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LIMITS IN SYMMETRIC CUBICAL CATEGORIES
(On weak cubical categories, II)

by Marco GRANDIS

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Résumé. Une catégorie cubique *symétrique* faible est équipée d'une action des groupes symétriques. Cette action, outre simplifier les conditions de cohérence, fournit une structure monoïdale fermée *symétrique* et un (*seul*) foncteur cocylindre, ce qui est essentiel pour définir les transformations cubiques. On étudie ici les limites cubiques symétriques, en prouvant qu'elles peuvent être construites à partir des produits, égalisateurs et tabulateurs du même genre. Les catégories doubles faibles sont un tronquement cubique des structures traitées ici, ce qui permet de comparer les limites doubles aux limites cubiques.

Abstract. A weak *symmetric* cubical category is equipped with an action of the symmetric groups. This action, besides simplifying the coherence conditions, yields a *symmetric* monoidal closed structure and *one* path functor – a crucial fact for defining cubical transformations. Here we deal with symmetric cubical limits, showing that they can be constructed from symmetric cubical products, equalisers and tabulators. Weak double categories are a cubical truncation of the present structures, so that double limits can be compared with the cubical ones.

Mathematics Subject Classifications (2000): 18D05, 55U10, 20B30

Key words: weak cubical category, weak double category, cubical set, symmetries.

Introduction

This is the second paper in a series on weak symmetric cubical categories. The first, referred to as Part I [G4], explored the role of symmetries in providing *one* path functor, whose 'homotopies' are the cubical transformations of cubical functors. The present paper, concerned with cubical limits, can also be viewed as a higher dimensional extension of the study of double limits in [GP1].

Weak cubical categories were introduced in [G1-G3], as a basis for the study of cubical cospans in algebraic topology and higher cobordism. They have a cubical structure, with faces and degeneracies, weak compositions in countably many directions (indexed as 1, 2,..., n,...) and a strict composition in *one* direction, called the *transversal* one (and indexed as 0).

As a leading example, one can think of the weak cubical category $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$ of cubical spans in a category with pullbacks $\mathbf{X}$. An n-dimensional *object* is a functor $x\colon \boldsymbol{\omega}^n \to \mathbf{X}$, where $\boldsymbol{\omega}$ is the 'formal span' category

$$(1) \qquad -1 \leftarrow 0 \rightarrow 1 \quad \boldsymbol{\omega} \qquad\qquad \begin{array}{ccccc} (-1,-1) & \longleftarrow & (0,-1) & \longrightarrow & (1,-1) \\ \uparrow & & \uparrow & & \uparrow \\ (-1,0) & \longleftarrow & (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \downarrow & & \downarrow \\ (-1,1) & \longleftarrow & (0,1) & \longrightarrow & (1,1) \end{array} \quad \boldsymbol{\omega}^2.$$

An n-dimensional *transversal map* is a natural transformation $f\colon x \to y\colon \boldsymbol{\omega}^n \to \mathbf{X}$ of such functors. The ordinary categories $\mathbb{S}\mathrm{p}_n(\mathbf{X}) = \mathbf{CAT}(\boldsymbol{\omega}^n, \mathbf{X})$ form a cubical object in $\mathbf{CAT}$, with obvious faces and degeneracies. Moreover, n-dimensional spans (and their maps) have *cubical composition laws* $x +_i y$ (or *concatenations*) in direction $i = 1,..., n$, that are computed with (a fixed choice of) pullbacks; these compositions are consistent with faces, *but only behave well up to invertible (transversal) comparisons*, for associativity, unitarity and interchange.

As already stressed in [G1], $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$ is a weak *symmetric* cubical category, when equipped with the action of the symmetric group S_n on $\mathbf{CAT}(\boldsymbol{\omega}^n, \mathbf{X})$ that permutes the factors of $\boldsymbol{\omega}^n$. These symmetries – *which permute the* weak *directions without modifying the transversal one* – reduce all faces, degeneracies and cubical compositions to the 1-indexed case (for instance), and allow us to simplify the coherence conditions. Notice also that cubical 1-truncation, that keeps *one* weak direction *and* the strict transversal one, yields the weak *double* category $\mathbb{S}\mathrm{p}(\mathbf{X}) = \mathrm{tr}_1(\omega\mathbb{S}\mathrm{p}(\mathbf{X}))$ of morphisms and ordinary spans, studied in [GP1]; symmetries 'disappear', since the groups S_0 and S_1 are trivial.

In section 1 we review more analytically the construction of $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$; this should be sufficient to clarify the general structure of symmetric cubical categories (a formal definition can be found in [G1] and Part I). Then we introduce *lax* symmetric cubical functors, with their transversal (or algebraic) and cubical (or geometric) transformations.

In the next two sections we deal with cubical limits in a weak *symmetric* cubical category $\mathbb{A}$. First, in Section 2, we consider symmetric cubical limits of *level* functors $F: \mathbf{X} \to tv_n\mathbb{A}$, with values in the ordinary category of n-cubes and n-maps: these are ordinary limits, required to be preserved by the functors $tv_n\mathbb{A} \to tv_m\mathbb{A}$ of the symmetric cubical structure. Then, in Section 3, we introduce general limits for a lax symmetric cubical functor $F: \mathbb{X} \to \mathbb{A}$; the definition takes advantage of the path functor P of such 'categories', *a consequence of the symmetric setting* (cf. 1.4.4; or 3.6 of Part I). The main theorem (3.7-3.8) reduces the existence of *symmetric cubical limits* to 'basic cases': products, equalisers and *tabulators* (always in the symmetric cubical sense); the tabulator of an n-cube x of $\mathbb{A}$ is an object x_0 with a universal n-map $e^n(x_0) \to x$ defined on the totally degenerate n-cube at x_0.

In Section 4 we compare the weak symmetric cubical categories $\wedge\mathbb{S}p = \wedge\mathbb{S}p(\mathbf{Set})$ and $\wedge\mathbb{C}osp$ of cubical spans and cospans of sets with their cubical truncations, the weak double categories $\mathbb{S}p$ and $\mathbb{C}osp$ already studied in [GP1-4]. Because of the previous construction theorem, comparing their limits amounts to comparing cubical tabulators of 1-cubes with double tabulators of vertical arrows, together with the limits of the 'transformations' of such data. At least in these basic situations, a cubical transformation of symmetric cubical functors seems to be a better notion than the various instances (lax, colax, strong) of vertical transformation of the corresponding truncated double functors (see 4.2, 4.3). Thus, a weak double category that *has a natural lifting as a weak sc-category* is perhaps better studied in this enrichment: truncation (at any degree) makes 'boundary problems'. However, there seems to be no way of reducing the general theory of double limits to that of cubical limits: the universal constructions of skeleton and coskeleton – adjoint to truncation – do not give good results in our basic examples (see 4.5, 4.6); this represents a negative answer to the use of the coskeletal construction, hypothetically suggested in Part I (3.9). For a more detailed analysis of these points, see 4.1.

Finally, in Section 5, we prove the main theorem on the construction of cubical limits from cubical products, cubical equalisers and cubical tabulators.

References to the rich literature on higher categories can be found in two recent books, by T. Leinster [Le] and E. Cheng - A. Lauda [CL]; but this literature is mostly concerned with the globular approach, rather than the cubical one. Strict cubical categories with 'connections' (higher degeneracies) have been studied by Al-Agl, Brown and Steiner [ABS], and proved to be equivalent to globular $\wedge$-categories. Monoidal n-categories of higher spans can be found in Batanin [Bt]. A structure for *cobordisms with corners*, using 2-cubical cospans, has been recently proposed by J. Morton [Mo] and J. Baez [Ba], in the form of a 'Verity double bicategory' [Ve]; see also Cheng and Gurski [CG]. For weak double categories see [GP1-4] and references therein.

The index α takes values 0, 1, which are also written as –, +. The prefix 'sc-' stands for 'symmetric cubical'. The reference I.2.3 applies to Section 2.3 of Part I.

1. Some points on weak symmetric cubical categories

Weak symmetric cubical categories (or weak sc-categories) have been introduced in [G1]; their definition is recalled in Part I [G4], Section 3, and is not repeated here. But we review some typical examples, that should be sufficient to make their structure clear. Then we introduce *lax* symmetric cubical functors, with transversal and cubical transformations. The index α takes values 0, 1, also written as –, +.

1.1. Cubical spans. We will use as a leading example the weak symmetric cubical category $\alpha\mathbb{S}p(\mathbf{X})$ of higher cubical spans. The weak double category $\mathbb{S}p(\mathbf{X})$ studied in [GP1] is a cubical truncation of the former (see Section 4).

Let $\mathbf{X}$ be a category *with a full choice of distinguished pullbacks*: in other words, to every cospan (f, g) we assign *one distinguished* pullback (f', g').

The 'geometric model' of cubical spans of dimension n is the category $\boldsymbol{\alpha}^n$, a cartesian power of the *formal span* $\boldsymbol{\alpha}$

$$(1) \qquad -1 \leftarrow 0 \rightarrow 1 \quad \boldsymbol{\alpha}, \qquad \begin{array}{ccccc} (-1,-1) & \longleftarrow & (0,-1) & \longrightarrow & (1,-1) \\ \uparrow & & \uparrow & & \uparrow \\ (-1,0) & \longleftarrow & (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \downarrow & & \downarrow \\ (-1,1) & \longleftarrow & (0,1) & \longrightarrow & (1,1) \end{array} \quad \boldsymbol{\alpha}^2. \qquad \begin{array}{cc} \bullet \longrightarrow & 1 \\ \downarrow 2 & \end{array}$$

An *n-cube* of $\alpha\mathbb{S}p(\mathbf{X})$ is a functor $x: \boldsymbol{\alpha}^n \to \mathbf{X}$; a 0-cube 'is' an object of $\mathbf{X}$, and will also be called an *object* of $\alpha\mathbb{S}p(\mathbf{X})$. A *transversal map* of n-cubes is a natural transformation $f: x \to y: \boldsymbol{\alpha}^n \to \mathbf{X}$; it is also called an *n-map*, but *should be viewed as an (n+1)-dimensional cell* (being represented by the associated functor $f: \mathbf{2}{\times}\mathbf{\vee}^n \to \mathbf{X}$, a diagram of dimension n+1).

These objects and maps form a category

$$(2) \quad \mathbb{S}p_n(\mathbf{X}) = \mathbf{CAT}(\mathbf{\vee}^n, \mathbf{X}),$$

whose composition law, written g.f or gf, is called the *transversal composition* of $\vee\mathbb{S}p(\mathbf{X})$ in degree n (and direction 0). The identity of x is written as id(x).

It is now easy to construct a symmetric cubical object in **CAT**, based on the structure of the category $\boldsymbol{\omega}$ as a formal symmetric interval (with respect to the cartesian product, in **CAT**)

$$(3) \quad \partial^\omega: \mathbf{1} \rightrightarrows \boldsymbol{\omega}, \qquad e: \boldsymbol{\omega} \to \mathbf{1}, \qquad s: \boldsymbol{\omega}^2 \to \boldsymbol{\omega}^2 \qquad (\omega = \pm),$$
$$\partial^\omega(*) = \omega 1, \qquad s(t_1, t_2) = (t_2, t_1).$$

Namely, faces, degeneracies and transpositions of n-cubes and n-maps are defined by pre-composition with the following maps between cartesian powers of $\boldsymbol{\omega}$ (for $\omega = \pm$ and $i = 1,..., n$)

$$(4) \quad \partial_i^\omega = \boldsymbol{\omega}^{i-1} \times \partial^\alpha \times \boldsymbol{\alpha}^{n-i}: \boldsymbol{\alpha}^{n-1} \to \boldsymbol{\alpha}^n, \qquad \partial_i^\alpha(t_1,..., t_{n-1}) = (t_1,..., \alpha 1,..., t_{n-1}),$$
$$e_i = \boldsymbol{\alpha}^{i-1} \times e \times \boldsymbol{\alpha}^{n-i}: \boldsymbol{\alpha}^n \to \boldsymbol{\alpha}^{n-1}, \qquad e_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n),$$
$$s_i = \boldsymbol{\alpha}^{i-1} \times s \times \boldsymbol{\alpha}^{n-i}: \boldsymbol{\alpha}^{n+1} \to \boldsymbol{\alpha}^{n+1}, \qquad s_i(t_1,..., t_{n+1}) = (t_1,..., t_{i+1}, t_i,..., t_n),$$

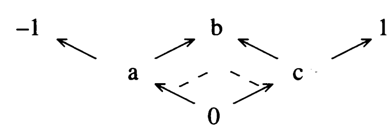
so that the $2n$ faces of an n-cube $x: \boldsymbol{\alpha}^n \to \mathbf{X}$ are $\partial_i^\alpha(x) = x \circ \partial_i^\alpha: \boldsymbol{\alpha}^{n-1} \to \mathbf{X}$, and so on.

An n-cube has 2^n *vertices*, the objects $\partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}(x)$. Similarly, a transversal n-map f has 2^n *vertices*, the 0-maps $\partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}(f)$; f is said to be *special* if its vertices are identities.

The *i-concatenation* (or *cubical composition in direction* i) $x +_i y$ of two n-cubes that are i-consecutive (i.e. $\partial_i^+(x) = \partial_i^-(y)$) is computed in the obvious way, by 3^{n-1} distinguished pullbacks whose 'vertices' are those of the common face (for $i = 1,..., n$).

This operation can be given a formal definition, based on the *model of binary composition* (for ordinary spans), the category $\boldsymbol{\alpha}_2$ displayed below, with one non-trivial distinguished pullback

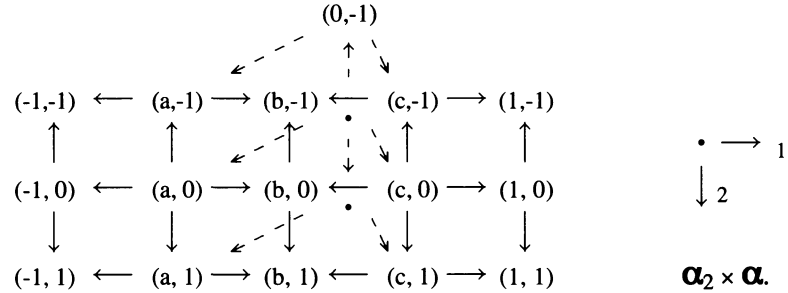
(5)



$\boldsymbol{\alpha}_2$.

Indeed, two consecutive spans x, y in **X** define a functor $[x, y]: \boldsymbol{\alpha}_2 \to \mathbf{X}$; the concatenation $x +_1 y: \boldsymbol{\alpha} \to \mathbf{X}$ is obtained by pre-composing $[x, y]$ with the *concatenation map* $c: \boldsymbol{\alpha} \to \boldsymbol{\alpha}_2$, already displayed in the diagram above, by the labels of the objects of $\boldsymbol{\alpha}_2$.

Then, i-concatenation of n-cubes is based on the cartesian product $\boldsymbol{\alpha}^{i-1} \times \boldsymbol{\alpha}_2 \times \boldsymbol{\alpha}^{n-i}$, as shown below for the concatenation of 2-cubes in direction $i = 1$

(6) 

Comparisons for associativity and interchange can be defined taking advantage of this formal construction (as in [G1], Section 3). These comparisons are invertible, special transversal maps:

(7) $\alpha_1 x\colon e_1\partial_1^- x +_1 x \to x$, $\alpha_1 x\colon x +_1 e_1\partial_1^+ x \to x$ (*unit 1-comparisons*),
$\alpha_1(x, y, x)\colon x +_1 (y +_1 z) \to (x +_1 y) +_1 z$ (*associativity 1-comparison*),
$\alpha_1(x, y, z, u)\colon (x +_1 y) +_2 (z +_1 u) \to (x +_2 z) +_1 (y +_2 u)$
(*interchange 1-comparison*).

Of course, we are assuming that all concatenations above are possible. The comparisons $\alpha_i, \alpha_i, \alpha_i, \alpha_i$ in the other directions are provided by transpositions, a fact that simplifies the structure and the coherence axioms. The comparison α_i deals with the interchange of $+_i$ and $+_{i+1}$. (Notice also that the 0-direction of $\alpha_1 x$ – which, of course, is inessential – is reversed, with respect to I.3.5.)

Cubical *co*spans are obtained by the dual procedure, over a category **X** with distinguished pushouts:

(8) $\alpha\mathbb{C}\mathrm{osp}(\mathbf{X}) = \alpha\mathbb{S}\mathrm{p}(\mathbf{X}^{op})$, $\mathbb{C}\mathrm{osp}_n(\mathbf{X}) = \mathbf{CAT}(\boldsymbol{\alpha}^n, \mathbf{X})$.

The category $\boldsymbol{\alpha}$ is the *formal cospan*, $-1 \leftarrow 0 \rightarrow 1$. (In [G1] and I.4.1, we have studied this case, of particular interest for higher cubical cobordism.)

1.2. Remarks. (a) Faces, degeneracies and concatenations can also be reduced to those in direction 1 (for instance), by means of transpositions

(1) $\partial_{i+1}^\alpha = \partial_i^\alpha s_i$, $e_{i+1} = s_i e_i$, $s_i(x) +_{i+1} s_i(y) = s_i(x +_i y)$,

but we only use such reductions *when* they do simplify things.

(b) The weak sc-category $\mathbb{A}$ is *unitary* when the unit comparisons $\alpha_1 x$ and $\alpha_1 x$ are identities, for all cubes x (which implies that this is true in every direction).

One can easily make $\wedge\mathbb{S}p(\mathbf{X})$ unitary, adopting a *unitarity constraint* for the choice of pullbacks in $\mathbf{X}$: the distinguished pullback of a cospan (f, 1) is (1, f), and symmetrically.

(c) More generally, we say that the weak sc-category $\mathbb{A}$ is *semi-unitary* when, for every n-cube x, $e_1(x) +_1 e_1(x) = e_1(x)$ and the unit comparisons $\wedge_1 e_1(x)$ and $\wedge_1 e_1(x)$ are identities. For the sake of simplicity, *we will always assume that this is the case* (cf. 3.2).

1.3. Other examples. We refer to Part I for more complex examples, like:

(a) the strict sc-category $\wedge\mathbb{R}$el *of cubical relations* of sets (I.4.2, I.5.6),

(b) the weak sc-category $\wedge\mathbb{C}$at *of cubical profunctors* (I.5.7).

An easier, if less representative, example is the *strict* symmetric cubical category $\wedge\,\mathbb{C}\mathrm{ub}(\mathbf{X})$ *of commutative cubes* on the arbitrary category $\mathbf{X}$ (I.3.3, I.3.4).

An n-cube is now a functor $x\colon \mathbf{2}^n \to \mathbf{X}$, where $\mathbf{2} = \{0 \to 1\}$ is the category corresponding to the ordinal two. A transversal map $f\colon x \to y$ of n-cubes is a natural transformation $f\colon x \to y\colon \mathbf{2}^n \to \mathbf{X}$ (and amounts to a cube of dimension n+1). The n-th component is the category

(1) $\quad \mathbb{C}\mathrm{ub}_n(\mathbf{X}) \;=\; \mathbf{CAT}(\mathbf{2}^n, \mathbf{X}).$

Again, we have a symmetric cubical object in $\mathbf{CAT}$, based on the structure of the category $\mathbf{2}$ as a formal symmetric interval, for the cartesian product (in $\mathbf{CAT}$)

(2) $\quad \wedge^{\wedge}\colon \mathbf{1} \rightrightarrows \mathbf{2} \qquad e\colon \mathbf{2} \to \mathbf{1}, \qquad s\colon \mathbf{2}^2 \to \mathbf{2}^2 \qquad (\wedge = 0, 1),$

$\qquad \wedge^{\wedge}(*) = \wedge, \qquad\qquad s(t_1, t_2) = (t_2, t_1).$

The concatenation $x +_i y$ of two n-cubes that are i-consecutive ($\wedge_i^+(x) = \wedge_i^-(y)$) is computed in the obvious way, by composing (in $\mathbf{X}$) the i-directed arrows of x and y (as below, for n = 2)

$$(3) \qquad \begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & x & \downarrow & y & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array} \qquad\qquad \bullet \longrightarrow i$$

Of course, these operations are strictly categorical, with a strict interchange.

1.4. Strict sc-functors and their transformations. A strict *symmetric cubical functor* $F\colon \mathbb{A} \to \mathbb{B}$ between weak sc-categories strictly preserves the whole structure: faces, degeneracies, transpositions, transversal composition and identities, concatenations and comparisons (cf. I.3.6)

(1) $\wedge_i^\wedge(Ff) = F(\wedge_i^\wedge f)$, $\quad e_i(Ff) = F(e_if)$, $\quad s_i(Ff) = F(s_if)$,
$F(gf) = Fg.Ff$, $\quad F(idx) = id(Fx)$, $\quad F(f +_i g) = F(f) +_i F(g)$,
$F(\wedge_1 x) = \wedge_1(Fx)$, $\quad F(\wedge_1 x) = \wedge_1(Fx)$,
$F(\wedge_1(x, y, z)) = \wedge_1(Fx, Fy, Fz)$, $\quad F(\wedge_1(x, y, z, u)) = \wedge_1(Fx, Fy, Fz, Fu)$.

(Again, we are assuming that the compositions above make sense.)

A *transversal* (or *algebraic*) *transformation* $h: F \to G: \mathbb{A} \to \mathbb{B}$ of such functors assigns to every n-cube x of $\mathbb{A}$ an n-map $hx: Fx \to Gx$ in $\mathbb{B}$; the family (hx) must commute with faces, degeneracies, transpositions and cubical compositions:

(2) $h(\wedge_i^\wedge x) = \wedge_i^\wedge(hx)$, $\quad h(e_ix) = e_i(hx)$, $\quad h(s_ix) = s_i(hx)$,
$h(x +_i y) = h(x) +_i h(y)$.

All this forms the 2-category wsc**CAT** of *weak sc-categories, strict sc-functors and their transversal transformations* (I.3.6).

A crucial fact, *depending on the symmetric setting*, is the presence of *one* path 2-functor (see Part I)

(3) P: wsc**CAT** $\to$ wsc**CAT**,

that shifts down all components, discarding the structure of index 1; the faces and degeneracies of index 1 are then used to build three transversal transformations, the *faces* and *degeneracy* of P

(4) $P\mathbb{A} = ((\mathbb{A}_{n+1}), (\wedge_{i+1}^\wedge), (e_{i+1}), (s_{i+1}), (+_{i+1}), \wedge_2, \wedge_2, \wedge_2, \wedge_2)$,
$\wedge^\wedge = \wedge_1^\wedge: P\mathbb{A} \to \mathbb{A}$, $\quad e = e_1: \mathbb{A} \to P\mathbb{A}$.

Here, $\wedge^\wedge$ and e are strict sc-functors: $\wedge_i^\wedge \wedge_1^\wedge = \wedge_1^\wedge \wedge_{i+1}^\wedge$, etc. A *cubical* (or *geometric*) *transformation of sc-functors* $F: F^- \to F^+: \mathbb{A} \to \mathbb{B}$ is an sc-functor $F: \mathbb{A} \to P\mathbb{B}$ with $\wedge^\wedge F = F^\wedge$ (cf. I.3.7).

1.5. Lax sc-functors. We will also need more general notions, that have not been explicitly defined in Part I.

A *lax symmetric cubical functor* $F: \mathbb{A} \to \mathbb{B}$ between weak sc-categories, or *lax sc-functor,* strictly preserves faces, transpositions, transversal composition and transversal identities, but has special transversal maps, called *comparisons*, for the cubical operations, namely degeneracies and concatenation in direction 1 (those of the other cubical directions being generated by transpositions):

(1) $F_1(x): e_1(Fx) \to F(e_1x)$ $\qquad$ (x in $\mathbb{A}$),
$F_1(x, y): Fx +_1 Fy \to F(x +_1 y)$ $\qquad$ (x, y in $\mathbb{A}$, $\wedge_1^+ x = \wedge_1^- y$).

(Recall that a transversal n-map is said to be *special* if its 2^n vertices are identities.) These comparisons must satisfy the following axioms of coherence:

(i) (*naturality*) for a transversal n-map $f: x \to x'$ in $\mathbb{A}$, and a cubical composition $f +_1 g$ (with $g: y \to y'$), we have the following commutative diagrams of transversal maps

$$
(2) \qquad
\begin{array}{ccc}
e_1(Fx) & \xrightarrow{e_1(Ff)} & e_1(Fx') \\
{\scriptstyle F_1(x)} \downarrow & & \downarrow {\scriptstyle F_1(x')} \\
F(e_1(x)) & \xrightarrow[F(e_1 f)]{} & F(e_1(x'))
\end{array}
\qquad
\begin{array}{ccc}
Fx +_1 Fy & \xrightarrow{Ff +_1 Fg} & Fx' +_1 Fy' \\
{\scriptstyle F_1(x,y)} \downarrow & & \downarrow {\scriptstyle F_1(x',y')} \\
F(x +_1 y) & \xrightarrow[F(f +_1 g)]{} & F(x' +_1 y')
\end{array}
$$

(ii) (*coherence laws for cubical identities*) for an n-cube x in $\mathbb{A}$, with 1-indexed faces $\wedge_1^- x = a$, $\wedge_1^+ x = b$, the following diagrams of transversal maps commute

$$
(3) \qquad
\begin{array}{ccc}
e_1(Fa) +_1 Fx & \xrightarrow{\wedge_1(Fx)} & Fx \\
{\scriptstyle F_1 a +_1 \mathrm{id}} \downarrow & & \uparrow {\scriptstyle F(\wedge_1 x)} \\
F(e_1 a) +_1 Fx & \xrightarrow[F_1(e_1 a,\, x)]{} & F(e_1 a +_1 x)
\end{array}
\qquad
\begin{array}{ccc}
Fx +_1 e_1(Fb) & \xrightarrow{\wedge_1(Fx)} & Fx \\
{\scriptstyle \mathrm{id} + F_1 b} \downarrow & & \uparrow {\scriptstyle F(\wedge_1 x)} \\
Fx +_1 F(e_1 b) & \xrightarrow[F(x,\, e_1 b)]{} & F(x +_1 e_1 b)
\end{array}
$$

(iii) (*coherence hexagon for associativity*) for 1-consecutive n-cubes $x, y\, z$ in $\mathbb{A}$, the following diagram of transversal maps is commutative (the index 1 is omitted in the labels of arrows)

$$
(4) \qquad
\begin{array}{ccc}
Fx +_1 (Fy +_1 Fz) & \xrightarrow{\wedge(Fx,Fy,Fz)} & (Fx +_1 Fy) +_1 Fz \\
{\scriptstyle \mathrm{id} + F(y,z)} \downarrow & & \downarrow {\scriptstyle F(x,y) + \mathrm{id}} \\
Fx +_1 F(y +_1 z) & & F(x +_1 y) +_1 Fz \\
{\scriptstyle F(x,\, y+z)} \downarrow & & \downarrow {\scriptstyle F(x+y,\, z)} \\
F((x +_1 (y +_1 z)) & \xrightarrow[F\wedge(x,y,z)]{} & F(x +_1 y) +_1 z
\end{array}
$$

(iv) (*coherence hexagon for interchange*) for n-cubes x, y, z, u in $\mathbb{A}$ making the following concatenations legitimate, the following diagram of transversal maps is commutative (omitting the indices 1, 2 in the labels of arrows)

$$
(5)\qquad
\begin{array}{ccc}
(Fx +_1 Fy) +_2 (Fz +_1 Fu) & \xrightarrow{\ \wedge(Fx,Fz,Fu,Fu)\ } & (Fx +_2 Fz) +_1 (Fy +_2 Fu) \\
\downarrow {\scriptstyle F(x,y) + F(z,u)} & & \downarrow {\scriptstyle F(x,z) + F(y,u)} \\
F(x +_1 y) +_2 F(z +_1 u) & & F(x +_2 z) +_1 F(y +_2 u) \\
\downarrow {\scriptstyle F(x+y,z+u)} & & \downarrow {\scriptstyle F(x+z,y+u)} \\
F((x +_1 y) +_2 (z +_1 u)) & \xrightarrow[\ F\wedge(x,y,z,u)\ ]{} & F((x +_2 z) +_1 (y +_2 u))
\end{array}
$$

A *pseudo sc-functor* is a lax sc-functor whose comparisons are invertible. A lax sc-functor F is said to be *unitary* if its unit comparisons $F_1(x)$ are identities. If $\mathbb{A}$, $\mathbb{B}$ and F are unitary, the cells $F_1(e_1\wedge_1^-x, x)$ and $F_1(x, e_1\wedge_1^+x)$ are also identities (by (ii)).

1.6. Transformations of lax sc-functors. (a) A *transversal transformation* of lax sc-functors $h: F \to G: \mathbb{A} \to \mathbb{B}$ assigns to every n-cube x of $\mathbb{A}$ an n-map $hx: Fx \to Gx$ in $\mathbb{B}$. This family must commute with faces and transpositions and satisfy the coherence conditions (ii) for degeneracies and cubical compositions:

(i) $\wedge_i^\wedge(hx) = h(\wedge_i^\wedge x)$, $\qquad h(s_ix) = s_i(hx)$,

(ii) for an n-cube x and a 1-consecutive n-cube y in $\mathbb{A}$

$$
(1)\qquad
\begin{array}{ccc}
e_1(Fx) & \xrightarrow{\ e_1(hx)\ } & e_1(Gx) \\
\downarrow {\scriptstyle F_1(x)} & & \downarrow {\scriptstyle G_1(x)} \\
G(e_1x) & \xrightarrow[\ h(e_1x)\ ]{} & G(e_1x)
\end{array}
\qquad\qquad
\begin{array}{ccc}
Fx +_1 Fy & \xrightarrow{\ hx +_1 hy\ } & Gx +_1 Gy \\
\downarrow {\scriptstyle F_1(x,\, y)} & & \downarrow {\scriptstyle G_1(x,\, y)} \\
F(x +_1 y) & \xrightarrow[\ h(x +_1 y)\ ]{} & G(x +_1 y)
\end{array}
$$

Weak sc-categories, lax sc-functors and their transversal transformations form a 2-category Lsc**CAT**.

(b) Using the path functor P: wsc**CAT** $\to$ wsc**CAT** of weak sc-categories (1.4), a *cubical* (or *geometric*) *transformation of lax sc-functors* $F: F^- \to F^+: \mathbb{A} \to \mathbb{B}$ will be a lax sc-functor $F: \mathbb{A} \to P\mathbb{B}$ with $\wedge^\wedge F = F^\wedge$ $(\wedge = \pm)$.

Thus, if x is an n-cube of $\mathbb{A}$, Fx is an (n+1)-cube of $\mathbb{B}$ with $\wedge_1^\wedge(Fx) = F^\wedge(x)$.

1.7. Level functors. We are also interested in the 2-functor

(1) tv_n: wsc**CAT** $\to$ **CAT**,

that sends a weak sc-category $\mathbb{A}$ to the ordinary category $\mathrm{tv}_n\mathbb{A}$ (often written as $\mathbb{A}_n$) of its n-cubes and n-transversal maps (I.3.3, I.3.4); in particular, $\mathrm{tv}_0 = \mathrm{tr}_0$. The left adjoint

(2) $\mathrm{w}\mathbb{SC}_n\colon \mathbf{CAT} \to \mathrm{wsc}\mathbf{CAT}, \qquad\qquad \mathrm{w}\mathbb{SC}_n \dashv \mathrm{tv}_n,$

sends a category $\mathbf{X}$ to the *free weak symmetric cubical category generated by* $\mathbf{X}$ *at level* n. (Its existence follows from the Freyd Adjoint Theorem.)

A functor $F\colon \mathbf{X} \to \mathrm{tv}_n\mathbb{A}$, or equivalently a symmetric cubical functor $\mathrm{w}\mathbb{SC}_n\mathbf{X} \to \mathbb{A}$, will be called an *n-level functor* with values in the weak sc-category $\mathbb{A}$.

2. Level limits in weak symmetric cubical categories

We deal here with symmetric cubical limits (or sc-limits) of *level* functors $F\colon \mathbf{X} \to \mathrm{tv}_n\mathbb{A}$,; these are ordinary limits that are required to be preserved by the functors $\mathrm{tv}_n\mathbb{A} \to \mathrm{tv}_m\mathbb{A}$ of the symmetric cubical structure. Thus, a product of n-cubes is an n-cube, and to say that $\mathbb{A}$ has symmetric cubical products (or sc-products) means that such products exist in every degree and are preserved by faces, degeneracies and transpositions. Of course, $\mathbb{A}$ has level symmetric cubical limits if and only if it has sc-products and sc-equalisers (2.2).

The parallel case without symmetries works in the same way and is only mentioned. (Crucial differences will appear in the next section, for general limits.)

2.1. Level limits. Let $\mathbb{A}$ be a weak symmetric cubical category.

An *n-level limit* in $\mathbb{A}$ will be the ordinary (1-categorical) limit of an n-level functor $F\colon \mathbf{X} \to \mathrm{tv}_n\mathbb{A}$, defined on a small category (cf. 1.7). This means an n-cube a of $\mathbb{A}$ equipped with a universal natural transformation $t\colon Da \to F\colon \mathbf{X} \to \mathrm{tv}_n\mathbb{A}$, where $Da\colon \mathbf{X} \to \mathrm{tv}_n\mathbb{A}$ is the constant functor at a (or, equivalently, a universal transversal transformation $Da \to F\colon \mathrm{w}\mathbb{SC}_n\mathbf{X} \to \mathbb{A}$ of the corresponding symmetric cubical functors).

Such a limit is called a *level symmetric cubical limit* if it is preserved by all functors $\mathrm{tv}_n\mathbb{A} \to \mathrm{tv}_m\mathbb{A}$ generated by faces, degeneracies and transpositions, for arbitrary m. (In other words, we want the limit to be preserved by all structural functors $\mathrm{tv}_n\mathbb{A} \to \mathrm{tv}_m\mathbb{A}$, corresponding to the maps $2^m \to 2^n$ of the symmetric cubical site $\mathbb{I}_s$, cf. I.2.1 or [GM].)

We say that $\mathbb{A}$ *has level limits*, or that it has *level symmetric cubical limits* (possibly *on a given* category $\mathbf{X}$), if all these exist. Obviously, an n-level limit is called a *product* (of level n) if $\mathbf{X}$ is discrete and an *equaliser* (of level n) if $\mathbf{X}$ is the category $0 \rightrightarrows 1$.

Plainly, $\mathbb{A}$ has level symmetric cubical limits on a given (small) category $\mathbf{X}$ if and only if:

(i) for every $n \geq 0$, every functor $F: \mathbf{X} \to tv_n\mathbb{A}$ has an (ordinary) limit;

(ii) such limits are preserved by all faces $\geq_i^{\geq}: tv_n\mathbb{A} \to tv_{n-1}\mathbb{A}$, degeneracies e_i: $tv_{n-1}\mathbb{A} \to tv_n\mathbb{A}$ and transpositions $s_i: tv_n\mathbb{A} \to tv_n\mathbb{A}$.

Level colimits and level sc-colimits are defined in the dual way.

Symmetries are not crucial in this section. If $\mathbb{A}$ is a weak cubical category, one can define in the same way *level (co)limits*, and – with obvious modifications (i.e. omitting transpositions) – *level cubical (co)limits.*

2.2. Theorem. (Construction and preservation of level limits). *Let* **A** *be a weak symmetric cubical category.*

(a) All level limits in $\mathbb{A}$ *can be constructed from products and equalisers.*

(b) All level symmetric cubical limits in $\mathbb{A}$ *can be constructed from symmetric cubical products and symmetric cubical equalisers.*

(c) If $\mathbb{A}$ *has all level limits (resp. level symmetric cubical limits), a symmetric cubical functor* $F: \mathbb{A} \to \mathbb{B}$ *with values in a weak symmetric cubical category preserves them if and only if it preserves products and equalisers (resp. the corresponding symmetric cubical limits).*

(d) Similar results hold in the non-symmetric case (omitting symmetries everywhere).

Proof. It is a straightforward consequence of a well-known theorem on ordinary limits. □

2.3. Examples. The following structures, introduced in [G1] or Part I and partially reviewed in Section 1, have all level symmetric cubical limits:

– the weak sc-category $\geq\mathbb{S}\mathrm{p}(\mathbf{X})$ of cubical spans on a complete category $\mathbf{X}$ equipped with a full choice of distinguished pullbacks (1.1);

– the weak sc-category $\geq\mathbb{C}\mathrm{osp}(\mathbf{X})$ of cubical cospans on a complete category $\mathbf{X}$ equipped with a full choice of distinguished pushouts (1.1);

– the strict sc-category $\geq\mathbb{C}\mathrm{ub}(\mathbf{X})$ of commutative cubes on a complete category $\mathbf{X}$ (1.3);

– the strict sc-category $\geq\mathbb{R}\mathrm{el}$ of cubical relations of sets (I.4.2, I.5.6);

– the weak sc-category $\geq\mathbb{C}\mathrm{at}$ of cubical profunctors (I.5.7).

For instance, if $\mathbb{A} = {\geq}\mathbb{S}p(\mathbf{X})$, the product $x = \geq x_i$ of a small family of n-cubical spans is the obvious n-cubical span with universal transversal maps $p_i: x \to x_i$; it is computed as a product in the functor category $\mathbf{CAT}(\geq^n, \mathbf{X})$, and obviously preserved by faces degeneracies, and transpositions. The equaliser of a pair of n-maps $f, g: x \to y$ is also computed as an equaliser of morphisms in $\mathbf{X}^{\geq^n}$, and preserved as above. Similarly in ${\geq}\mathbb{C}osp(\mathbf{X})$ and ${\geq}\mathbb{C}ub(\mathbf{X})$.

If $\mathbb{A} = {\geq}\mathbb{R}el$, the product $x = \geq x_i$ of a small family of n-cubical relations is computed, again, as a cartesian product of the 'graphs' of the relations that intervene in the factors x_i.

2.4. Level limits as lax cubical functors. Condition (i) of the definition of level sc-limit (2.1) says that, for every $n \geq 0$, the diagonal functor $D_n: tv_n\mathbb{A} \to \mathbf{CAT}(\mathbf{X}, tv_n\mathbb{A})$ has a right adjoint

(1) $\lim_n: \mathbf{CAT}(\mathbf{X}, tv_n\mathbb{A}) \to tv_n\mathbb{A}$.

Condition (ii) says that these functors are the components of a *morphism of symmetric cubical objects* in **CAT**

(2) $\lim = (\lim_n)_{n\geq 0}: \mathbf{CAT}(\mathbf{X}, |\mathbb{A}|) \to |\mathbb{A}|$.

Here, $|\mathbb{A}|$ denotes the underlying sc-object, where we forget the concatenation laws. Taking also such compositions into account, the universal property yields a *unitary lax symmetric cubical functor*, defined on the weak symmetric cubical category $\mathbb{L}v(\mathbf{X}, \mathbb{A})$ of level functors and their natural transformations (I.3.7)

(3) $\lim = (\lim_n)_{n\geq 0}: \mathbb{L}v(\mathbf{X}, \mathbb{A}) \to \mathbb{A}$.

Therefore, if $\mathbb{A}$ has all level sc-limits, we will also say that it has *lax functorial* level sc-limits. More particularly, we say that it has *pseudo functorial level sc-limits* if (3) happens to be a *pseudo* cubical functor.

A similar terminology will be used for products, equalisers, or any 'type' of limit. Colimits and the non-symmetric case give rise to a similar terminology.

It is easy to see that sc-limits are pseudo-functorial in ${\geq}\mathbb{S}p(\mathbf{X})$ and lax-functorial in ${\geq}\mathbb{C}osp(\mathbf{X})$.

2.5. Remarks. For the extensions in the next section, it will be useful to review the definition of the n-level limit $(a, t: Da \to F)$ of a functor $F: \mathbf{X} \to tv_n\mathbf{A}$ in a different form, internal to (weak) symmetric cubical categories. We replace:

– $\mathbf{X}$ with the weak sc-category $\mathbb{X} = w\mathbb{S}\mathbb{C}_0\mathbf{X}$ freely generated by the category $\mathbf{X}$ *at level* 0,

– the functor $F: \mathbf{X} \to tv_n\mathbf{A} = tv_0P^n\mathbb{A}$ with the corresponding sc-functor $F: \mathbb{X} \to P^n\mathbb{A}$;
– the n-cube a of $\mathbb{A}$ (a 0-cube of $P^n\mathbb{A}$) with the corresponding constant sc-functor $Da: \mathbb{X} \to P^n\mathbb{A}$;
– the natural transformation t with the corresponding transversal transformation of sc-functors $t: Da \to F: \mathbb{X} \to P^n\mathbb{A}$.

Now, the weak sc-categories $P^n\mathbb{A}$ form a symmetric cubical object $P^\bullet\mathbb{A}$ in wsc**CAT**, with the obvious faces, degeneracies and transpositions (I.3.7.3).

Therefore, saying that **A** has *level sc-limits* on **X** also amounts to saying that the limit functors

(1) $\lim_n$: wsc**CAT**$(\mathbb{X}, P^n\mathbb{A}) \to tv_n\mathbb{A}$,

produce a lax sc-functor defined on the weak sc-category $\mathbb{A}^{\mathbb{X}}$ of higher sc-functors from $\mathbb{X}$ to $\mathbb{A}$ and their transversal transformations (I.3.7(c))

(2) $\lim = (\lim_n)_{n\in\mathbb{N}}: \mathbb{A}^{\mathbb{X}} = \mathbb{W}sc(\mathbb{X}, \mathbb{A}) \to \mathbb{A}$.

3. General limits in weak symmetric cubical categories

We now consider general limits in weak *symmetric* cubical categories, taking advantage of their path functor P (1.4, I.3.6). $\mathbb{X}$ is assumed to be a small weak sc-category, while $F: \mathbb{X} \to \mathbb{A}$ is a lax sc-functor, viewed as an object in the category Lsc**CAT**$(\mathbb{X}, \mathbb{A})$ of lax sc-functors $\mathbb{X} \to \mathbb{A}$ and transversal transformations (1.6).

3.1. Motivation. Limits of lax sc-functors with values in $P^p\mathbf{A}$ will be called *sc-limits of degree p in* $\mathbb{A}$. Let us begin with some simple examples, based on a 2-cube x in the weak sc-category $\mathbb{A}$, introducing definitions that will be made precise below (in 3.4, 3.5).

(a) The *tabulator of degree zero* of the 2-cube x will be an object T_2x (i.e. a 0-cube) with a universal 2-map $h: e^2(T_2x) \to x$ (where $e^2 = e_1e_1 = e_2e_1: \mathbb{A}_0 \to \mathbb{A}_2$ is the composed degeneracy). For instance:

– for $\mathbb{A} = \sqcap\mathbb{S}p(\mathbf{Set})$, T_2x is the central object x_{00} of the 2.cube $x: \mathbf{\Pi}^2 \to \mathbf{Set}$;
– for $\mathbb{A} = \sqcap\mathbb{C}osp(\mathbf{Set})$, T_2x is the limit in **Set** of the diagram $x: \mathbf{\Pi}^2 \to \mathbf{Set}$.

(b) But the 2-cube x can also be viewed as a 1-cube of $P\mathbb{A}$. Its *tabulator of degree one* will be the tabulator of degree zero of x as a 1-cube of $P\mathbb{A}$; this amounts to a 1-cube $T_{2,1}x$ of $\mathbb{A}$ with a universal 2-map $h: e_2(T_{2,1}x) \to x$ (where $e_2: \mathbb{A}_1 \to \mathbb{A}_2$ is *the* degeneracy $(P\mathbb{A})_0 \to (P\mathbb{A})_1$). For instance:

– for $\mathbb{A} = \sqcap\mathbb{S}p(\mathbf{Set})$, the span $T_{2,1}x$ is the central part of the 2-cubical span $x: \mathbf{\Pi}^2 \to \mathbf{Set}$ with respect to direction 2;

– for $\mathbb{A} = \sqcap\mathbb{C}osp(\mathbf{Set})$, $T_{2,1}x$ is computed by taking the limit in **Set** of the three cospans obtained from $x: \mathbf{\Pi}^2 \to \mathbf{Set}$, by restriction to $\{i\}\bullet\mathbf{\Pi}$, for $i = -1, 0, 1$.

(Notice that the tabulator of degree one of the symmetric 2-cube s_1x is a 1-cube a with a universal 2-map $e_2(a) \to s_1(x)$, i.e. $e_1(a) \to x$. We need not consider and *name* universal problems that can be reduced to some previous case by the use of symmetries.)

(c) Finally, the 2-cube x is a 0-cube of $P^2\mathbb{A}$. Its *tabulator of degree two* is x itself. Notice that this is a (trivial) level limit, while the previous limits are not level: the data and the solution are not contained in one transversal category $tv_n\mathbb{A}$.

3.2. Cones. Let $\mathbb{X}$ and $\mathbb{A}$ be weak sc-categories, and let $\mathbb{X}$ be small. Consider the *diagonal* functor

(1) $D: tv_0\mathbb{A} \to \mathrm{wsc}\mathbf{CAT}(\mathbb{X}, \mathbb{A})$.

D takes each 0-object a to the constant sc-functor, defined as follows on n-objects and n-maps of $\mathbb{X}$

(2) $Da: \mathbb{X} \to \mathbb{A}, \qquad Da(x) = e^n(a), \qquad Da(u) = id(e^na) \qquad (x, u \text{ in } tv_n\mathbb{X}),$

and every 0-map $f: a \to b$ in $\mathbb{A}$ to the diagonal transversal transformation

(3) $Df: Da \to Db: \mathbb{X} \to \mathbb{A}, \qquad (Df)(x) = e^n(f): e^n(a) \to e^n(b) \quad (x \text{ in } tv_n\mathbb{X}).$

Da is a *strict* sc-functor, because $\mathbb{A}$ is assumed to be semi-unitary (1.2(c)).

Let $F: \mathbb{X} \to \mathbb{A}$ be a *lax* sc-functor (1.5), with comparison special cells $F_1(x): e_1(Fx) \to F(e_1(x))$ and $F_1(x, y): Fx +_1 Fy \to F(x +_1 y)$. A *(transversal) sc-cone* for F is a transversal transformation $h: Da \to F: \mathbb{X} \to \mathbb{A}$, where a (the *vertex* of the cone) is in $tv_0\mathbb{A}$. By definition (1.6), this amounts to assigning the following data:

– a transversal n-map $hx: e^n(a) \to Fx$, for every n-object x in $\mathbb{X}$,

subject to the following axioms:

(scc.1) $Ff.hx = hy \qquad (f: x \to y \text{ in } \mathbb{X});$

(scc.2) h commutes with faces and transpositions and $h(e_1(x)) = F_1(x).(e_1(h(x));$

(scc.3) $h(x +_1 y) = F_1(x, y).(hx +_1 hy): e^n(a) \to F(x +_1 y) \qquad (\partial_1^+ x = \partial_1^- y),$

$$
(4)\qquad
\begin{array}{ccc}
e^{n+1}(a) & \xrightarrow{e_1(hx)} & e_1(Fx) \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle F_1(x)} \\
e^{n+1}(a) & \xrightarrow[h(e_1x)]{} & F(e_1(x))
\end{array}
\qquad
\begin{array}{ccc}
e^n(a) +_1 e^n(a) & \xrightarrow{hx +_1 hy} & Fx +_1 Fy \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle F_1(x,y)} \\
e^n(a) & \xrightarrow[h(x +_1 y)]{} & F(x +_1 y)
\end{array}
$$

(Again, we are using the semi-unitarity of $\mathbb{A}$ in the right diagram above.) More precisely (as $\mathbb{X}$ might be empty, in which case a is not determined by Da), a cone of F is a *pair* (a, h: Da → F) as above, i.e. an object of the ordinary comma category (D↓F), where F is viewed as an object of the category Lsc**CAT**($\mathbb{X}$, $\mathbb{A}$).

3.3. Definition (Limits and cubical limits). A *(transversal) limit* lim(F) = (a, h) of the lax sc-functor F ∈ Lsc**CAT**($\mathbb{X}$, $\mathbb{A}$) is a universal cone (a, h: Da → F). In other words:

(tl.0) a is an object of $\mathbb{A}$ and h: Da → F is a transversal transformation of lax sc-functors;

(tl.1) for every cone (a', h': Da' → F) there is precisely one 0-map t: a' → a in $\mathbb{A}$ such that h.Da = h'.

We say that $\mathbb{A}$ has *limits of degree zero* on $\mathbb{X}$ if all these exist. We say that $\mathbb{A}$ has *limits of all degrees* on $\mathbb{X}$ if all sc-categories $P^n\mathbb{A}$ satisfy this condition, for n ∈0.

We say that $\mathbb{A}$ has *symmetric cubical limits on* $\mathbb{X}$, or *lax functorial sc-limits based on* $\mathbb{X}$, if:

(i) $\mathbb{A}$ has limits of all degrees on $\mathbb{X}$;

(ii) the limit-functors $\lim_n$: Lsc**CAT**($\mathbb{X}$, $P^n\mathbb{A}$) → $tv_n\mathbb{A}$ commute with faces, degeneracies and transpositions.

Then the universal property gives a unitary *lax* sc-functor

(1) lim = $(\lim_n)_{n\in 0}$: Lsc**CAT**($\mathbb{X}$, $P^\bullet\mathbb{A}$) → $\mathbb{A}$.

We say that $\mathbb{A}$ has *pseudo functorial sc-limits on* $\mathbb{X}$ if this lax sc-functor happens to be a pseudo sc-functor.

In particular, if $\mathbb{X} = \mathrm{w}\mathbb{S}\mathbb{C}_0\mathbf{X}$ is the weak sc-category freely generated by a category **X**, *at level* 0, then $\mathbb{A}$ has sc-limits on **X** if and only if it has level sc-limits on the category **X** (cf. 2.5).

Without symmetries, things would become complicated. While the condition of having limits of degree zero can be directly extended to cubical categories, *the conditions (i), (ii) should (perhaps) be rewritten replacing each* P^n *with the family* *of all path functors of degree n,* namely $P_i^n = P^{n-i}.SP^iS$ for $i = 0,..., n$ (I.1.8). We will not deal with such a situation, except in the particular case of level limits (2.5), where these conditions have already been expressed in a simpler form.

3.4. Tabulators of degree zero. Let $\mathbb{A}$ be a weak symmetric cubical category. The 'total' degeneracy

(1) $(e_1)^n = e_1 \dots e_1 = \dots = e_n \dots e_1 \colon \mathbb{A}_0 \to \mathbb{A}_n,$

will be written as e^n (it is the unique composed degeneracy $\mathbb{A}_0 \to \mathbb{A}_n$).

An n-cube x of $\mathbb{A}$ can be viewed as a strict sc-functor $x \colon \mathbf{u}_n \to \mathbb{A}$, where $\mathbf{u}_n$ is the strict sc-category freely generated by one n-cube $u^{(n)}$. The *tabulator of degree zero* of x in $\mathbb{A}$ is the limit of this sc-functor $x \colon \mathbf{u}_n \to \mathbb{A}$. (The term 'degree zero' refers to the degree of the solution.)

The tabulator is thus an object $x_0 = T_n x$ equipped with an n-map $t \colon e^n(x_0) \to x$, universal in the obvious sense: the pair $(x_0, t_x \colon e^n(x_0) \to x)$ is a universal arrow from the functor $e^n \colon tv_0\mathbb{A} \to tv_n\mathbb{A}$ to the object x of $tv_n\mathbb{A}$. Explicitly, this means that, for every object x_1 and every n-map $f \colon e^n(x_1) \to x$ there is a unique 0-map h such that

(2) $h \colon x_1 \to x_0$

$t_x.e^n(h) = f$

$$\begin{array}{ccc} e^n(x_1) & \xrightarrow{e^n(h)} & e^n(x_0) \\ & \searrow^{f} & \downarrow t_x \\ & & x \end{array}$$

We say that $\mathbb{A}$ *has tabulators of degree zero* if all these exist, for arbitrary $n \in \mathbb{N}$. Obviously, the tabulator of an object always exists, and is the object itself.

When such tabulators exist, we can form for every $n \in \mathbb{N}$ a functor

(3) $T_n \colon \mathrm{wsc}\mathbf{CAT}(\mathbf{u}_n, \mathbb{A}) \to tv_0\mathbb{A}$ $\qquad\qquad (T_0 = \mathrm{id}).$

The *projection* $p_i^{\epsilon}x$ of $T_n x$ will be the following 0-map of $\mathbb{A}$ (for $i = 1,..., n$ and $\epsilon = \pm$)

(4) $p_i^{\epsilon}x \colon T_n x \to T_{n-1}(\partial_i^{\epsilon}x)$

$t_z.e^{n-1}(p_i^{\epsilon}x) = \partial_i^{\epsilon}(t_x)$

$$\begin{array}{ccc} e^{n-1}T_n x & \xrightarrow{e^{n-1}(p_i^{\epsilon}x)} & e^{n-1}T_{n-1}(\partial_i^{\epsilon}x) \\ & \searrow^{\partial_i^{\epsilon}(t_x)} & \downarrow t_z \\ & & z = \partial_i^{\epsilon}x \end{array}$$

One can use these projections to 'map' the sc-category $\mathbf{A}$ to the sc-category of spans $\mathbf{Sp}(\mathbb{A}_0)$, provided that $\mathbb{A}$ has all tabulators of degree zero and $\mathbb{A}_0$ has pullbacks.

3.5. Tabulators of higher degree. Now, given an n-cube x *of* $P^p\mathbb{A}$ (of degree $n+p$ in $\mathbb{A}$), its tabulator – if extant – is an object of $P^p\mathbb{A}$. This amounts to a p-cube $x_p = T_{np}x$ of $\mathbf{A}$ with an (n+p)-map $t_x \colon (e_{p+1})^n(x) \to x$ that is a universal arrow from the functor $(e_{p+1})^n \colon tv_p\mathbb{A} \to tv_{n+p}\mathbb{A}$ to the object x of $tv_{n+p}\mathbb{A}$.

We say that $\mathbb{A}$ *has tabulators of all degrees* if, for every $p \geq 0$, the sc-category $P^p\mathbb{A}$ has tabulators of degree zero. Then we can form for every $n, p \geq 0$ a functor

$$(1)\quad T_{n,p} \colon \mathrm{wsc}\mathbf{CAT}(\mathbf{u}_n, P^p\mathbb{A}) \to tv_p\mathbb{A}, \qquad (T_{n,0} = T_n).$$

We say that $\mathbb{A}$ has *sc-tabulators*, or *lax functorial sc-tabulators*, if:

(i) $\mathbf{A}$ has tabulators of all degrees;

(ii) for every $n \geq 0$, the functors $T_{n,p} \colon \mathrm{wsc}\mathbf{CAT}(\mathbf{u}_n, P^p\mathbb{A}) \to tv_p\mathbb{A}$ commute with faces, degeneracies and transpositions.

Then, for every $n \geq 0$, the universal property gives a unitary *lax* sc-functor

$$(2)\quad T_{n,\bullet} = (T_{n,p})_{p\geq 0} \colon \mathrm{wsc}\mathbf{CAT}(\mathbf{u}_n, P^\bullet\mathbb{A}) \to \mathbb{A}.$$

Again, $\mathbb{A}$ has *pseudo functorial sc-tabulators* when these lax sc-functors are pseudo..

3.6. Tabulators and concatenation. First, if the (n–1)-cube a and the degenerate n-cube $x = e_ia$ have tabulators in $\mathbb{A}$, these are linked by a *diagonal* transversal 0-map d_ia, defined as follows

$$(1)\quad d_ia \colon T_{n-1}a \to T_n(e_ia), \qquad t_x.e^n(d_ia) = e_i(t_a)$$

$$\begin{array}{ccc} e^n(T_{n-1}a) & \xrightarrow{\;e^n(d_ia)\;} & e^n(T_n(e_ia)) \\ & {\scriptstyle e_i t_a}\searrow & \downarrow {\scriptstyle t_x} \\ & & e_ia = x \end{array}$$

Given now a cubical composite $z = x +_i y$, the three tabulators of x, y, z are also related. The link goes through the ordinary pullback $T_{ni}(x, y)$ of the objects T_nx and T_ny, over the tabulator $T_{n-1}a$ of the (n–1)-cube $a = \partial^+_i x = \partial^-_i y$ (provided such pullback exists).

More precisely, let $p_{ix} \colon T_nx \to T_{n-1}a$ and $q_{iy} \colon T_ny \to T_{n-1}a$ be defined by the universal property of t_a, as in the left diagram below; then $T_{ni}(x, y)$ is the pullback of the span (p_{ix}, q_{iy})

$$
(2)\qquad
\begin{array}{ccc}
e^{n-1}T_nx & \searrow^{\partial_i^+t_x} & \\
{\scriptstyle e^{n-1}p_{ix}}\downarrow & & \\
e^{n-1}T_{n-1}a & \xrightarrow{\ t_a\ } & a \\
{\scriptstyle e^{n-1}q_{iy}}\uparrow & & \\
e^{n-1}T_ny & \nearrow_{\partial_i^-t_y} &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & T_nx & & \\
 & {\scriptstyle p_i(x,y)}\nearrow & & \searrow{\scriptstyle p_{ix}} & \\
T_{ni}(x,y) & & & & T_{n-1}a \\
 & {\scriptstyle q_i(x,y)}\searrow & & \nearrow{\scriptstyle q_{iy}} & \\
 & & T_ny & &
\end{array}
$$

We now have the *diagonal* transversal 0-map $d_i(x, y)$ given by the universal property of T_nz

(3) $d_i(x, y): T_{ni}(x, y) \to T_nz, \qquad t_z.e^n(d_i(x, y)) = t_x.e^np_i(x, y) +_i t_y.e^nq_i(x, y),$

The i-concatenation above is legitimate, because of the previous construction:

$$\partial_i^+(t_x.e^np_i(x, y)) = \partial_i^+(t_x).e^{n-1}p_i(x, y) = t_a.e^{n-1}(p_{ix}).e^{n-1}p_i(x, y)$$
$$= t_a.e^{n-1}(q_{iy}).e^{n-1}q_i(x, y) = \partial_i^-(t_y).e^{n-1}q_i(x, y) = \partial_i^-(t_y.e^nq_i(x, y)).$$

It is easy to show (and it also follows from the construction theorem below) that $T_{ni}(x, y)$ is the transversal limit of the diagram 'formed' of $z = x +_i y$ (based on the weak sc-category freely generated by two i-consecutive n-cubes).

3.7. Theorem (Construction and preservation of cubical limits, I). *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be weak sc-categories.*

(a) All transversal limits of degree 0 in $\mathbb{A}$ *can be constructed from level limits and tabulators of degree 0, or also from products, equalisers and tabulators of degree* 0.

(b) *If* $\mathbb{A}$ *has all transversal limits of degree 0, an sc-functor* $F: \mathbb{A} \to \mathbb{B}$ *preserves them if and only if it preserves products, equalisers and tabulators of degree 0.*

Proof. See Section 5.

3.8. Main Theorem (Construction and preservation of cubical limits, II). *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be weak sc-categories.*

(a) All transversal limits in $\mathbb{A}$ *can be constructed from level limits and tabulators, or also from products, equalisers and tabulators. If* $\mathbb{A}$ *has all transversal limits, an sc-functor* $F: \mathbb{A} \to \mathbb{B}$ *preserves them if and only if it preserves products, equalisers and tabulators.*

(b) All lax-functorial (resp. pseudo-functorial) sc-limits in $\mathbb{A}$ *can be constructed from lax-functorial (resp. pseudo-functorial) sc-products, sc-equalisers and sc-*

tabulators. If $\mathbb{A}$ *has all sc-limits, an sc-functor* $F\colon \mathbb{A} \to \mathbb{B}$ *preserves them if and only if it preserves sc-products, sc-equalisers and sc-tabulators.*

Proof. It follows from the previous theorem. For (a), apply 3.7 to the family of sc-categories $P^n\mathbb{A}$, and sc-functors $P^nF\colon P^n\mathbb{A} \to P^n\mathbb{B}$. For (b), apply the previous point to the structural sc-functors

(1) $\partial_i^\partial\colon P^n\mathbb{A} \rightleftarrows P^{n-1}\mathbb{A} :e_i, \qquad s_i\colon P^n\mathbb{A} \to P^n\mathbb{A}.$ □

3.9. Corollary. *If* **X** *is a complete category, the weak sc-category* $\partial\,\mathbb{S}\mathrm{p}(\mathbf{X})$ *has pseudo-functorial sc-limits. If* **X** *is a complete category with pushouts,* $\partial\,\mathbb{C}\mathrm{osp}(\mathbf{X})$ *has lax-functorial sc-limits.*

Proof. The construction of sc-tabulators was shown in 3.1; their pseudo or lax behaviour is easily verified. □

4. Comparing cubical and double limits

We now compare the symmetric cubical limits studied here with the double limits of [GP1], via truncation and its adjoint functors. Essentially, this means comparing cubical tabulators of 1-cubes and double tabulators of vertical arrows, since the comparison of level limits is obvious.

4.1. Comments. Our comparison will be based on structures of spans and cospans of *sets*, namely:

– the weak sc-categories $\partial\,\mathbb{S}\mathrm{p} = \partial\,\mathbb{S}\mathrm{p}(\mathbf{Set})$ and $\partial\,\mathbb{C}\mathrm{osp} = \partial\,\mathbb{C}\mathrm{osp}(\mathbf{Set})$ of cubical spans and cospans;

– their 1-truncation, the weak double categories $\mathbb{S}\mathrm{p} = \mathrm{tr}_1\partial\,\mathbb{S}\mathrm{p}$ and $\mathbb{C}\mathrm{osp} = \mathrm{tr}_1\partial\,\mathbb{C}\mathrm{osp}$.

(Replacing the ground-category **Set** with any 'non-trivial' category with suitable limits and colimits would give similar results.)

In both cases, a cubical transformation of 1-cubes (and its limit) seems to be a good notion. For *spans* (resp. *cospans*), this amounts to a *colax* (resp. *lax*) vertical transformation of vertical arrows, as shown in 4.2 (resp. 4.3). On the other hand, a lax (resp. colax) vertical transformation of spans (resp. cospan) appears to be an ill-formed notion of little interest; the second does not even have a limit.

Now, if we start from a weak double category, there seems to be no general procedure that would be able to reconstruct $\partial\,\mathbb{S}\mathrm{p}$ and $\partial\,\mathbb{C}\mathrm{osp}$ from their truncations: in fact, by the previous argument, a 2-cube of $\omega\mathbb{S}p$ corresponds to *colax* vertical transformation of vertical arrows, while a 2-cube of $\omega\mathbb{C}osp$ agrees with the *lax* case. Furthermore, the universal constructions adjoint to truncation – examined in 4.4-4.6 – are not of much help: skeleton, the left adjoint, only adds degenerate cubes; coskeleton gives a less trivial weak sc-category, but $cosk_1\mathbb{C}osp$ does not have tabulators of degree 1 (4.6).

As a conclusion of this analysis, a weak double category that has a natural lifting as a weak sc-category is perhaps better studied in this enrichment. But there seems to be no way of reducing the general theory of double limits to that of cubical limits.

4.2. Truncating cubical spans. The weak symmetric cubical category $\omega\mathbb{S}p$ determines, by 1-truncation, the weak double category $\mathbb{S}p = tr_1(\omega\mathbb{S}p)$ of sets, mappings and spans:

– objects and horizontal arrows are small sets and mappings between them;

– vertical arrows are ordinary spans $u: \boldsymbol{\omega} \to \mathbf{Set}$;

– double cells are natural transformations $f = (f_{-1}, f_0, f_1): u \to v: \boldsymbol{\omega} \to \mathbf{Set}$,

$$(1) \qquad \begin{array}{ccc} X & \xrightarrow{f_{-1}} & X' \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle v'} \\ U & \xrightarrow{f_0} & V \\ {\scriptstyle u''}\downarrow & & \downarrow{\scriptstyle v''} \\ Y & \xrightarrow[f_1]{} & Y' \end{array} \qquad\qquad \begin{array}{ccc} \bullet & \longrightarrow & 0 \\ \downarrow{\scriptstyle 1} & & \end{array}$$

We denote with $\mathbf{u}_1$ the strict sc-category freely generated by a 1-cube (as in 3.4). An sc-functor $u: \mathbf{u}_1 \to \omega\mathbb{S}p$ amounts to an ordinary span $u = (u', u'')$ of sets; its tabulator is simply the central object of the span. The truncation $\mathbf{u} = tr_1\mathbf{u}_1$ is the strict double category freely generated by a vertical arrow $-1 \twoheadrightarrow 1$. A double functor $u: \mathbf{u} \to \mathbb{S}p$ 'is' a span $u = (u', u'')$ of sets; its tabulator, as a double limit, is again the central object of the span.

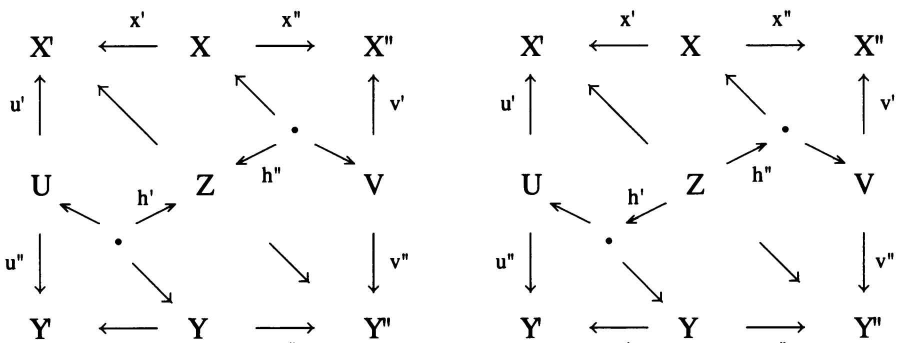
Differences appear when we consider 'transformations' of such sc-functors *or* double functors.

A *cubical transformation* $\omega: u \to v: \mathbf{u}_1 \to \omega\mathbb{S}p$ 'is' a 2-cube of $\omega\mathbb{S}p$ with 1-indexed faces $\partial^-_1\omega = u$, $\partial^+_1\omega = v$

$$(2)\qquad \begin{array}{ccccc} X & \xrightarrow{x'} & X & \xleftarrow{x''} & X' \\ {\scriptstyle u'}\downarrow & & \downarrow & & \downarrow{\scriptstyle v'} \\ U & \xrightarrow{z'} & Z & \xleftarrow{z''} & V \\ {\scriptstyle u''}\uparrow & & \uparrow & & \uparrow{\scriptstyle v''} \\ Y & \xrightarrow[y']{} & Y & \xleftarrow[y'']{} & Y' \end{array} \qquad\qquad \begin{array}{cc} \bullet & \longrightarrow 1 \\ \downarrow{\scriptstyle 2} & \end{array}$$

The tabulator of degree 1 of Φ (3.5) is a span w with a universal 2-map $e_2(w) \to \Phi$; this means $w = (z', z'')$, the central span of Φ with respect to direction 2. (The transformation $\Phi: x \to y$ defined by the transposed 2-cube would give the central span of Φ with respect to the other direction.)

Now, a *unitary lax* (resp. *colax*) *vertical transformation* $\Phi: u \twoheadrightarrow v: \mathbf{u} \to \mathbb{S}p$, as defined in [GP4], 4.4, is a unitary lax (resp. colax) double functor $\Phi: \mathbf{u}\times\mathbf{u} \to \mathbb{S}p$ that restricts to u, v on $\{\mp 1\}\times\mathbf{u}$. (We only consider the unitary case for the sake of simplicity). Therefore, it corresponds to the left (resp. right) commutative diagram below, where the upper-right and the lower-left quadrilateral are pullbacks

(3) 

Such pullbacks yield the vertical composites $x\otimes v, u\otimes y: X' \twoheadrightarrow Y''$; the mappings h', h'' are their comparisons, in the lax or colax direction. (If they are bijective, we get the same notion, essentially equivalent to a *strong* vertical transformation, as defined in [GP1], 7.4).

In the lax case, the limit is the span $(w', w'') = (U \leftarrow W \to V)$, where W is the pullback of the cospan (h', h'') (or, equivalently, the limit in **Set** of the whole left diagram); w', w'' are the induced mappings. The colax case is essentially equivalent to a cubical transformation (2), and has the same limit of the latter: the span $U \to Z \leftarrow V$.

As a synopsis of this first analysis, a *colax* vertical transformation of spans amounts to a cubical transformation of spans viewed as 1-cubes of the weak sc-category ⊗𝕊p; the two notions have the same 'limit'. On the other hand, a *lax* vertical transformation of spans, represented in the left diagram (3), seems to be a not well-formed notion – even if it does possess a limit. Finally, a *strong* vertical transformation amounts to a commutative diagram (2) where the upper-right and the lower-left square are pullbacks, a condition which seems to be of little interest.

4.3. Truncating cubical cospans. Let us now consider tabulators in the weak symmetric cubical category ⊗ℂosp and its 1-truncation, the weak double category ℂosp = tr_1(⊗ℂosp) of sets, mappings and spans.

A general double cell in ℂosp is a natural transformation $f = (f_{-1}, f_0, f_1)\colon u \to v\colon \otimes \to$ **Set**

$$(1)\qquad \begin{array}{ccc} X & \xrightarrow{f_{-1}} & X' \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle v'} \\ U & \xrightarrow{f_0} & V \\ {\scriptstyle u''}\uparrow & & \uparrow{\scriptstyle v''} \\ Y & \xrightarrow[f_1]{} & Y' \end{array} \qquad\qquad \begin{array}{cc} \bullet & \longrightarrow\ 0 \\ \downarrow{\scriptstyle 1} & \end{array}$$

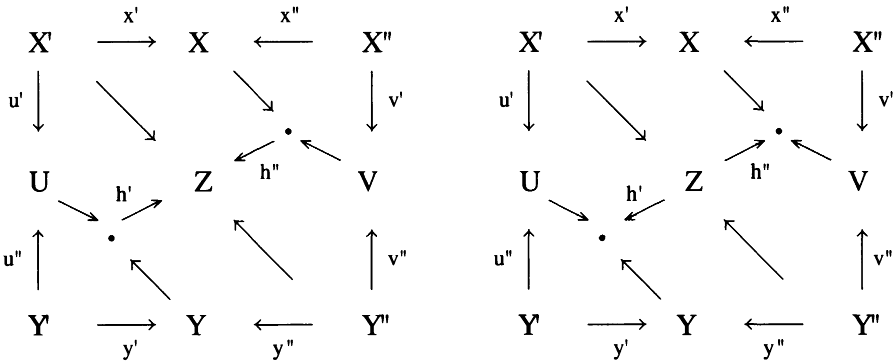
An sc-functor $u\colon \mathbf{u}_1 \to$ ⊗ℂosp 'is' an ordinary cospan u = (u', u") of sets, and its tabulator is the pullback of the cospan (in **Set**). Similarly, a double functor $u\colon \mathbf{u} \to$ ℂosp 'is' a cospan u = (u', u") of sets, and its tabulator – as a double limit – is the pullback of the span.

A *cubical transformation* ⊗: $u \to v\colon \mathbf{u}_1 \to$ ⊗ℂosp is now expressed by a 2-cubical cospan with 1-indexed faces u, v

$$(2)\qquad \begin{array}{ccccc} X & \xrightarrow{x'} & X & \xleftarrow{x''} & X' \\ {\scriptstyle u'}\downarrow & & \downarrow & & \downarrow{\scriptstyle v'} \\ U & \xrightarrow{z'} & Z & \xleftarrow{z''} & V \\ {\scriptstyle u''}\uparrow & & \uparrow & & \uparrow{\scriptstyle v''} \\ Y & \xrightarrow[y']{} & Y & \xleftarrow[y'']{} & Y' \end{array} \qquad\qquad \begin{array}{cc} \bullet & \longrightarrow\ 1 \\ \downarrow{\scriptstyle 2} & \end{array}$$

The tabulator of degree 1 of Λ is the cospan $(w', w'') = (Tu \to T_2\Lambda \leftarrow Tv)$, where $T_2\Lambda$ is the pullback of the cospan $X \to Z \leftarrow Y$ and w', w'' are the induced mappings.

Working as above, this amounts to a *unitary lax vertical transformation* Λ : $u \twoheadrightarrow v$: $\mathbf{u} \to \mathbb{C}\mathrm{osp}$, represented in the left (commutative) diagram below, where the upper-right and the lower-left quadrilateral are pushouts; the limit is again the cospan $Tu \to T_2\Lambda \leftarrow Tv$

(3) 

A colax vertical transformation is represented in the right diagram above. Besides being a 'strange' notion, it seems to have no limit.

4.4. Weak double categories and coskeleton. Backwards, from weak double categories to weak symmetric cubical categories, we have two functors, called 1-*skeleton* and 1-*coskeleton*, that are, respectively, left and right adjoint to 1-truncation (cf. I.3.9)

(1) $\mathrm{sk}_1, \mathrm{cosk}_1$: $\mathrm{w}\mathbf{DBL} \to \mathrm{wsc}\mathbf{CAT}$, $\qquad \mathrm{sk}_1 \dashv \mathrm{tr}_1 \dashv \mathrm{cosk}_1$.

The skeleton-procedure just adds degenerate cubes (under an equivalence relation determined by the cubical relations). It may be more interesting to view a weak double category in wsc**CAT** *via the 1-coskeleton functor.* Concretely, if $\mathbb{D}$ is a weak double category, the weak sc-category $\mathbb{A} = \mathrm{cosk}_1(\mathbb{D})$ coincides with $\mathbb{D}$ in cubical degree 0 and 1 (according to the previous translation of terminology, at the beginning of 4.2). Then, a 2-cube is a 'shell' of 1-cubes of $\mathbb{D}$ (i.e. vertical arrows) under no further condition

(2)
$$\begin{array}{ccc} A & \xrightarrow{v} & B \\ {\scriptstyle u}\downarrow & \# & \downarrow{\scriptstyle u'} \\ C & \xrightarrow[v']{} & D \end{array} \qquad \begin{array}{cc} \bullet \longrightarrow 1 \\ \downarrow 2 \end{array} \qquad \begin{array}{ll} \wedge_1^- u = \wedge_1^- v, & \wedge_1^+ u' = \wedge_1^+ v', \\ \wedge_1^+ u = \wedge_1^- v', & \wedge_1^+ v = \wedge_1^- u', \end{array}$$

Notice that the *#-marked* square is *not* assumed to commute under concatenation of 1-cubes, in any sense (strict, weak or lax). A transversal 2-map is a similar 'shell' of 1-maps of $\mathbb{D}$. Similarly, one defines all the higher components, by n-dimensional shells of 1-cubes or 1-maps of $\mathbb{D}$.

Faces and degeneracies are obvious. Concatenations are also obvious, and computed with the vertical composition of vertical arrows or double cells in $\mathbb{D}$. Finally, the comparisons for associativity and units are families of comparisons of $\mathbb{D}$, while interchange is necessarily strict.

Viewing weak double categories in this way leads us to define a *coskeletal vertical transformation of double functors* (between weak double categories) $F\colon F^- \to F^+\colon \mathbb{D} \to \mathbb{E}$ as a cubical transformation of the corresponding 1-coskeletons

(3) $F\colon \mathrm{cosk}_1F^- \to \mathrm{cosk}_1F^+\colon \mathrm{cosk}_1\mathbb{D} \to \mathrm{cosk}_1\mathbb{E}$.

Explicitly, this means to assign:

(a) to every object A of $\mathbb{D}$ a 1-cube $FA\colon F^-A \twoheadrightarrow F^+A$ of $\mathrm{cosk}_1\mathbb{E}$ (i.e. a vertical arrow of $\mathbf{E}$),

(b) to every horizontal map (0-map) $f\colon A \to A'$ of $\mathbb{D}$, a 1-map $Ff\colon F^-f \twoheadrightarrow F^+f$ (a double cell of $\mathbf{E}$),

consistently with the transversal structure (faces, identities and composition):

(4) $F(\wedge_0^\wedge f) = \wedge_0^\wedge(Ff)$, $\qquad F(\mathrm{id}X) = \mathrm{id}(FX)$, $\qquad F(gf) = Fg.Ff$.

Notice that there is *no* 'naturality' condition based on a 1-cube $u\colon A \twoheadrightarrow A'$ of $\mathbb{D}$: the latter is simply sent to a 2-dimensional *shell*, with 1-indexed faces $F^\wedge(u)$ and 2-indexed faces FA, FA'

(5)
$$\begin{array}{ccc} F^-A & \xrightarrow{FA} & F^+A \\ {\scriptstyle F^-u}\downarrow & \# & \downarrow{\scriptstyle F^+u} \\ F^-A' & \xrightarrow[FX']{} & F^+A' \end{array} \qquad \begin{array}{cc} \bullet \longrightarrow 1 \\ \downarrow 2 \end{array}$$

Moreover, the consistency with concatenation of 1-cubes is simply 'managed' by the cubical functors F^-, F^+.

More generally, we define in the same way a *coskeletal vertical transformation* $F^- \to F^+: \mathbb{D} \to \mathbb{E}$ *of* lax *double functors* $F^\wedge$: the only comparisons that we need are those of the latter

(6) $F_1^\wedge(A): e_1(F^\wedge A) \to F^\wedge(e_1(A))$ (A in $\mathbb{D}$),

$F_1^\wedge(u, v): F^\wedge u +_1 F^\wedge v \to F^\wedge(u +_1 v)$ (u, v in $\mathbb{D}$, $\wedge_1^+ u = \wedge_1^- v$).

Indeed, defining $F_1(A)$ as the pair $(F_1^-(A), F_1^+(A))$ we automatically get the unit comparisons of F; similarly for $F_1(u, v)$.

The present notion is compared below with a lax (resp. colax) vertical transformation of lax double functors: the latter requires to insert in each diagram (5) a vertical 'filler' $F^-A \twoheadrightarrow F^+A'$, with comparisons coming from (resp. going to) the vertical composites $FA \wedge F^+u$, $F^+u \wedge FA'$.

4.5. From ordinary to cubical spans. The unit of the adjunction $tr_1 \dashv cosk_1$, evaluated on the weak sc-category $\wedge\mathbb{S}p$

(1) $\wedge: \wedge\mathbb{S}p \to cosk_1\mathbb{S}p = cosk_1 tr_1(\wedge\mathbb{S}p)$,

maps $\wedge\mathbb{S}p$ to a 'poorer' weak sc-subcategory, where an n-cube is a shell of ordinary spans, as in the following *solid* diagram (for $n = 2$); the sc-functor $\wedge$ forgets the dashed arrows

$$(2) \qquad \begin{array}{ccccc} X' & \xleftarrow{x'} & X & \xrightarrow{x''} & X'' \\ {\scriptstyle u'}\uparrow & & \uparrow & & \uparrow{\scriptstyle v'} \\ U & \xleftarrow{z'} & Z & \xrightarrow{z''} & V \\ {\scriptstyle u''}\downarrow & & \downarrow & & \downarrow{\scriptstyle v''} \\ Y' & \xleftarrow[y']{} & Y & \xrightarrow[y'']{} & Y'' \end{array} \qquad\qquad \begin{array}{ll} \bullet \longrightarrow 1 \\ \downarrow 2 \end{array}$$

Notice that $tr_1\wedge = \mathrm{id}\mathbb{S}p$. An sc-functor $u: \mathbf{u}_1 \to cosk_1\mathbb{S}p$ is again an ordinary span $u = (u', u'')$ of sets, with tabulator given by its central object.

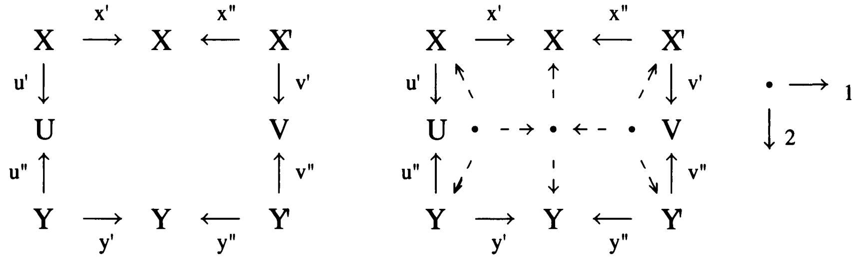
A *coskeletal vertical transformation* $\wedge: u \twoheadrightarrow v: \mathbf{u} \to \mathbb{S}p$, as defined in 4.4, is a cubical transformation $\wedge: u \to v: \mathbf{u}_1 \to cosk_1\mathbb{S}p$, and amounts to the solid shell above. The tabulator of degree 1 of $\wedge$ is the dashed span (z', z''), where Z is the limit in **Set** of the solid diagram $\wedge$ and z', z'' are part of the limit-maps.

4.6. From ordinary to cubical cospans. We are again interested in the unit of the adjunction $tr_1 \dashv cosk_1$, evaluated now on the weak sc-category $\wedge\mathbb{C}osp$

(1) $\wedge: \wedge\mathbb{C}osp \to cosk_1\mathbb{C}osp = cosk_1 tr_1(\wedge\mathbb{C}osp)$.

An n-cube of the weak sc-subcategory $\mathrm{cosk}_1\mathbb{C}\mathrm{osp}$ is a shell of ordinary cospans, as in the left *solid* diagram below (for $n = 2$)

(2)



An sc-functor $u: \mathbf{u}_1 \to \mathrm{cosk}_1\mathbb{C}\mathrm{osp}$ is now an ordinary cospan $u = (u', u'')$ of sets, and its tabulator is the pullback of the cospan (in **Set**).

A *coskeletal vertical transformation* $\eta : u \twoheadrightarrow v: \mathbf{u} \to \mathbb{C}\mathrm{osp}$ is a cubical transformation $\eta : u \to v: \mathbf{u}_1 \to \mathrm{cosk}_1\mathbb{C}\mathrm{osp}$, and amounts to the left diagram (2). For the tabulator of degree 1 of η one should insert dashed arrows forming the commutative right-hand diagram above; plainly, *there is no universal solution* for this problem.

5. Proof of the theorem on the construction of cubical limits

We end with a proof of Theorem 3.7. The argument is similar to the proof of the corresponding theorem for double limits [GP1].

5.1. Comments. Of course one needs only to prove the 'sufficiency' part of the statement. We write down the argument for the construction of limits; the preservation property is proved in the same way.

The solution is based on transforming F into a graph-morphism $G: \mathbf{X} \to \mathrm{tv}_0\mathbb{A}$, and taking its limit. The graph $\mathbf{X}$ is a sort of 'transversal subdivision' of $\mathbb{X}$, where every n-cube of $\mathbb{X}$ is replaced with an object *simulating its tabulator* (of level 0). The procedure is similar to computing the end of a functor $S: \mathbf{C}^{op}{\times}\mathbf{C} \to \mathbf{D}$ as the limit of the associated functor $S^§: \mathbf{C}^§ \to \mathbf{D}$ based on Kan's *subdivision category* of $\mathbf{C}$ ([Ka], 1.10; [Ma], IX.5).

5.2. Transversal subdivision. The *transversal subdivision* $\mathbf{X}$ of $\mathbb{X}$ is a graph, formed by the following objects and arrows (and is finite whenever $\mathbb{X}$ is).

(a) For every n-cube x of $\mathbb{X}$, there is an object x in $\mathbf{X}$. For every n-map $f: x \to y$ of $\mathbb{X}$, there is an arrow $f: x \to y$ in $\mathbf{X}$.

(b) For every n-cube x of $\mathbb{X}$, we also add $2n$ arrows $p_i^\eta(x): x \to \eta_i^\eta x$ (that simulate the projections 3.4.4 of a tabulator) and n arrows $d_i x: x \to e_i x$ (that simulate the diagonal maps 3.6.1).

(c) For every i-concatenation of n-cubes $z = x +_i y$ in $\mathbb{X}$, we also add an object $(x, y)_i$ in $\mathbf{X}$ and three arrows

(1) $p_i(x, y): (x, y)_i \to x, \qquad q_i(x, y): (x, y)_i \to y, \qquad d_i(x, y): (x, y)_i \to z,$

that simulate the object $T_{ni}(x, y)$ of 3.6.2, its projections and its diagonal map.

5.3. The associated morphism of graphs. We now construct a graph-morphism $G: \mathbf{X} \to tv_0\mathbb{A}$ that naturally comes from F and the tabulator-construction in $\mathbb{A}$.

(a) For every n-cube x of $\mathbb{X}$, we define Gx as the following object (0-cube) of $\mathbb{A}$

(1) $G(x) = T_n(Fx) \qquad\qquad (t_{Fx}: e^nG(x) \to F(x)).$

For every n-map $f: x \to y$ of $\mathbb{X}$, we define Gf as the 0-map of $\mathbb{A}$ determined by the universal property of t_{Fy}, as follows:

(2) $Gf: T_n(Fx) \to T_n(Fy),$

$t_{Fy}.e^n(Gf) = Ff.t_{Fx}$

$$\begin{array}{ccc} e^nT_n(Fx) & \xrightarrow{e^n(Gf)} & e^nT_n(Fy) \\ {\scriptstyle t_{Fx}}\downarrow & & \downarrow{\scriptstyle t_{Fy}} \\ Fx & \xrightarrow[Ff]{} & Fy \end{array}$$

(b) We define $G(p_i^\eta x): Gx \to Gz$ as the following 0-map of $\mathbb{A}$ (writing $z = \eta_i^\eta x$)

(3) $G(p_i^\eta x): T_n(Fx) \to T_{n-1}(Fz)$

$t_{Fz}.e^{n-1}(G(p_i^\eta x)) = \eta_i^\eta(t_{Fx})$

$$\begin{array}{ccc} e^{n-1}T_n(Fx) & \xrightarrow{e^{n-1}(Gp_i^\eta x)} & e^{n-1}T_{n-1}(Fz) \\ & \searrow{\scriptstyle \eta_i^\eta(t_{Fx})} & \downarrow{\scriptstyle t_{Fz}} \\ & & Fz \end{array}$$

Furthermore, $G(d_ix): Gx \to G(e_ix)$ is a modification of the diagonal map of type 3.6.1, using the comparison $F_i(x): e_iFx \to Fe_ix$ of the lax sc-functor F

(4) $G(d_ix): T_nFx \to T_{n+1}(Fe_ix)$

$t_{Fe_ix}.e^{n+1}(G(d_ix)) = F_i(x).e_i(t_{Fx})$

$$\begin{array}{ccc} e^{n+1}T_n(Fx) & \xrightarrow{e^{n+1}(Gd_ix)} & e^{n+1}T_{n+1}(Fz) \\ {\scriptstyle e_it_{Fx}}\downarrow & & \downarrow{\scriptstyle t_{Fe_ix}} \\ e_iFx & \xrightarrow[F_i(x)]{} & Fe_ix \end{array}$$

(c) Now, $G(x, y)_i = \mathsf{T}_{ni}(Fx, Fy)$ is the transversal limit of the i-composable pair Fx, Fy (3.6).

The arrows $p_i(x, y)$ and $q_i(x, y)$ of $\mathbf{X}$ are taken by G to the projections 3.6.2 of $\mathsf{T}_{ni}(Fx, Fy)$

(5) $G(p_i(x, y)): G(x, y)_i \to Gx, \qquad G(q_i(x, y)): G(x, y)_i \to Gy,$

so that $(G(x, y)_i, p_i(x, y), q_i(x, y))$ is the pullback of (q_{ix}, p_{ix}) in $tv_0\mathbb{A}$.

Finally, the arrow $d_i(x, y)$ of $\mathbf{X}$ is sent by G to the following modification of the diagonal 3.6.3 of $G(x, y)_i$, taking into account the comparison $F_i(x, y)$ of the lax sc-functor F (with $z = x +_i y$)

(6) $G(d_i(x, y)): \mathsf{T}_{ni}(Fx, Fy) \to \mathsf{T}_n F(z),$

$t_{Fz}.e^n(G(d_i(x, y)) = F_i(x,y).(t_{Fx}.e^n p_i(x,y) +_i t_{Fy}.e^n q_i(x,y)): e^n(G(x,y)_i) \to F(z).$

The limit of this diagram $G: \mathbf{X} \to tv_0\mathbb{A}$ exists, by hypotheses and Theorem 2.2.

5.4. From sc-cones to cones. In order to prove that the limit of G gives the limit of degree 0 of F, we construct an isomorphism

$(D \downarrow F) \to (D' \downarrow G),$

from the comma category of sc-cones of F to the comma category of ordinary cones of the graph-morphism G. We proceed first in this direction, and then backwards.

Let $(a, h: Da \to F)$ be an sc-cone of F. For every n-cube x of $\mathbb{X}$, we define $k(x): a \to Gx = \mathsf{T}_n(Fx)$ as the 0-map of $\mathbb{A}$ determined by the n-map hx, via the tabulator-property

(1) $t_{Fx}.e^n(kx) = hx.$

Further, we define $k(x, y)_i: a \to G(x, y)_i$ by means of the pullback-property of $G(x, y)_i$

(2) $p_i(x, y).k(x, y)_i = kx: a \to Gx \qquad q_i(x, y).k(x, y)_i = ky: a \to Gy.$

Let us verify that this family k is indeed a cone of $G: \mathbf{X} \to tv_0\mathbb{A}$.

(a) Coherence with a map $f: x \to y$ means that $Gf.kx = ky$, which follows from the cancellation property of t_{Fy}

(3) $t_{Fy}.e^n(Gf.kx) = Ff.t_{Fx}.e^n(kx) = Ff.hx = hy = t_{Fy}.e^n(ky).$

(b) Coherence with the arrows $p_i^{\downarrow}(x): x \to \downarrow_i^{\downarrow} x$ and $d_i x: x \to e_i x$ follows from 5.3.3 and 5.3.4 (we write $z = e_i x$ in the second case)

(4) $G(p_i^{\downarrow}(x)).kx = k(\downarrow_i^{\downarrow}x)$,

$t_{Fz}.e^{n+1}(G(d_ix).kx) = F_i(x).e_i(t_{Fx}).e^{n+1}(kx) = F_i(x).e_i(t_{Fx}.e^n(kx))$
$= F_i(x).e_ihx = h(z) = t_{Fz}.e^n(kz)$.

(c) Coherence with the arrows $p_i(x, y)$ and $q_i(x, y)$ holds by definition 5.3.5. For $d_i(x, y)$ and $z = x +_i y$ we have:

(5) $t_{Fz}.e^n(G(d_i(x, y).k(x, y)_i) = F_i(x, y)(t_{Fx}.e^n p_i(x, y) +_i t_{Fy}.e^n q_i(x, y)).e^n k(x, y)_i$
$= F_i(x, y).(hx +_i hy) = hz = t_{Fz}.e^n(kx)$.

Finally, a map of sc-cones $f\colon (a, h\colon Da \to F) \to (b, h'\colon Db \to F)$ determines a map of G-cones $f\colon (a, k) \to (b, k')$, since:

(6) $t_{Fx}.e^n(k'x.f) = h'x.e^n(f) = hx = t_{Fx}.e^n(kx)$.

5.5. From cones to sc-cones. In the reverse direction $(D' \downarrow G) \to (D \downarrow F)$, we just specify the procedure on cones. Given a cone $(a, k\colon D'a \to G)$ of G, one forms an sc-cone $(a, h\colon Da \to F)$ by letting

(1) $hx = t_{Fx}.e^n(kx)\colon e^n(a) \to x$ $\qquad (x \downarrow \mathbb{A}_n)$.

This satisfies (scc.1) since, for $f\colon x \to y$ in $\mathbb{X}$

(2) $Ff.hx = Ff.t_{Fx}.e^n(kx) = t_{Fy}.e^n(Gf.kx) = t_{Fy}.e^n(ky) = hy$.

It also satisfies the conditions (scc.2, 3) concerning the 1-concatenation in $\mathbb{X}$; this proceeds much as above (with $z = e_1x$ in the first case, and $z = x +_1 y$ in the second)

(3) $F_1(x).e_1(hx) = F_1(x).e_1(t_{Fx}.e^n(kx)) = F_1(x).e_1(t_{Fx}).e^{n+1}(kx)$
$= t_{Fz}.e^{n+1}(G(d_1x).kx) = t_{Fz}.e^{n+1}(kz) = hz$.

(4) $F_1(x, y).(hx +_1 hy) = F_1(x, y).(t_{Fx}.e^n p_1(x, y) +_1 t_{Fy}.e^n q_1(x, y)).e^n k(x, y)_1$
$t_{Fz}.e^n(G(d_1(x, y)).k(x, y)_1) = t_{Fz}.e^n(kz) = hz$.

This completes the proof.

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