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ON SYNCHRONIZED RELATIVELY FULL EMBEDDINGS AND Q-UNIVERSALITY

To Jiří Adámek on his 60th birthday

by V. KOUBEK and J. SICHLER

Abstract

M. E. Adams et W. Dziobiak ont démontré que toute quasi-variété ff-algébrique universelle de systèmes algébriques de signature finie est Q-universelle. Dans cet article on introduit la notion de plongement synchronisé relativement plein qu'on utilise ensuite afin de modifier leur résultat pour les quasi-variétés d'algèbres.

1 Introduction

We aim to show a new connection between two algebraic structures associated with quasivarieties of algebras. All needed definitions are given in the next section.

First, for any quasivariety $\mathbb{Q}$, the homomorphisms between its members form a concrete category. The richness of the categorical structure is reflected in the notion of algebraic universality studied in the monograph [18] by A. Pultr and V. Trnková.

When ordered by inclusion, the subquasivarieties of a given quasivariety $\mathbb{Q}$ form a lattice we denote $\mathrm{QLat}(\mathbb{Q})$. This is the second algebraic structure associated with $\mathbb{Q}$. Questions about the size of $\mathrm{QLat}(\mathbb{Q})$ or lattice identities satisfied in $\mathrm{QLat}(\mathbb{Q})$ motivated M. V. Sapir [19] to define and exhibit Q-universal quasivarieties, and W. Dziobiak [9, 10] to introduce what is now called an A-D family of objects of $\mathbb{Q}$. A survey of these notions and results concerning them is given in [2]. M. E. Adams

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and W. Dziobiak [3] linked the latter two properties by showing that every quasivariety $\mathbb{Q}$ containing an A-D family is also Q-universal. The converse implication is still an open problem, originally stated by M. E. Adams and W. Dziobiak.

Problem 1.1. Is there a Q-universal quasivariety containing no A-D family?

In [4], M. E. Adams and W. Dziobiak proved the following remarkable and quite surprising result connecting the two algebraic structures associated with a quasivariety of algebraic systems.

Theorem 1.2 [4]. *Any finite-to-finits algebraically universal (ff-alg-universal) quasivariety of algebraic systems of finite similarity type contains an A-D family and hence it is Q-universal.* □

In [16], the present authors extended this result as follows.

Theorem 1.3 [16]. *Any almost ff-alg-universal quasivariety of algebraic systems of finite similarity type contains an A-D family and hence it is Q-universal.* □

Almost universality is a special case of relative universality, see Section 2. Here we aim to modify the latter result for quasivarieties of algebras. We assume that

($*$) $\mathbb{Q}$ is a quasivariety of finitary algebras and $\mathbb{V}$ is a proper subvariety of $\mathbb{Q}$ such that there exists a synchronized $\mathcal{I}(\mathbb{V})$-relatively full embedding F from the category of all undirected graphs into $\mathbb{Q}$ such that Ff is surjective for every graph quotient homomorphism f and $F\mathbf{G}$ is finite for every finite graph $\mathbf{G}$.

Theorem 1.4. *Any quasivariety $\mathbb{Q}$ satisfying ($*$) contains an A-D family and hence it is Q-universal.*

As already noted, all needed notions are reviewed in Section 2 below, and the proof of Theorem 1.4 is given in Section 3. It is based on the fact that any subquasivariety $\mathbb{R}$ of a quasivariety $\mathbb{Q}$ is an epireflective full subcategory of $\mathbb{Q}$. In Section 3, it is also shown how Theorem 1.4 incorporates earlier results of [6, 7, 8].

2 Basic notions and their context

Alg-universality. A category $\mathbb{K}$ is alg-universal if any category of algebras and all homomorphisms between them can be fully embedded into $\mathbb{K}$. This is equivalent to the fact that there exists a full embedding from the category $\mathbb{GRA}$ of all undirected graphs and all graph homomorphisms into $\mathbb{K}$. Moreover, if $\mathbb{K}$ is a concrete category and there exists a full embedding $F : \mathbb{GRA} \to \mathbb{K}$ such that the underlying set of $F\mathbf{G}$ for every finite graph is finite then we say that F preserves finiteness and that $\mathbb{K}$ is ff-alg-universal. If $\mathbb{K}$ is a concrete category then any $\mathbb{K}$-object A with a finite underlying set is called finite. Next we give several well-known properties of alg-universal categories. To do this, we say that a category $\mathbb{K}$ is a monoid universal if for every monoid $\mathbf{M}$ there exists a $\mathbb{K}$-object A such that the endomorphism monoid of A is isomorphic to $\mathbf{M}$.

Theorem 2.1 [18]. *(a) Any concrete alg-universal category $\mathbb{K}$ is monoid universal; and if $\mathbb{K}$ is ff-alg-universal, then for every finite monoid $\mathbf{M}$ there exists a finite $\mathbb{K}$-object A such that the endomorphism monoid of A is isomorphic to $\mathbf{M}$.*

(b) If $\mathbb{K}$ is alg-universal, then for a proper class I there exists a family $\{F_i : \mathbb{K} \to \mathbb{K} \mid i \in I\}$ of full embeddings such that F_iA is not isomorphic to F_jB for any $\mathbb{K}$-objects A and B and for any distinct $i, j \in I$. For any set I there exists a family $\{F_i : \mathbb{K} \to \mathbb{K} \mid i \in I\}$ of full embeddings such that there exists no $\mathbb{K}$-morphism between F_iA and F_jB for any $\mathbb{K}$-objects A and B and for any distinct $i, j \in I$.

(c) If $\mathbb{K}$ is ff-alg-universal and I is a countable set, then there exists a family $\{F_i : \mathbb{K} \to \mathbb{K} \mid i \in I\}$ of full embeddings F_i preserving finiteness such that there exists no $\mathbb{K}$-morphism between F_iA and F_jB for any $\mathbb{K}$-objects A and B and any distinct $i, j \in I$. □

Theorem 2.1 provides a tool for proving that a given category $\mathbb{K}$ is not alg-universal. For example, if $\mathbb{K}$ is a concrete category such that for every set X there exists only a set of non-isomorphic $\mathbb{K}$-objects with a given underlying set X and if there exists a cardinal α such that every $\mathbb{K}$-object whose underlying set has cardinality greater than α has a non-identity endomorphism, then $\mathbb{K}$ is not alg-universal.

Hence for example the variety of lattices or the variety of monoids or the category of topological spaces and continuous mappings are not alg-universal because of the existence of constant morphisms. On the other hand, both the variety of semigroups [13] and the variety of (0,1)-lattices ([11] or [12]) are alg-universal.

Thus we can say that monoids or lattices have sufficiently rich structure to be 'close' to being alg-universal while still permitting constant morphisms, although these categories are not alg-universal in the strict sense. This motivates a notion of almost alg-universality that ignores the constant morphisms. Next we define a more general concept expressing this idea.

Let $\mathbb{K}$ be a category. A class $\mathcal{C}$ of $\mathbb{K}$-morphisms is an ideal if $f \circ g \in \mathcal{C}$ for $\mathbb{K}$-morphisms $f : a \to b$, $g : b \to c$ whenever $f \in \mathcal{C}$ or $g \in \mathcal{C}$. A faithful functor $F : \mathbb{L} \to \mathbb{K}$ is called $\mathcal{C}$-relatively full embedding if

(•) $Ff \notin \mathcal{C}$ for any $\mathbb{L}$-morphism f;

(•) if $f : Fa \to Fb$ is a $\mathbb{K}$-morphism for $\mathbb{L}$-objects a and b then either $f \in \mathcal{C}$ or $f = Fg$ for some $\mathbb{K}$-morphism $g : a \to b$.

Thus F is a full embedding exactly when it is $\mathcal{C}$-relatively full embedding for $\mathcal{C} = \emptyset$. Observe that, if $F : \mathbb{L} \to \mathbb{K}$ is a $\mathcal{C}$-relatively full embedding for some ideal $\mathcal{C}$ then f is an $\mathbb{L}$-isomorphism if and only if Ff is a $\mathbb{K}$-isomorphism. If there exists a $\mathcal{C}$-relatively full embedding $F : \mathbb{GRA} \to \mathbb{K}$ then we say that $\mathbb{K}$ is $\mathcal{C}$-relatively alg-universal. If, moreover, $\mathbb{K}$ is concrete and F preserves finiteness, then $\mathbb{K}$ is called $\mathcal{C}$-relatively ff-alg-universal. Clearly, $\mathbb{K}$ is $\mathcal{C}$-relatively alg-universal (or $\mathcal{C}$-relatively ff-alg-universal) for $\mathcal{C} = \emptyset$ just when $\mathbb{K}$ is alg-universal (or ff-alg-universal, respectively). If $\mathbb{K}$ is conrete category and $\mathcal{C}$ is the ideal consisting of all $\mathbb{K}$-morphisms with constant underlying mapping then we say that $F : \mathbb{L} \to \mathbb{K}$ is almost full embedding instead of $\mathcal{C}$-relatively full embedding and that $\mathbb{K}$ is almost alg-universal or almost ff-alg-universal instead of $\mathcal{C}$-relatively alg-universal or $\mathcal{C}$-relatively ff-alg-universal. The variety of lattices [20] and the variety of monoids [17] or [15] are almost alg-universal but not alg-universal. A second consequence of Theorem 2.1 is that a category $\mathbb{K}$ which is not monoid-universal is not alg-universal. This fact was exploited by M. E. Adams and W. Dziobiak in [5], where they proved that the variety of monadic Boolean algebras is not alg-universal, yet contains a proper class of non-isomorphic algebras whose endomorphism monoids consist of the identity map alone.

Theorem 2.1 naturally leads to the following question.

Problem 2.2. Is there a variety $\mathbb{V}$ of algebras which is monoid universal but not alg-universal?

We shall consider ideals of a special type. Let O be a class of $\mathbb{K}$-objects. Then $\mathcal{I}(O)$ denotes a class of all $\mathbb{K}$-morphisms $f : a \to b$ such that there exist $\mathbb{K}$-morphisms $g : a \to c$ and $h : c \to b$ with $c \in O$ and $f = h \circ g$. Clearly, $\mathcal{I}(O)$ is an ideal of $\mathbb{K}$ called an object ideal of O. In what follows, we shall consider even more specific object ideals.

Q-universality. A class $\mathbb{Q}$ of algebraic systems of a finitary type Δ is a quasivariety if it is closed under all products, all ultraproducts, all subsystems and all isomorphic images. For any class $\mathbb{K}$ of algebraic systems of type Δ, there exists the least quasivariety $\mathbb{Q}$ containing $\mathbb{K}$, which we shall denote $\mathbb{Q} = \text{Qua}(\mathbb{K})$. Quasivarieties will be viewed as categories whose morphisms are all homomorphisms, that is, mappings preserving all operations and relations.

M. V. Sapir [19] defined a quasivariety $\mathbb{Q}$ of finite type Δ as Q-universal if for every quasivariety $\mathbb{R}$ of finite type the lattice $\mathrm{QLat}(\mathbb{R})$ is a homomorphic image of a sublattice of $\mathrm{QLat}(\mathbb{Q})$.

Let $\mathcal{P}(\omega_0)$ be the set of all finite subsets of natural numbers and $\mathcal{P}(\omega) = \mathcal{P}(\omega_0)\setminus \{\emptyset\}$ the set of all finite non-empty subsets of natural numbers. W. Dziobiak [9, 10] studied families $\{\mathbf{S}_A \mid A \in \mathcal{P}(\omega_0)\}$ of finite algebraic systems of a given type Δ we now call Adams-Dziobiak families (or A-D families) defined by these four conditions:

(p1) $\mathbf{S}_\emptyset$ is the terminal algebraic system;

(p2) if $A = B \cup C$ for $A, B, C \in \mathcal{P}(\omega_0)$, then $\mathbf{S}_A \in \mathrm{Qua}(\{\mathbf{S}_B, \mathbf{S}_C\})$;

(p3) if $A \in \mathcal{P}(\omega)$ and $B \in \mathcal{P}(\omega_0)$ with $\mathbf{S}_A \in \mathrm{Qua}(\{\mathbf{S}_B\})$, then $A = B$;

(p4) if $\mathbf{U}, \mathbf{V} \in \mathrm{Qua}(\{\mathbf{S}_A \mid A \in \mathcal{P}\})$ are finite algebraic systems for some finite $\mathcal{P} \subset \mathcal{P}(\omega)$ and if there exists an injective homomorphism $f : \mathbf{S}_A \to \mathbf{U} \times \mathbf{V}$ for some $A \in \mathcal{P}(\omega)$, then there exists an injective homomorphism $g : \mathbf{S}_A \to \mathbf{U}$ or there exists an injective homomorphism $g : \mathbf{S}_A \to \mathbf{V}$ or there exist $B, C \in \mathcal{P}(\omega)$ and injective homomorphisms $g_B : \mathbf{S}_B \to \mathbf{U}$ and $g_C : \mathbf{S}_C \to \mathbf{V}$ with $A = B \cup C$.

We recall some known results.

Theorem 2.3. *(a) If $\mathbb{Q}$ is a Q-universal quasivariety then* $\mathrm{QLat}(\mathbb{Q})$ *has cardinality* $2^{\aleph_0}$ *and the free lattice over a countable set can be embedded into* $\mathrm{QLat}(\mathbb{Q})$. *Thus* $\mathrm{QLat}(\mathbb{Q})$ *satisfies no non-trivial lattice identity* [2].

(b) If a quasivariety $\mathbb{Q}$ contains an A-D family, then the lattice of all ideals of the free lattice over a countable set can be embedded into $\mathrm{QLat}(\mathbb{Q})$ [3]. □

Thus to prove that a quasivariety $\mathbb{Q}$ of finite type is Q-universal, it suffices to prove that $\mathbb{Q}$ has an A-D family. We shall study only quasivarieties $\mathbb{Q}$ of algebras.

In Section 3 we give certain conditions sufficient for the existence of an A-D family in a quasivariety of algebras of finite type. For this we use factorization systems and epireflection.

Factorization systems and epireflections. For a category $\mathbb{K}$, let $\mathcal{E}$ be a class of $\mathbb{K}$-epimorphisms and let $\mathcal{M}$ be a class of $\mathbb{K}$-monomorphisms. We say that $(\mathcal{E}, \mathcal{M})$ is a factorization system of $\mathbb{K}$ if $\mathcal{E}$ and $\mathcal{M}$ are closed under composition, $f \in \mathcal{E} \cap \mathcal{M}$ if and only if f is a $\mathbb{K}$-isomorphism, and for every $\mathbb{K}$-morphism $f : a \to b$ there exist unique, up to a commuting isomorphism, $g : a \to c \in \mathcal{E}$ and $h : c \to b \in \mathcal{M}$ with $f = h \circ g$, see [1]. Any factorization system has the diagonalization property. We formulate it for categories with products. If $\mathbb{K}$ is a category with products and an $(\mathcal{E}, \mathcal{M})$-factorization system, then we write $\{f_i : a \to b_i \mid i \in I\} \in \mathcal{M}$ if the morphism $f : a \to \prod_{i \in I} b_i$ such that $f_i = \pi_i \circ f$ for all $i \in I$ where $\pi_i : \prod_{j \in I} b_j \to b_i$ is the i-th projection belongs to $\mathcal{M}$. Then the diagonalization property says: if $g_i \circ f = k_i \circ h$ for all $i \in I$ where $f : a \to b \in \mathcal{E}$, $\{g_i : b \to c_i \mid i \in I\}$ is a family of $\mathbb{K}$-morphisms, $h : a \to d$ is a $\mathbb{K}$-morphism and $\{k_i : d \to c_i \mid i \in I\} \in \mathcal{M}$ then there exists a $\mathbb{K}$-morphism $l : b \to d$ such that $h = l \circ f$ and $g_i = k_i \circ l$ for all $i \in I$. If $h \in \mathcal{E}$ then $l \in \mathcal{E}$, and if $\{g_i \mid i \in I\} \in \mathcal{M}$ then $l \in \mathcal{M}$.

We say that a family $\{f_i : A \to A_i \mid i \in I\}$ is separating if for distinct $a, b \in A$ there exists $i \in I$ with $f_i(a) \neq f_i(b)$. If $\mathbb{K}$ is a concrete category then a family $\{f_i : a \to b_i \mid i \in I\}$ of $\mathbb{K}$-morphisms is separating if the family of underlying mapping is separating. For concrete categories $\mathbb{K}$ and $\mathbb{L}$ we say that a functor $F : \mathbb{K} \to \mathbb{L}$ preserves separating families if $\{Ff_i : Fa \to Fb_i \mid i \in I\}$ is a separating family in $\mathbb{L}$ whenever $\{f_i : a \to b_i \mid i \in I\}$ is a separating family in $\mathbb{K}$.

For a concrete category $\mathbb{K}$, let $\mathrm{Inj}_{\mathbb{K}}$ consist of all $\mathbb{K}$-homomorphisms such that the underlying mapping is injective and $\mathrm{Surj}_{\mathbb{K}}$ consist of all $\mathbb{K}$-morphisms such that the underlying mapping is surjective. Clearly, every morphism from $\mathrm{Inj}_{\mathbb{K}}$ is a monomorphism of $\mathbb{K}$ and every morphism from $\mathrm{Surj}_{\mathbb{K}}$ is an epimorphism of $\mathbb{K}$. If $(\mathrm{Surj}_{\mathbb{K}}, \mathrm{Inj}_{\mathbb{K}})$ is a factorization system of $\mathbb{K}$ then we say $\mathbb{K}$ has a concrete factorization system and $(\mathrm{Surj}_{\mathbb{K}}, \mathrm{Inj}_{\mathbb{K}})$ is a concrete factorization system of $\mathbb{K}$. Clearly, for every quasivariety $\mathbb{Q}$ of algebras $(\mathrm{Surj}_{\mathbb{Q}}, \mathrm{Inj}_{\mathbb{Q}})$ is a concrete factorization system of $\mathbb{Q}$ (because every bijective homomorphism is an isomorphism). Observe that a family $\{f_i : \mathbf{A} \to \mathbf{B}_i \mid i \in I\}$ of $\mathbb{Q}$-homomorphisms is separating if and only if it belongs to $\mathrm{Inj}_{\mathbb{Q}}$, i.e. if the homomorphism $f : \mathbf{A} \to \prod_{i \in I} \mathbf{B}_i$ with $f_i = f \circ \pi_i$ has an injective underlying mapping where $\pi_i : \prod_{j \in I} \mathbf{B}_j \to \mathbf{B}_i$ is the i-th projection for all $i \in I$. Thus for a concrete category $\mathbb{K}$ we shall say that a family $\{f_i : A \to B_i \mid i \in I\}$ of $\mathbb{K}$-morphisms belong to $\mathrm{Inj}_{\mathbb{K}}$ just when its corresponding family of underlying mappings is separating. A functor $F : \mathbb{Q} \to \mathbb{R}$ between quasivarieties $\mathbb{Q}$ and $\mathbb{R}$ preserves surjectivity if $F(\mathrm{Surj}_{\mathbb{Q}}) \subseteq \mathrm{Surj}_{\mathbb{R}}$.

If $\mathbb{Q}$ is a quasivariety of algebraic systems and $\mathbb{R}$ is a subquasivariety of $\mathbb{Q}$ (of the same type) then, by Theorem 10.1.2 from [14], $\mathbb{R}$ is an epireflective subcategory of $\mathbb{Q}$. This means that for every algebraic system $A \in \mathbb{Q}$ there exists a surjective homomorphism $\rho_A : A \to RA$ where $RA \in \mathbb{R}$ such that for every homomorphism $f : A \to C$ where $C \in \mathbb{R}$ there exists exactly one homomorphism $f^* : RA \to C$ with $f = f^* \circ \rho_A$. Since $\mathbb{R}$ is a full subcategory of $\mathbb{Q}$ then ρ_A is the identity morphism exactly when $A \in \mathbb{R}$. Then $R : \mathbb{Q} \to \mathbb{R}$ such that $Rf = (\rho_B \circ f)^*$ for every homomorphism $f : A \to B$ in $\mathbb{Q}$ is a functor which is a left adjoint to the inclusion functor from $\mathbb{R}$ to $\mathbb{Q}$. We say that R is an epireflection. Observe that $R(\mathrm{Surj}_{\mathbb{Q}}) \subseteq \mathrm{Surj}_{\mathbb{R}}$.

A quasivariety $\mathbb{Q}$ of algebras closed under homomorphic images is a variety. If $\mathbb{Q}$ is a quasivariety of algebras and $\mathbb{V}$ is a subvariety of $\mathbb{Q}$ then a homomorphism $f : \mathbf{A} \to \mathbf{B} \in \mathbb{Q}$ belongs to the ideal $\mathcal{I}(\mathbb{V})$ if and only if $\mathrm{Im}(f) \in \mathbb{V}$.

3 Sufficient conditions for Q-universality

Definition. Let $\mathbb{Q}$ be a quasivariety of finitary algebraic systems, let $\mathbb{V}$ be a proper subvariety of $\mathbb{Q}$ and let $R : \mathbb{Q} \to \mathbb{V}$ be the corresponding epireflection. For any object $\mathbf{A} \in \mathbb{Q}$, let A denote the underlying set of $\mathbf{A}$ and let $\rho_{\mathbf{A}} : \mathbf{A} \to R\mathbf{A}$ denote the surjective $\mathbb{Q}$-morphism from the epitransformation ρ. Let $F : \mathbb{K} \to \mathbb{Q}$ be a $\mathcal{I}(\mathbb{V})$-relatively full embedding. Let $\mathbf{S} \in \mathbb{V}$ be an algebraic system with the underlying set S. We say that F is $\mathbf{S}$-synchronized and call $\mathbf{S}$ its synchronizer if for every $\mathbb{K}$-object k there exists an injective mapping μ_k from S to the underlying set of RFk such that $\mathrm{Im}(\mu_k)$ is an induced subobject of RFk and μ_k is an isomorphism of $\mathbf{S}$ onto the subobject of RFk on the set $\mathrm{Im}(\mu_k)$, and for every $\mathbb{K}$-morphism $f : k_1 \to k_2$ we have

(s1) if Ff is injective on $(\rho_{Fk_1})^{-1}(\mathrm{Im}(\mu_{k_1}))$, then Ff is injective;

(s2) $RFf \circ \mu_{k_1} = \mu_{k_2}$;

(s3) if $Ff \in \mathrm{Surj}_{\mathbb{Q}}$ and A_i is the underlying set of RFk_i for $i = 1, 2$, then every mapping $h : A_2 \to A_1$ such that $RFf \circ h = 1_{A_2}$ is a homomorphism from RFk_2 to RFk_1;

(s4) for every $\mathbb{K}$-object k, if s is an element of the underlying set of RFk such that $s \notin \mathrm{Im}(\mu_k)$ then $\rho_{Fk}^{-1}\{s\}$ is a singleton.

Next we interpret the condition (s3) for algebras.

Proposition 3.1. *Let $\mathbb{Q}$ be a quasivariety of algebras of a finitary similarity type Δ, let $\mathbb{V}$ be a proper subvariety of $\mathbb{Q}$ and let $F : \mathbb{K} \to \mathbb{Q}$ be a functor. Then (s3) holds* *exactly when*

($\bullet$) *if Ff is surjective and for every $s \in A_2$ with $|RFf^{-1}\{s\}| > 1$, if $\sigma_{RFk_2}(a_1, a_2, \dots, a_n) = s$ for an n-ary operation σ and $a_1, a_2, \dots, a_n \in A_2$, then $s = a_{i_0}$ for some $i_0 \in \{1, 2, \dots, n\}$ and $k(s) = \sigma_{RFk_1}(k(a_1), k(a_2), \dots, k(a_n))$ for every mapping $k : \{a_1, a_2, \dots, a_n\} \to A_1$ such that $RFf \circ k(a_i) = a_i$ for all $i \in \{1, 2, \dots, n\}$.*

Proof. Assume (s3). Let $s = \sigma_{RFk_2}(a_1, a_2, \dots, a_n)$ for some $\sigma \in \Delta$, let $a_1, a_2, \dots, a_n, s \in A_2$ and $|RFf^{-1}\{s\}| > 1$. Let $h : A_2 \to A_1$ be a mapping such that $RFf \circ h$ is the identity mapping. Then $h(s) = \sigma_{RFk_1}(h(a_1), h(a_2), \dots, h(a_n))$. If $s \notin \{a_1, a_2, \dots, a_n\}$ then there exists a mapping $h' : A_2 \to A_1$ with $RFf \circ h = RFf \circ h'$, $h(s) \neq h'(s)$ and $h(t) = h'(t)$ for all $t \in A_2 \setminus \{s\}$. Hence $h'(s) \neq \sigma_{RFk_1}(h'(a_1), h'(a_2), \dots, h'(a_n))$ and this contradicts the fact that $h' : RFk_2 \to RFk_1$ is a homomorphism. Thus there exists $i_0 \in \{1, 2, \dots, n\}$ with $a_{i_0} = s$. If $k : \{a_1, a_2, \dots, a_n\} \to A_1$ is a mapping such that $RFf \circ k(a_i) = a_i$ for every $i \in \{1, 2, \dots, n\}$ then there exists a mapping $h : A_2 \to A_1$ such that $RFf \circ h$ is the identity mapping of A_2 and $h(a_i) = k(a_i)$ for all $i = \{1, 2, \dots, n\}$. But $h : RFk_2 \to RFk_1$ is a homomorphism, by (s3), and hence $k(s) = \sigma_{RFk_1}(k(a_1), k(a_2), \dots, k(a_n))$ because $s = a_{i_0}$. Whence the condition ($\bullet$) holds.

For the converse, assume ($\bullet$) and let $h : A_2 \to A_1$ be a mapping such that $RFf \circ h$ is the identity of A_2. Choose an n-ary operation σ of type Δ and $a_1, a_2, \dots, a_n \in A_2$. Write $s = \sigma_{RFk_2}(a_1, a_2, \dots, a_n)$. First we assume that $|RFf^{-1}(s)| > 1$. Then ($\bullet$) gives an $i_0 \in \{1, 2, \dots, n\}$ with $s = a_{i_0}$ and $h(s) = \sigma_{RFk_1}(h(a_1), h(a_2), \dots, h(a_n))$, as required. From $Ff \in \mathrm{Surj}_{\mathbb{Q}}$ we infer that $RFf \in \mathrm{Surj}_{\mathbb{Q}}$, and hence $|RFf^{-1}\{s\}| = 1$ is the only remaining case. If

$$t = \sigma_{RFk_1}(h(a_1), h(a_2), \dots, h(a_n))$$

then

$$\begin{aligned} RFf(t) &= \sigma_{RFk_2}(RFf(h(a_1)), RFf(h(a_2)), \dots, RFf(h(a_n))) \\ &= \sigma_{RFk_2}(a_1, a_2, \dots, a_n) = s \end{aligned}$$

and hence $t = h(s)$. Thus h is a homomorphism, and the proof is complete. □

Remark. Observe that if $F : \mathbb{K} \to \mathbb{Q}$ is an almost full embedding then F is synchronized $\mathcal{I}(\mathbb{T})$-relatively full embedding for the trivial variety $\mathbb{T}$. Indeed, its synchronizer **S** is a singleton algebra and μ_k is the identity automorphism of **S** for every

$\mathbb{K}$-object k. Clearly, the conditions (s1)-(s4) are satisfied. And $F : \mathbb{K} \to \mathbb{Q}$ is a full embedding exactly when F is an almost full embedding and for every $\mathbb{K}$-object k there exists no $\mathbb{Q}$-morphism from the terminal object of $\mathbb{Q}$ into Fk.

Let $\mathbb{N}_0$ be a poset viewed as a category whose objects are sets from the set $\mathcal{P}(\omega_0)$ of all finite subsets of ω and there exists an $\mathbb{N}_0$-morphism from $A \in \mathcal{P}(\omega_0)$ into $B \in \mathcal{P}(\omega_0)$ if and only if $B \subseteq A$. Let $\mathbb{N}$ be the full subcategory of $\mathbb{N}_0$ whose objects belong to the set $\mathcal{P}(\omega) = \mathcal{P}(\omega_0) \setminus \{\emptyset\}$ of all non-void subsets of ω. For $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$, let $\eta_{A,B}$ denote the unique $\mathbb{N}$-morphism from A to B.

Theorem 3.2. *Let $\mathbb{Q}$ be a quasivariety of finitary algebras and let $\mathbb{V}$ be a subvariety of $\mathbb{Q}$. If there exists a synchronized $\mathcal{I}(\mathbb{V})$-relatively full embedding $F : \mathbb{N} \to \mathbb{Q}$ such that*

(1) *FA is a finite algebra for every $A \in \mathcal{P}(\omega)$;*

(2) *$F\eta_{A,B} \in \mathrm{Surj}_{\mathbb{Q}}$ for every $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$ (then $RF\eta_{A,B}$ is a retract);*

(3) *if $A = B \cup C$ for $A, B, C \in \mathcal{P}(\omega)$ then $\{F\eta_{A,B}, F\eta_{A,C}\}$ is a separating family.*

Then $\{\mathbf{S}_A \mid A \in \mathcal{P}(\omega_0)\}$ is an A-D family where $\mathbf{S}_\emptyset$ is a singleton algebra in $\mathbb{Q}$ and $\mathbf{S}_A = FA$ for all $A \in \mathcal{P}(\omega)$.

Proof. We need to prove (p1)–(p4). Clearly, (p1) is satisfied. To prove (p2), consider sets $A, B, C \in \mathcal{P}(\omega)$ with $A = B \cup C$. By (3), $\{F\eta_{A,B}, F\eta_{A,C}\}$ is a separating family and thus FA is a subobject of $FB \times FC$. Hence we obtain $FA \in \mathrm{Qua}\{FB, FC\}$ and the proof of (p2) is complete.

For every $A \in \mathcal{F}(\omega)$, let $\rho_A : FA \to RFA$ denote the epireflection homomorphism of FA into $\mathbb{V}$. Then $\rho_A \in \mathrm{Surj}_{\mathbb{Q}}$.

To prove (p3), let $A, B \in \mathcal{P}(\omega)$ be such that $FA \in \mathrm{Qua}\{FB\}$. By the hypothesis, FB is finite, so that the family of all homomorphisms from FA to FB is separating. Since F is $\mathcal{I}(\mathbb{V})$-relatively full embedding we infer that if $B \not\subseteq A$ then every homomorphism from FA into FB factorizes through ρ_A. Since $FA \notin \mathbb{V}$ and $RFA \in \mathbb{V}$, the mapping ρ_A is not injective and thus $FA \notin \mathrm{Qua}\{FB\}$ – a contradiction. Thus we can assume that $B \subseteq A$. If $f : FA \to FB$ is a homomorphism then either $h = F\eta_{A,B}$ or h factorizes through ρ_A because F is $\mathcal{I}(\mathbb{V})$-relatively full embedding. Since the family of all homomorphisms from FA to FB is separating, the pair $\{F\eta_{A,B}, \rho_A\}$ must be a separating family. We claim that this is impossible when $B \neq A$. Indeed, if $B \neq A$ then $F\eta_{A,B}$ is not injective; this is because from (2) it would follow that $F\eta_{A,B}$ is an isomorphism, contrary to the relative fulness of F. But then $F\eta_{A,B}$ is not injective on $(\rho_A)^{-1}(\mathrm{Im}(\mu_A))$ by (s1) and hence, by (s2), for some $s \in S$ there are distinct $a, b \in \rho_A^{-1}\{s\}$ with $F\eta_{A,B}(a) = F\eta_{A,B}(b)$. Hence $\{F\eta_{A,B}, \rho_A\}$ is not a separating family, a contradiction. Thus $A = B$, and (p3) follows.

To prove (p4), let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a finite set and let $\mathbf{B}, \mathbf{C} \in \mathrm{Qua}\{FX \mid X \in \mathcal{F}\}$ be finite algebras such that there exist $A \in \mathcal{P}(\omega)$ and an injective homomorphism $f : FA \to \mathbf{B} \times \mathbf{C}$. Hence there exist finite separating families $\{g_i : \mathbf{B} \to FX_i \mid i \in I\}$ and $\{h_j : \mathbf{C} \to FY_j \mid j \in J\}$ such that $X_i, Y_j \in \mathcal{P}(\omega)$ for all $i \in I$ and $j \in J$. Let $\pi_1 : \mathbf{B} \times \mathbf{C} \to \mathbf{B}$, $\pi_2 : \mathbf{B} \times \mathbf{C} \to \mathbf{C}$ be projections.

First we prove that we can assume that $\pi_1 \circ f, \pi_2 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$. So assume that (p4) is satisfied if $\pi_1 \circ f, \pi_2 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$. By the factorization property, there exist homomorphisms

$$f_1' : FA \to \mathbf{B}' \in \mathrm{Surj}_{\mathbb{Q}}, f_1'' : \mathbf{B}' \to \mathbf{B} \in \mathrm{Inj}_{\mathbb{Q}},$$
$$f_2' : FA \to \mathbf{C}' \in \mathrm{Surj}_{\mathbb{Q}}, f_2'' : \mathbf{C}' \to \mathbf{C} \in \mathrm{Inj}_{\mathbb{Q}}$$

with $\pi_1 \circ f = f_1'' \circ f_1'$ and $\pi_2 \circ f = f_2'' \circ f_2'$. Since f is injective we infer that $\{\pi_1 \circ f, \pi_2 \circ f\}$ is separating and hence $\{f_1', f_2'\}$ is also separating. Thus there exists an injective homomorphism $f' : FA \to \mathbf{B}' \times \mathbf{C}'$ with $\pi_1' \circ f' = f_1'$ and $\pi_2' \circ f' = f_2'$ where $\pi_1' : \mathbf{B}' \times \mathbf{C}' \to \mathbf{B}'$ and $\pi_2' : \mathbf{B}' \times \mathbf{C}' \to \mathbf{C}'$ are projections. Then $\{g_i \circ f_1'' : \mathbf{B}' \to FX_i \mid i \in I\}$ and $\{h_j \circ f_2'' : \mathbf{C}' \to FY_j \mid j \in J\}$ are separating families and, by the assumption, the condition (p4) is satisfied for $f', \mathbf{B}'$ and $\mathbf{C}'$ because $\pi_1' \circ f', \pi_2' \circ f' \in \mathrm{Surj}_{\mathbb{Q}}$. Then (p4) is also satisfied for f, $\mathbf{B}$ and $\mathbf{C}$ because $f_1'' : \mathbf{B}' \to \mathbf{B}, f_2'' : \mathbf{C}' \to \mathbf{C} \in \mathrm{Inj}_{\mathbb{Q}}$. Thus with no loss of generality we can assume that $\pi_1 \circ f, \pi_2 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$.

Let us define $I' = \{i \in I \mid g_i \circ \pi_1 \circ f = F\eta_{A,X_i}\}$ and $J' = \{j \in J \mid g_j \circ \pi_2 \circ f = F\eta_{A,Y_j}\}$. Then $X_i \subseteq A$ and $Y_j \subseteq A$ for all $i \in I$ and $j \in J$. Observe that $g_i \circ \pi_1 \circ f$ and $g_j \circ \pi_2 \circ f$ factorize through ρ_A for all $i \in I \setminus I'$ and $j \in J \setminus J'$ because F is $\mathcal{I}(\mathbb{V})$-relatively full embedding. Hence $I' \neq \emptyset$ or $J' \neq \emptyset$. Set $U = \bigcup_{i \in I'} X_i$ and $V = \bigcup_{j \in J'} Y_j$. Then $U \cup V \subseteq A$, and $g_i \circ \pi_1 \circ f$ factorizes through $F(\eta_{A,U})$ for all $i \in I'$ and $g_j \circ \pi_2 \circ f$ factorizes through $F(\eta_{A,V})$ for all $j \in J'$. Since $\{g_i \circ \pi_1 \circ f \mid i \in I\} \cup \{g_j \circ \pi_2 \circ f \mid j \in J\} \in \mathrm{Inj}_{\mathbb{Q}}$ we infer, by (p3), that if $J' = \emptyset$ then $U = A$, if $I' = \emptyset$ then $V = A$, if $I' \neq \emptyset \neq J'$ then $A = U \cup V$. Assume that $I' \neq \emptyset$. Since $\pi_1 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$, $\{F\eta_{U.X_i} \mid i \in I'\} \in \mathrm{Inj}_{\mathbb{Q}}$ by (3) and $g_i \circ \pi_1 \circ f = F\eta_{U,X_i} \circ F\eta_{A,U}$ for all $i \in I$, by the diagonalization property there exists a homomorphism $\psi : \mathbf{B} \to FU$ with $\psi \circ \pi_1 \circ f = F\eta_{A,U}$ and $F\eta_{U,X_i} \circ \psi = g_i$

for all $i \in I'$. From $F\eta_{A,U} \in \mathrm{Surj}_{\mathbb{Q}}$ it follows that $\psi \in \mathrm{Surj}_{\mathbb{Q}}$.

Since $\{g_i \mid i \in I\}$ is a separating family, for distinct $u, v \in FA$ we have $\pi_1 \circ f(u) \neq \pi_1 \circ f(v)$ if and only if there exists $i \in I$ with $g_i \circ \pi_1 \circ f(u) \neq g_i \circ \pi_1 \circ f(v)$. If $i \in I'$ then $g_i \circ \pi_1 \circ f = F\eta_{A,X_i} = F\eta_{U,X_i} \circ F\eta_{A,U}$. Thus if $F\eta_{A,U}(u) \neq F\eta_{A,U}(v)$ for $u, v \in FA$ then $\pi_1 \circ f(u) \neq \pi_1 \circ f(v)$. If $i \in I \setminus I'$ then $g_i \circ \pi_1 \circ f = h \circ \rho_A$ for some homomorphism h and thus $\pi_1 \circ f(u) \neq \pi_1 \circ f(v)$ implies that $\rho_A(u) \neq \rho_A(v)$ or $F\eta_{A,U}(u) \neq F\eta_{A,U}(v)$ because $\{F\eta_{U,X_i} \mid i \in I'\}$ is a separating family.

Let $\mathbf{S}$ be a synchronizer of F. Consider $t \in \rho_A^{-1}(\mathrm{Im}(\mu_A))$ and $u \in FA \setminus \rho_A^{-1}(\mathrm{Im}(\mu_A))$. Then $\rho_A(t) = \mu_A(s)$ for some $s \in S$. By (s2), $\rho_U \circ \psi \circ \pi_1 \circ f(t) = \rho_U \circ F\eta_{A,U}(t) = \mu_U(s)$ and $\rho_U \circ \psi \circ \pi_1 \circ f(u) = \rho_U \circ F\eta_{A,U}(u) \notin \mathrm{Im}(\mu_U)$. Hence $\psi^{-1}(\rho_U^{-1}(\mathrm{Im}(\mu_U)) = \pi_1 \circ f(\rho_A^{-1}(\mathrm{Im}(\mu_A))$. If we combine this fact with the foregoing argument we conclude that for $u, v \in \rho_A^{-1}(\mathrm{Im}(\mu_A))$ we have $\pi_1 \circ f(u) = \pi_1 \circ f(v)$ if and only if $F\eta_{A,U}(u) = F\eta_{A,U}(v)$. From (s2) it follows that $(RF\eta_{A,U})^{-1}(\mu_U(s)) = \{\mu_A(s)\}$ for all $s \in S$. Thus $(R\psi)^{-1}(\mu_U(s)) = \{R(\pi_1 \circ f(\mu_A(s))\}$ for every $s \in S$ because $\psi \circ \pi_1 \circ f = F\eta_{A,U}$. Since $F\eta_{A,U}$ is surjective, by (s3), every mapping ν' from the underlying set of RFU into the underlying set of RFA such that $RF\eta_{A,U} \circ \nu'$ is the identity mapping is a homomorphism from RFU into RFA. From $\psi \circ \pi_1 \circ f = F\eta_{A,U}$ we conclude $R(\psi \circ \pi_1 \circ f) = RF\eta_{A,U}$. For a homomorphism $\nu' : RFU \to RFA$ such that $RF\eta_{A,U} \circ \nu'$ is the identity automorphism of RFU we set $\nu = R(\pi_1 \circ f) \circ \nu'$ and hence $\nu : RFU \to R\mathbf{B}$ is a homomorphism such that $R\psi \circ \nu$ is the identity homomorphism of RFU. Since ν' exists by (s3), we can assume that we have a homomorphism $\nu : RFU \to R\mathbf{B}$ such that $R\psi \circ \nu$ is the identity homomorphism of RFU.

For every $i \in I \setminus I'$ there exists a homomorphism $\bar{g}_i : RFA \to FX_i$ with $g_i \circ \pi_1 \circ f = \bar{g}_i \circ \rho_A$. By the properties of factorization system, there exist homomorphisms $\sigma : RFA \to \mathbf{D} \in \mathrm{Surj}_{\mathbb{Q}}$ and $\sigma_i : \mathbf{D} \to FX_i$ for $i \in I \setminus I'$ such that $g_i \circ \pi_1 \circ f = \sigma_i \circ \sigma \circ \rho_A$ for all $i \in I \setminus I'$ and $\{\sigma_i \mid i \in I \setminus I'\} \in \mathrm{Inj}_{\mathbb{Q}}$. By the diagonalization property, there exists a homomorphism $\phi' : \mathbf{B} \to \mathbf{D}$ such that $\phi' \circ \pi_1 \circ f = \sigma \circ \rho_A$ and $\sigma_i \circ \phi' = g_i$ for all $i \in I \setminus I'$. From $\rho_A, \sigma \in \mathrm{Surj}_{\mathbb{Q}}$ it follows that $\phi' \in \mathrm{Surj}_{\mathbb{Q}}$. From $RFA \in \mathbb{V}$ and $\sigma : RFA \to \mathbf{D} \in \mathrm{Surj}_{\mathbb{Q}}$ it follows that $\mathbf{D} \in \mathbb{V}$ and if $\rho_{\mathbf{B}} : \mathbf{B} \to R\mathbf{B}$ is the epireflection morphism of $\mathbf{B}$ into $\mathbb{V}$, then there exists a homomorphism $\phi : R\mathbf{B} \to \mathbf{D} \in \mathrm{Surj}_{\mathbb{Q}}$ with $\phi' = \phi \circ \rho_{\mathbf{B}}$. Then

$$\sigma \circ \rho_A = \phi' \circ \pi_1 \circ f = \phi \circ \rho_B \circ \pi_1 \circ f = \phi \circ R(\pi_1 \circ f) \circ \rho_A$$

and $\sigma = \phi \circ R(\pi_1 \circ f)$ follows because $\rho_A \in \mathrm{Surj}_{\mathbb{Q}}$. Since $\{g_i \mid i \in I\} \in \mathrm{Inj}_{\mathbb{Q}}$ we infer that the family $\{\psi, \rho_B\}$ is separating. Hence there exists a homomorphism

$\omega : \mathbf{B} \to FU \times R\mathbf{B} \in \mathrm{Inj}_{\mathbb{Q}}$ such that $\tau_1 \circ \omega = \psi$ and $\tau_2 \circ \omega = \rho_B$ where $\tau_1 : FU \times R\mathbf{B} \to FU$ and $\tau_2 : FU \times R\mathbf{B} \to R\mathbf{B}$ are projections. Then $\tau_1 \circ \omega \circ \pi_1 \circ f = \psi \circ \pi_1 \circ f = F\eta_{A,U}$ and $\tau_2 \circ \omega \circ \pi_1 \circ f = \rho_B \circ \pi_1 \circ f = R(\pi_1 \circ f) \circ \rho_A$. Hence for every $b \in \mathbf{B}$ and $a \in FA$ with $\pi_1 \circ f(a) = b$ we have $\omega(b) = (F\eta_{A,U}(a), R(\pi_1 \circ f) \circ \rho_A(a))$. By the property of products, there exists a homomorphism $\lambda : FU \to FU \times R\mathbf{B}$ such that $\tau_1 \circ \lambda$ is the identity morphism of FU and $\tau_2 \circ \lambda = \nu \circ \rho_U$, hence $\lambda \in \mathrm{Inj}_{\mathbb{Q}}$. Select $u \in FU$. If $\rho_U(u) \in \mathrm{Im}(\mu_U)$ then, by (s2), $\rho_A((F\eta_{A,U})^{-1}(u)) = (RF\eta_{A,U})^{-1}(\rho_U(u))$ is a singleton and hence for every $a \in FA$ with $F\eta_{A,U}(a) = u$ we have $\{R(\pi_1 \circ f) \circ \rho_A(a)\} = (R\psi)^{-1}(\rho_U(u)) = \{\nu(\rho_U(u))\}$. Thus $\lambda(u) = (F\eta_{A,U}(a), R(\pi_1 \circ f) \circ \rho_A(a)) \in \mathrm{Im}(\omega)$. If $\rho_U(u) \notin \mathrm{Im}(\mu_U)$, then there exists $a \in FA$ such that $\rho_{\mathbf{B}} \circ \pi_1 \circ f(a) = \nu(\rho_U(u))$ because $\rho_{\mathbf{B}}, \pi_1 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$. Then

$$R\psi \circ \rho_{\mathbf{B}} \circ \pi_1 \circ f(a) = R\psi \circ \nu(\rho_U(u) = \rho_U(u).$$

Since

$$R\psi \circ \rho_{\mathbf{B}} \circ \pi_1 \circ f = \rho_U \circ \psi \circ \pi_1 \circ f = \rho_U \circ F\eta_{U,A}$$

we conclude that $\rho_U(u) = \rho_U(F\eta_{A,U}(a))$ and, by (s4), $u = F\eta_{A,U}(a)$. Thus $\lambda(u) = (F\eta_{A,U}(a), R(\pi_1 \circ f) \circ \rho_A(a)) \in \mathrm{Im}(\omega)$ because $R(\pi_1 \circ f) \circ \rho_A = \rho_{\mathbf{B}} \circ \pi_1 \circ f$. Thus $\mathrm{Im}(\lambda) \subseteq \mathrm{Im}(\omega)$, so that there exists an injective homomorphism from FU to $\mathbf{B}$.

If $J' \neq \emptyset$ then the same proof gives the existence of an injective $\nu : FV \to \mathbf{C}$, and (p4) follows. □

The technical statement below enables us to prove a generalized version of Theorem 3.2. We say that a surjective homomorphism $f : \mathbf{A} \to \mathbf{B}$ of algebraic systems of similarity type Δ is a quotient if for every relation $r \in \Delta$ we have that $(b_0, b_1, \ldots, b_k) \in r_{\mathbf{B}}$ if and only if there exists $(a_0, a_1, \ldots, a_k) \in r_{\mathbf{A}}$ with $f(a_i) = b_i$ for all $i = 0, 1, \ldots, k$. A quasivariety $\mathbb{Q}$ is closed under quotients if algebraic system $\mathbf{A} \in \mathbb{Q}$ whenever there exist an algebraic system $\mathbf{B} \in \mathbb{Q}$ and a quotient $f : \mathbf{B} \to \mathbf{A}$. Let $\mathrm{Quot}_{\mathbb{Q}}$ denote the class of all quotients of $\mathbb{Q}$. It is well-known [1] that $(\mathrm{Quot}_{\mathbb{Q}}, \mathrm{Inj}_{\mathbb{Q}})$ is a factorization system in $\mathbb{Q}$, and that $\mathrm{Surj}_{\mathbb{Q}} = \mathrm{Quot}_{\mathbb{Q}}$ if $\mathbb{Q}$ is a quasivariety of algebras. If $\mathbb{Q}$ is clear from the context, we write Quot instead of $\mathrm{Quot}_{\mathbb{Q}}$.

Proposition 3.3. *Let $\mathbb{Q}$ be a quasivariety of algebraic systems and let $\mathbb{R}$ be a proper subquasivariety of $\mathbb{Q}$ closed under quotients. If there exists an $\mathcal{I}(\mathbb{R})$-relatively full embedding $F : \mathbb{N} \to \mathbb{Q}$ such that FA is finite for all $A \in \mathcal{P}(\omega)$ and $F\eta_{A,B} \in \mathrm{Quot}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$, then there exists an $\mathcal{I}(\mathbb{R})$-relatively full embedding $G : \mathbb{N} \to \mathbb{Q}$ such that*

(1) *GA is finite for all $A \in \mathcal{P}(\omega)$;*

(2) *if $A, B, C \in \mathcal{P}(\omega)$ satisfy $B \cup C \subseteq A$, then $\{G\eta_{A,B}, G\eta_{A,C}\}$ is a separating family if and only if $A = B \cup C$;*

(3) *$G\eta_{A,B} \in \mathrm{Quot}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$.*

Moreover, if $\mathbb{Q}$ is a quasivariety of algebras and F is synchronized then G is synchronized.

The fairly technical proof of this Proposition can be found in the Appendix.

Proof of Theorem 1.4 completed. Let $\mathbb{GRA}$ denote the (concrete) category of all undirected graphs and compatible mappings. We recall that there exists a full embedding Φ of $\mathbb{N}$ into $\mathbb{GRA}$ such that ΦA is a finite graph of every $A \in \mathbb{N}$ and $\Phi\eta_{A,B} \in \mathrm{Quot}_{\mathbb{GRA}}$ for every $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$, see [7]. Let $F : \mathbb{GRA} \to \mathbb{Q}$ satisfy the hypothesis of Theorem 1.4. Then the composite $F \circ \Phi : \mathbb{N} \to \mathbb{Q}$ satisfies the hypothesis of Proposition 3.3, and hence $\mathbb{Q}$ contains an A-D family, by Theorem 3.2. This concludes the proof of Theorem 1.4. □

Remark. The embeddings from $\mathbb{GRA}$ into the variety of semigroups generated by $\mathbf{M}_2$ or $\mathbf{M}_3$ or $\mathbf{M}_3^d$ or $\mathbf{M}_4$ or $\mathbf{M}_4^d$ constructed in [6, 7, 8] are synchronized (here for a semigroup $\mathbf{S} = (S, \cdot)$, its dual is defined as $\mathbf{S}^d = (S, \odot)$ with $s \odot t = t \cdot s$ for all $s, t \in S$) and constitute special cases of Theorem 3.2. The semigroups $\mathbf{M}_2$, $\mathbf{M}_3$ and $\mathbf{M}_4$ are defined in Table 1.

$\mathbf{M}_2$	a	b	c	0
a	0	c	0	0
b	c	0	0	0
c	0	0	0	0
0	0	0	0	0

$\mathbf{M}_3$	d	a	b	c
d	a	a	a	b
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

$\mathbf{M}_4$	t	u	v	s	0
t	t	u	s	s	0
u	t	u	0	s	0

Table 1: The semigroups $\mathbf{M}_2$, $\mathbf{M}_3$ and $\mathbf{M}_4$

Finally, we show that for quasivarieties of algebras Theorem 1.4 generalizes Theorem 1.3 of [16]. So let $\mathbb{Q}$ be a quasivariety of algebras and let $\mathbb{V}$ be a proper subvariety of $\mathbb{Q}$. We say that an epireflection $R : \mathbb{Q} \to \mathbb{V}$ is constant on a functor $F : \mathbb{N} \to \mathbb{Q}$ if the composite $R \circ F$ is a constant functor. It is then clear that if the epireflection R is constant on an $\mathcal{I}(\mathbb{V})$-relatively full embedding F, then F is synchronized. Thus we immediately obtain

Corollary 3.4. *Let $\mathbb{Q}$ be a quasivariety of algebras and let $\mathbb{V}$ be a proper subvariety of $\mathbb{Q}$. If $F : \mathbb{N} \to \mathbb{Q}$ is an $\mathcal{I}(\mathbb{V})$-relatively full embedding such that the epireflection of $\mathbb{Q}$ into $\mathbb{V}$ is constant on F, $F\eta_{A,B} \in \mathrm{Surj}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$ and FA is finite for all $A \in \mathcal{P}(\omega)$ then there exists an A-D family in $\mathbb{Q}$, and thus $\mathbb{Q}$ is Q-universal.* □

Thus, in particular, the object ideal $\mathcal{I}(\mathbb{V})$ associated with such an $\mathcal{I}(\mathbb{V})$-relatively full embedding F is *principal* in the sense that it is determined by a single object of $\mathbb{V}$ and includes the case when the synchronizer is a singleton algebra, that is, the case of an almost full embedding.

Appendix

Proof of Proposition 3.3. Consider a functor $H : \mathbb{N}_0 \to \mathbb{N}$ defined by $H\emptyset = \{0\}$ and $HA = \{0\} \cup \{n+1 \mid n \in A\}$ for all $A \in \mathcal{P}(\omega)$ and $H\eta_{A,B} = \eta_{HA,HB}$ for $A, B \in \mathcal{P}(\omega_0)$ with $B \subseteq A$. Then H is a full embedding (since $A \subseteq B$ if and only if $HA \subseteq HB$ for $A, B \in \mathcal{P}(\omega_0)$, it is correctly defined). Thus the composite $F' = F \circ H : \mathbb{N}_0 \to \mathbb{Q}$ is an $\mathcal{I}(\mathbb{R})$-relatively full embedding such that $F'A$ is finite for all $A \in \mathcal{P}(\omega)$ and $F'\eta_{A,B} = F\eta_{HA,HB} \in \mathrm{Quot}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega_0)$ with $B \subseteq A$.

Since F' is an $\mathcal{I}(\mathbb{R})$-relatively full embedding, $F'A \notin \mathbb{R}$ for all $A \in \mathcal{P}(\omega_0)$. For $n \in \omega$, set $G\{n\} = F'\{n\}$. For $A \in \mathcal{P}(\omega)$, define $\Pi(A) = \prod_{a \in A} F'\{a\}$ and let $\pi_a : \Pi(A) \to F'\{a\}$ be the a-th projection for each $a \in A$. By the universal property of products, there exists a unique homomorphism $\tau_A : F'A \to \Pi(A)$ such that $F'\eta_{A,\{a\}} = \pi_a \circ \tau_A$ for every $a \in A$. Factorizing τ_A in $\mathbb{Q}$ in the factorization system $(\mathrm{Quot}_{\mathbb{Q}}, \mathrm{Inj}_{\mathbb{Q}})$, we obtain homomorphisms (unique up to an isomorphism) $\chi_A : F'A \to GA \in \mathrm{Quot}_{\mathbb{Q}}$ and $\mu_A : GA \to \Pi(A) \in \mathrm{Inj}_{\mathbb{Q}}$ such that $\tau_A = \mu_A \circ \chi_A$. Since the underlying set of $F'A$ is finite and since χ_A is a quotient, the underlying set of GA is finite for all $A \in \mathcal{P}(\omega)$. This proves (1).

Consider $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$. By the universal property of products, there exists a unique homomorphism $\Pi(\eta_{A,B}) : \Pi(A) \to \Pi(B)$ such that $\pi_b = \kappa_b \circ \Pi(\eta_{A,B})$ for all $b \in B \subseteq A$, where $\kappa_b : \Pi(B) \to F'\{b\}$ is the b-th projection for $b \in B$. Then for every $b \in B$ we have

$$\begin{aligned}\kappa_b \circ \Pi(\eta_{A,B}) \circ \tau_A = \pi_b \circ \tau_A = F'\eta_{A,\{b\}} &= F'\eta_{B,\{b\}} \circ F'\eta_{A,B} \\ &= \kappa_b \circ \tau_B \circ F'\eta_{A,B}\end{aligned}$$

because $\kappa_b \circ \tau_B = F'\eta_{B,\{b\}}$, and hence

$$\Pi(\eta_{A,B}) \circ \mu_A \circ \chi_A = \Pi(\eta_{A,B}) \circ \tau_A = \tau_B \circ F'\eta_{A,B} = \mu_B \circ \chi_B \circ F'\eta_{A,B}$$

because the family $\{\kappa_b \mid b \in B\}$ of projections is separating.

By the diagonalization property, there exists a homomorphism $G\eta_{A,B} : GA \to GB$ with $G\eta_{A,B} \circ \chi_A = \chi_B \circ F'\eta_{A,B}$ and $\Pi(\eta_{A,B}) \circ \mu_A = \mu_B \circ G\eta_{A,B}$ because $\mu_B \in \text{Inj}$ and $\chi_A \in \text{Quot}$. From $\chi_B \circ F'\eta_{A,B} \in \text{Quot}$ it follows that $\chi_B \circ F'\eta_{A,B} \in \text{Quot}$ and $G\eta_{A,B} \in \text{Quot}$, and (3) is proved. Note the diagram below, commuting for every $b \in B \subseteq A$.

$$\begin{array}{ccccccc}
F'A & \xrightarrow{\chi_A} & GA & \xrightarrow{\mu_A} & \prod(A) & \xrightarrow{\pi_b} & G\{b\} = F'\{b\} \\
{\scriptstyle F'\eta_{A,B}}\downarrow & & {\scriptstyle G\eta_{A,B}}\downarrow & & {\scriptstyle \prod(\eta_{A,B})}\downarrow & & \| \\
F'B & \xrightarrow{\chi_B} & GB & \xrightarrow{\mu_B} & \prod(B) & \xrightarrow{\kappa_b} & G\{b\} = F'\{b\}
\end{array}$$

To prove that G is a functor, let $A, B, C \in \mathcal{P}(\omega)$ satisfy $C \subseteq B \subseteq A$. Then

$$\begin{aligned}
G\eta_{B,C} \circ G\eta_{A,B} \circ \chi_A &= G\eta_{B,C} \circ \chi_B \circ F'\eta_{A,B} \\
&= \chi_C \circ F'\eta_{B,C} \circ F'\eta_{A,B} \\
&= \chi_C \circ F'\eta_{A,C} = G\eta_{A,C} \circ \chi_A
\end{aligned}$$

and because $\chi_A \in \text{Quot}$ we conclude that $G\eta_{B,C} \circ G\eta_{A,B} = G\eta_{A,C}$. Since $F'\eta_{A,A}$ is the identity homomorphism, from $G\eta_{A,A} \circ \chi_A = \chi_A \circ F'\eta_{A,A} = \chi_A \in \text{Quot}$ it follows that $G\eta_{A,A}$ is also the identity homomorphism. Altogether, G is a functor.

We turn to (2). Note that $F'\eta_{A,\{a\}} = \pi_a \circ \tau_A = \pi_a \circ \mu_A \circ \chi_A$ and $G\eta_{A,\{a\}} \circ \chi_A = F'\eta_{A,\{a\}}$ for every $a \in A$ because $\chi_{\{a\}}$ is the identity morphism of $F'\{a\} = G\{a\}$. From $\chi_A \in \text{Quot}$ we then obtain $G\eta_{A,\{a\}} = \pi_a \circ \mu_A$ for every $a \in A$. But then $\{G\eta_{A,\{a\}} \mid a \in A\}$ is a separating family because $\mu_A \in \text{Inj}$ and the family $\{\pi_a \mid a \in A\}$ of projections is separating. Hence $\{G\eta_{A,B}, G\eta_{A,C}\}$ is a separating family for any $A, B, C \in \mathcal{P}(\omega)$ with $A = B \cup C$. Conversely, assume that $B \cup C \subsetneq A$ and $\{G\eta_{A,B}, G\eta_{A,C}\}$ is a separating family. Then $\{G\eta_{A,\{a\}} \mid a \in B \cup C\}$ is clearly a separating family. Set $A' = B \cup C$. Then $G\eta_{A,A'} \in \text{Inj}$ and thus from the already proved (3) it follows that $G\eta_{A,A'}$ is an isomorphism. Choose $a \in A \setminus A'$. Since $G\eta_{A,\{a\}} \circ \chi_A = F'\eta_{A,\{a\}} \in \text{Quot}$, we have $G\eta_{A,\{a\}} = \pi_a \circ \mu_A \in \text{Quot}$. But then $\pi_a \circ \mu_A \circ (G\eta_{A,A'})^{-1} \circ \chi_{A'} : F'A' \to F'\{a\}$ is a quotient because $(G\eta_{A,A'})^{-1}, \chi_{A'} \in \text{Quot}$. This is a contradiction because F' is an $\mathcal{I}(\mathbb{R})$-relatively full embedding, $F'\{a\}$ does not belong to $\mathbb{R}$ and $\{a\} \nsubseteq A'$. Hence (2) follows.

To prove that G is an $\mathcal{I}(\mathbb{R})$-relatively full embedding consider $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$. Then $\eta_{A,B}$ is a morphism of $\mathbb{N}$ and we must prove that $\text{Im}(G\eta_{A,B}) \notin$

$\mathbb{R}$. For every $b \in B$, $G\eta_{B,\{b\}} \in$ Quot and $G\{b\} \notin \mathbb{R}$. Since $\mathbb{R}$ is closed under Quot we infer that $GB \notin \mathbb{R}$ and because $G\eta_{A,B} \in$ Quot we conclude that $\mathrm{Im}(G\eta_{A,B}) \notin \mathbb{R}$. Conversely, let $f : GA \to GB$ for $A, B \in \mathcal{P}(\omega)$ be a homomorphism such that $\mathrm{Im}(f) \notin \mathbb{R}$. To complete the proof it suffices to prove that $B \subseteq A$ and $f = G\eta_{A,B}$. Let $f' : A \to C \in$ Quot and $f'' : C \to B \in$ Inj be homomorphisms with $f = f'' \circ f'$ then C is isomorphic to $\mathrm{Im}(f)$. Since $\{G\eta_{B,\{b\}} \mid b \in B\}$ is a separating family, we infer that $\{G\eta_{B,\{b\}} \circ f'' \mid b \in B\}$ is a separating family and, by the universal property of products, the morphism $h : C \to \prod_{b\in B} \mathrm{Im}(G\eta_{B,\{b\}} \circ f'') \in$ Inj. Since $\mathrm{Im}(f) \notin \mathbb{R}$ we conclude that $\prod_{b\in B} \mathrm{Im}(G\eta_{B,\{b\}} \circ f'') \notin \mathbb{R}$, and thus there exists $b \in B$ such that $\mathrm{Im}(G\eta_{B,\{b\}} \circ f'') = \mathrm{Im}(G\eta_{B,\{b\}} \circ f) \notin \mathbb{R}$. Thus $G\eta_{B,\{b\}} \circ f \notin \mathcal{I}(\mathbb{R})$. Since $\chi_A \in$ Quot we conclude that $G\eta_{B,\{b\}} \circ f \circ \chi_A : F'A \to F'\{b\} \notin \mathcal{I}(\mathbb{R})$ and thus $b \in A$ and $G\eta_{B,\{b\}} \circ f \circ \chi_A = F'\eta_{A,\{b\}}$ because F' is an $\mathcal{I}(\mathbb{R})$-relatively full embedding. Then

$$F'\eta_{\{b\},\emptyset} \circ G\eta_{B,\{b\}} \circ f \circ \chi_A = F'\eta_{\{b\},\emptyset} \circ F'\eta_{A,\{b\}} = F'\eta_{A,\emptyset}.$$

Since for every $b' \in B$ we have

$$\begin{aligned} F'\eta_{\{b'\},\emptyset} \circ G\eta_{B,\{b'\}} \circ \chi_B &= F'\eta_{\{b'\},\emptyset} \circ F'\eta_{B,\{b'\}} \\ &= F'\eta_{B,\emptyset} = F'\eta_{\{b\},\emptyset} \circ F'_{B,\{b\}} \\ &= F'\eta_{\{b\},\emptyset} \circ G\eta_{B,\{b\}} \circ \chi_B \end{aligned}$$

we infer that $F'\eta_{\{b'\},\emptyset} \circ G\eta_{B,\{b'\}} = F'\eta_{\{b\},\emptyset} \circ G\eta_{B,\{b\}}$ for all $b' \in B$ because $\chi_B \in$ Quot. From this it follows that

$$F'\eta_{A,\emptyset} = F'\eta_{\{b\},\emptyset} \circ G\eta_{B,\{b\}} \circ f \circ \chi_A = F'\eta_{\{b'\},\emptyset} \circ G\eta_{B,\{b'\}} \circ f \circ \chi_A$$

for all $b' \in B$. Since $F'\eta_{A,\emptyset} \notin \mathcal{I}(\mathbb{R})$ we conclude that $G\eta_{B,\{b'\}} \circ f \circ \chi_A \notin \mathcal{I}(\mathbb{R})$ for all $b' \in B$ because $F'\eta_{\{b'\},\emptyset} \in$ Quot and $\mathbb{R}$ is closed under Quot. Hence $b' \in A$ and $G\eta_{B,\{b'\}} \circ f \circ \chi_A = F'\eta_{A,\{b'\}}$ for all $b' \in B$ because F' is an $\mathcal{I}(\mathbb{R})$-relatively full embedding. Thus $B \subseteq A$ and

$$\begin{aligned} G\eta_{B,\{b'\}} \circ G\eta_{A,B} \circ \chi_A &= G\eta_{B,\{b'\}} \circ \chi_B \circ F'\eta_{A,B} \\ &= F'\eta_{B,\{b'\}} \circ F'\eta_{A,B} \\ &= F'\eta_{A,\{b'\}} = G\eta_{B,\{b'\}} \circ f \circ \chi_A \end{aligned}$$

for all $b' \in B$. By (2), $\{G\eta_{B,\{b'\}} \mid b' \in B\} \in$ Inj and thus $G\eta_{A,B} \circ \chi_A = f \circ \chi_A$. But $\chi_A \in$ Quot, and this completes the proof that $f = G\eta_{A,B}$.

It remains to prove that if $\mathbb{Q}$ is a quasivariety of algebras and F is synchronized then also G is synchronized. First observe that F' is also synchronized. For $A \in$

$\mathcal{P}(\omega)$ let $\rho_{F'A}$ and ρ_{GA} be the respective epireflection morphisms of $F'A$ and GA. Let S be an algebra and for $A \in \mathcal{P}(\omega)$ let $\nu_A : \mathbf{S} \to RF'A$ witness the fact that F' is synchronized. Since for every $a \in A$ we have $F'\eta_{A,\{a\}} = G\eta_{A,\{a\}} \circ \chi_A$ we conclude that $RF'\eta_{A,\{a\}} = R(G\eta_{A,\{a\}} \circ \chi_A)$. Set $\zeta_A = R\chi_A \circ \nu_A : \mathbf{S} \to RGA$, then the property that for every $s \in S$ and $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$ we have $RF'\eta_{A,B}(\nu_A(s)) = \nu_B(s)$ implies $RG\eta_{A,B}(\zeta_A(s)) = \zeta_B(s)$ and the fact that ν_A is injective for every $A \in \mathcal{P}(\omega)$ and $\chi_{\{a\}}$ is the identity mapping for every $a \in \omega$ imply that ζ_A is injective for all $A \in \mathcal{P}(\omega)$. The validity of (s1) and (s2) for F' implies that G also satisfies (s1) and (s2). From the facts that F' satisfies (s4) and $\zeta_{\{a\}} = \nu_{\{a\}}$ for all $a \in \omega$ and $\{G\eta_{A,\{a\}} \mid a \in A\}$ is a separating family for all $A \in \mathcal{P}(\omega)$ it follows that (s4) holds for G. Indeed, if u and v are distinct elements of RGA with $\rho_{GA}(u), \rho_{GA}(v) \notin \mathrm{Im}(\zeta_A)$ then there exists $a \in A$ with $F'\eta_{A,\{a\}}(u) \neq F'\eta_{A,\{a\}}(v)$ and hence $\rho_{G\{a\}} \circ F'\eta_{A,\{a\}}(u) \neq \rho_{G\{a\}} \circ F'\eta_{A,\{a\}}(v)$. Then $\rho_{G\{a\}} \circ F'\eta_{A,\{a\}} = RF'\eta_{A,\{a\}} \circ \rho_{GA}$ implies that $\rho_{GA}(u) \neq \rho_{GA}(v)$. If u and v are elements of RGA with $\rho_{GA}(u) \notin \mathrm{Im}(\zeta_A)$ and $v \in \mathrm{Im}(\zeta_A)$ then, by the same argument, we obtain that $\rho_{GA}(u) \neq \rho_{GA}(v)$ and hence GA satisfies (s4). To prove (s3) consider $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$. Choose $b \in B$. Since F' satisfies (s3), the condition (•) from Proposition 3.1 is satisfied for $F'\eta_{A,B}$ and $F'\eta_{B,\{b\}}$. Since χ is a surjective natural transformation from F' onto G and since $F'\{b\} = G\{b\}$ we conclude, by Proposition 3.1, that every mapping h from the underlying set of RGB into RGA such that $RG\eta_{A,B} \circ h$ is the identity mapping is a homomorphism from RGA into RGB. Thus G satisfies (s3) and whence G is synchronized. □

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