

# DIAGRAMMES

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**Foundations for stepwise refinement of program specifications  
via cylindric algebra theory**

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Foundations for  
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via  
Cylindric Algebra Theory

I. Németi

1. Introduction.

To investigate connections between different theories formulated in completely different languages is a problem in many branches of computer science , e. g. in structured programming, in structuring program specifications, Burstall-Goguen (77),(79), Goguen-Burstall (78),(80),(79), Dömölki (79), Mosses (79), in data types and semantics, Blum-Estes (77), in A. I. , Andréka-Gergely-Németi (72), McCarthy-Hayes (69), Andréka-Németi (79a) etc... The connections between the different theories and languages can be called interpretations or translations but translations would suggest something much simpler than the thing we have in mind. Most often they are called Theory Morphisms. One point to be stressed is that between two theories there are usually many theory morphisms. The subject of investigation here is actually a category consisting of theories and their morphisms. The notion of a theory morphism from the theory  $T$  into the theory  $T'$  was defined for example in Winkowski (78) §1 (p. 277) and there it was called a modelling  $\mu$  of the theory  $T$  in the other one  $T'$  . In that paper theory morphisms are used to study Computer Simulation.

There is a related branch of "standard" Universal Algebra called Lattice of Varieties, see Grätzer (79) p. 389. That lattice

is a special subcategory of the Category of Theories considered here. For the purposes reported here, it is too special for two reasons:

1. The main point in the presently reviewed field is that the theories are of different similarity types.
2. Between two theories there are many morphisms.

(In some Computer Science papers the Category of Theories was called "Hierarchy of Languages" to emphasize that the underlying languages, similarity types, or even logics are usually different, see Andréka-Gergely-Német (72), Rattray-Rus (77), Andréka-Németi (79a), Sain (79b).)

Here we quote three approaches which are complementary (and are most useful when applied together).

(1.1) Let the theories  $Th_1$  and  $Th_2$  be equational but possibly heterogenous (many sorted). Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be the countably generated free algebras of the varieties  $Mod(Th_1)$  and  $Mod(Th_2)$  respectively. Then a suitable homomorphism  $h: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$  could establish a connection between the two theories i. e.

between the two varieties. The problem is that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are usually of completely different similarity types ! Hence the notion of a homomorphism between them is just meaningless.

To alleviate this problem, Blum-Estes (77) generalized the usual notion of homomorphism to be defined between algebras of different similarity types. One definition could be to say that  $f: \mathcal{A} \longrightarrow \mathcal{B}$  is a generalized homomorphism if  $f$  takes every term function of  $\mathcal{A}$  into some term function of  $\mathcal{B}$ . That is, the image of an operation of  $\mathcal{A}$  is required to be a function definable in  $\mathcal{B}$ . One can then make restrictions (or generalizations) on the notion of definability used e. g. term-definable, first-order definable, implicitly definable by first order formulas etc... Then the Theory of Definability which is a branch of Model Theory, see for example the Chang-Keisler monograph, can be used as a guide

for choosing the notion with the desired properties from a fairly broad spectrum of existing and well understood ones. For some varying choices see Blum-Estes (77), Blum-Lynch (79a),(79b).

(1.2) One of the main aims of Algebraic Theories of Lawvere is to deal with the present problem. I. e. the aim is to investigate the "structure" or "system" consisting of several theories of several similarity types and several possible "interpretations" (i. e. connections) between these theories and languages. Hence this is not a mere translation of Universal Algebra into category theoretical language, but instead this is an approach to a problem inherent both in Universal Algebra and in Model Theory: to a problem which has not been attacked in "standard" Universal Algebra or Model Theory yet. Though it should be mentioned here that: the Lattice of varieties, Reducts, Clone Algebras, see Grätzer (79), are branches in "standard" Universal Algebra which might be applicable to the present problem. Our reason for pointing this out is that there have been misinterpretations saying Lawvere's Algebraic Theories were merely "new bottles for old wine".

Algebraic Theories have been widely applied to the quoted Computer Science problems and from these applications the theory itself benefited considerably. Some of the references are: Elgot (71),(75), Tiuryn (79a), Wand (75a),(77a), Goguen-Burstall (78), Wagner et al (77),(76), and other works of the ADJ team. The field is too extensive to make proper references here: our references are samples chosen in a random manner.

Burstall and Goguen have made a distinction here which is worth of emphasizing. Namely: Stepwise refinement of programs and stepwise refinement of specifications are two different matters which need different tools. The former needs Rational Algebraic Theories, ADJ (76), or iterative ones, Elgot (75), while the second needs only plain Algebraic Theories. In program specifications, in their stepwise refinement, etc..., no algorithms are involved. Specifications are only declarative statements. Hence in the

theory (or foundations) of Specifications no algorithmic or iterative notions are needed.

Specifications are important in themselves. Specifications are answers to "what" while programs are answers to "how". Sometimes it is more important to understand clearly what we are trying to do than to understand how we are trying to do it. It is valuable to be aware of what one really does intend to do and what one does not (and what one just happens to do without really intending).

An autonomous theory of specifications is badly needed. An approach to that is Burstall-Goguen (77),(79), Goguen-Burstall (78),(80),(79), Mosses (79) etc...

(1.3) The category of all first-order theories and their morphisms can be treated naturally by using Cylindric Algebras, Néméti-Sain (78). Here a theory morphism correspond to an homomorphism between two cylindric algebras. This correspondence works both ways. A representation theorem to this effect was formulated in Néméti-Sain (78), and Sain (79b). See Thm. 5 in the present paper. The category of theories obtained this way is complete and cocomplete, thence "Theory Procedures" of Burstall-Goguen (77) do work in this setting too, see Thm. 4 in this paper. Here, the so-called "Regular Cylindric Algebras" are the main tool for handling all first order theories see Prop. 1 in Néméti-Sain (78). For regular cylindric algebras see Henkin-Monk-Tarski (79), Andréka-Gergely-Néméti (77), Néméti-Sain (78). In the latter two references they were called "i-finite" instead of "regular". "Base homomorphisms" of cylindric set algebras do represent theory morphisms see Sain (79b), Andréka-Néméti (79b).

By this kind of Algebraic Logic one can go beyond classical first order logic and languages as was shown in Andréka-Gergely-Néméti (77) and Néméti-Sain (78).

In this paper we shall not explain in more detail than it was done in sec. 1 , 2 , how and why the category of all first

order theories and theory morphisms is considered to be the basic device for stepwise refinement of program specifications. For detailed explanations the reader is kindly referred to e. g. Burstall-Goguen (77),(79) and Goguen-Burstall (78)-(80). However, if we keep in mind that algebraic theories (and their morphisms) are special cases of first order theories (and their morphisms) then reading sec. 1 , 2 again might be sufficient to see the point, i. e. to see that an appropriate study of the category of all first order theories and their morphisms is somehow a mathematical foundation for an autonomous theory of specifications (in the sense outlined in sec. 1 , 2 ).

## 2. The category of all first order theories.

Definition 1 (theories, concepts).

- (i) By a similarity type  $t$  we understand a pair  $t = \langle H, t' \rangle$  such that  $t'$  is a function  $t': \text{Dom}(t') \longrightarrow N$  where  $N$  is the set of natural numbers and  $0 \notin N$ , and  $H \subseteq \text{Dom}(t')$ . Let  $\langle r, n \rangle \in t'$  (i. e. let  $t'(r) = n$ ). If  $r \in H$  then  $r$  is said to be an  $(n - 1)$ -ary function symbol, if  $r \notin H$  then  $r$  is said to be an  $n$ -ary relation symbol.
- (ii) Throughout the paper  $\alpha: \text{Ord} \longrightarrow \text{Sets}$  is a fixed one-one function defined on the class  $\text{Ord}$  of all ordinals. I. e. if  $\alpha < \beta$  are two ordinals then  $x_\alpha \neq x_\beta$ . Let  $\alpha \in \text{Ord}$  and let  $t$  be a similarity type.  $F_t^\alpha$  denotes the set of all first order formulas with equality and with variables in  $\{x_i : i < \alpha\}$ , see e. g. Chang-Keisler (73), Monk (76).

(iii) Let  $\alpha$  be an ordinal,  $Ax \subseteq F_t^\alpha$  and  $\Psi \in F_t^\alpha$ .

The semantic consequence  $Ax \models \Psi$  is defined the usual way, see e. g. the references in (ii).

(iv) By a theory  $T$  in  $\alpha$  variables we understand a pair  $T = \langle Ax, F_t^\alpha \rangle$  such that  $t$  is a similarity type and  $Ax \subseteq F_t^\alpha$ . See Def. 11.29. of Monk (76).

(v) Let  $T$  be a theory. Then the set  $C_T$  of the concepts of  $T$  is defined as follows. Let  $T = \langle Ax, F_t^\alpha \rangle$ . Then

$\equiv_T \subseteq F_t \times F_t$  is defined by

$$(\forall \varphi, \psi \in F_t^\alpha) \left[ \varphi \equiv_T \psi \iff Ax \models (\varphi \longleftrightarrow \psi) \right].$$

Now  $C_T \stackrel{\text{df}}{=} F_t^\alpha / \equiv_T$ .

End of Definition 1.

Convention.

Throughout,  $\alpha$  denotes an arbitrary ordinal.

Let  $i$  be a number.

Then  $t_i = \langle H_i, t_i \rangle$ ,  $Ax_i \subseteq F_{t_i}^\alpha$ ,  $T_i = \langle Ax_i, F_{t_i}^\alpha \rangle$ .

Definition 2 (interpretations - cf. Monk (65), Def. 11.43, theory morphisms, concept-interpretations).

Let  $T_1$  and  $T_2$  be theories in  $\alpha$  variables.

(i) Let  $m : F_{t_1}^\alpha \longrightarrow F_{t_2}^\alpha$ .

We define  $\langle T_1, m, T_2 \rangle$  to be an interpretation going from  $T_1$  into  $T_2$  (or an interpretation of  $T_1$  in  $T_2$ ) iff conditions (a)-(c) below hold:

$$(a) \quad m(x_i = x_j) = (x_i = x_j) \quad \text{for every } i, j < \alpha,$$

$$(b) \quad m(\varphi \wedge \psi) = m(\varphi) \wedge m(\psi),$$

$$m(\neg \varphi) = \neg m(\varphi),$$

$$m(\exists x_i \varphi) = \exists x_i m(\varphi),$$

for all  $\varphi, \psi \in F_{t_1}^\alpha$  and  $i < \alpha$ ,

(c)  $Ax_2 \models m(\Psi)$  for all  $\Psi \in F_{t_1}^\alpha$  such that  $Ax_1 \models \Psi$ .

We shall often say that  $m$  is an interpretation but in these cases we actually mean  $\langle T_1, m, T_2 \rangle$ .

(ii) Let  $m, n$  be two interpretations of  $T_1$  in  $T_2$ . The interpretations  $\langle T_1, m, T_2 \rangle$ ,  $\langle T_1, n, T_2 \rangle$  are defined to be semantically equivalent, in symbols  $m \equiv n$ , iff condition (a) below holds.

(a)  $Ax_2 \models \left[ m(\Psi) \longleftrightarrow n(\Psi) \right]$  for all  $\Psi \in F_{t_1}^\alpha$ .

(iii) Let  $\langle T_1, m, T_2 \rangle$  be an interpretation.

We define the equivalence class  $m/\equiv$  of  $m$  (or more precisely  $\langle T_1, m, T_2 \rangle / \equiv$ ) to be:

$m/\equiv \stackrel{\text{df}}{=} \{ \langle T_1, n, T_2 \rangle : n \equiv m \}$ .

(iv) By a theory morphism  $\mu: T_1 \longrightarrow T_2$  going from  $T_1$  into  $T_2$  we understand an equivalence class of interpretations of  $T_1$  in  $T_2$ , i. e.  $\mu$  is a theory morphism  $\mu: T_1 \longrightarrow T_2$  iff  $\mu = m/\equiv$  for some interpretation  $\langle T_1, m, T_2 \rangle$ .

End of Definition 2.

Definition 3 (presentations of theory morphisms).

Let  $T_1$  and  $T_2$  be two theories in  $\alpha$  variables.

(i) By a presentation of interpretations from  $T_1$  to  $T_2$  we understand a mapping  $p: t_1^! \longrightarrow F_{t_2}^\alpha$ .

(ii) The interpretations  $\langle T_1, m, T_2 \rangle$  satisfies the presentation  $p: t_1^! \longrightarrow F_{t_2}^\alpha$  iff for every  $\langle r, n \rangle \in t_1^!$  conditions

(a) and (b) below hold.

(a) If  $r \in H_1$  then  $m(r(x_0, \dots, x_{n-2}) = x_{n-1}) = p(r, n)$ .

(b) If  $r \notin H_1$  then  $m(r(x_0, \dots, x_{n-1})) = p(r, n)$ .

We define the theory morphism  $\mu$  to satisfy the presentation  $p$  if  $\langle T_1, m, T_2 \rangle$  satisfies  $p$  for some  $\langle T_1, m, T_2 \rangle \in \mu$ .

End of Definition 3.

Proposition 1 (presentations determine morphisms uniquely).

Let  $T_1$  and  $T_2$  be two theories in  $\alpha$  variables.

Let  $p: t_1^! \longrightarrow F_{t_2}^\alpha$  be a presentation of interpretations from

$T_1$  to  $T_2$ .

Then there is at most one theory morphism which satisfies  $p$ .

QED

Definition 4.

(i)  $TH^\alpha$  is defined to be the pair  $TH^\alpha \stackrel{df}{=} \langle ObTH^\alpha, MorTH^\alpha \rangle$  of classes where

$ObTH^\alpha \stackrel{df}{=} \{ T : T \text{ is a theory in } \alpha \text{ variables} \}$  (i. e.

$ObTH^\alpha = \{ \langle Ax, F_t^\alpha \rangle : t \text{ is an arbitrary similarity type and } Ax \subseteq F_t^\alpha \}$  )

$MorTH^\alpha \stackrel{df}{=} \{ \langle T_1, \mu, T_2 \rangle : \mu \text{ is a theory morphism } \mu: T_1 \longrightarrow T_2 \text{ and } T_1, T_2 \in ObTH^\alpha \}$ .

(ii) Let  $\mu: T_1 \longrightarrow T_2$  and  $\nu: T_2 \longrightarrow T_3$  be two theory morphisms. We define the composition  $\nu, \mu: T_1 \longrightarrow T_3$  to be the (unique) theory morphism for which

$$(\exists m \in \mu, \exists n \in \nu) \nu, \mu = (n.m)/\cong$$

where the function  $(n.m): F_{t_1}^\alpha \longrightarrow F_{t_3}^\alpha$  is defined by

$$(n.m)(\Psi) = n(m(\Psi)) \text{ for all } \Psi \in F_{t_1}^\alpha.$$

(iii) Let  $T = \langle Ax, F_t^\alpha \rangle$  be a theory. The identity function

$Id_{F_t^\alpha}$  is defined to be  $Id_{F_t^\alpha} \stackrel{df}{=} \{ \langle \Psi, \Psi \rangle : \Psi \in F_t^\alpha \}$ .

The identity morphism  $Id_T$  on  $T$  is defined to be

$$\text{Id}_T \stackrel{\text{df}}{=} (\text{Id}_{F_t}^\alpha) / \cong .$$

End of Definition 4.

From now on we shall use the basic notions of category theory, see Herrlich-Strecker (73).

Proposition 2.

$\text{TH}^\alpha$  is a category with objects  $\text{ObTH}^\alpha$ , morphisms  $\text{MorTH}^\alpha$ , composition  $\nu \cdot \mu$  for any  $\mu, \nu \in \text{MorTH}^\alpha$  and identity morphisms  $\text{Id}_T$  for all  $T \in \text{ObTH}^\alpha$ .

Proof.

Fact 2.1. Let  $\mu: T_1 \longrightarrow T_2$  be a theory morphism. Then there is a unique mapping  $\bar{\mu}: C_{T_1} \longrightarrow C_{T_2}$  between the sets of concepts (of  $T_1$  and  $T_2$  resp.) such that

$$(\forall m \in \mu)(\forall \varphi \in F_{t_1}^\alpha) \bar{\mu}(\varphi / \cong_{T_1}) = m(\varphi) / \cong_{T_2} .$$

Fact 2.2.  $\langle \text{ObTH}^\alpha, \text{Interpretations} \rangle$  is a category with usual set theoretic composition  $\langle T_1, (n.m), T_3 \rangle$  of interpretations  $\langle T_1, m, T_2 \rangle$  and  $\langle T_2, n, T_3 \rangle$  and with identity  $\langle T, \text{Id}_{F_t}^\alpha, T \rangle$  for every theory  $T = \langle \text{Ax}, F_t^\alpha \rangle$ .

QED

Now we shall use the category theoretic notion of isomorphism between theories. Two theories  $T_1$  and  $T_2$  are defined to be isomorphic iff in the category  $\text{TH}^\alpha$  the objects  $T_1$  and  $T_2$  are isomorphic in the category theoretical sense, i. e. iff there are two morphisms  $\mu: T_1 \longrightarrow T_2$  and  $\nu: T_2 \longrightarrow T_1$  such that  $\mu \cdot \nu = \text{Id}_{T_2}$  and  $\nu \cdot \mu = \text{Id}_{T_1}$ .

Proposition 3.

Let  $T = \langle Ax, F_t^\alpha \rangle$  be an arbitrary theory with  $t = \langle H, t' \rangle$ . Then there exists a theory  $T^+ = \langle Ax^+, F_{t^+}^\alpha \rangle$  such that (i)-(iv) below hold.

- (i)  $t^+ = \langle 0, t' \rangle$  (i. e.  $t^+$  is the same as  $t'$  except that there is no function symbol in  $t^+$  where  $0$  is the empty set.
- (ii)  $T$  and  $T^+$  are isomorphic, i. e.  $T \cong T^+$ .
- (iii) Let the presentation  $p: t' \longrightarrow F_{t^+}^\alpha$  be defined as  $p(r, n) \stackrel{\text{df}}{=} r(x_0, \dots, x_{n-1})$  for every  $\langle r, n \rangle \in t'$ . Then there is an isomorphism  $\mu: T \longrightarrow T^+$  which satisfies  $p$ , i. e.
 
$$(\exists m \in \mu)(\forall \langle r, n \rangle \in t') r(x_0, \dots, x_{n-1}) = \begin{cases} m(r(x_0, \dots, x_{n-2})=x_{n-1}) & \text{if } r \in H \\ m(r(x_0, \dots, x_{n-1})) & \text{if } r \notin H \end{cases} .$$
- (iv) The above interpretation  $m: F_t^\alpha \longrightarrow F_{t^+}^\alpha$  is effective (is a computable function).

Proof.

The proof is based on Thm. 10.5 of Bell-Machover (77).

QED

To formulate Theorem 4 below we use the notion of completeness of categories in the sense of Herrlich-Strecker (73) or equivalently MacLane (71). For those unfamiliar with these notions we shall recall them after having formulated Thm. 4 .

Theorem 4.

- (i) The category  $TH^\alpha$  of all theories is complete and cocomplete.
- (ii) There is an effective procedure to construct the limits and

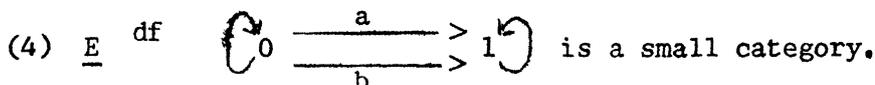
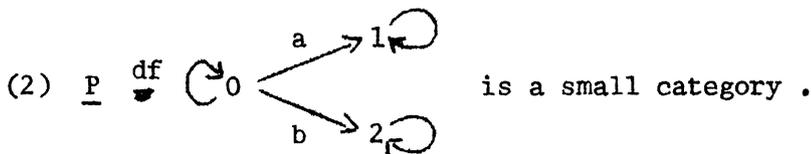
colimits of the effectively given diagrams in  $TH^\alpha$ .

Before proving Thm. 4 we shall recall the basic notions used in its formulation, and shall try to illustrate its meaning.

By a small category we understand a category  $\underline{C} = \langle \text{Ob}\underline{C}, \text{Mor}\underline{C} \rangle$  such that Mor $\underline{C}$  is a set.

Examples.

- (1) Every partially ordered set is a small category: let  $\langle P, \leq \rangle$  be a partial order, then  $\langle P, \{ \langle a, b \rangle \in P \times P : a \leq b \} \rangle$  is a small category.



End of examples.

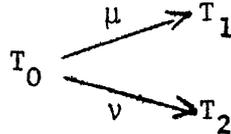
By a diagram in the category  $TH^\alpha$  we understand a functor  $D: \underline{C} \longrightarrow TH^\alpha$  where  $\underline{C}$  is a small category, and a functor is a pair  $D = \langle D_0, D_1 \rangle$  of functions  $D_0: \text{Ob}\underline{C} \longrightarrow \text{Ob}TH^\alpha$ ,  $D_1: \text{Mor}\underline{C} \longrightarrow \text{Mor}TH^\alpha$  such that  $D$  preserves the basic structure of  $\underline{C}$  i. e. (i)-(iii) below hold.

- (i) If  $f: A \longrightarrow B$  in  $\underline{C}$ , then  $D_1(f): D_0(A) \longrightarrow D_0(B)$  in  $TH^\alpha$ .
- (ii)  $D_1(f \cdot g) = D_1(f) \cdot D_1(g)$  for all  $f, g \in \text{Mor}\underline{C}$ .
- (iii)  $D_1(\text{Id}_A) = \text{Id}_{D_0(A)}$  for all  $A \in \text{Ob}\underline{C}$ .

The category  $\underline{C}$  is called the index category of the diagram  $D$ .

Examples.

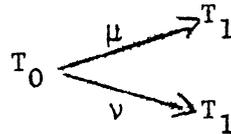
(5) Let  $\underline{P}$  be the category in Example (2). Let  $D: \underline{P} \longrightarrow \text{TH}^\alpha$  be any functor. Let  $T_i = D_0(i)$  for  $i < 3$  and let  $\mu \stackrel{\text{df}}{=} D_1(a)$ ,  $\nu \stackrel{\text{df}}{=} D_1(b)$ . By definition,  $T_0, T_1, T_2$  are theories and  $\mu, \nu$  are theory morphisms. Then the diagram  $D$  is usually illustrated as



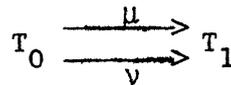
We do not indicate the identity morphisms  $D_1(i,i)$ ,  $i < 3$ , if not needed.

The historical reason for calling these functors diagrams is the possibility of illustrating them as we have just done.

(6) In the above diagram we may choose  $D_0(1) = D_0(2)$ . Then we obtain



(7) If we replace the index category of the above diagram by  $\underline{E}$  given in Example (4) then we obtain



It is important to keep in mind that the diagrams (6) and (7) above are different !

End of Examples.

Definition 5 (cone, colimit, limit).

Let  $D: \underline{I} \longrightarrow \text{TH}^\alpha$  be a diagram. Let  $\underline{I} = \langle I, M \rangle$ .

(i) A cone over  $D$  is a system  $\langle \langle h_i : i \in I \rangle, T \rangle$  such

that  $T \in \text{ObTH}^\alpha$ ,  $(\forall i \in I) h_i \in \text{MorTH}^\alpha$  and for every  $f \in M$  if  $f: i \longrightarrow j$  in  $\underline{I}$  then  $h_i = h_j \cdot D_1(f)$  in  $\text{TH}^\alpha$ .

(ii) The colimit of  $D$  in  $\text{TH}^\alpha$  is a cone  $\langle \langle g_i : i \in I \rangle, G \rangle$  over  $D$  such that for every cone  $\langle \langle h_i : i \in I \rangle, T \rangle$  over  $D$  there is a unique morphism  $\mu: G \longrightarrow T$  such that  $(\forall i \in I) h_i = \mu \cdot g_i$ .

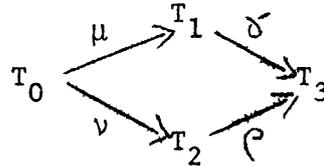
(iii) The limit of  $D$  is defined exactly as above but all the arrows are reversed.

End of Definition 5.

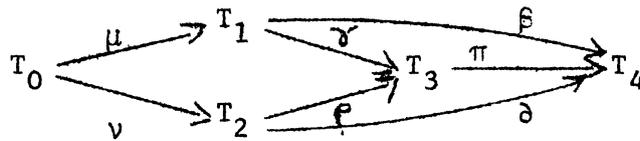
Examples.

(8) The colimit of the diagram  $T_0 \begin{matrix} \xrightarrow{\mu} T_1 \\ \xrightarrow{\nu} T_2 \end{matrix}$  given in

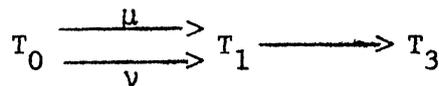
Example (5) is a theory  $T_3$  and two theory morphisms  $\gamma, \rho$  such that



and  $\gamma \cdot \mu = \rho \cdot \nu$ , and for any other  $T_4$  and  $\beta, \partial$  if  $\beta \cdot \mu = \partial \cdot \nu$  then there is a unique  $\pi: T_3 \longrightarrow T_4$  such that  $\pi \cdot \gamma = \pi \cdot \rho$



(9) The colimit of the diagram in Example (7) is a single theory morphism  $\gamma$  and a theory  $T_3$  such that



and  $\gamma \cdot \mu = \gamma \cdot \nu$  etc...

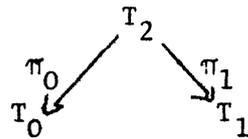
Note that the difference between diagrams (6) and (7) shows itself in that in the colimit of (6) there are two morphisms  $\gamma, \rho$  from  $T_1$  into  $T_3$  while in the colimit of (7) there is only one.

(10) Let the index category  $\underline{I}$  of the diagram  $D$  be the one in Example (3). Then our diagram  $D$  has the shape

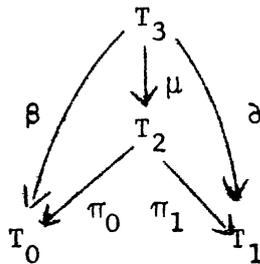


i. e. no morphisms except the identity ones.

The limit of this diagram is a theory  $T_2$  and two morphisms  $\pi_0, \pi_1$  such that

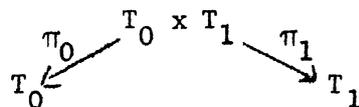


and for every similar cocone  $\langle T_3, \beta, \partial \rangle$  there is a unique morphism  $\mu: T_3 \longrightarrow T_2$  such that  $\beta = \pi_0 \cdot \mu$  and  $\partial = \pi_1 \cdot \mu$ .



End of Examples.

Limits of the kind of Example (10) are called products and are denoted by  $T_0 \times T_1$ . If  $T_0$  and  $T_1$  are two theories then



is defined to be the limit in Example (10) and is called the product of  $T_0$  and  $T_1$ . The morphisms  $\pi_0$  and  $\pi_1$  are called the projections of the product.

Definition 6.

A category  $\underline{K}$  is said to be complete and cocomplete if for every small category  $\underline{I}$  and for every diagram (i. e. functor)  $D: \underline{I} \longrightarrow \underline{K}$  both the limit and the colimit of  $D$  exist in  $\underline{K}$ .

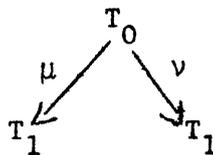
End of Definition 6.

By the aboves we see that Theorem 4 says that every diagram  $D: \underline{I} \longrightarrow TH^\alpha$  has both limit and colimit in  $TH^\alpha$ . I. e. in the category  $TH^\alpha$  of all theories all possible limits and colimits exist (and can be constructed).

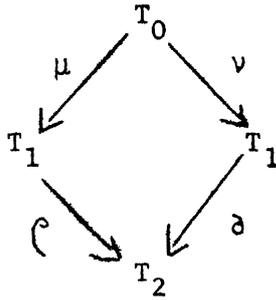
Examples.

(14) Let  $T_0 \stackrel{df}{=} \langle \emptyset, F_{t_0}^\omega \rangle$ ,  $T_1 \stackrel{df}{=} \langle Ax_1, F_{t_1}^\omega \rangle$ , where  $t_0 = \langle \emptyset, \{ \langle R, 2 \rangle \} \rangle$ ,  $t_1 = \langle \{ + \}, \{ \langle +, 3 \rangle \} \rangle$  and  $Ax_1 \stackrel{df}{=} \{ (x_0 + x_0 = x_0), ((x_0 + x_1) + x_2 = x_0 + (x_1 + x_2)), x_0 + x_1 = x_1 + x_0 \}$ . Let  $\mu: T_0 \longrightarrow T_1$  and  $\nu: T_0 \longrightarrow T_1$  be two theory morphisms such that for some  $m \in \mu$  and  $n \in \nu$  we have  $m(R(x_0, x_1)) = x_0 + x_1 = x_1$  and  $n(R(x_0, x_1)) = x_0 + x_1 = x_0$ .

(a) Consider the diagram



The colimit of this diagram is



where  $T_2 =$  "Lattice theory" , i. e.

$$T_2 = \langle Ax_2, F_{t_2}^\omega \rangle , \text{ where } t_2 = \langle \{+, \cdot\}, \{ \langle +, 3 \rangle, \langle \cdot, 3 \rangle \} \rangle$$

and

$$Ax_2 = \{ (x_0 + (x_0 \cdot x_1) = x_0), (x_0 \cdot (x_0 + x_1) = x_0) \} \\ \cup \{ (x_0 \cdot x_0 = x_0), ((x_0 \cdot x_1) \cdot x_2 = x_0 \cdot (x_1 \cdot x_2)), (x_0 \cdot x_1 = x_1 \cdot x_0) \} \\ \cup Ax_1 .$$

$\rho$  and  $\delta$  are such that  $r(x_0 + x_1 = x_2) = x_0 + x_1 = x_2$  and  $d(x_0 + x_1 = x_2) = x_0 \cdot x_1 = x_2$  for some  $r \in \rho$  and  $d \in \delta$  .

(b) Consider the diagram

$$T_0 \begin{array}{c} \xrightarrow{\mu} \\ \xrightarrow{\nu} \end{array} T_1$$

The colimit of this diagram is

$$T_0 \begin{array}{c} \xrightarrow{\mu} \\ \xrightarrow{\nu} \end{array} T_1 \xrightarrow{\rho} T_2$$

where  $T_2 = \langle Ax_2, F_{t_1}^\omega \rangle$  and

$$Ax_2 = \{ x_0 + x_1 = x_1 + x_0, (x_0 + x_1) + x_2 = x_0 + (x_1 + x_2), x_0 + x_0 = x_0, \\ x_0 + x_1 = x_0 \longrightarrow x_1 = x_0 \}$$

Proof.

Proof of (a).

1). Proof of  $\rho \cdot \mu = \delta \cdot \nu$  :

$$r(m(R(x_0, x_1))) = r(x_0 + x_1 = x_1) = x_0 + x_1 = x_1 ,$$

$$d(n(R(x_0, x_1))) = d(x_0 + x_1 = x_0) = x_0 \cdot x_1 = x_0 .$$

We have to prove  $r.m = d.n$  , i. e. we have to show

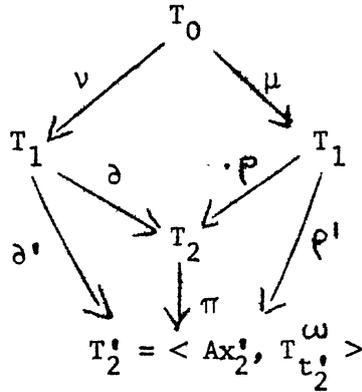
$$(x_0 + x_1 = x_1) / \cong_{T_2} = (x_0 \cdot x_1 = x_0) / \cong_{T_2} , \text{ i. e. that}$$

$$Ax_2 \models (x_0 + x_1 = x_1 \iff x_0 \cdot x_1 = x_0) .$$

Suppose  $x_0 + x_1 = x_1$  . Then  $x_0 \cdot x_1 = x_0 \cdot (x_0 + x_1) = x_0$  by  $(x_0 \cdot (x_0 + x_1) = x_0) \in Ax_2$  .

We obtain  $Ax_2 \models (x_0 \cdot x_1 = x_0 \implies x_0 + x_1 = x_1)$  similarly.

2). Suppose  $\rho' \cdot \mu = \partial' \cdot \nu$  . We have to show  $\pi \cdot \rho = \rho'$  and  $\pi \cdot \partial = \partial'$  for some theory morphism  $\pi$  .



Let  $r' \in \rho'$  and  $d' \in \partial'$  .

$$Ax_2' \models (r'(x_0 + x_1 = x_1) \iff d'(x_0 + x_1 = x_0)) \text{ by } \rho' \cdot \mu = \partial' \cdot \nu .$$

$$\text{Let } p(x_0 + x_1 = x_2) \stackrel{df}{=} r'(x_0 + x_1 = x_2) \text{ and}$$

$$p(x_0 \cdot x_1 = x_2) \stackrel{df}{=} d'(x_0 + x_1 = x_2) .$$

We have to show that  $p$  determines a theory morphism

$\pi : T_2 \longrightarrow T_2'$  . I. e. we have to show that

$$(\forall \varphi \in Ax_2) Ax_2' \models p(\varphi) .$$

Notation:  $r'(+) = \oplus$  ,  $d'(+) = \otimes$  .

We know that

$$Ax_2' \models \{ x_0 \oplus x_0 = x_0, (x_0 \oplus x_1) \oplus x_2 = x_0 \oplus (x_1 \oplus x_2), \\ x_0 \otimes x_1 = x_1 \iff x_0 \otimes x_1 = x_0 \} .$$

$$\text{Now } p(x_0 + (x_0 \cdot x_1) = x_0) = x_0 \oplus (x_0 \otimes x_1) = x_0 .$$

We have to show  $Ax_2 \models x_0 \oplus (x_0 \otimes x_1) = x_0$  .  
 $x_0 \oplus (x_0 \otimes x_1) = (x_0 \oplus x_0) \otimes x_1 = x_0 \otimes x_1$  and therefore  
 $x_0 \otimes (x_0 \otimes x_1) = x_0$  .  
 Similarly for the other elements of  $Ax_2$  .

Proof of (b).

The proof is based on the fact that  
 $\text{Th}(Ax_2) = \text{Th}(Ax_1 \cup \{x_0 + x_1 = x_0 \longleftrightarrow x_0 + x_1 = x_1\})$  .

QED

End of Examples.

Definition 7.

Let  $\alpha$  be any ordinal. The class  $Lf_\alpha$  of Cylindric Algebras was defined in the monograph Henkin-Monk-Tarski (71). Here  $Lf_\alpha$  is considered a category the usual way:

The objects of  $Lf_\alpha$  are the cylindric algebras  $Lf_\alpha$  and the morphisms of  $Lf_\alpha$  are the homomorphisms between algebras in  $Lf_\alpha$  .

In more detail:

$Lf_\alpha$  is defined the following way.

First a similarity type  $\mathcal{L}_\alpha$  of algebras is fixed with  $\alpha+2$  operations symbols. Then 8 simple schemes C0-C7 of equations are postulated. Then  $CA_\alpha$  is defined to be the class of all algebras of type  $\mathcal{L}_\alpha$  satisfying the equations C0-C7 . Clearly  $CA_\alpha$  is a variety.

Then  $Lf_\alpha \subseteq CA_\alpha$  is defined as follows:

Let  $\mathcal{A} \in CA_\alpha$  . Then  $\mathcal{A} \in Lf_\alpha$  iff

$(\forall x \in A) \{f \in \text{Dom}(\mathcal{L}'_\alpha) : \mathcal{L}'_\alpha(f) > 0 \text{ and } f^{\mathcal{A}}(x, \dots, x) \neq x\}$  is finite .

I. e.  $\mathcal{A} \in Lf_\alpha$  iff

$(\forall x \in A) \left[ \text{the number of those basic operations of } \mathcal{A} \text{ which } x \text{ is not a fixed point is finite} \right]$  .

I. e.  $\mathcal{A} \in \text{Lf}_\alpha$  iff

$(\forall x \in A)$  [x is a fixed point of every basic operation of  $\mathcal{A}$   
with finitely many exceptions]

By this the class  $\text{Lf}_\alpha$  of algebras has been defined.

The category  $\underline{\text{Lf}}_\alpha$  is the usual one, namely:

$\text{Ob}\underline{\text{Lf}}_\alpha \stackrel{\text{df}}{=} \text{Lf}_\alpha$

and

$\text{Mor}\underline{\text{Lf}}_\alpha \stackrel{\text{df}}{=} \text{All the homomorphisms between the algebras in } \text{Lf}_\alpha$   
in the usual sense.

End of Definition 7.

From the above definition it should be clear that the definition of  $\underline{\text{Lf}}_\alpha$  is much simpler than that of  $\text{TH}^\alpha$ , at least for certain kinds of investigations. Certainly the standard tools of abstract algebra and universal algebra can be directly applied to  $\text{Lf}_\alpha$ .

Theorem 5.

Let  $\alpha$  be an arbitrary ordinal. Assume  $\alpha \geq \omega$ . Then the categories  $\text{TH}^\alpha$  and  $\underline{\text{Lf}}_\alpha$  are isomorphic in the sense of MacLane (71).

QED of Theorem 5.

Parts and some semantic aspects of the above Theorem 5 are illustrated and proved in Némethi-Sain (78). See also Andr eka-Gergely-Némethi (77),(80).

Theorem 5 above can be used to apply the theory of Cylindric Algebras (CA-theory) to investigate the category  $\text{TH}^\alpha$  of theories and theory morphisms. Hence CA-theory can be used as a theoretical foundation for stepwise refinement of program specifications along the lines outlined e. g. in Burstall-Goguen (77),(79), Goguen-Burstall (78)-(80).

In the remaining part of this paper we shall concentrate on the category  $TH^\alpha$  of all first order theories and interpretations between them. We have seen that  $TH^\alpha$  is isomorphic with the category  $\underline{Lf}_\alpha$  of all locally finite cylindric algebras and their homomorphisms.  $\underline{Lf}_\alpha$  can be defined in a clear abstract style and nice representation theorems simplify our work with  $\underline{Lf}_\alpha$ . We shall prove that  $\underline{Lf}_\alpha$  and  $TH^\alpha$  are strongly algebraïdal categories (in the sense of e. g. Banaschewski-Herrlich (76), Andréka-Németi (79) ) and they have enough projectives, are complete and cocomplete etc... Hence the results of category theoretic Universal Algebra in the sense of e. g. Andréka-Németi (79),(78), Németi-Sain (77) can be applied to  $TH^\alpha$  and  $\underline{Lf}_\alpha$ . The strongly small objects (or compact objects) and projective objects of  $TH^\alpha$  and  $\underline{Lf}_\alpha$  will be characterized.  $TH^\alpha$  is a locally finitely presented category in the sense of Gabriel-Ulmer. Specially ultraproducts exist in  $TH^\alpha$  and  $\underline{Lf}_\alpha$ . It turns out that ultraproducts of theories (objects of  $TH^\alpha$ ) commute with ultraproducts of their models. I. e. let  $\langle T_i : i \in I \rangle$  be a system of theories and  $F$  be an ultrafilter on  $I$ . Let  $T$  be the ultraproduct of  $\langle T_i : i \in I \rangle$  modulo  $F$  in the category  $TH^\alpha$ . Then a structure  $\mathcal{A}$  is a model of the ultraproduct theory  $T$  iff  $\mathcal{A}$  is an ultraproduct of some models  $\langle \mathcal{A}_i : i \in I \rangle$  modulo  $F$  such that  $(\forall i \in I) [\mathcal{A}_i \text{ is a model of the theory } T_i]$ . This result extends to reduced products if and only if the cylindric algebras  $\underline{Lf}_\alpha$  are replaced with cylindric (meet) semilattices.

Parts of the results reported here are joint results of the author with H. Andréka and T. Gergely.

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