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COVERING MORPHISMS OF CROSSED COMPLEXES AND OF CUBICAL OMEGA-GROUPOIDS ARE CLOSED UNDER TENSOR PRODUCT

by Ronald BROWN and Ross STREET

Résumé

Le but de cet article est de démontrer les théorèmes mentionnés dans le titre, ainsi que le corollaire disant que le produit tensoriel de deux résolutions croisées libres, en groupes ou en groupoïdes, est aussi une résolution croisée libre, en groupes ou en groupoïdes. Ce corollaire est obtenu en utilisant l'équivalence entre la catégorie des complexes croisés et celle des omega-groupoïdes cubiques, avec connexion, dans laquelle on donne la définition initiale du produit tensoriel. D'autre part, c'est dans cette deuxième catégorie qu'on peut appliquer les techniques de sous-catégories denses pour reconnaitre qu'un produit tensoriel de revêtements est un revêtement.

Abstract

The aim is the proof of the theorems of the title and the corollary that the tensor product of two free crossed resolutions of groups or groupoids is also a free crossed resolution of the product group or groupoid. The route to this corollary is through the equivalence of the category of crossed complexes with that of cubical ω-groupoids with connections where the initial definition of the tensor product lies. It is also in the latter category that we are able to apply techniques of dense subcategories to identify the tensor product of covering morphisms as a covering morphism.

Mots-clés / Keywords: crossed complexes, cubical omega-groupoids, monoidal closed, density, covering morphisms.
Classification MSC 2010: 18B40, 18D05, 18D10, 18G40, 55U40

Introduction

A series of papers by R. Brown and P.J. Higgins, surveyed in [Bro99, Bro09], has shown how the category Crs of crossed complexes is a useful tool for certain nonabelian higher dimensional local-to-global problems in algebraic topology, for example the calculation of homotopy 2-types of unions of spaces; and also that crossed complexes are suitable coefficients for nonabelian cohomology, generalising an earlier use of crossed modules as coefficients. While crossed complexes have a long history in algebraic topology, particularly in the reduced case, i.e. when C_0 is a singleton, the extended use in these papers made them a tool whose properties could be developed independently of classical tools in algebraic topology such as simplicial approximation. A key new tool for this approach was cubical, using the notion of cubical ω–groupoids with connections. A book is in press on these topics, [BHS11].

One aspect of this work is that it leads to specific calculations of homotopical and group theoretical invariants; as an example, the notion of identities among relations for a presentation of groups combines both of these fields, since it also concerns the second homotopy group $\pi_2(K(\mathcal{P}))$ of the 2-complex determined by a presentation $\mathcal{P}$ of a group. Calculations of this module were obtained in [BRS99] not through 'killing homotopy groups', or its homological equivalent, finding generators of a kernel, but through the notion of 'constructing a home for a contracting homotopy'. To this end we had to work by constructing a free crossed resolution $\widetilde{F}$ of the universal covering crossed complex of a group or groupoid. Any construction of a contracting homotopy of $\widetilde{F}$ breaks the symmetry of the situation, as is necessary, and also may rely on rewriting methods, such as determining a maximal tree in the Cayley graph. Thus we see covering crossed complexes as a basic tool in the application of crossed complex methods, in analogy to the application of covering spaces in algebraic topology.

A major tool for dealing with homotopies is the construction of a monoidal closed structure on the category Crs of crossed complexes giving an exponential law of the form

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C))$$

for all crossed complexes A, B, C, [BH87].

This monoidal closed structure and the notion of classifying space BC of a crossed complex C is applied in [BH91] to give the homotopy classification result

$$[X, BC] \cong [\Pi X_*, C]$$

where on the left hand side with X a CW-complex, we have topology, and on the right hand side, with ΠX_* the fundamental crossed complex of the skeletal filtration of X, we have the algebra of crossed complexes.

Tonks proved in [Ton94, Theorem 3.1.5] that the tensor product of free crossed resolutions of a group is a free crossed resolution: his proof used the crossed complex Eilenberg-Zilber Theorem, [Ton94, Theorem 2.3.1], which was published in [Ton03]. The result on resolutions is applied in for example [BP96] to construct some small free crossed resolutions of a product of groups. We give here an alternative approach to this result.

The PhD thesis [Day70] of Brian Day addressed the problem of extending a promonoidal structure on a category $\mathcal{A}$ along a dense functor $J : \mathcal{A} \to \mathcal{X}$ into a suitably complete category $\mathcal{X}$ to obtain a closed monoidal structure on $\mathcal{X}$. The two published papers [Day70a, Day72] are only part of the thesis and represent components towards the density result. The formulas in, and the spirit of, Day's work suggested our approach to the present paper. However, here the category $\mathcal{A}$ is actually small (consisting of cubes) and monoidal, and so is an easy case of Day's general setting. The same simplification occurs in the approach to the Gray tensor product of 2-categories in [Str88], and of globular ∞-categories in [Cra99, Proposition 4.1].

One advantage of cubical methods is the standard formula

$$I^m_* \otimes I^n_* \cong I^{m+n}_* \tag{1}$$

where I^m_* is the standard topological m-cube with its standard skeletal filtration. This equation is modelled in the category ω-Gpd by the formula

$$\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n} \tag{2}$$

where for $m \geqslant 0$ $\mathbb{I}^m$ is the free ω-Gpd on one generator c_m of dimension m. We apply (2) by proving in Theorem 5.1 that the full subcategory of ω-Gpd on these objects $\mathbb{I}^m, m \geqslant 0$, is dense in ω-Gpd. The proof requires a

further property of ω-groupoids, that they are T-complexes [BH81, BH81c]. We then use the methods of Brian Day [Day72] to characterise the tensor product on ω-Gpd as determined by the formula (2).

We use freely the notions and properties of ends and coends, for which see [ML71].

The final ingredient we need is the fact that if $p\colon \widetilde{C} \to C$ is a covering morphism of crossed complexes then $p^*\colon \mathsf{Crs}/C \to \mathsf{Crs}/\widetilde{C}$ preserves colimits, since it has a right adjoint. This result is due to Howie [How79], in fact for the case of a fibration rather than just a covering morphism. Because of the equivalence of categories, this applies also to the case of the category ω-Gpd. However we need to characterise fibrations and coverings in the category ω-Gpd. This is done in Section 4. It is possible that the covering morphisms are part of a factorization system as are the discrete fibrations in the contexts of [Bou87] and [SV10].

The use of crossed complexes continues work of J.H.C. Whitehead, [Whi49, Whi50], and of J. Huebschmann, [Hue80], all for the single vertex case.

1 Crossed complexes

For the purposes of algebraic topology the most important feature of the category Crs of crossed complexes is the *fundamental crossed complex functor*, [BH81a],

$$\Pi\colon \mathsf{FTop} \to \mathsf{Crs}$$

from the category of filtered spaces

$$X_*\colon X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty.$$

An extra assumption is commonly made that X_∞ is the union of all the X_n, but we do not use that condition. For such a filtered space X_*, various relative homotopy groups

$$(\Pi X_*)_n(x) = \pi_n(X_n, X_{n-1}, x)$$

for $x \in X_0$ and $n \geqslant 2$, may be combined with the fundamental groupoid $(\Pi X_*)_1 = \pi_1(X_1, X_0)$ on the set X_0 to give a crossed complex ΠX_*. There are boundary operations $\delta_n \colon (\Pi X_*)_n \to (\Pi X_*)_{n-1}$ and operations of $(\Pi X_*)_1$ on $(\Pi X_*)_n, n \geqslant 2$, satisfying axioms which are characteristic for crossed complexes. This last fact follows because for every crossed complex C there is a filtered space X_* such that $C \cong \Pi X_*$ [BH81a, Corollary 9.3].

2 Fibrations and covering morphisms of crossed complexes

The definition of fibration of crossed complexes we are using is due to Howie in [How79]; it requires the definition of fibration of groupoids given in [Bro70, Bro06], generalising the definition of covering morphism of groupoids given in [Hig71]. The notion of fibration of crossed complexes given in this Section leads to a Quillen model structure on the category Crs, as shown by Brown and Golasiński in [BG89], and compared with model structures on related categories in [ArMe10].

First recall that for a groupoid G and object x of G we write $\mathrm{Cost}_G\, x$ for the union of the $G(u, x)$ for all objects u of G. A morphism of groupoids $p \colon H \to G$ is called a *fibration (covering morphism)*, [Bro70], if the induced map $\mathrm{Cost}_H\, y \to \mathrm{Cost}_G\, py$ is a surjection (bijection) for all objects y of H. (Here we use the conventions of [BHS11] rather than of [Bro06].)

Definition 2.1 A morphism $p \colon D \to C$ of crossed complexes is a *fibration (covering morphism)* if

(i) the morphism $p_1 \colon D_1 \to C_1$ is a *fibration (covering morphism)* of groupoids;

(ii) for each $n \geqslant 2$ and $y \in D_0$, the morphism of groups $p_n \colon D_n(y) \to C_n(py)$ is surjective (bijective).

The morphism p is a *trivial fibration* if it is a fibration, and also a weak equivalence, by which is meant that p induces a bijection on π_0 and isomorphisms

$\pi_1(D, y) \to \pi_1(C, py)$, $H_n(D, y) \to H_n(C, py)$ for all $y \in D_0$ and $n \geqslant 2$. □

Remark 2.2 It is worth remarking that the notion of covering morphism of groupoids appears in the paper [Smi51, (7.1)] under the name 'regular morphism'. Strong applications of covering morphisms to combinatorial group theory are given in [Hig71], and a full exposition is also given in [Bro06, Chapter 10].

A fibration of groupoids gives rise to a family of exact sequences, [Bro70, Bro06], which are extended in [How79] to a family of exact sequences arising from a fibration of crossed complexes. These latter exact sequences have been applied to the classification of nonabelian extensions of groups in [BM94], and to the homotopy classification of maps of spaces in [Bro08a]. □

In Section 4 we will need the following result, which is an analogue for crossed complexes of known results for groupoids [Bro06, 10.3.3] and for spaces.

Proposition 2.3 *Let $p\colon \widetilde{C} \to C$ be a covering morphism of crossed complexes, and let $y \in \widetilde{C}_0$. Let F be a connected crossed complex, let $x \in F_0$, and let $f\colon F \to C$ be a morphism of crossed complexes such that $f(x) = p(y)$. Then the following are equivalent:*

(i) *f lifts to a morphism $\widetilde{f}\colon F \to \widetilde{C}$ such that $\widetilde{f}(x) = y$ and $p\widetilde{f} = f$;*

(ii) $f(F_1(x)) \subseteq p(\widetilde{C}_1(y))$;

(iii) $f_*(\pi_1(F, x)) \subseteq p_*(\pi_1(\widetilde{C}, y))$.

Further, if the lifted morphism as above exists, then it is unique.

Proof That (i)⇒ (ii) ⇒ (iii) is clear.

So we assume (iii) and prove (i).

We first assume F_0 consists only of x. Then the value of $\widetilde{f}$ on x is by assumption defined to be y.

Next let $a \in F_1(x)$. By the assumption (iii) there is $c \in C_2(py)$ and $b \in \widetilde{C}(y)$ such that $f(a) = p(b) + \delta_2(c)$. Since p is a covering morphism there is a unique $d \in \widetilde{C}_2(y)$ such that $p(d) = c$. Thus $f(a) = p(b + \delta_2(d))$. So we define $\widetilde{f}(a) = b + \delta_2(d) \in \widetilde{C}_2(y)$. It is easy to prove from the definition of covering morphism of groupoids that this makes $\widetilde{f}$ a morphism $F_1(x) \to \widetilde{C}_1(y)$ such that $p\widetilde{f} = f$.

For $n \geqslant 2$ we define $\widetilde{f}\colon F_n(x) \to \widetilde{C}_n(y)$ to be the composition of f in dimension n and the inverse of the bijection $p\colon \widetilde{C}_n(y) \to C_n(py)$.

It is now straightforward to check that this defines a morphism $\widetilde{f}\colon F, x \to \widetilde{C}, y$ of crossed complexes as required.

If F_0 has more than one point, then we choose for each u in F_0 an element $\tau_u \in F_1(u, x)$ with $\tau_x = 1_x$. Then $f(\tau_u)$ lifts uniquely to $\bar{\tau}_u \in \mathrm{Cost}_{\widetilde{C}}\, y$: any lift $\widetilde{f}\colon F, x \to \widetilde{C}, y$ of f must satisfy $\widetilde{f}(\tau_u) = \bar{\tau}_u$ so we take this as a definition of $\widetilde{f}$ on these elements.

If $a \in F_1(u, v)$ then $a = \tau_u + a' - \tau_v$ where $a' \in F_1(x)$ and so we define $\widetilde{f}(a) = \bar{\tau}_u + \widetilde{f}(a') - \bar{\tau}_v$. If $n \geqslant 2$ and $\alpha \in F_n(u)$ then $\alpha^{\tau_u} \in F_n(x)$ and we define $\widetilde{f}(\alpha) = \widetilde{f}(\alpha^{\tau_u})^{-\bar{\tau}_u}$.

It is straightforward to check that these definitions give a morphism $\widetilde{f}$: $F, x \to \widetilde{C}, y$ of crossed complexes lifting f, and the uniqueness of such a lift is also easy to prove. □

We will use the above result in the following form.

Corollary 2.4 *Let $p\colon \widetilde{C} \to C$ be a covering morphism of crossed complexes, and let F be a connected and simply connected crossed complex. Then the following diagram, in which each ε is an evaluation morphism, is a pullback in the category of crossed complexes:*

$$\begin{array}{ccc} \mathsf{Crs}(F, \widetilde{C}) \times F & \xrightarrow{\varepsilon} & \widetilde{C} \\ p_* \times 1 \downarrow & & \downarrow p \\ \mathsf{Crs}(F, C) \times F & \xrightarrow[\varepsilon]{} & C, \end{array}$$

where the sets of morphisms of crossed complexes have the discrete crossed complex structure.

Proof This is simply a restatement of a special case of the existence and uniqueness of liftings of morphisms established in the Proposition. □

Remark 2.5 Because the category Crs is equivalent to that of strict globular ω-groupoids, as shown in [BH81b], the methods of this paper are also relevant to that category; see also [Bro08b]. However we are not able to make use of the globular case, nor even the 2-groupoid case. □

Let C be a crossed complex. We write CrsCov/C for the full subcategory of the slice category Crs/C whose objects are the covering morphisms of C. The following Theorem, which is proved in [BRS99], shows that the classification of covering morphisms of crossed complexes, reduces to that of covering morphisms of groupoids.

Theorem 2.6 *If C is a crossed complex, then the functor $\pi_1 \colon \mathsf{Crs} \to \mathsf{Gpd}$ induces an equivalence of categories*

$$\pi_1' \colon \mathsf{CrsCov}/C \to \mathsf{GpdCov}/(\pi_1 C).$$

An alternative descriptions of the category GpdCov/G for a groupoid G in terms of actions of G on sets is well known and of course gives the classical theory of covering maps of spaces, see [Bro06, Chapter 10]. Consequently, if the crossed complex C is connected, and $x \in C_0$, then connected covering morphisms of C are determined up to isomorphism by conjugacy classes of subgroups of $\pi_1(C, x)$. In particular, a universal cover $\widetilde{C} \to C$ of a connected crossed complex is constructed up to isomorphism from a base point $x \in C_0$ and the trivial subgroup of $\pi_1(C, x)$.

The monoidal closed structure and many other major properties of crossed complexes are obtained by working through another algebraic category, that of *cubical ω-groupoids* with connections which we abbreviate here to *ω-groupoids*. The category of these, which we write ω-Gpd, is a natural home for these deeper properties. The equivalence with crossed complexes proved in [BH81] is a foundation for this whole project. Indeed the definition of tensor product for ω-groupoids is much easier to deal with than that for crossed complexes, and we find it easier to give a dense subcategory for ω-groupoids than for crossed complexes.

3 Cubical omega-groupoids with connection

We recall from [BH81] that a *cubical ω-groupoid with connection* is in the first instance a cubical set $\{K_n \mid n \geqslant 0\}$, so that it has face maps $\{\partial_i^{\pm}\colon K_n \to K_{n-1} \mid i = 1,\dots,n; n \geqslant 1\}$ and degeneracy maps $\{\varepsilon_i\colon K_n \to K_{n+1} \mid i = 1,\dots,n; n \geqslant 0\}$ satisfying the usual rules. Further there are *connections* $\{\Gamma_i^{\pm}\colon K_n \to K_{n+1} \mid i = 1,\dots,n; n \geqslant 1\}$ which amount to an additional family of 'degeneracies' and which in the case of the singular cubical complex of a space derive from the monoid structures $\max, \min$ on the unit interval $[0,1]$. Finally there are n groupoid structures $\{\circ_i \mid i = 1,\dots,n\}$, defined on K_n with initial, final and identity maps $\partial_i^-, \partial_i^+, \varepsilon_i$ maps respectively.

The laws satisfied by all these structures are given in several places, such as [AABS02, GM03], and we do not repeat them here. Note that because we are dealing with groupoid operations $\circ_i$ we can set $\Gamma_i = \Gamma_i^-$ so that $\Gamma_i^+ = -_i -_{i+1} \Gamma_i$. In this case the laws were first given in [BH81].

A major example of this structure is constructed from a filtered space X_* as follows. One first forms the cubical set with connections RX_* which in dimension n is the set of filtered maps $I_*^n \to X_*$ where I_*^n is the standard n-cube with its skeletal filtration. Then ρX_* is the quotient of RX_* by the relation of homotopy through filtered maps and relative to the vertices of I^n. It is easy to see that ρX_* inherits the structure of cubical set with connection, and it is proved in [BH81a, Theorem A] that the obvious compositions on RX_* are also inherited by ρX_* to make it what is called the fundamental ω-groupoid ρX_* of the filtered space X_*.

The main result of [BH81] is that the category ω-Gpd is equivalent to the category Crs of crossed complexes, and in [BH81a, Theorem 5.1] it is proved that this equivalence takes ρX_* to ΠX_*.

As said in the Introduction, the free ω-groupoid on a generator c_n of dimension n is written $\mathbb{I}^n$. More generally, the free ω-groupoid on a cubical set K is written $\rho' K$: this is a purely algebraic definition. A major result is that $\rho' K$ is equivalent to $\rho|K|_*$ where $|K|_*$ is the skeletal filtration of the geometric realisation of K and ρ is defined above; so we write both as ρK. This equivalence is proved in [BH81a, Proposition 9.5] for the case $K = \mathsf{I}^n$,

and the general case follows by similar methods.

We shall also need the properties of thin elements in an ω–groupoid G. An element t of G_n is called *thin* if it has a decomposition as a multiple composition of elements $\varepsilon_i x, \Gamma_j y$, or their repeated negatives in various directions. Clearly a morphism of ω–groupoids preserves thin elements.

A family B of elements of $\mathbb{I}^n$ is called an $(n-1)$-*box in* $\mathbb{I}^n$ if they form all faces $\partial_i^{\pm} c_n$ but one of c_n. An element x is called a *filler* of the box if these all-but-one faces $\partial_i^{\pm} x$ are exactly the elements of B.

Then B generates a sub-ω-groupoid $\bar{B}$ of $\mathbb{I}^n$. The image family $\widehat{b}(B)$ of this by a morphism of ω–groupoids $\widehat{b}\colon \bar{B} \to G$ is called an $(n-1)$-box in G. Again we have the notion of a filler of a box in G. A basic result on ω–groupoids [BH81, Proposition 7.2] is:

Proposition 3.1 (Uniqueness of thin fillers) *A box in an ω–groupoid has a unique thin filler.*

The thin elements in an ω-groupoid satisfy Keith Dakin's axioms, [Dak77]:

D1) a degenerate element is thin;

D2) every box has a unique thin filler;

D3) if all faces but one of a thin element are thin, then so is the remaining face.

These axioms for a thin structure in fact give a structure equivalent to that of an ω–groupoid, as shown in [BH81c]. That is, the connections *and the compositions* are determined by the thin structure: we will use this fact in the proof of Theorem 5.1. The following Lemma is also used there.

Lemma 3.2 *If $t \in G_n$ is a thin element of an ω–groupoid G, then there is a thin element $b_t \in \mathbb{I}^n$ such that $\widehat{t}(b_t) = \widehat{t}(c_n)$.*

Proof Let $\widehat{t} : \mathbb{I}^n \to G$ be the morphism such that $\widehat{t}(c_n) = t$. We can find a box B in $\mathbb{I}^n$ and such that t is a filler of $\widehat{t}\,|\colon \bar{B} \to G$. This box B in $\mathbb{I}^n$ also has a unique thin filler b_t in $\mathbb{I}^n$. Since $\widehat{t}$ is a morphism of ω-groupoids,

it preserves thin elements and so $\widehat{t}(b_t)$ is thin and also a filler of the box B in G. By uniqueness of thin fillers $\widehat{t}(b_t) = t = \widehat{t}(c_n)$. □

Remark 3.3 Thin elements in higher categorical rather than groupoid situations are also used in [Str87, Hig05, Ste06, Ver08]. □

4 Fibrations and coverings of omega-groupoids

We now transfer to cubical ω–groupoids the definition in Section 2 of fibration and covering morphism of crossed complex.

Theorem 4.1 *Let $p\colon G \to H$ be a morphism of ω-Gpds. Then the corresponding morphism of crossed complexes $\gamma(p)\colon \gamma(G) \to \gamma(H)$ is a fibration (covering morphism) if and only if $p\colon G \to H$ is a Kan fibration (covering map) of cubical sets.*

Proof Let $J^n_{\varepsilon,i}$ for $\varepsilon = \pm, i = 1, \ldots, n$, be the subcubical set of the cubical set I^n generated by all faces of I^n except ∂^ε_i.

We consider the following diagrams:

$$\begin{array}{ccc} \Pi J^n_{\varepsilon,i} & \longrightarrow & \gamma G \\ \downarrow & \nearrow & \downarrow{\scriptstyle \gamma(p)} \\ \Pi I^n & \longrightarrow & \gamma H \end{array} \qquad \begin{array}{ccc} \rho J^n_{\varepsilon,i} & \longrightarrow & G \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \rho I^n & \longrightarrow & H \end{array} \qquad \begin{array}{ccc} J^n_{\varepsilon,i} & \longrightarrow & UG \\ \downarrow & \nearrow & \downarrow{\scriptstyle Up} \\ I^n & \longrightarrow & UH \end{array}$$

(i) (ii) (iii)

By a simple modification of the simplicial argument in [BH91], we find that the condition that diagrams of the first type have the completion shown by the dotted arrow is necessary and sufficient for γp to be a fibration of crossed complexes (with uniqueness for a covering morphism). In the second diagram, $\rho(K)$ is the free cubical ω-groupoid on the cubical set K, and the equivalence of the first and the second diagram is one of the results of

[BH81a, Section 9]. Finally, the equivalence with the third diagram, in which U gives the underlying cubical set, follows from freeness of ρ. □

Corollary 4.2 *Let $p\colon K \to L$ be a morphism of ω-Gpds such that the underlying map of cubical sets is a Kan fibration. Then the pullback functor*

$$f^*\colon \omega\text{-Gpd}/L \to \omega\text{-Gpd}/K$$

has a right adjoint and so preserves colimits.

Proof This is immediate from Theorem 4.1 and the main result of Howie [How79]. □

Corollary 4.3 *A covering crossed complex of a free crossed complex is also free.*

Proof A free crossed complex is given by a sequence of pushouts, analogously to the definition of CW-complexes, see [BH91, BHS11]. □

5 Dense subcategories

Our aim in this section is to explain and prove the theorem:

Theorem 5.1 *The full subcategory $\mathcal{I}$ of ω-Gpd on the objects $\mathbb{I}^n$ is dense in ω-Gpd.*

We recall from [ML71] the definition of a dense subcategory. First, in any category C, a morphism $f\colon C \to D$ induces a natural transformation $f_*\colon \mathsf{C}(-,C) \Rightarrow \mathsf{C}(-,D)$ of functors $\mathsf{C}^{op} \to \mathsf{Set}$. Conversely, any such natural transformation is induced by a (unique) morphism $C \to D$.

If $\mathcal{I}$ is a subcategory of C, then each object C of C gives a functor

$$\mathsf{C}^{|\mathcal{I}}(-,C)\colon \mathcal{I}^{op} \to \mathsf{Set}$$

and a morphism $f\colon C \to D$ of C induces a natural transformation of functors $f_*\colon \mathsf{C}^{|\mathcal{I}}(-,C) \Rightarrow \mathsf{C}^{|\mathcal{I}}(-,D)$. The subcategory $\mathcal{I}$ is *dense* in C if every such natural transformation arises from a morphism. More precisely, there is a functor $\eta\colon \mathsf{C} \to \mathsf{CAT}(\mathcal{I}^{op}, \mathsf{Set})$ defined in the above way, and $\mathcal{I}$ is *dense* in C if η is full and faithful.

Example 5.2 Consider the Yoneda embedding

$$\Upsilon\colon \mathsf{C} \to \mathsf{C}^{op}\text{-}\mathsf{Set} = \mathsf{CAT}(\mathsf{C}^{op}, \mathsf{Set})$$

where C is a small category. Then each object $K \in \mathsf{C}^{op}\text{-}\mathsf{Set}$ is a colimit of objects in the image of Υ and this is conveniently expressed in terms of coends as that the natural morphism

$$\int^c (\mathsf{C}^{op}\text{-}\mathsf{Set}(\Upsilon c, K) \times \Upsilon c) \quad \to \quad K$$

is an isomorphism. Thus the Yoneda image of C is dense in $\mathsf{C}^{op}\text{-}\mathsf{Set}$. For more on the relation between density and the Yoneda Lemma, see [Pra09]. □

Example 5.3 Let $\mathbb{Z}$ be the cyclic group of integers. Then $\{\mathbb{Z}\}$ is a generating set for the category Ab of abelian groups, but the full subcategory of Ab on this set is not dense in Ab. In order for a natural transformation to specify not just a function $f\colon A \to B$ but a morphism in Ab, we have to enlarge this to a full subcategory including $\mathbb{Z} \oplus \mathbb{Z}$. □

Proof of Theorem 5.1 We will use the main result of [BH81c], that the compositions in a cubical ω-groupoid are determined by its thin elements.

Let G, H be ω–groupoids and let $\mathsf{f}\colon \omega\text{-}\mathsf{Gpd}_{\mathcal{I}}(-,G) \to \omega\text{-}\mathsf{Gpd}_{\mathcal{I}}(-,H)$ be a natural transformation. We define $f\colon G \to H$ as follows.

Let $x \in G_n$. Then x defines $\widehat{x}\colon \mathbb{I}^n \to G$. We set $f(x) = \mathsf{f}(\widehat{x})(c_n) \in H_n$. We have to prove f preserves all the structure.

For example, we prove that $f(\partial_i^{\pm} x) = \partial_i^{\pm} f(x)$. Let $\bar{\partial}_i^{\pm}\colon \mathbb{I}^{n-1} \to \mathbb{I}^n$ be given by having value $\partial_i^{\pm} c_n$ on c_{n-1}. The natural transformation condition implies that $\mathsf{f}(\bar{\partial}_i^{\pm})^* = (\bar{\partial}_i^{\pm})^* \mathsf{f}$. On evaluating this on $\widehat{x}$ we obtain

$f(\partial_i^{\pm} x) = \partial_i^{\pm} f(x)$ as required. In a similar way, we prove that f preserves the operations ε_i, Γ_i.

Now suppose that $t \in G_n$ is thin in G. We prove that $f(t)$ is thin in H. By Lemma 3.2, there is a thin element $b_t \in \mathbb{I}^n$ such that $\widehat{t}(b_t) = t$. Let $\bar{b}\colon \mathbb{I}^n \to \mathbb{I}^n$ be the unique morphism such that $\bar{b}(c_n) = b_t$. Then the natural transformation condition implies $f(t) = \mathsf{f}(\widehat{t})(c_n) = \mathsf{f}(\widehat{t})(b_t)$. Since b_t is thin, it follows that $f(t)$ is thin. Thus f preserves the thin structure.

The main result of [BH81c] now implies that the operations $\circ_i$ are preserved by f. □

We can also conveniently represent each ω–groupoid as a coend.

Corollary 5.4 *The subcategory $\mathcal{I}$ of ω-*Gpd *is dense and for each object G of ω-*Gpd *the natural morphism*

$$\int^n \omega\text{-}\mathsf{Gpd}(\mathbb{I}^n, G) \times \mathbb{I}^n \to G$$

is an isomorphism.

Proof This is a standard consequence of the property of $\mathcal{I}$ being dense. □

Corollary 5.5 *The full subcategory of* Crs *generated by the objects ΠI_*^n is dense in* Crs.

Proof This follows from the fact that the equivalence $\gamma\colon \omega\text{-}\mathsf{Gpd} \to \mathsf{Crs}$ takes $\mathbb{I}^n$ to ΠI_*^n, [BH81a, Theorem 5.1]. □

Remark 5.6 The paper [BH81b] gives an equivalence between the category Crs of crossed complexes and the category there called ∞-groupoids and now commonly called globular ω-groupoids. Thus the above Corollary yields also a dense subcategory, based on models of cubes, in the latter category. □

Remark 5.7 It is easy to find a generating set of objects for the category Crs, namely the free crossed complexes on single elements, given in fact by

ΠE^n_*, where E^n_* is the usual cell decomposition of the unit ball, with one cell for $n = 0$ and otherwise three cells. It is not so obvious how to construct directly from this generating set a dense subcategory closed under tensor products. □

6 The tensor product of covering morphisms

Our aim is to prove the following:

Theorem 6.1 *The tensor product of two covering morphisms of crossed complexes is a covering morphism.*

Remark 6.2 The reason why we have to give an indirect proof of this result is that the definition of covering morphism involves *elements* of crossed complexes; but it is difficult to specify exactly the elements of a tensor product whose definition is perforce by generators and relations. □

It is sufficient to assume that all the crossed complexes involved are connected. We will also work in the category of ω–groupoids, and prove the following:

Theorem 6.3 *Let G, H be connected ω–groupoids with base points x, y respectively, and let $p : \widetilde{G} \to G$ be the covering morphism determined by the subgroup M of $\pi_1(G, x)$. Let $\phi : C \to G \otimes H$ be the covering morphism determined by the subgroup $M \times \pi_1(H, y)$ of*

$$\pi_1(G \otimes H, (x, y)) \cong \pi_1(G, x) \times \pi_1(H, y).$$

Then there is an isomorphism $\psi : C \to \widetilde{G} \otimes H$ such that $(p \otimes 1_H)\psi = \phi$, and, consequently,

$$p \otimes 1_H : \widetilde{G} \otimes H \to G \otimes H$$

is a covering morphism.

Proof Here we were inspired by the formulae of Brian Day [Day70].

First we know from [BH87] that the tensor product of ω-Gpds satisfies $\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$, showing that $\mathcal{I}$ is a full monoidal subcategory of ω-Gpds.

Since also from [BH87] the tensor preserves colimits in each variable, it follows from Corollary 5.4 that the tensor product $G \otimes H$ of ω-groupoids G and H satisfies

$$G \otimes H \cong \int^{m,n} \omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, G) \times \omega\text{-}\mathsf{Gpd}(\mathbb{I}^n, H) \times (\mathbb{I}^m \otimes \mathbb{I}^n). \qquad (3)$$

Let $p\colon \widetilde{G} \to G$ be the covering morphism determined by the subgroup M and let $\phi\colon C \to G \otimes H$ be the covering morphism determined by the subgroup $M \times \pi_1(H, y)$ of

$$\pi_1(G, x) \times \pi_1(H, y) \cong \pi_1(G \otimes H, (x, y)).$$

By Corollary 4.2, pullback ϕ^* by ϕ preserves colimits. Hence

$$\begin{aligned} C &\cong \phi^* \left(\int^{m,n} \omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, G) \times \omega\text{-}\mathsf{Gpd}(\mathbb{I}^n, H) \times (\mathbb{I}^m \otimes \mathbb{I}^n) \right) \\ &\cong \int^{m,n} \phi^*(\omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, G) \times \omega\text{-}\mathsf{Gpd}(\mathbb{I}^n, H)) \times (\mathbb{I}^m \otimes \mathbb{I}^n) \end{aligned}$$

and so because of the construction of C by the specified subgroup:

$$\begin{aligned} &\cong \int^{m,n} \omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, \widetilde{G}) \times \omega\text{-}\mathsf{Gpd}(\mathbb{I}^m, H) \times (\mathbb{I}^m \otimes \mathbb{I}^n) \\ &\cong \widetilde{G} \otimes H. \end{aligned}$$

□

Corollary 6.4 *The tensor product of covering morphisms of ω-groupoids is again a covering morphism.*

Proof Because tensor product commutes with disjoint union, it is sufficient to restrict to the connected case. Since the composition of covering morphisms is again a covering morphism, it is sufficient to restrict to the case of $p \otimes 1_H$, and that is proved in Theorem 6.3. □

The proof of Theorem 6.1 follows immediately.

Corollary 6.5 *If F, F' are free and aspherical crossed complexes, then so also is $F \otimes F'$.*

Proof It is sufficient to assume F, F' are connected. Since F, F' are aspherical, their universal covers $\widetilde{F}, \widetilde{F'}$ are acyclic. Since they are also free, they are contractible, by a Whitehead type theorem, [BG89, Theorem 3.2]. But the tensor product of free crossed complexes is free, by [BH91, Cor. 5.2]. Therefore $\widetilde{F} \otimes \widetilde{F'}$ is contractible, and hence acyclic. Therefore $F \otimes F'$ is aspherical. □

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