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REPRESENTABILITY RELATIVE TO A DOCTRINE

To Jirka Adámek on the occasion of his 60^{th} *birthday.*

by Panagis KARAZERIS and Jiří VELEBIL*

Abstract

Nous proposons la notion de doctrine en vue de fournir un environnement uniforme pour l'étude des concepts de représentabilité faible. Puisque les (co)limites sont des notions de représentabilité, ceci nous permet de définir et d'étudier des concepts affaiblis de (co)limites. Par exemple dans le cas où la doctrine en question est celle des cocomplétions libres pour les colimites d'une certaine classe, l'existence de limites affaiblies dans la catégorie ambiante est étroitement liée aux limites usuelles dans la complétion libre. De manière analogue, noux pouvons relier certaines structures promonoidales faibles sur une catégorie à de vraies structures monoidales sur une cocomplétion libre.

1 Introduction

Many "classical" notions of category theory are in fact *representability* notions. For example, the existence of a left adjoint of $U : \mathscr{A} \longrightarrow \mathscr{B}$ is the assertion that the functor $\mathscr{B}(B, U-) : \mathscr{A} \longrightarrow \mathsf{Set}$ is *representable* for each B. This means that there is a natural isomorphism

$$\mathscr{B}(B, U-) \cong \mathscr{A}(FB, -)$$

for each B. The assignment $B \mapsto FB$ then extends to a functor $F : \mathscr{B} \longrightarrow \mathscr{A}$ — the desired left adjoint of U.

However, it is often fruitful to *weaken* the representability concept and study weaker notions. Probably the best known instance of weakened representability notion is the case of *weak limits*, studied, e.g., in [FS].

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Example 1.1. A functor $F : \mathscr{A}^{op} \longrightarrow \mathsf{Set}$ is called *weakly representable*, if there exists an epimorphism $e : \mathscr{A}(-, A) \longrightarrow F$ for some A. The epimorphism e is often called a *weak representation of* F.

A diagram $D : \mathscr{D} \longrightarrow \mathscr{A}$ is said to have a *weak limit*, if there exists a weak representation

$$e : \mathscr{A}(-, A) \longrightarrow \mathsf{Cone}(D) \tag{1.1}$$

where $\mathsf{Cone}(D) : \mathscr{A}^{op} \longrightarrow \mathsf{Set}$ is the *cone functor* of D, i.e., $\mathsf{Cone}(D)X$ is the set of all D-cones having X as a vertex.

Thus, the weak representation (1.1) picks up a *distinguished cone*

$$\ell_d : A \longrightarrow Dd$$

obtained as $e(\mathrm{id}_A)$ with the following *weak universal property*: for any D-cone

$$c_d : X \longrightarrow Dd$$

there exists some (not necessarily unique) $f : X \longrightarrow A$ such that $c_d = \ell_d \cdot f$ for all d in $\mathscr{D}$.

The weak representability concept allows one to define *weak right adjoints* as those functors $U : \mathscr{A} \longrightarrow \mathscr{B}$ for which every functor $\mathscr{B}(B, U-) : \mathscr{A} \longrightarrow \mathsf{Set}$ is weakly representable, see [BTh].

The above example is quite typical: one defines the weakened representability concept first and infers the concept of a weakened (co)limit as a representability notion.

The following notion of a *multilimit* is due to Yves Diers [Di]:

Example 1.2. A functor $F : \mathscr{A}^{op} \longrightarrow \mathsf{Set}$ is called *multirepresentable* if there exists a natural isomorphism

$$\coprod_i \mathscr{A}(-, K_i) \cong F$$

for some diagram $K : \mathscr{K} \longrightarrow \mathscr{A}$ with $\mathscr{K}$ discrete. Multirepresentability of the cone functor $\mathsf{Cone}(D)$ then establishes the notion of a *multilimit* of a diagram D.

A somewhat more elaborate notion of a "weak" limit is the notion of a *finite plurilimit of a finite diagram* introduced in [KRV] in connection to the existence of limits in free cocompletions.

Example 1.3. Let $D : \mathscr{D} \longrightarrow \mathscr{A}$ be a finite diagram. A *finite plurilimit of* D is a finite diagram

$$K : \mathscr{K} \longrightarrow \mathscr{A}$$

together with an isomorphism

$$\mathrm{colim}_i\, \mathscr{A}(-, K_i) \cong \mathsf{Cone}(D)$$

In elementary terms this means that there is a distinguished finite family

$$\ell^i_d : K_i \longrightarrow Dd$$

of D-cones such that each cone

$$c_d : X \longrightarrow Dd$$

factors through some distinguished cone and any two such factorizations are connected by a zig-zag (in $\mathscr{K}$).

Thus, a *finite plurirepresentability* of $F : \mathscr{A}^{op} \longrightarrow \mathsf{Set}$ is the existence of a finite diagram $\mathscr{A}(-, A_i)$ of representables together with a natural isomorphism

$$\mathrm{colim}_i\, \mathscr{A}(-, A_i) \cong F.$$

In other words, finitely plurirepresentable presheaves F are exactly the finitely presentable objects of the presheaf category $[\mathscr{A}^{op}, \mathsf{Set}]$, see [AR$_1$].

Let us observe that the examples of "weak" representability notions of F above share the following feature:

> *The functor F is an object of a category $\mathbb{C}(\mathscr{A})$ lying "in between" $\mathscr{A}$ and the presheaf category $[\mathscr{A}^{op}, \mathsf{Set}]$.*

In Example 1.1 we take for $\mathbb{C}(\mathscr{A})$ the full subcategory of $[\mathscr{A}^{op}, \mathsf{Set}]$ spanned by quotients of representables, in Example 1.3 we take $\mathbb{C}(\mathscr{A})$ to be the full subcategory of $[\mathscr{A}^{op}, \mathsf{Set}]$ spanned by finite colimits of representables.

In general, it seems reasonable that such a category $\mathbb{C}(\mathscr{A})$, measuring the "degree of representability", should have the following properties:

> *There is a fully faithful dense functor $\gamma_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathbb{C}(\mathscr{A})$. Moreover, γ should be (pseudo)natural in $\mathscr{A}$.*

Such a pair $(\mathbb{C}, \gamma)$ is what we call a *doctrine* in Definition 3.1 below. To allow a wider scope of applications, we prefer to work with categories enriched over some suitable monoidal closed base $\mathscr{V}$. Since limits form only one instance of representability, we will study first representability in general and then turn our attention to limits, monoidal structures, etc., as special cases of a general notion.

Organization of the Paper

We work in enriched category theory and we gather the necessary notions in Section 2. Section 3 is devoted to the basic definition of weakened representability. We derive a result characterizing weak representability in terms of the existence and "absoluteness" of certain colimits in Theorem 3.7. The main result of the paper connecting weak limits in a category to honest limits in its free cocompletion is formulated in Theorem 4.7 of Section 4. Finally, in Section 5 we derive that the existence of a class of limits in a free cocompletion under a given class of colimits amounts to some kind of a distributive law, Theorem 5.4. A question related to weakened representability is the existence of monoidal structures on a free cocompletion. This is the topic of Section 6.

Related Work

The question of the existence of limits in the free cocompletion under *all small* colimits has been extensively studied by Brian Day and Steve Lack in [DL]. In many cases our results are easy extensions of theirs.

2 Preliminary Notions

For details on the basic notions of enriched category theory we refer to the monograph [K_1].

Assumption 2.1. Throughout the paper, $\mathscr{V} = (\mathscr{V}_o, \otimes, I, [-,-])$ is a fixed symmetric monoidal closed category that is complete and cocomplete. When we say category, functor, natural, etc., we mean a $\mathscr{V}$-category, $\mathscr{V}$-functor, $\mathscr{V}$-natural, etc., unless we explicitly say an ordinary category, ordinary functor, ordinary natural, etc.

Notation 2.2. To any functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ we associate its

tilde-conjugate $\widetilde{F} : \mathscr{B} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$ sending B to $\mathscr{A}(F-, B)$

hat-conjugate $\widehat{F} : \mathscr{B} \longrightarrow [\mathscr{A}, \mathscr{V}]^{op}$ sending B to $\mathscr{A}(B, F-)$

where the functor categories are assumed to exist in a higher universe when $\mathscr{A}$ is large.

We will often work with weighted (co)limit notions, we recollect the notions here. However we want to work in a slightly bigger generality than in, e.g., [K_1].

Therefore our weights are "collections of weights in the sense of [K1]", see Remark 2.4. In fact, our notion of weights and weighted (co)limits comes from [SW].

Definition 2.3. A functor $W : \mathscr{M} \longrightarrow [\mathscr{D}^{op}, \mathscr{V}]$ will be called a *weight* (*small*, if both $\mathscr{M}$ and $\mathscr{D}$ are small). A *diagram* in $\mathscr{A}$ is a functor $D : \mathscr{D} \longrightarrow \mathscr{A}$. A *colimit* of D weighted by W is a functor $W \star D : \mathscr{M} \longrightarrow \mathscr{A}$ together with an isomorphism

$$\mathscr{A}((W \star D)M, A) \cong [\mathscr{D}^{op}, \mathscr{V}](WM, \widetilde{D}A)$$

natural in M and A.

Remark 2.4. The usual definition of a weighted colimit (as described, e.g., in [K1]) deals with weights of the form $W : \mathscr{I} \longrightarrow [\mathscr{D}^{op}, \mathscr{V}]$ where $\mathscr{I}$ is the *unit* category with $\mathscr{I}(*.*) = I$. Thus a weight is then identified with a mere functor $W : \mathscr{D}^{op} \longrightarrow \mathscr{V}$. We find the generalized notion more suitable for our purposes.

Clearly, for any weight $W : \mathscr{M} \longrightarrow [\mathscr{D}^{op}, \mathscr{V}]$ and any $D : \mathscr{D} \longrightarrow \mathscr{A}$, there is an isomorphism (with either side existing if the other does)

$$(W \star D)M \cong WM \star D$$

natural in M, where the expression on the right is the "classical" notion of a colimit of $D : \mathscr{D} \longrightarrow \mathscr{A}$ weighted by $WM : \mathscr{D}^{op} \longrightarrow \mathscr{V}$.

Of course, a limit in $\mathscr{A}$ is just a colimit in $\mathscr{A}^{op}$. We spell out the limit concept explicitly to bring attention to the variances of weights.

Definition 2.5. A *limit* of $D : \mathscr{D} \longrightarrow \mathscr{A}$ weighted by $W : \mathscr{M} \longrightarrow [\mathscr{D}, \mathscr{V}]^{op}$ is a functor $\{W, D\} : \mathscr{M} \longrightarrow \mathscr{A}$ together with an isomorphism

$$\mathscr{A}(A, \{W, D\}M) \cong [\mathscr{D}, \mathscr{V}]^{op}(\widehat{D}A, WM) = [\mathscr{D}, \mathscr{V}](WM, \widehat{D}A)$$

natural in M and A.

Remark 2.6. Certainly, analogous remarks to those we made on colimits can be made on limits.

3 Weakened Representability

In this section we formulate the weakened representability notion and formulate its basic properties.

Recall that a functor F is called *dense* if its tilde-conjugate $\widetilde{F}$ (Notation 2.2) is fully faithful.

Definition 3.1. A pair $(\mathbb{C}, \gamma)$ consisting of a pseudofunctor $\mathbb{C}$ on $\mathscr{V}$-CAT (the 2-category of categories, functors and natural transformations) and a (pointwise) fully faithful dense pseudonatural transformation $\gamma : \mathrm{Id} \longrightarrow \mathbb{C}$ is called a *doctrine*.

Remark 3.2. By Proposition 5.16 of [K$_1$], the existence of a fully faithful dense $\gamma_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathbb{C}(\mathscr{A})$ is equivalent to the fact that $\mathbb{C}(\mathscr{A})$ is a full subcategory of $[\mathscr{A}^{op}, \mathscr{V}]$ containing the representable functors (as a full subcategory). Thus, our concept of a doctrine captures precisely the idea that $\mathbb{C}(\mathscr{A})$ should lie "in between" $\mathscr{A}$ and $[\mathscr{A}^{op}, \mathscr{V}]$, that we expressed in the introduction.

In most situations below we will suppress γ and refer to a doctrine just by $\mathbb{C}$.

Example 3.3.

1. The *identity doctrine* $(\mathrm{Id}, \mathrm{id})$.

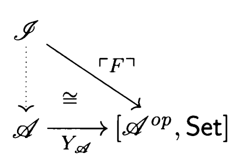
   Observe that $\widetilde{\mathrm{id}_{\mathscr{A}}}$ is the Yoneda embedding $Y_{\mathscr{A}} : \mathscr{A} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$ for every category $\mathscr{A}$.

2. Any KZ-doctrine of a *free cocompletion under a class of colimits* is a doctrine.

   More precisely, for every class $\mathbb{C}$ of small weights, we denote by $\gamma_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathbb{C}(\mathscr{A})$ the free cocompletion of $\mathscr{A}$ under colimits weighted by members of $\mathbb{C}$. It is well-known that each $\gamma_{\mathscr{A}}$ is fully faithful and dense.

3. When $\mathscr{V} = \mathrm{Set}$: the *doctrine of quotients* $(\mathbb{Q}, \gamma)$, where $\mathbb{Q}(\mathscr{A})$ consists of quotients of representables in $[\mathscr{A}^{op}, \mathrm{Set}]$.

As a motivation of the definition of weakened representability we choose the existence of a factorization



where $\mathscr{I}$ denotes the one-morphism (ordinary) category. Instead of representability of a (name of) a single functor $F : \mathscr{A}^{op} \longrightarrow \mathrm{Set}$ we will however study weakened representability of $G : \mathscr{M} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$. Weakened representability of such "diagrams of presheaves" will widen the scope of applications.

Definition 3.4. Let $(\mathbb{C}, \gamma)$ be a doctrine. A functor $G : \mathscr{M} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$ is called *representable relative to* $(\mathbb{C}, \gamma)$ when there is a functor $\mathrm{rep}(G) : \mathscr{M} \longrightarrow \mathbb{C}(\mathscr{A})$ and a natural isomorphism α

$$\begin{array}{ccc} \mathscr{M} & & \\ {\scriptstyle \mathrm{rep}(G)}\downarrow & \overset{G}{\searrow} & \\ & \cong & \\ \mathbb{C}(\mathscr{A}) & \xrightarrow[\widetilde{\gamma_{\mathscr{A}}}]{} & [\mathscr{A}^{op}, \mathscr{V}] \end{array} \tag{3.1}$$

The natural isomorphism $\alpha : G \longrightarrow \widetilde{\gamma_{\mathscr{A}}} \cdot \mathrm{rep}(G)$ is called the *representation of* G.

Remark 3.5. In elementary terms, representability relative to $(\mathbb{C}, \gamma)$ means that

$$GM \cong \mathbb{C}(\mathscr{A})(\gamma_{\mathscr{A}}-, \mathrm{rep}(G)M)$$

holds naturally in M. Observe further that, if it exists, $\mathrm{rep}(G)$ is itself determined to within an isomorphism since $\widetilde{\gamma_{\mathscr{A}}}$ is fully faithful. For the same reason it suffices to speak only of values of $\mathrm{rep}(G)$ on objects.

Example 3.6.

1. Representability of $G : \mathscr{I} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$ relative to $(\mathrm{Id}, \mathrm{id})$, where $\mathscr{I}$ is the unit category, is the usual concept of representability of a single functor $G* = F : \mathscr{A}^{op} \longrightarrow \mathscr{V}$.

2. When $\mathscr{V} = \mathsf{Set}$, representability of $G : \mathscr{I} \longrightarrow [\mathscr{A}^{op}, \mathsf{Set}]$ relative to $\mathbb{Q}$ is the concept of *weak representability* of $G* = F : \mathscr{A}^{op} \longrightarrow \mathsf{Set}$, see Example 1.1.

3. Representability of $G = \widetilde{F} : \mathscr{M} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$ for some $F : \mathscr{A} \longrightarrow \mathscr{M}$ relative to $(\mathbb{C}, \gamma)$ is precisely the notion of a $\gamma_{\mathscr{A}}$*-comodel* in the terminology of [K$_1$]. Since in such a situation there is an isomorphism

$$\mathscr{M}(FA, M) \cong \mathbb{C}(\mathscr{A})(\gamma_{\mathscr{A}}A, \mathrm{rep}(\widetilde{F})M)$$

natural in M and A, representability of $\widetilde{F}$ asserts the existence of an *adjunction relative to* $\gamma_{\mathscr{A}}$, denoted by

$$F \dashv_{\gamma_{\mathscr{A}}} \mathrm{rep}(\widetilde{F})$$

See also [Th].

The following theorem characterizes representability in the spirit of certain "absolute" colimits, compare with Theorem 4.80 of [K1].

Theorem 3.7. *For $G : \mathscr{M} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$, the following are equivalent:*

1. *G is representable relative to $(\mathbb{C}, \gamma)$.*
2. *The colimit $G \star \gamma_{\mathscr{A}}$ of $\gamma_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathbb{C}(\mathscr{A})$ weighted by $G : \mathscr{M} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$ exists and is preserved by $\widetilde{\gamma_{\mathscr{A}}}$ (i.e., the colimit $G \star \gamma_{\mathscr{A}}$ is $\gamma_{\mathscr{A}}$-*absolute *in the terminology of Section 5.4 of [K1]).*

Then $\mathrm{rep}(G) \cong G \star \gamma_{\mathscr{A}}$.

Proof. (1) implies (2): The functor $\mathrm{rep}(G) : \mathscr{M} \longrightarrow \mathbb{C}(\mathscr{A})$ together with the isomorphism

$$\mathbb{C}(\mathscr{A})(\mathrm{rep}(G), X) \cong [\mathscr{A}^{op}, \mathscr{V}](\widetilde{\gamma_{\mathscr{A}}}\mathrm{rep}(G), \widetilde{\gamma_{\mathscr{A}}}X) \cong [\mathscr{A}^{op}, \mathscr{V}](GM, \widetilde{\gamma_{\mathscr{A}}}X),$$

naturally in X and M, (where the first isomorphism is due to the fact that $\widetilde{\gamma_{\mathscr{A}}}$ is fully faithful and the second is due to representability of G w.r.t. $(\mathbb{C}, \gamma)$) exhibits $\mathrm{rep}(G)$ as $G \star \gamma_{\mathscr{A}}$. Moreover, this colimit is clearly preserved by $\widetilde{\gamma_{\mathscr{A}}}$, since we have isomorphisms

$$\widetilde{\gamma_{\mathscr{A}}}(G \star \gamma_{\mathscr{A}}) \cong \widetilde{\gamma_{\mathscr{A}}} \cdot \mathrm{rep}(G) \cong G \cong G \star Y \cong G \star (\widetilde{\gamma_{\mathscr{A}}}\gamma_{\mathscr{A}})$$

where we used that $(\widetilde{\gamma_{\mathscr{A}}}\gamma_{\mathscr{A}} \cong Y$, since $\gamma_{\mathscr{A}}$ is fully faithful dense.

(2) implies (1): This is trivial. Put $\mathrm{rep}(G) = G \star \gamma_{\mathscr{A}}$ and commutativity of (3.1) up to isomorphism follows from the fact that $G \star \gamma_{\mathscr{A}}$ is preserved by $\widetilde{\gamma_{\mathscr{A}}}$. ☐

Remark 3.8. The above theorem indeed reduces to Theorem 4.80 of [K1] when $(\mathbb{C}, \gamma)$ is the identity doctrine: if $\gamma_{\mathscr{A}} = \mathrm{id}_{\mathscr{A}}$, every $(GM)^{op} : \mathscr{A} \longrightarrow \mathscr{V}^{op}$ is a left Kan extension of itself along $\mathrm{id}_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathscr{A}$ and such Kan extensions preserve $\mathrm{id}_{\mathscr{A}}$-absolute colimits by Theorem 5.29 of [K1]. Hence every $(GM)^{op}$ preserves the colimit $G \star \mathrm{id}_A$, which was to be proved.

4 Weakened Limit Notions

The relevance of classes of weighted (co)limits even in the case of $\mathscr{V} = \mathsf{Set}$ has been discussed in detail in [AK]. In the rest of the paper whenever we speak of classes of (co)limits, we always have on mind a *saturated class* of small weights in the sense of [KS]. This means the following (we formulate it for classes of colimits, the case of limits is analogous):

Definition 4.1. The class $\mathbb{C}$ of small colimit weights is *saturated* if, for every small category $\mathscr{D}$, the class

$$\mathbb{C}[\mathscr{D}] = \{W : \mathscr{D}^{op} \longrightarrow \mathscr{V} \mid W \in \mathbb{C}\}$$

considered as a full subcategory of $[\mathscr{D}^{op}, \mathscr{V}]$ is a free cocompletion of $\mathscr{D}$ under $\mathbb{C}$-colimits.

The fact that we restrict ourselves to saturated classes is nothing grave: each class can be made saturated. Saturated classes, however, enjoy nice properties, e.g., one can prove the following ([KS]):

> *A presheaf $X : \mathscr{A}^{op} \longrightarrow \mathscr{V}$ belongs to $\mathbb{C}(\mathscr{A})$ if and only if there is a small $\mathbb{C}$-weight $W : \mathscr{D}^{op} \longrightarrow \mathscr{V}$ and a functor $J : \mathscr{D} \longrightarrow \mathscr{A}$ such that $X \cong \mathrm{Lan}_{J^{op}} W$ holds.*

Definition 4.2. For a weight $W : \mathscr{M} \longrightarrow [\mathscr{D}, \mathscr{V}]^{op}$ and a diagram $D : \mathscr{D} \longrightarrow \mathscr{A}$, define a *cylinder functor* as follows

$$\mathsf{Cyl}(W, D) : \mathscr{M} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}] \quad M \mapsto [\mathscr{D}, \mathscr{V}]^{op}(\widehat{D}-, WM)$$

Recall that a *limit* $\{W, D\}$ *of* D *weighted by* W exists, if $\mathsf{Cyl}(W, D)$ is representable in the usual sense, i.e., when there is a natural isomorphism

$$\begin{array}{ccc} \mathscr{M} & & \\ {\scriptstyle \{W,D\}}\big\downarrow & \cong \quad \searrow^{\mathsf{Cyl}(W,D)} & \\ \mathscr{A} & \xrightarrow[Y_{\mathscr{A}}]{} & [\mathscr{A}^{op}, \mathscr{V}] \end{array}$$

Definition 4.3. Provided $\mathsf{Cyl}(W, D)$ is representable relative to $(\mathbb{C}, \gamma)$, we say that a *limit of D weighted by W exists relative to* $(\mathbb{C}, \gamma)$. The functor $\mathrm{rep}(\mathsf{Cyl}(W, D)) : \mathscr{M} \longrightarrow \mathbb{C}(\mathscr{A})$ is then denoted by $\{W, D\}_{(\mathbb{C},\gamma)}$:

$$\begin{array}{ccc} \mathscr{M} & & \\ {\scriptstyle \{W,D\}_{(\mathbb{C},\gamma)}}\big\downarrow & \cong \quad \searrow^{\mathsf{Cyl}(W,D)} & \\ \mathbb{C}(\mathscr{A}) & \xrightarrow[\widetilde{\gamma_{\mathscr{A}}}]{} & [\mathscr{A}^{op}, \mathscr{V}] \end{array} \tag{4.1}$$

Limits relative to $(\mathbb{C}, \gamma)$ are limits of representables in $\mathbb{C}(\mathscr{A})$, as the next result shows:

Lemma 4.4. *For any weight $W : \mathscr{M} \longrightarrow [\mathscr{D}, \mathscr{V}]^{op}$ and any diagram $D : \mathscr{D} \longrightarrow \mathscr{A}$ the isomorphism*

$$\{W, D\}_{(\mathbb{C},\gamma)} \cong \{W, \gamma_{\mathscr{A}} D\} : \mathscr{M} \longrightarrow \mathbb{C}(\mathscr{A})$$

holds, either side existing when the other does.

Proof. Let $W : \mathscr{M} \longrightarrow [\mathscr{D}, \mathscr{V}]^{op}$ be a weight and let $D : \mathscr{D} \longrightarrow \mathscr{A}$ be a diagram.
Since $\gamma_{\mathscr{A}}$ is fully faithful we have an isomorphism

$$[\mathscr{D}, \mathscr{V}]^{op}(\widehat{\gamma_{\mathscr{A}} D}\gamma_{\mathscr{A}} A, WM) \cong [\mathscr{D}, \mathscr{V}]^{op}(\widehat{D}A, WM)$$

natural in m and A, which proves:

1. In case $\{W, D\}_{(\mathbb{C},\gamma)}$ exists:

$$\begin{aligned} \mathbb{C}(\mathscr{A})(\gamma_{\mathscr{A}} A, \{W, D\}_{(\mathbb{C},\gamma)} M) &\cong \mathsf{Cyl}(W, D)(M)(A) \\ &\cong [\mathscr{D}, \mathscr{V}]^{op}(\widehat{\gamma_{\mathscr{A}} D}\gamma_{\mathscr{A}} A, WM) \end{aligned}$$

   Thus $\{W, \gamma_{\mathscr{A}} D\}$ exists and is isomorphic to $\{W, D\}_{(\mathbb{C},\gamma)}$.

2. In case $\{W, \gamma_{\mathscr{A}} D\}$ exists:

$$\begin{aligned} \mathbb{C}(\mathscr{A})(\gamma_{\mathscr{A}} A, \{W, \gamma_{\mathscr{A}} D\} M) &\cong [\mathscr{D}, \mathscr{V}]^{op}(\widehat{\gamma_{\mathscr{A}} D}\gamma_{\mathscr{A}} A, WM) \\ &\cong [\mathscr{D}, \mathscr{V}]^{op}(\widehat{D}A, WM) \end{aligned}$$

   Hence $\{W, D\}_{(\mathbb{C},\gamma)}$ exists and is isomorphic to $\{W, \gamma_{\mathscr{A}} D\}$.

□

Notation 4.5. By $\mathbb{L}$ we denote a saturated *doctrine of free completion under small limits of a certain class* and by $\lambda_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathbb{L}(\mathscr{A})$ we denote the fully faithful codense embedding into a free $\mathbb{L}$-completion of $\mathscr{A}$.

Definition 4.6. A small weight $W : \mathscr{M} \longrightarrow [\mathscr{D}, \mathscr{V}]^{op}$ (i.e., one where both $\mathscr{M}$ and $\mathscr{D}$ are small) is called an $\mathbb{L}$-*weight* provided it factors through $\widehat{\lambda_{\mathscr{D}}} : \mathbb{L}(\mathscr{D}) \longrightarrow [\mathscr{D}, \mathscr{V}]^{op}$.

Now comes the main result of this section.

Theorem 4.7. *For any $\mathscr{A}$ the following are equivalent:*

1. *$\mathscr{A}$ is $\mathbb{L}$-complete relative to $(\mathbb{C}, \gamma)$, i.e., $\{W, D\}_{(\mathbb{C},\gamma)}$ exists for any $\mathbb{L}$-weight W and any diagram D.*

2. *$\mathbb{C}(\mathscr{A})$ has $\mathbb{L}$-limits of representables.*

3. *There is an adjunction $\lambda_{\mathscr{A}} \dashv_{\gamma_{\mathscr{A}}} \operatorname{rep}(\widetilde{\lambda_{\mathscr{A}}}) : \mathbb{L}(\mathscr{A}) \longrightarrow \mathbb{C}(\mathscr{A})$.*

If, moreover, $\mathbb{C}$ is a colimit doctrine, the above are further equivalent to

4. *For every object X in $\mathbb{L}(\mathscr{A})$ there is a $\mathbb{C}$-weight $W_X : \mathscr{K}_X^{op} \longrightarrow \mathscr{V}$ and a diagram $J_X : \mathscr{K}_X \longrightarrow \mathscr{A}$ such that the isomorphism*

$$\mathbb{L}(\mathscr{A})(\lambda_{\mathscr{A}} A, X) \cong \int^{K \in \mathscr{K}_X^{op}} W_X K \otimes \mathscr{A}(A, J_X K) \tag{4.2}$$

holds.

5. *There is an honest adjunction $\mathbb{C}(\lambda_{\mathscr{A}}) \dashv R : \mathbb{C}\mathbb{L}(\mathscr{A}) \longrightarrow \mathbb{C}(\mathscr{A})$ with R preserving $\mathbb{C}$-colimits.*

Proof. (1) implies (2): Use Lemma 4.4.

(2) implies (3): It suffices to define the desired functor $\mathbb{L}(\mathscr{A}) \longrightarrow \mathbb{C}(\mathscr{A})$ on objects. To this end, express an object X as a limit $\{W, \lambda_{\mathscr{A}} D\} : \mathscr{I} \longrightarrow \mathbb{L}(\mathscr{A})$ for some small $\mathbb{L}$-weight $W : \mathscr{I} \longrightarrow [\mathscr{D}, \mathscr{V}]^{op}$ and a diagram $D : \mathscr{D} \longrightarrow \mathscr{A}$. (Here, $\mathscr{I}$ denotes the category on one object $*$ with $\mathscr{I}(*, *) = I$.)

Using Lemma 4.4 there exists a limit $\{W, \gamma_{\mathscr{A}} D\} : \mathscr{I} \longrightarrow \mathbb{C}(\mathscr{A})$. The assignment

$$\{W, \lambda_{\mathscr{A}} D\}(*) \mapsto \{W, \gamma_{\mathscr{A}} D\}(*)$$

is the object assignment of a functor $\mathbb{L}(\mathscr{A}) \longrightarrow \mathbb{C}(\mathscr{A})$ that clearly is a right adjoint to $\lambda_{\mathscr{A}}$ relative to $\gamma_{\mathscr{A}}$.

(3) implies (1): Suppose $W : \mathscr{M} \longrightarrow [\mathscr{D}, \mathscr{V}]^{op}$ is an $\mathbb{L}$-weight and $D : \mathscr{D} \longrightarrow \mathscr{A}$ a diagram. Since $\mathbb{L}(\mathscr{A})$ has $\mathbb{L}$-limits, there is a natural isomorphism

$$\begin{array}{ccc} \mathscr{M} & & \\ {\scriptstyle \{W,\lambda_{\mathscr{A}} D\}} \big\downarrow & \overset{\mathsf{Cyl}(W,\lambda_{\mathscr{A}} D)}{\searrow} \ \cong & \\ \mathbb{L}(\mathscr{A}) & \xrightarrow[Y_{\mathbb{L}(\mathscr{A})}]{} & [\mathbb{L}(\mathscr{A})^{op}, \mathscr{V}] \end{array}$$

Since $[\lambda_{\mathscr{A}}^{op}, \mathscr{V}] \cdot Y_{\mathbb{L}(\mathscr{A})} = \widetilde{\lambda_{\mathscr{A}}}$, we have the square

$$\begin{array}{ccc} \mathbb{L}(\mathscr{A}) & \xrightarrow{Y_{\mathbb{L}(\mathscr{A})}} & [\mathbb{L}(\mathscr{A})^{op}, \mathscr{V}] \\ {\scriptstyle \operatorname{rep}(\widetilde{\lambda_{\mathscr{A}}})} \big\downarrow & \cong & \big\downarrow {\scriptstyle [\lambda_{\mathscr{A}}^{op}, \mathscr{V}]} \\ \mathbb{C}(\mathscr{A}) & \xrightarrow[\widetilde{\gamma_{\mathscr{A}}}]{} & [\mathscr{A}^{op}, \mathscr{V}] \end{array}$$

and by pasting with the above triangle we obtain

$$\begin{array}{ccc}
\mathscr{M} & & \\
{\scriptstyle \{W,\lambda_{\mathscr{A}} D\}}\downarrow & \searrow^{\mathsf{Cyl}(W,\lambda_{\mathscr{A}} D)} & \\
 & \cong & \\
\mathbb{L}(\mathscr{A}) & \xrightarrow[Y_{\mathbb{L}(\mathscr{A})}]{} & [\mathbb{L}(\mathscr{A})^{op}, \mathscr{V}] \\
{\scriptstyle \mathrm{rep}(\widetilde{\lambda_{\mathscr{A}}})}\downarrow & \cong & \downarrow{\scriptstyle [\lambda_{\mathscr{A}}^{op},\mathscr{V}]} \\
\mathbb{C}(\mathscr{A}) & \xrightarrow[\widetilde{\gamma_{\mathscr{A}}}]{} & [\mathscr{A}^{op}, \mathscr{V}]
\end{array}$$

The proof will be finished once we show that

$$[\lambda_{\mathscr{A}}^{op}, \mathscr{V}] \cdot \mathsf{Cyl}(W, \lambda_A D) \cong \mathsf{Cyl}(W, D)$$

holds. But this is straightforward:

$$\begin{aligned}
[\lambda_{\mathscr{A}}^{op}, \mathscr{V}](\mathsf{Cyl}(W, \lambda_A D)m) &= [\mathscr{D}, \mathscr{V}]^{op}(\widehat{\lambda_A D}\lambda_A -, Wm) \\
&\cong [\mathscr{D}, \mathscr{V}]^{op}(\widehat{D}-, Wm) \\
&= \mathsf{Cyl}(W, D)(m)
\end{aligned}$$

where the isomorphism is due to the fact that λ_A is fully faithful.

(3) is equivalent to (4), since the latter condition just asserts that $\widetilde{\lambda_{\mathscr{A}}}$ lands in $\mathbb{C}(\mathscr{A})$.

(3) implies (5): Define $R = \mathrm{Lan}_{\gamma_{\mathscr{A}}}(\mathrm{rep}(\widetilde{\lambda_{\mathscr{A}}})) : \mathbb{C}(\mathscr{A}) \longrightarrow \mathbb{CL}(\mathscr{A})$. Then R preserves $\mathbb{C}$-colimits by definition and the adjunction $\mathbb{C}(\lambda_{\mathscr{A}}) \dashv R$ follows from $\lambda_{\mathscr{A}} \dashv_{\gamma_{\mathscr{A}}} \mathrm{rep}(\widetilde{\lambda_{\mathscr{A}}})$ and the properties of left Kan extensions.

(5) implies (3): The restriction $R \cdot \gamma_{\mathscr{A}} : \mathbb{L}(\mathscr{A}) \longrightarrow \mathbb{C}(\mathscr{A})$ clearly makes the diagram

$$\begin{array}{ccc}
\mathbb{L}(\mathscr{A}) & & \\
{\scriptstyle R\cdot\gamma_{\mathscr{A}}}\downarrow & \searrow^{\widetilde{\lambda_{\mathscr{A}}}} & \\
 & \cong & \\
\mathbb{C}(\mathscr{A}) & \xrightarrow[\widetilde{\gamma_{\mathscr{A}}}]{} & [\mathscr{A}^{op}, \mathscr{V}]
\end{array}$$

commutative. □

Remark 4.8. Observe that $\mathbb{C}(\lambda_{\mathscr{A}}) : \mathbb{C}(\mathscr{A}) \longrightarrow \mathbb{CL}(\mathscr{A})$ is always fully faithful, thus, by Theorem 4.7, for every category $\mathscr{A}$ that is $\mathbb{L}$-complete relative to $\mathbb{C}$, the category $\mathbb{C}(\mathscr{A})$ has those limits that $\mathbb{CL}(\mathscr{A})$ has.

Remark 4.9. The implication (2) $\Rightarrow$ (5) above is used implicitly in the proof of Theorem 3.8 of [DL] for the case $\mathbb{C}$ = small colimits and $\mathbb{L}$ = small limits.

Example 4.10. In this example we fix $\mathscr{V} = \mathsf{Set}$ and recover the plurilimit concept of [KRV] as finite completeness relative to finite cocompleteness.

To prove it, denote by $\mathbb{C} = \mathsf{colex}$ the doctrine of finite colimits and by $\mathbb{L} = \mathsf{lex}$ the doctrine of finite limits. More precisely, colex-weights are those functors $W : \mathscr{K}^{op} \longrightarrow \mathsf{Set}$ where the category $\mathscr{K}$ is finite and every WK is a finite set.

Condition (4.2) of Theorem 4.7 then says the following:

For every object X in $\mathsf{lex}(\mathscr{A})$ there exist a finite weight $W_X : \mathscr{K}_X^{op} \longrightarrow \mathsf{Set}$ and a functor $J_X : \mathscr{K}_X \longrightarrow \mathscr{A}$ such that the isomorphism

$$\mathsf{lex}(\mathscr{A})(\lambda_{\mathscr{A}} A, X) \cong \int^{K \in \mathscr{K}_X^{op}} W_X K \times \mathscr{A}(A, J_X K)$$

holds.

This is precisely the concept of a plurilimit: an object X is a finite limit of representables $X = \lim_i \mathscr{A}(A_i, -)$ and every $W_D K'$ is a finite set of cones for A_i's having $J_X K'$ as a vertex. More precisely, every element of $W_X K'$ is such a cone by virtue of the map

$$\begin{aligned} W_X K' &\cong \int^{K \in \mathscr{K}_X^{op}} W_X K \times \mathscr{K}_X(K', K) \longrightarrow \\ &\longrightarrow \int^{K \in \mathscr{K}_X^{op}} W_X K \times \mathscr{A}(J_X K', J_X K) \cong \mathsf{lex}(\mathscr{A})(\lambda_{\mathscr{A}} J_X K', X) \end{aligned}$$

where the first isomorphism is due to Yoneda lemma, the second map is given by the action of J_X on hom-sets, and the final isomorphism is an instance of (4.2) for $A := J_X K'$.

Having identified the elements of W_X as a (finite!) family of distinguished cones for A_i's, we see that the isomorphism (4.2) says that every cone for A_i's having A as a vertex factors through some distinguished cone and every two such factorizations are connected via a zig-zag in $\mathscr{K}_X$.

Thus, condition (4) of Theorem 4.7 expresses precisely the concept of a plurilimit of a finite diagram, as defined in [KRV].

Example 4.11. Take $\mathbb{C} = \mathsf{colim}$, the doctrine of all small colimits, and $\mathbb{L} = \mathsf{lim}$, the doctrine of all small limits (i.e., $\mathsf{lim}(\mathscr{A}) = (\mathsf{colim}(\mathscr{A}^{op}))^{op}$, then $\mathscr{A}$ is lim-complete relative to colim if and only if the *Isbell conjugate* $\mathsf{lim}(\mathscr{A})(\lambda_{\mathscr{A}} -, F) : \mathscr{A}^{op} \longrightarrow \mathscr{V}$ of any small $F : \mathscr{A} \longrightarrow \mathscr{V}$ is small.

Example 4.12. Left-coherent rings. Here we take $\mathscr{V} = \mathsf{Ab}$, $\mathbb{C} = \mathsf{colex}$ the doctrine of finite Ab-colimits and $\mathbb{L} = \mathsf{lex}$ the doctrine of finite Ab-limits. Thus, to be a colex-weight $W : \mathscr{K}^{op} \longrightarrow \mathsf{Ab}$ means that $\mathscr{K}$ has finitely many objects and each $\mathscr{K}(K, K')$ and each WK is an finitely presentable Abelian group. The lex-weights are characterized in the same way.

Let R be a ring with a unit, considered as an Ab-category $\mathscr{R}$ on one object in the usual way. Then condition (3) of Theorem 4.7 translates as the condition that the R-duality functor $\mathrm{Hom}(-, R)$ restricts to a functor from the category of finitely presentable right R-modules to the category of finitely presentable left R-modules. Thus, by (the dual of) Proposition 1 of [C] we obtain the result

> *A ring R is left-coherent if and only if the category $\mathscr{R}$ is* lex*-complete relative to* colex.

The following is well-known in additive category theory but indicates the applicability of Theorem 4.7. Recall from [Be], Corollary 3.2 and Corollary 3.9 that $\mathsf{colex}\,\mathscr{A}$, in the sense of Example 4.12 above, is the free cocompletions of $\mathscr{A}$ under cokernels. Also recall from [Kr], Lemma 1.6(1), that for an additive $\mathscr{A}$ we have $\mathsf{colex}\,\mathsf{lex}\,\mathscr{A} \cong \mathsf{lex}\,\mathsf{colex}\,\mathscr{A}$. Hence our Theorem 4.7 ((3) implies (5)) yields:

Corollary 4.13. *If $\mathscr{A}$ is a (right) coherent additive category (i.e., $\mathscr{A}$ has weak kernels), then its completion* $\mathsf{colex}\,\mathscr{A}$ *under cokernels has kernels (hence is Abelian).*

5 Limits of Representables in Free Cocompletions

The proof of the following result is trivial.

Proposition 5.1. *If $\mathbb{C}(\mathscr{A})$ has $\mathbb{L}$-limits and $\widetilde{\gamma_{\mathscr{A}}}$ preserves them, then $\mathscr{A}$ is $\mathbb{L}$-complete relative to $\mathbb{C}$.*

Remark 5.2. The converse of the preceding proposition does not hold in general, see Example 5.4 of [KRV] for a category $\mathscr{A}$ having finite limits such that $\mathbb{C}(\mathscr{A})$, the cocompletion of $\mathscr{A}$ under finite colimits, does not have finite limits. See, however, Theorem 5.4 below.

Recall that a *lifting* $\mathbb{C}^*$ *of* $\mathbb{C}$ to the category $\mathscr{V}\text{-}\mathsf{CAT}^{\mathbb{L}}$ of $\mathbb{L}$-algebras is a natural isomorphism

$$\begin{array}{ccc} \mathscr{V}\text{-}\mathsf{CAT}^{\mathbb{L}} & \xrightarrow{\mathbb{C}^*} & \mathscr{V}\text{-}\mathsf{CAT}^{\mathbb{L}} \\ {\scriptstyle U^{\mathbb{L}}}\downarrow & \cong & \downarrow{\scriptstyle U^{\mathbb{L}}} \\ \mathscr{V}\text{-}\mathsf{CAT} & \xrightarrow[\mathbb{C}]{} & \mathscr{V}\text{-}\mathsf{CAT} \end{array} \tag{5.1}$$

Such a lifting is well-known to be equivalent to the existence of a pseudonatural transformation $\delta : \mathbb{L}\mathbb{C} \longrightarrow \mathbb{C}\mathbb{L}$ satisfying the following two axioms: the diagrams

$$\begin{array}{ccc} \mathbb{L}\mathbb{C}(\mathscr{A}) & \xrightarrow{\quad\delta_{\mathscr{A}}\quad} & \mathbb{C}\mathbb{L}(\mathscr{A}) \\ & \nwarrow^{\lambda_{\mathbb{C}(\mathscr{A})}} \quad \cong \quad \nearrow^{\mathbb{C}(\lambda_{\mathscr{A}})} & \\ & \mathbb{C}(\mathscr{A}) & \end{array} \tag{5.2}$$

$$\begin{array}{ccc} \mathbb{L}\mathbb{L}\mathbb{C}(\mathscr{A}) \xrightarrow{\mathbb{L}\delta_{\mathscr{A}}} \mathbb{L}\mathbb{C}\mathbb{L}(\mathscr{A}) \xrightarrow{\delta_{\mathbb{L}(\mathscr{A})}} & \mathbb{C}\mathbb{L}\mathbb{L}(\mathscr{A}) \\ {\scriptstyle m^{\mathbb{L}}_{\mathbb{C}(\mathscr{A})}}\downarrow \qquad \cong & \downarrow{\scriptstyle \mathbb{C}(m^{\mathbb{L}}_{\mathscr{A}})} \\ \mathbb{L}\mathbb{C}(\mathscr{A}) \xrightarrow[\delta_{\mathscr{A}}]{\qquad\qquad\qquad} & \mathbb{C}\mathbb{L}(\mathscr{A}) \end{array} \tag{5.3}$$

are commutative to within an isomorphism, satisfying some coherence conditions, see Section 5.2 of [Ta]. We call such data a *lifting law of* $\mathbb{C}$ *to* $\mathbb{L}$.

There are cases, however, when the above axioms boil down to just one triangle, since $\mathbb{L}$ is co-KZ by [PCW]:

Lemma 5.3. *If every instance of the triangle of* (5.2) *is a right Kan extension, then the right-hand square commutes to within an isomorphism and we have a lifting law.*

Proof. Define the isomorphism in the right-hand square as the extension of the identity 2-cell on $\delta_{\mathscr{A}}$ along $\lambda_{\mathbb{L}\mathbb{C}(\mathscr{A})} : \mathbb{L}\mathbb{C}(\mathscr{A}) \longrightarrow \mathbb{L}\mathbb{L}\mathbb{C}(\mathscr{A})$:

$$\begin{array}{ccccc} \mathbb{L}\mathbb{C}(\mathscr{A}) & & \xrightarrow{\qquad\delta_{\mathscr{A}}\qquad} & & \mathbb{C}\mathbb{L}(\mathscr{A}) \\ & \searrow^{\lambda_{\mathbb{L}\mathbb{C}(\mathscr{A})}} & & \swarrow^{\lambda_{\mathbb{C}\mathbb{L}(\mathscr{A})}} & \downarrow{\scriptstyle \mathbb{C}(\lambda_{\mathbb{L}(\mathscr{A})})} \\ {\scriptstyle \mathrm{id}} & \mathbb{L}\mathbb{L}\mathbb{C}(\mathscr{A}) & \xrightarrow{\mathbb{L}\delta_{\mathscr{A}}} \mathbb{L}\mathbb{C}\mathbb{L}(\mathscr{A}) \xrightarrow{\delta_{\mathbb{L}(\mathscr{A})}} & & \mathbb{C}\mathbb{L}\mathbb{L}(\mathscr{A}) \quad {\scriptstyle \mathrm{id}} \\ & {\scriptstyle m^{\mathbb{L}}_{\mathbb{C}(\mathscr{A})}}\downarrow & & & \downarrow{\scriptstyle \mathbb{C}(m^{\mathbb{L}}_{\mathscr{A}})} \\ \longrightarrow & \mathbb{L}\mathbb{C}(\mathscr{A}) & \xrightarrow[\delta_{\mathscr{A}}]{\qquad\qquad} & & \mathbb{C}\mathbb{L}(\mathscr{A}) \end{array}$$

(This is can be done: all morphisms in the square preserve $\mathbb{L}$-limits — $\mathbb{C}(m^{\mathbb{L}}_{\mathscr{A}})$ does since it is a $\mathbb{C}$-image of a right adjoint $m^{\mathbb{L}}_{\mathscr{A}}$ and $\delta_{\mathbb{L}(\mathscr{A})}$ preserves $\mathbb{L}$-limits, since it arises as a right Kan extension.) □

The lifted pseudofunctor $\mathbb{C}^*$ sends an algebra $(\mathscr{A}, a)$ to $(\mathbb{C}(\mathscr{A}), \mathbb{C}(a)\delta_{\mathscr{A}})$, as usual.

We obtain thus the following:

Theorem 5.4. *The following are equivalent:*

1. *Every $\mathbb{C}(\mathscr{A})$ has $\mathbb{L}$-limits whenever $\mathscr{A}$ has $\mathbb{L}$-limits.*
2. *Every $\mathbb{C}(\mathscr{A})$ has $\mathbb{L}$-limits whenever $\mathscr{A}$ is $\mathbb{L}$-complete relatively to $\mathbb{C}$.*
3. *Every $\mathbb{CL}(\mathscr{A})$ has $\mathbb{L}$-limits.*
4. *There exists a lifting law $\delta : \mathbb{LC} \longrightarrow \mathbb{CL}$ of $\mathbb{C}$ to $\mathbb{L}$.*

Proof. For the implication $(1) \Rightarrow (2)$ use Remark 4.8 and the implication $(2) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (4)$: Define $\delta_{\mathscr{A}} : \mathbb{LC}(\mathscr{A}) \longrightarrow \mathbb{CL}(\mathscr{A})$ as a right Kan extension of $\mathbb{C}(\lambda_{\mathscr{A}})$ along $\lambda_{\mathbb{C}(\mathscr{A})}$ (it exists, since $\mathbb{CL}(\mathscr{A})$ is assumed to have $\mathbb{L}$-limits). Then use Lemma 5.3.

$(4) \Rightarrow (1)$: $\mathbb{L}$-algebras are precisely the categories having $\mathbb{L}$-limits and the lifting δ gives us the square (5.1). Thus, if $\mathscr{A}$ has $\mathbb{L}$-limits, so does every $\mathbb{C}(\mathscr{A})$, being the $\mathbb{C}^*$-image of an $\mathbb{L}$-algebra $\mathscr{A}$. □

Corollary 5.5. *The equivalent conditions of Theorem 5.4 are satisfied in the presence of an honest full distributive law $\delta : \mathbb{LC} \longrightarrow \mathbb{CL}$.*

Example 5.6. Let $\mathscr{V} = \mathsf{Set}$ and $\mathbb{L}$ be the doctrine of small limits and $\mathbb{C}$ the doctrine of small $\mathbb{D}$-filtered colimits for a sound limit doctrine $\mathbb{D}$ in the sense of [ABLR].

Then, as proved in Theorem 6.3 of [ABLR], there is a distributive law $\delta : \mathbb{LC} \longrightarrow \mathbb{CL}$. By choosing various sound limits doctrines $\mathbb{D}$ we obtain the following results:

1. Corollary 3.9 of [DL]: In case $\mathbb{D}$ is empty, $\mathbb{C}$ is the doctrine colim of all small colimits.

   Hence $\mathsf{colim}(\mathscr{A})$, the category of small presheaves on $\mathscr{A}$, has small limits, whenever $\mathscr{A}$ does.

2. In case $\mathbb{D}$ is the doctrine of α-small limits, $\mathbb{C}$ is the doctrine of α-filtered colimits in the usual sense.

   Hence $\mathbb{C}(\mathscr{A})$, the α-inductive cocompletion of $\mathscr{A}$, has small limits, whenever $\mathscr{A}$ has small limits.

3. In case $\mathbb{D}$ is the doctrine of finite products, $\mathbb{C}$ is the doctrine of sifted colimits (see [ABLR]).

   Hence $\mathbb{C}(\mathscr{A})$, the cocompletion of $\mathscr{A}$ under sifted colimits, has small limits, whenever $\mathscr{A}$ has them. (Categories of the form $\mathbb{C}(\mathscr{A})$ are called *generalized varieties* in [AR_2].)

4. In case $\mathbb{D}$ is the doctrine of finite conected limits, $\mathbb{C}$ is the doctrine of small coproducts of filtered categories (see [ABLR]).

   Hence $\mathbb{C}(\mathscr{A}) = \mathsf{Fam}(\mathsf{Ind}(\mathscr{A}))$ has small limits, whenever $\mathscr{A}$ does.

Remark 5.7. In view of Theorems 4.7 and 5.4, the cases (2), (3) and (4) of Example 5.6 have the expected generalizations over a base $\mathscr{V}$, at least when $\mathscr{V}$ is cartesian closed. Details will appear elsewhere.

6 Monoidal Structures on $\mathbb{C}(\mathscr{A})$

The results of this section are easy generalizations of results from Section 7 of [DL].

Recall that *promonoidal structure* on $\mathscr{A}$ consists of a pair $P : \mathscr{A} \otimes \mathscr{A} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$ and $J : \mathscr{I} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$ satisfying associativity and unit constraints up to coherent isomorphisms, see [D]. In fact, the axioms express exactly the fact that the triple $(\mathscr{A}, P, J)$ (called a *promonoidal category*) is exactly a pseudomonoid in the monoidal bicategory MOD of modules, see [DS].

Definition 6.1. A promonoidal category $(\mathscr{A}, P, J)$ is called $\mathbb{C}$-*representable* if both P and J are functors representable relatively to $\mathbb{C}$.

Example 6.2.

1. If we take the identity doctrine Id for $\mathbb{C}$, then Id-representable promonoidal categories $(\mathscr{A}, P, J)$ are precisely the monoidal categories $(\mathscr{A}, \square, E)$, since we must have representations

$$\begin{array}{ccc} \mathscr{A} \otimes \mathscr{A} & & \\ {\scriptstyle \square}\downarrow & \cong \searrow^{P} & \\ \mathscr{A} & \xrightarrow[Y_{\mathscr{A}}]{} & [\mathscr{A}^{op}, \mathscr{V}] \end{array} \qquad \begin{array}{ccc} \mathscr{I} & & \\ {\scriptstyle E}\downarrow & \cong \searrow^{J} & \\ \mathscr{A} & \xrightarrow[Y_{\mathscr{A}}]{} & [\mathscr{A}^{op}, \mathscr{V}] \end{array}$$

   The associativity and unit constraints on P, J then assert precisely that $(\mathscr{A}, \square, E)$ is a monoidal category.

2. Clearly, every monoidal category $(\mathscr{A}, \square, E)$ is $\mathbb{C}$-representable promonoidal, for every doctrine $(\mathbb{C}, \gamma)$.

3. If we take colim (the doctrine of all small colimits) for $\mathbb{C}$, then colim-representable promonoidal categories are precisely promonoidal categories with both P and J small, see [DL].

Given a promonoidal category $(\mathscr{A}, P, J)$, there is a canonical *convolution monoidal structure* on $[\mathscr{A}^{op}, \mathscr{V}]$, provided each coend

$$F \otimes_P G = \int^{A,B} P(-; A, B) \otimes FA \otimes GB, \quad F, G \text{ in } [\mathscr{A}^{op}, \mathscr{V}]$$

exists in $[\mathscr{A}^{op}, \mathscr{V}]$. Then $\otimes_P$ is a tensor product on $[\mathscr{A}^{op}, \mathscr{V}]$ having $J(*) : \mathscr{A}^{op} \longrightarrow \mathscr{V}$ as a unit. See [D].

Lemma 6.3. *The following facts are equivalent:*

1. *$\mathbb{C}(\mathscr{A})$ has a monoidal structure.*

2. *$\mathscr{A}$ has a $\mathbb{C}$-representable promonoidal structure P, J.*

Moreover, every monoidal structure on $\mathbb{C}(\mathscr{A})$ is a convolution monoidal structure for some $\mathbb{C}$-representable promonoidal structure on $\mathscr{A}$.

Proof. (1) implies (2): Suppose $(\mathbb{C}(\mathscr{A}), \Box, J)$ is a monoidal category. Define $P' : \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathbb{C}(\mathscr{A})$ as the restriction of the tensor product

$$-\Box- : \mathbb{C}(\mathscr{A}) \otimes \mathbb{C}(\mathscr{A}) \longrightarrow \mathbb{C}(\mathscr{A})$$

along $\gamma_{\mathscr{A}} \otimes \gamma_A$ and put $P = \widetilde{\gamma_{\mathscr{A}}} P' : \mathscr{A} \otimes \mathscr{A} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$. Then $(\mathscr{A}, P, J)$ is promonoidal and $\mathbb{C}$-representable.

(2) implies (1): Let the following diagrams

$$\begin{array}{ccc} \mathscr{A} \otimes \mathscr{A} & & \\ {\scriptstyle P'}\downarrow & \searrow^{P} & \\ \mathbb{C}(\mathscr{A}) & \xrightarrow[\widetilde{\gamma_{\mathscr{A}}}]{} & [\mathscr{A}^{op}, \mathscr{V}] \end{array} \qquad \begin{array}{ccc} \mathscr{I} & & \\ {\scriptstyle J'}\downarrow & \searrow^{J} & \\ \mathbb{C}(\mathscr{A}) & \xrightarrow[\widetilde{\gamma_{\mathscr{A}}}]{} & [\mathscr{A}^{op}, \mathscr{V}] \end{array}$$

commutative to within isomorphisms witness $\mathbb{C}$-representability of a promonoidal category $(\mathscr{A}, P, J)$.

Define $-\otimes_P- : \mathbb{C}(\mathscr{A}) \otimes \mathbb{C}(\mathscr{A}) \longrightarrow \mathbb{C}(\mathscr{A})$ to be the colimit of $P' : \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathbb{C}(\mathscr{A})$ weighted by

$$W : \mathbb{C}(\mathscr{A}) \otimes \mathbb{C}(\mathscr{A}) \longrightarrow [(\mathscr{A} \otimes \mathscr{A})^{op}, \mathscr{V}], \quad (F, G) \mapsto ((A, B) \mapsto FA \otimes GB)$$

This colimit clearly exists (it exists pointwise, in fact, W is a $\mathbb{C}$-weight) and it defines a (convolution) tensor product, since $(\mathscr{A}, P, J)$ was promonoidal. Thus, $(\mathbb{C}(\mathscr{A}), \otimes_P, J')$ is a monoidal category.

The last assertion is obvious. $\Box$

Remark 6.4. By adapting the results of Brian Day [D], the closedness of the convolution monoidal structure on $\mathbb{C}(\mathscr{A})$ would require the existence of certain limits in $\mathbb{C}(\mathscr{A})$ preserved by $\widetilde{\gamma_{\mathscr{A}}}$. Clearly, such limits need not exist in $\mathbb{C}(\mathscr{A})$: take the identity doctrine for $\mathbb{C}$ and any monoidal category $\mathscr{A}$ that is not closed.

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