mathcal{C}_\alpha$ of algebras is fixed with $\alpha+2$ operations symbols. Then 8 simple schemes C0–C7 of equations are postulated. Then CA_α is defined to be the class of all algebras of type $\mathcal{C}_\alpha$ satisfying the equations C0–C7 . Clearly CA_α is a variety.
Then $Lf_\alpha \subsetneq CA_\alpha$ is defined as follows:
Let $\mathfrak{A} \in CA_\alpha$. Then $\mathfrak{A} \in Lf_\alpha$ iff
$(\forall x \in A)$ $\{f \in Dom(\mathcal{C}_\alpha^!) : \mathcal{C}_\alpha^!(f) > 0$ and $f^{\mathfrak{A}}(x,\dots,x) \neq x\}$ is finite .
I. e. $\mathfrak{A} \in Lf_\alpha$ iff
$(\forall x \in A)$ [the number of those basic operations of $\mathfrak{A}$ which x is not a fixed point is finite] .

I. e. $\mathfrak{A} \in Lf_\alpha$ iff

$(\forall x \in A)$ [x is a fixed point of every basic operation of $\mathfrak{A}$ with finitely many exceptions] .

By this the class Lf_α of algebras has been defined.

The category $\underline{Lf}_\alpha$ is the usual one, namely:

$Ob\underline{Lf}_\alpha \stackrel{df}{=} Lf_\alpha$

and

$Mor\underline{Lf}_\alpha \stackrel{df}{=}$ All the homomorphisms between the algebras in Lf_α in the usual sense.

End of Definition 7.

From the above definition it should be clear that the definition of $\underline{Lf}_\alpha$ is much simpler than that of TH^α , at least for certain kinds of investigations. Certainly the standard tools of abstract algebra and universal algebra can be directly applied to Lf_α .

Theorem 5.

Let α be an arbitrary ordinal. Assume $\alpha \geq \omega$. Then the categories TH^α and $\underline{Lf}_\alpha$ are isomorphic in the sense of MacLane (71).

QED of Theorem 5.

Parts and some semantic aspects of the above Theorem 5 are illustrated and proved in Németi-Sain (78). See also Andréka-Gergely-Németi (77),(80).

Theorem 5 above can be used to apply the theory of Cylindric Algebras (CA-theory) to investigate the category TH^α of theories and theory morphisms. Hence CA-theory can be used as a theoretical foundation for stepwise refinement of program specifications along the lines outlined e. g. in Burstall-Goguen (77),(79), Goguen-Burstall (78)-(80).

In the remaining part of this paper we shall concentrate on the category TH^{α} of all first order theories and interpretations between them. We have seen that TH^{α} is isomorphic with the category $\underline{Lf}_{\alpha}$ of all locally finite cylindric algebras and their homomorphisms. Lf_{α} can be defined in a clear abstract style and nice representation theorems simplify our work with Lf_{α} . We shall prove that $\underline{Lf}_{\alpha}$ and TH^{α} are strongly algebroïdal categories (in the sense of e. g. Banaschewski-Herrlich (76), Andréka-Németi (79)) and they have enough projectives, are complete and cocomplete etc... Hence the results of category theoretic Universal Algebra in the sense of e. g. Andréka-Németi (79),(78), Németi-Sain (77) can be applied to TH^{α} and $\underline{Lf}_{\alpha}$. The strongly small objects (or compact objects) and projective objects of TH^{α} and $\underline{Lf}_{\alpha}$ will be characterised. TH^{α} is a locally finitely presented category in the sense of Gabriel-Ulmer. Specially ultraproducts exist in TH^{α} and $\underline{Lf}_{\alpha}$. It turns out that ultraproducts of theories (objects of TH^{α}) commute with ultraproducts of their models. I. e. let $\langle T_i : i \in I \rangle$ be a system of theories and F be an ultrafilter on I . Let T be the ultraproduct of $\langle T_i : i \in I \rangle$ modulo F in the category TH^{α} . Then a structure $\mathfrak{A}$ is a model of the ultraproduct theory T iff $\mathfrak{A}$ is an ultraproduct of some models $\langle \mathfrak{A}_i : i \in I \rangle$ modulo F such that $(\forall i \in I)\ [\mathfrak{A}_i$ is a model of the theory $T_i]$. This result extends to reduced products if and only if the cylindric algebras Lf_{α} are replaced with cylindric (meet) semilattices.

Parts of the results reported here are joint resuls of the author with H. Andréka and T. Gergely.

References.

ADJ (76): J. A. Goguen, J. W. Thatcher, E. G. Wagner, J. B. Wright, Rational Algebraic Theories and Fixed-Point Solutions, Proc. IEEE 17th Symp. on Foundations of Computer Science (Houston, Texas), 1976, pp. 147-158.

H. Andréka, T. Gergely, I. Németi (72): Hierarchy of languages for A.I. , Hungarian Pub. of Central Res. Inst. Phys. Budapest, No KFKI-72-46, 1972.

H. Andréka, T. Gergely, I. Németi (77): On Universal Algebraïc Construction of Logics, Studia Logica XXXVI,1-2 , 1977, pp. 9-47.

H. Andréka, T. Gergely, I. Németi (80): Model theoretic semantics for many-purpose languages and language hierarchies, Proc. of COLING'80 Tokio, 1980 .

H. Andréka, I. Németi (78): Łoś lemma holds in every category, Studia Sci. Math. Hung., To appear.

H. Andréka, I. Németi (79) Formulas and Ultraproducts in Categories, Beiträge z. Algebra u. Geom. , 1979, pp. 133-151.

H. Andréka, I. Németi (79a): Néhány magyarorszagi kutatás a számitástudomány matematikai alapjai terén, Proc. of the "NJSZT Első Országos Kongresszusa", 1979.

H. Andréka, I. Németi (79b): Base homomorphisms of generalized cylindric set algebras, Manuscript, 1979.

B. Banaschewski, H. Herrlich (76): Subcategories defined by implications, Houston J. of Math., 1976/2 , pp. 149-171.

J. L. Bell, M. Machover (77): A course in Mathematical Logic, North-Holland, 1977.

E. K. Blum, D. R. Estes (77): A generalization of the homomorphism concept, Alg. Univ. 7 , 1977, pp. 143-161.

E. K. Blum, N. A. Lynch (79a): A difference in expressive power between flowcharts and recursive schemes, Math. Syst. Theory 12, 1979, pp. 205-211.

E. K. Blum, K. A. Lynch (79b): Relative complexity operation sets for numeric and bit string algebras, To appear in Math. Syst. Theory.

R. M. Burstall, J. A. Goguen (77): Putting theories together to make specifications, Proc. of 5th Int. Joint Conf. on Artificial Intelligence, MIT, Cambridge, Mass. , 1977, pp. 1045-1058.

R. M. Burstall, J. A. Goguen (79): Some Fundamental Properties of Algebraic Theories: A Tool for Semantics of Computation, accepted by Theoretical Computer Science.

C. C. Chang, H. J. Keisler (73): Model Theory, North-Holand, 1973.

B. Dömölki (79): An example of hierarchical program specification, to appear in Proc. Abstract Software Specifications, Jan. 1979, Copenhagen, Ed. D. Bjørner and C. Jones.

C. C. Elgot (71): Algebraic theories and program schemes, Symp. on Semantics of Algorithmic Languages, ed. E. Engeler, Springer-Verlag, 1971, pp. 71-88.

C. C. Elgot (75): Monadic computation and iterative algebraic theories, IBM Res. Report RC 4564, Oct. 1973 and Proc. of Logic Coll.'73, North-Holland, 1975, p. 230.

J. A. Goguen, R. M. Burstall (78): Some fundamental properties of algebraic theories: A tool for semantics of computation, preprint 1978, submitted to Theor. Comput. Sci. .

J. A. Goguen, R. M. Burstall (79): CAT, a system for the structured elaboration of correct programs from structured specifications, in preparation.

J. A. Goguen, R. M. Burstall (80): The semantics of CLEAR, a specification language, to appear in Proc. of 1979 Copenhagen Winter School on Abstract Software Specifications.

G. Grätzer (79): Universal Algebra, 2nd Ed., Springer-Verlag, 1979.

L. Henkin, J. D. Monk, A. Tarski (71): Cylindric Algebras Part I, North-Holland, 1971.

L. Henkin, J. D. Monk, A. Tarski (79): Cylindric set algebras and related structures I , submitted to Alg. Univ. .

H. Herrlich, G. E. Strecker (73): Category Theory, Allyn and Bacon Inc. , Boston, 1973.

S. MacLane (71): Categories for the Working Mathematician, Springer-Verlag, 1971.

J. McCarthy, P. J. Hayes (69): Some philosophical problems from the standpoint of artificial intelligence, Machine Intelligence 4, 1969.

J. D. Monk (76): Mathematical Logic, Springer-Verlag, 1976.

P. Mosses (79): Modular Denotational Semantics, Rep. of the 2nd Int. Workshop on the Semantics of Programming Languages, presented in the Rundbrief d. Fachgruppe Künstliche Intelligenz in der Gesellschaft für Informatik, No. 17 , 1979, pp. 50-51, pub. P. Raufels (Instit. für Informatik III, Univ. Bonn, Kurtfürstenstr. 74, 5300 Bonn 1).

I. Németi, I. Sain (77): Cone-injectivity and some Birkhoff-type theorems in categories, contributions to Universal Algebra, Proc. Coll. Esztergom, 1977, Colloq. Math. Soc. J. Bolyai, Noth-Holland, 1981.

I. Németi, I. Sain (78): Connections between Algebraïc Logic and Initial Algebra Semantics of CF Languages, to appear in Proc. Coll. Logic in Programming, Salgótarján, 1978, Colloq. Math. Soc. J. Bolyai, North-Holland.

C. M. I. Rattray, T. Rus (77): Hash-Hierarchy, a mathematical device for computer system modelling, Proc. 1st Symp. on Math. Modelling, Missouri, 1977, pp. 1-15.

I. Sain (79b): Theories, Theory Morphisms and Cylindric Algebras, Manuscript, 1979.

J. Tiuryn (79a): Connections between regular algebras and rational algebraïc theories, Proc. 2nd Workshop categorical and algebraïc methods in comp. sci. and syst. theory, Dortmund, 1979.

E. G. Wagner, J. B. Wright, J. A. Goguen, J. W. Thatcher (76):

Some fundamentals of order-algebraïc semantics, IBM Res. Rep. RC 6020, June 1976 and Proc. of 5th Int. Symp. on Math. Found. of Comp. Sci. , Gdansk, Poland, LNCS 45, pp. 153-168.

E. G. Wagner, J. W. Thatcher, J. B. Wright (77): Free continuous theories, IBM Th. Watson Res. Cent., RC 6906, 1977.

M. Wand (75a): Fixed-point constructions in orderenriched categories, Techn. Rep. 23, Comp. Sci. Dept., Indiana Univ., Apr. 1975, to appear in Theor. Comput. Sci., 1979.

M. Wand (77a): Algebraïc theories and tree rewriting systems, Techn. Rep. No. 66, Comp. Sci. Dept., Indiana Univ., Bloomington, Indiana 47401, 1977.

J. Winkowski (78): Towards an understanding of computer simulation, Ann. Soc. Math. Pol., Series IV: Fundamenta Informaticae I, 1978, pp. 277-289.