CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

HORST HERRLICH
KYRIAKOS KEREMEDIS
ELEFTHERIOS TACHTSIS

Tychonoff products of super second countable and super separable metric spaces

Cahiers de topologie et géométrie différentielle catégoriques, tome 49, nº 4 (2008), p. 267-279

<http://www.numdam.org/item?id=CTGDC_2008__49_4_267_0>

© Andrée C. Ehresmann et les auteurs, 2008, tous droits réservés.

L'accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. XLIX-4 (2008)

TYCHONOFF PRODUCTS OF SUPER SECOND COUNTABLE AND SUPER SEPARABLE METRIC SPACES

Dedicated to Jiři Adámek on the occasion of his 60th birthday

by ***Horst HERRLICH, Kyriakos KEREMEDIS, and Eleftherios TACHTSIS***

Abstract

Nous montrons que dans ZF, i.e. la théorie des ensembles de Zermelo-Fraenkel sans l'axiome du choix, les conditions suivantes sont équivalentes.

1. CAC($\mathbb{R}$), i.e. *l'axiome du choix restreint aux familles dénombrables de sous-ensembles non vides des réels.*
2. *Le produit de Tychonoff d'un nombre fini d'espaces métriques à bases super-dénombrables est encore à bases super-dénombrables.*

Donc, la condition 2. n'est pas un théorème de la thorie ZF des ensembles.

Nous montrons aussi que l'énoncé: *Le produit de Tychonoff de deux espaces métriques Cantor-complets super-séparables (héréditairement séparables) est super-séparable (resp. héréditairement séparable)* est démontrable dans ZF.

1 Introduction, terminology, and preliminary results

The present paper is a continuation of the research in [2] on super second countable (SSC) and super separable (SS) metric spaces (complete definitions are given below). The study of SSC and SS spaces in ZF was initiated in [1]. In [2] we elucidated on the interrelation between the aforementioned properties in the realm of metric spaces. In particular, we showed that every SSC metric space is SS, is provable without employing any choice principle. In contrast to this result, we established that the converse implication, i.e. every SS metric space is SSC, is not

MSC (2000): 03E25, 54B10, 54D65, 54D70, 54E35.

Keywords: Axiom of choice, weak axioms of choice, super second countable metric spaces, super separable metric spaces, Tychonoff products.

provable in ZF set theory. In [2] we also examined whether the properties of SSC and SS are hereditary among metric spaces. We showed that the statement "every SSC metric space is hereditarily SSC" is provable in ZF, whereas the corresponding statement for SS metric spaces is not deducible from the ZF axioms alone.

Naturally, one having defined the notions of SSC and SS, may ask whether these properties are preserved under Tychonoff products of finitely many such spaces. In Theorem 2 we show that in ZF if the Tychonoff product of two non-empty metric spaces (and by induction of finitely many) is SSC then each one of the coordinate spaces is SSC. However, the situation for the converse implication is totally different. In particular, in Theorem 8 we prove that the weak choice axiom CAC($\mathbb{R}$) is equivalent to the statement "Tychonoff products of finitely many SSC metric spaces are SSC". Therefore, the latter proposition is not a theorem of ZF.

Regarding Tychonoff products of finitely many SS metric spaces, we establish (see the forthcoming Theorem 7) that if the coordinate spaces are in addition Cantor-complete, then their product is SS and the proof requires no choice principles. The same result applies for hereditarily separable metric spaces which are Cantor-complete (see Theorem 7). However, it still eludes us whether it is provable in ZF that the Tychonoff product of two SS metric spaces (or hereditarily separable metric spaces) is again SS (resp. hereditarily separable).

Finally, we would like to note that since the axioms of countability play a prominent role in the theory of metric spaces we believe that studying their set theoretic strength as well as the interrelation between them in the absence of AC is important so that we may conceive better our limitations without AC within this part of topology.

Definition 1

1. A topological space (X, T) is called *super second countable* (SSC) if every base for T has a countable subfamily which is a base.

2. A topological space (X, T) is called *super first countable* (SFC) if for every $x \in X$ every neighbourhood base $\mathcal{V}(x)$ of x there is a countable subfamily $\mathcal{B}(x)$ of $\mathcal{V}(x)$ which is a neighbourhood base of x.

3. A topological space (X, T) is called *super separable* (SS) if every dense subspace of X is separable.

4. A topological space (X, T) is called *dense-in-itself* iff X has no isolated points.

5. If (X, T) is a topological space, then a set $Y \subset X$ is called *nowhere dense* if the closure of Y has empty interior.

6. A metric space (X, d) is called *(Cantor) complete* if $\bigcap_{n\in\omega} F_n \neq \emptyset$ for every descending family $\{F_n : n \in \omega\}$ of non-empty closed sets such that $\lim_{n\to\infty} \text{diameter}(F_n) = 0$.

7. CAC($\mathbb{R}$): The axiom of choice restricted to countable families of non-empty subsets of $\mathbb{R}$.

8. M(P,Q): Every metric space having the property P has also the property Q.

9. For any topological space (X, T), let

$$\text{Iso}(X) = \{x \in X : x \text{ is isolated in } X\}.$$

By transfinite recursion we define a decreasing sequence $(X_\alpha)_{\alpha\in\text{Ord}}$ of closed subspaces of X as follows:

$$\begin{aligned} X_0 &= X, \\ X_{\alpha+1} &= X_\alpha \setminus \text{Iso}(X_\alpha), \\ X_\alpha &= \bigcap\{X_\beta : \beta < \alpha\} \text{ for limit } \alpha. \end{aligned}$$

The set X_α, $\alpha \in \text{Ord}$, is called the *αth Cantor-Bendixson derivative* of X.

10. A topological space (X, T) is called *scattered* iff $\text{Iso}(Y) \neq \emptyset$ for each non-empty closed subspace Y of X. Clearly, X is scattered iff there exists an ordinal α_0 such that $X_{\alpha_0} = \emptyset$. If X is scattered, then the ordinal number $\min\{\alpha : X_\alpha = \emptyset\}$ is called the *Cantor-Bendixson rank* of the scattered space X and it is denoted by $|X|_{CB}$.

In what follows, "S" stands for separable, "hSSC" stands for hereditarily super second countable, "hSS" stands for hereditarily super separable, and "hS" stands for hereditarily separable.

Proposition 1 *([3]) CAC($\mathbb{R}$) iff every family $\mathcal{A} = \{A_n : n \in \omega\}$ of non-empty subsets of reals has a partial choice function (i.e., $\mathcal{A}$ has an infinite subfamily with a choice function).*

Proposition 2 *([6]) The following statements are equivalent:*

(i) CAC($\mathbb{R}$).

(ii) A metric space is second countable iff it is separable.

Proposition 3 *([1]) The following statements are equivalent:*

(i) CAC($\mathbb{R}$).

(ii) $\mathbb{R}$ is hS.

(iii) $\mathbb{R}$ is SS.

Proposition 4 *([1]) The following statements are equivalent:*

(i) CAC($\mathbb{R}$).

(ii) Every second countable topological space is SSC.

(iii) $\mathbb{R}$ (with the standard topology) is SSC.

Proposition 5 *([2]) The following statements are equivalent:*

(i) CAC($\mathbb{R}$).

(ii) $\mathbb{R}$ is hSSC.

(iii) $\mathbb{R}$ is hSS.

(iv) Every countable subspace of $\mathbb{R}$ is SSC.

Clearly, if (X, d) is a hS metric space then it is also hSS. Thus, M(hS, hSS) and consequently, M(hS, SS), are true in ZF. However, the converse need not be true. Indeed, letting $X = \mathbb{C} \cup \mathbb{Q}$, where $\mathbb{C}$ denotes the Cantor ternary set, carry the subspace topology, we can easily verify that X is SS ($\mathbb{C}$ is nowhere dense in X). Now, if X is hS, then $\mathbb{C}$ is hS. Since the proposition "$\mathbb{C}$ is hereditarily separable" is equivalent to CAC($\mathbb{R}$) (recall that $|\mathbb{C}| = |\mathbb{R}|$ and follow the proof of Proposition 3 in [1]), it follows that M(SS, hS) implies CAC($\mathbb{R}$) and consequently it is not a theorem of ZF. However, it is easily seen that if X is hSS then X is also hS. Hence, M(hSS, hS) $\Leftrightarrow$ M(hS, hSS) and we have completed the proof of the following proposition:

Proposition 6 *Let (X, d) be a metric space. Then:*

(i) *If X is hS, then X is SS.*

(ii) *X is hSS iff X is hS.*

(iii) *M(SS, hS) iff CAC($\mathbb{R}$).*

(iv) *M(hSS, hS) $\Leftrightarrow$ M(hS, hSS).*

Theorem 1 *([2]) The following statements are provable in ZF:*

(i) *Every SSC metric space is SS.*

(ii) *Every closed subspace of a SSC metric space is SSC.*

(iii) *If (X, d) is a SS metric space and O is an open subset of X, then the closure of O in X is SS.*

(iv) *Every dense subspace of a SS metric space is SS.*

(v) *Every open subset of a SS metric space is SS.*

Proposition 7 *Let X be a scattered topological space having a base $\mathcal{B} = \{B_v : v \in \aleph\}$, where $\aleph$ is a well ordered cardinal number. Then X is a well-orderable set.*

Proof. Let α be the Cantor-Bendixson rank of X. Clearly, $X = \cup\{\mathrm{Iso}(X_i) : i \in \alpha\}$ and for $i \neq j$, $\mathrm{Iso}(X_i) \neq \mathrm{Iso}(X_j)$. Since for every $i \in \alpha$ the restriction of $\mathcal{B}$ to X_i is a base we may use the well ordering of $\mathcal{B}$ to define a well ordering on $\mathrm{Iso}(X_i)$. Then, X being the union of a well ordered family of well ordered sets is well ordered. ■

Remark 1. From Proposition 7 it follows that a second countable scattered metric space (X, d) is well orderable. Hence, (X, d) is hereditarily separable and hereditarily super separable. Thus, in the sequel, when dealing with the properties of SS, hSS and hS we shall always assume that (X, d) is **not** scattered.

2 Main results

Theorem 2 *(ZF) Let X, Y be two non-empty metric spaces such that $X \times Y$ is SSC (SS). Then, each one of X and Y is SSC (SS).*

Proof. Assume that $X \times Y$ is SSC. We show that X is SSC (similarly one proves that Y is SSC). To see this, fix $\mathcal{B}$ and $\mathcal{C}$ two bases for X and Y, respectively. Then $\mathbf{B} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$ is a base for $X \times Y$ and consequently there exists a countable subfamily $\mathbb{B} = \{(B_n \times C_n) \in \mathbf{B} : n \in \omega\}$ which is a base for $X \times Y$. Clearly, $\{B_n : n \in \omega\} \subset \mathcal{B}$ is a base for X and X is SSC as required.

Assume now that $X \times Y$ is SS. We show that X is SS (similarly one proves that Y is SS). Fix a dense subset D of X. Then, $D \times Y$ is a dense subset of $X \times Y$ and consequently there is a countable dense subset $\{(d_n, y_n) : n \in \omega\} \subset D \times Y$ of $X \times Y$. Clearly, $\{d_n : n \in \omega\} \subset D$ is a countable dense subset of D and X is SS as promised. ∎

Theorem 3 *The following statements are equivalent:*

(i) For every SS metric space X, $X \times \mathbb{N}$ is SS.

(ii) For every SS metric space X, for every family $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ of dense subsets of X there exists a countable subset $G \subset \cup\mathcal{A}$ such that for every $i \in \mathbb{N}$, $\overline{G \cap A_i} = X$.

(iii) For all SS metric spaces X and Y, $X \times Y$ is SS.

Proof. (i) $\Rightarrow$ (ii) Fix a family $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ of dense subsets of the SS metric space X. Clearly, $D = \cup\{A_i \times \{i\} : i \in \mathbb{N}\}$ is a dense subset of $X \times \mathbb{N}$ and consequently there exists a countable dense subset $H = \{h_n : n \in \mathbb{N}\}$ of D. Clearly, $G = \{g_n = Dom(h_n) : n \in \mathbb{N}\} \subset \cup\mathcal{A}$ and it satisfies: $\forall i \in \mathbb{N}$, $\overline{G \cap A_i} = X$.

(ii) $\Rightarrow$ (iii) Let X, Y be two SS metric spaces and fix a dense subset D of $X \times Y$. Let $\mathcal{B} = \{B_n : n \in \omega\}$, $\mathcal{C} = \{C_n : n \in \omega\}$ be bases for X and Y, respectively. For each $n \in \mathbb{N}$ we let:

$$A_n = \{x \in X : \exists y \in C_n, (x, y) \in D\}.$$

Then A_n is dense in X. Indeed, let U be a non-empty open subset of X. Since D is dense in $X \times Y$, $(U \times C_n) \cap D \neq \emptyset$. Let (x, y) be a member of the latter intersection. Clearly, $x \in U \cap A_n$ and A_n is dense in X. Put $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$. By (ii) there exists a countable set $G_X \subset \cup\mathcal{A}$ satisfying: $\forall n \in \mathbb{N}$, $\overline{G_X \cap A_n} = X$. Then the set $(G_X \times Y) \cap D$ is dense in $X \times Y$. To see this, let $O = B_n \times C_m$ for some $n, m \in \mathbb{N}$. Then $B_n \cap G_X \cap A_m \neq \emptyset$. Let x be a member of the latter intersection. By the definition of A_m there exists $y \in C_m$ such that $(x, y) \in D$. Clearly, $(x, y) \in O \cap (G_X \times Y) \cap D$.

For each $n \in \mathbb{N}$ we let:

$$W_n = \{y \in Y : \exists x \in B_n, (x, y) \in (G_X \times Y) \cap D\}.$$

Then W_n is dense in Y. Indeed, let U be a non-empty open subset of Y. Then $(B_n \times U) \cap (G_X \times Y) \cap D \neq \emptyset$, so let (x, y) be an element of the latter intersection. Clearly, $y \in U \cap W_n$ and W_n is dense in Y as required. Put $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$. By (ii) there exists a countable set $G_Y \subset \cup\mathcal{W}$ satisfying: $\forall n \in \mathbb{N}, \overline{G_Y \cap W_n} = Y$.

We assert that the countable set $(G_X \times G_Y) \cap D$ is dense in $X \times Y$, hence it is dense in D. Let $O = B_n \times C_m$ for some $n, m \in \mathbb{N}$. Then $C_m \cap G_Y \cap W_n \neq \emptyset$. Let y be an element in the latter intersection. By the definition of W_n there exists an element $x \in B_n$ such that $(x, y) \in (G_X \times Y) \cap D$. Clearly, $(x, y) \in O \cap (G_X \times G_Y) \cap D$. Thus, D is separable and $X \times Y$ is SS finishing the proof of (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) This is straightforward ■

Theorem 4 *Let X, Y be two non-empty metric spaces. Then $X \times Y$ is SS iff $X \times \mathbb{N}$ and $Y \times \mathbb{N}$ are SS.*

Proof. ($\Rightarrow$) Assume that $X \times Y$ is SS and without loss of generality that both X and Y are infinite sets. By Theorem 2 X is SS and Y is SS. We show that $X \times \mathbb{N}$ is SS (and similarly $Y \times \mathbb{N}$ is SS). If $\overline{\text{Iso}(Y)} = Y$ then $X \times \mathbb{N}$ embeds as a dense subspace of $X \times Y$ and consequently by Theorem 1 (iv), $X \times \mathbb{N}$ is SS. So, we may assume that $\overline{\text{Iso}(Y)} \neq Y$. Fix D a dense subset of $X \times \mathbb{N}$. Then,

$$\mathcal{A} = \{A_n = Dom(D \cap (X \times \{n\})) : n \in \mathbb{N}\}$$

is a family of dense subsets of X. Since $Y \backslash \overline{\text{Iso}(Y)}$ is a second countable dense-in-itself metric space, we may fix a maximal infinite antichain $\mathcal{O}$ consisting of basic open sets in $Y \backslash \overline{\text{Iso}(Y)}$ (hence of open sets in Y). Then $\mathcal{V} = \mathcal{O} \cup \{\{y\} : y \in \text{Iso}(Y)\}$ is clearly countable (Y is second countable, hence $\text{Iso}(Y)$ is countable), so let $\{O_n : n \in \mathbb{N}\}$ be an enumeration of $\mathcal{V}$. It can be readily verified that

$$\mathbf{D} = \cup\{A_n \times O_n : n \in \mathbb{N}\}$$

is a dense subset of $X \times Y$ and consequently it contains a countable dense subset $\mathbf{G}$. It is also easy to see that for every $n \in \mathbb{N}$, $\mathbf{G}_n = \mathbf{G} \cap (A_n \times O_n)$ is dense in the open subspace $X \times O_n$ of $X \times Y$. Hence, $G_n = \pi_X(\mathbf{G}_n) \subset A_n$ is dense in X and consequently for every $n \in \mathbb{N}$, $G_n \times \{n\} \subset A_n \times \{n\} \subset D$ is dense in $X \times \{n\}$. Thus, $G = \cup\{G_n \times \{n\} : n \in \mathbb{N}\} \subset D$ is a countable dense subset of $X \times \mathbb{N}$ (hence of D) and $X \times \mathbb{N}$ is SS as required.

($\Leftarrow$) From Theorem 2 we deduce that X and Y are SS and from our hypothesis and the proof of Theorem 3 we conclude that $X \times Y$ is SS as required. ■

The proofs of the subsequent two theorems are similar to the proofs of Theorems 3 and 4, respectively, so we leave them as an easy exercise for the reader.

Theorem 5 *The following statements are equivalent:*

(i) For every hS metric space X, $X \times \mathbb{N}$ is hS.

(ii) For every hS metric space X, for every family $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ of subsets of X there exists a countable subset $G \subset \cup\mathcal{A}$ such that for every $i \in \mathbb{N}$, $G \cap A_i$ is dense in A_i.

(iii) For all hS metric spaces X and Y, $X \times Y$ is hS.

Theorem 6 *Let X, Y be two non-empty metric spaces. Then $X \times Y$ is hS iff $X \times \mathbb{N}$ and $Y \times \mathbb{N}$ are hS.*

We show next that the statement "*the product of two complete hS (SS) metric spaces is hS (SS)*" is provable in ZF set theory. First we need to establish the subsequent two Lemmas.

Lemma 1 *Let X be a SS metric space. Then, every family $\mathcal{A} = \{A_n : n \in \omega\}$ of dense subsets of X has a choice function.*

Proof. Let $\mathcal{B}$ be a countable base for the metric topology on X and let $\mathcal{O} = \{O_n : n \in \omega\} \subset \mathcal{B}$ be an antichain. Put

$$Y = \bigcup\{O_n \cap A_n : n \in \omega\} \cup (\bigcap\{(O_n)^c : n \in \omega\}).$$

Since A_n is dense in X, it follows that $O_n \cap A_n \neq \emptyset$ for all $n \in \omega$. Furthermore, Y is dense in X. Indeed, let V be a non-empty open subset of X. If V does not meet any O_n, then clearly $V \cap Y \neq \emptyset$. If $V \cap O_n \neq \emptyset$ for some $n \in \omega$, then since A_n is dense in X, $V \cap O_n \cap A_n \neq \emptyset$, hence $V \cap Y \neq \emptyset$. Thus, $\overline{Y} = X$.

Since X is SS, let $D = \{d_n : n \in \omega\}$ be a dense subset of Y. For every $n \in \omega$, $O_n \cap Y$ is a non-empty open subset of Y and since $\mathcal{O}$ is an antichain,

$O_n \cap Y = O_n \cap A_n$. Therefore, $(O_n \cap A_n) \cap D \neq \emptyset$ for every $n \in \omega$. Then $f = \{(n, d_{m_n}) : n \in \omega\}$, $m_n = \min\{m \in \omega : d_m \in O_n \cap A_n\}$, is clearly a choice function for $\mathcal{A}$, finishing the proof of the Lemma. ■

Lemma 2 *The following statements are equivalent:*

(i) CAC($\mathbb{R}$).

(ii) There exists a complete dense-in-itself hS metric space.

(iii) There exists a complete dense-in-itself SS metric space.

Proof. (i) $\Rightarrow$ (ii) $\mathbb{R}$ with the standard metric is complete and dense-in-itself. By Proposition 3, $\mathbb{R}$ is hS.

(ii) $\Rightarrow$ (iii) This follows from Proposition 6.

(iii) $\Rightarrow$ (i) Fix a complete dense-in-itself SS metric space (X, d). Fix a neighbourhood base $\mathcal{B} = \{B_n : n \in \omega\}$ for X consisting of closed sets. For every $n \in \omega$, via a straightforward induction we construct a π-base[1] $\Pi_n = \cup\{L_{ni} : i \in \omega\}$ of B_n such that:

(a) Each $L_{ni} \subset \mathcal{B}$ is a maximal antichain in B_n and each member of L_{ni} has diameter less that $1/(i+1)$.

(b) $\forall m \in \mathbb{N}, \forall O \in L_{nm}, \exists Q \in L_{n(m-1)}, O \subset Q$.

We do this as follows: For $i = 0$, by using the well ordering of $\mathcal{B}$, we let L_{n0} be a maximal antichain in B_n with each of its members having diameter less than 1.

For $i = m+1$ and using the well-ordering of $\mathcal{B}$ again, we fix for each $O \in L_{nm}$ a maximal antichain C_O in O with each of its members having diameter less that $1/(i+1)$. Put $L_{ni} = \cup\{C_O : O \in L_{nm}\}$.

Clearly, $\Pi_n = \cup\{L_{ni} : i \in \omega\}$ is a π-base for B_n. Furthermore, as B_n is closed and for every $i \in \omega$, each member of L_{ni} has diameter less than $1/(i+1)$, and for every maximal chain $\mathcal{C}$ of Π_n, $\mathcal{C} \cap L_{ni}$ is a singleton, it follows that $\cap\mathcal{C}$ is a singleton. For every $n \in \omega$, let $Y_n = \cup\{\cap\mathcal{C} : \mathcal{C}$ is a maximal chain of $\Pi_n\}$. We assert that for every $n \in \omega$, $|Y_n| = |\omega^\omega|$ ($= |\mathbb{R}|$ in ZF). Indeed, we define a function $H_n : \omega^\omega \to Y_n$ by recursion as follows: Let $f \in \omega^\omega$. Let $V_{f(0)}$ be the $f(0)$ element of L_{n0} in the numbering that L_{n0} inherits as a subset of $\mathcal{B}$. Assume that we have

[1]Let (X, T) be a topological space. $B \subset T \setminus \{\emptyset\}$ is called a *π-base* if every non-empty element of T contains an element of B.

defined a chain $V_{f(0)} \supset V_{f(1)} \supset \ldots \supset V_{f(k)}$ of Π_n. Let $V_{f(k+1)}$ be the $f(k+1)$ element of $C_{V_{f(k)}}$ (the maximal antichain in $V_{f(k)}$ in the above construction of Π_n) in $L_{n(k+1)}$. Clearly, $V_{f(k+1)} \subset V_{f(k)}$, $\{V_{f(k)} : k \in \omega\}$ is a maximal chain of Π_n, and $\cap\{V_{f(k)} : k \in \omega\}$ is a singleton, say $\{a_{nf}\}$. Finally, define $H_n(f) = a_{nf}$. It can be readily verified that H_n is a one-to-one function from ω^ω onto Y_n. Thus, $|Y_n| = |\omega^\omega|$ as claimed.

We show next that CAC($\mathbb{R}$) holds. Fix a disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty subsets of ω^ω. It is straightforward to verify that for every $i \in \omega$, $D_i = \cup\{H_n(A_i) : n \in \omega\}$ is a dense subset of X. By Lemma 1 let f be a choice function of the family $\mathcal{D} = \{D_i : i \in \omega\}$. For every $i \in \omega$, let $g_i = H_{n_i}^{-1}(f(i))$, $n_i = \min\{n \in \omega : f(i) \in H_n(A_i)\}$. Clearly, $F = \{(i, g_i) : i \in \omega\}$ is a choice function of $\mathcal{A}$, finishing the proof of (iii) $\Rightarrow$ (i) and of the Lemma. ∎

Theorem 7 *Let X, Y be two non-empty complete metric spaces. Then,*

(i) $X \times Y$ is SS iff X and Y are SS.

(ii) $X \times Y$ is hS iff X and Y are hS.

Proof. (i) ($\Rightarrow$) This follows from Theorem 2.

(i) ($\Leftarrow$) Assume that X and Y are SS. In view of the proof of Theorem 3 it suffices to show that $X \times \mathbb{N}$ is SS. We consider the following two cases:

(a) $\overline{\mathrm{Iso}(X)} = X$. Clearly, $\mathrm{Iso}(X)$ is countable and consequently $\mathrm{Iso}(X) \times \mathbb{N}$ is a countable discrete dense subset of $X \times \mathbb{N}$. Since every dense subspace of $X \times \mathbb{N}$ includes $\mathrm{Iso}(X) \times \mathbb{N}$ it follows that $X \times \mathbb{N}$ is a SS space.

(b) $\overline{\mathrm{Iso}(X)} \neq X$. Then $Z = X \backslash \overline{\mathrm{Iso}(X)}$, being open in X, is dense-in-itself and since X is SS it follows from Theorem 1 (v) that Z is SS. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a base for X and let $x \in Z$. Then there exists $n \in \mathbb{N}$ such that $x \in \overline{B_n} \subset Z$. Clearly, $\overline{B_n}$ is a complete dense-in-itself subspace of X, and by Theorem 1 (iii), $\overline{B_n}$ is SS. Thus, by Lemma 2, CAC($\mathbb{R}$) holds. Since $X \times \mathbb{N}$ is second countable, it follows from Proposition 4 that $X \times \mathbb{N}$ is SSC. By Theorem 1 (i), $X \times \mathbb{N}$ is SS.

(ii) ($\Rightarrow$) This is straightforward.

(ii) ($\Leftarrow$) Assume that X and Y are hS metric spaces. If X and Y are scattered then, in view of Remark 1, X and Y are well orderable. Thus, $X \times Y$ is a well orderable separable metric space and consequently $X \times Y$ is hS. So assume that X is not scattered. This means that for some ordinal number α, the subspace X_α of X is a complete dense-in-itself hS metric space. Thus, by Lemma 2, CAC($\mathbb{R}$) holds. Since $X \times Y$ is second countable, it is hereditarily second countable as well. Thus, by CAC($\mathbb{R}$), $X \times Y$ is hS. ∎

Theorem 8 *The following statements are equivalent:*

(i) CAC($\mathbb{R}$).

(ii) Tychonoff products of finitely many SSC metric spaces are SSC.

Proof. (i) $\Rightarrow$ (ii) Let $\{(X_i, d_i) : i \leq n\}$, $n \in \mathbb{N}$, be a family of SSC metric spaces, and let X be its Tychonoff product. Clearly, X is a second countable space, hence by CAC($\mathbb{R}$), X is SSC.

(ii) $\Rightarrow$ (i) Fix $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ a disjoint family of non-empty subsets of $\wp(\mathbb{N})$. It suffices, in view of Proposition 1, to show that some infinite subfamily $\mathcal{E}$ of $\mathcal{A}$ has a choice function. Without loss of generality we may assume that for every $n \in \mathbb{N}$ and for every $X \in A_n$, $X \cap [1, n] = \emptyset$. Consider $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ as a subspace of $\mathbb{R}$ equipped with the standard metric.

Claim. X is SSC.

Proof of the Claim. Fix a base $\mathcal{B}$ for X. Since for every $n \in \mathbb{N}$, $1/n$ is an isolated point of X, it follows that $\{1/n\} \in \mathcal{B}$. Let $\{S_n : n \in \mathbb{N}\}$ be an enumeration of the set of all finite subsets of $\{1/n : n \in \mathbb{N}\}$. Since for every neighbourhood $O \in \mathcal{B}$ of 0, $X \backslash O$ is some S_n, it follows that $\mathcal{C} = \{\{1/n\} : n \in \mathbb{N}\} \cup \{O \in \mathcal{B} : 0 \in O\} \subset \mathcal{B}$ is a countable subset of $\mathcal{B}$ and is a base for X. Hence, X is SSC as required.

By the Claim and our hypothesis we have that $Y = X \times X$ is SSC. For every $n \in \mathbb{N}$ and every $Z \in A_n$ let

$$O_{nZ} = (([0, 1/n] \times [0, 1/n]) \backslash (\{(1/n, 1/z) : z \in Z\} \cup \{(1/n, 0)\})) \cap Y.$$

It can be readily verified that $\mathcal{C} = \{O_{nZ} : n \in \mathbb{N}, Z \in A_n\} \cup \{\{(1/n, 1/m)\} : n, m \in \mathbb{N}\} \cup \{(\{1/n\} \times [0, 1/m]) \cap Y : n, m \in \mathbb{N}\} \cup \{([0, 1/m] \times \{1/n\}) \cap Y : m, n \in \mathbb{N}\}$ is a base for the topology of Y. Let $\mathcal{D} \subset \mathcal{B}$ be a countable base for Y. Let $\{O_{n_i Z_{n_i}} : i \in \mathbb{N}\} \subset \mathcal{D}$ be an enumeration of a strictly decreasing neighbourhood base of the point $(0, 0)$ of Y. It is straightforward to verify that $\{(n_i, Z_{n_i}) : i \in \mathbb{N}\}$ is a choice function of the infinite subfamily $\mathcal{E} = \{A_{n_i} : i \in \mathbb{N}\}$ of $\mathcal{A}$, finishing the proof of the theorem. ■

Theorem 9 *The following statements are equivalent:*

(i) CAC($\mathbb{R}$).

(ii) Tychonoff products of finitely many SFC metric spaces are SFC.

Proof. (i) $\Rightarrow$ (ii) This can be established similarly to the proof of (i) $\Rightarrow$ (ii) of Theorem 8.

(ii) $\Rightarrow$ (i) Simply note that the metric space X of the proof of Theorem 8 is trivially SFC. ■

Theorem 10 *The following statements are equivalent:*

(i) CAC($\mathbb{R}$).

(ii) X^ω is SSC (resp. SS) for every metrizable SSC (resp. SS) space (X, T).

(iii) The Baire space ω^ω is SSC (resp. SS).

Proof. (i) $\Rightarrow$ (ii) Let (X, T) be a metrizable SSC (resp. SS) space. Then in ZF, X^ω is a metrizable second countable space, hence by CAC($\mathbb{R}$), X^ω is SSC (thus by Theorem 1 (i), it is SS) as required.

(ii) $\Rightarrow$ (iii) ω, as a subspace of $\mathbb{R}$, is SSC (SS) in ZF. By our hypothesis, the Baire space ω^ω is SSC (SS).

(iii) $\Rightarrow$ (i) ω^ω is (topologically) homeomorphic to the set I of the irrational numbers (see [5]). If I is SSC, then from Theorem 1 (i) we deduce that I is SS. The conclusion follows from the fact that CAC($\mathbb{R}$) holds iff I is SS (the latter equivalence can be proved as in Lemma 2.2 in [1] but with I instead of $\mathbb{R}$). ■

Question. In ZF, are Tychonoff products of finitely many hS (or SS) metric spaces hS (or SS)?

References

[1] G. Gutierres, *On first and second countable spaces and the axiom of choice*, Topology Appl. **143** (2004), 93-103.

[2] H. Herrlich, K. Keremedis, and E. Tachtsis, *On Super Second Countable and Super Separable Metric Spaces*, accepted in Topology Proceedings.

[3] P. Howard, K. Keremedis, J.E. Rubin, A. Stanley, and E. Tachtsis, *Non-constructive properties of the real numbers*, Math. Logic Quart. **47** (2001), 423-431.

[4] P. Howard and J.E. Rubin, *Consequences of the Axiom of Choice*, Amer. Math. Soc. Math. Surveys and Monographs **59** (1998).

[5] T. Jech, *Set Theory*, Academic Press, New-York, 1978.

[6] K. Keremedis, *Countable Disjoint Unions in Topology and Some Weak Forms of the Axiom of Choice*, submitted.

Horst Herrlich, Feldhäuser Str. 69, 28865 Lilienthal, Germany.
E-mail address: horst.herrlich@t-online.de

Kyriakos Keremedis, Department of Mathematics, University of the Aegean, Karlovassi 83200, Samos, Greece.
E-mail address: kker@aegean.gr

Eleftherios Tachtsis, Department of Statistics and Actuarial-Financial Mathematics, University of the Aegean, Karlovassi 83200, Samos, Greece.
E-mail address: ltah@aegean.gr