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CAHIERS DE TOPOLOGIE ET
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ON FILTERED WEIGHTED COLIMITS OF PRESHEAVES

To Jiří Adámek, on his sixtieth birthday.

by F. BORCEUX and J. ROSICKÝ*†

Abstract

Quand un topos $\widehat{\mathcal{C}}$ de préfaisceaux est localement finiment présentable au sens des catégories enrichies, nous prouvons que la notion de $\widehat{\mathcal{C}}$-colimite pondérée filtrante se réduit au caractère filtrant au sens usuel, lorsque l'on évalue la situation en chaque objet $C \in \mathcal{C}$.

Introduction

The present paper has been motivated by the interest of the second author in homotopy theory and its relations with the theory of locally finitely presentable or accessible categories. In this context algebraic problems enriched in simplicial sets appear quite naturally and thus a careful analysis of filtered colimits enriched in simplicial sets is needed.

In enriched category theory over a symmetric monoidal closed category $\mathcal{V}$, the correct notions of limit and colimit to consider are those of *weighted limit* and *weighted colimit*:

$$\mathsf{lim}_W F \in \mathcal{E}, \quad \mathsf{colim}_W F \in \mathcal{E}$$

where W and F are $\mathcal{V}$-functors of the form

$$W\colon \mathcal{D} \longrightarrow \mathcal{V}, \quad F\colon \mathcal{D} \longrightarrow \mathcal{E}$$

with W covariant in the limit case and contravariant in the colimit case. In the case $\mathcal{V} = \mathsf{Set}$, this reduces simply to the limit or the colimit of the composite

$$\mathsf{Elts}(W) \longrightarrow \mathcal{D} \xrightarrow{\ F\ } \mathcal{E}.$$

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where $\mathsf{Elts}(W)$ indicates the category of elements of W. Choosing as weight W the constant functor on the singleton recaptures then the usual notions $\lim F$ and $\operatorname{colim} F$.

It has been proved in [7] for the theory of locally finitely presentable categories, and in [4] for the theory of finitely accessible categories, that these theories can be generalized to the enriched context, provided that the base category $\mathcal{V}$ is itself locally finitely presentable, with moreover the property that finitely presentable objects are stable in $\mathcal{V}$ under finite tensor products. Such a $\mathcal{V}$ is called a *locally finitely presentable base*.

Consider now a small category $\mathcal{C}$ and the corresponding topos $\widehat{\mathcal{C}} = [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$ of presheaves, viewed as a cartesian closed category. The topos $\widehat{\mathcal{C}}$ is always locally finitely presentable, but in general a finite product of finitely presentable presheaves is not finitely presentable. We prove that this last property holds as soon as $\mathcal{C}$ admits *finite weak multilimits*. This very mild requirement is of course satisfied when $\mathcal{C}$ is finitely complete, or when $\mathcal{C}$ is finite. But the category $\mathbf{\Delta}$ used to define simplicial sets admits also that property. Thus the topos of simplicial sets is a locally finitely presentable base.

We develop our study not just in the case of simplicial sets, but for an arbitrary presheaf topos $\widehat{\mathcal{C}}$ which is a locally finitely presentable base. Conical limits and conical colimits in $\widehat{\mathcal{C}}$ are well-known to be computed pointwise as in Set. But what about the weighted limits and colimits? Let us thus consider $\widehat{\mathcal{C}}$-functors F, W on a small $\widehat{\mathcal{C}}$-category $\mathcal{D}$. We can of course evaluate this situation at each object $C \in \mathcal{C}$

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "F"] \arrow[d, "W"'] & \widehat{\mathcal{C}} \\
\widehat{\mathcal{C}} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{ev}_C(\mathcal{D}) \arrow[r, "\mathsf{ev}_C(F)"] \arrow[d, "\mathsf{ev}_C(W)"'] & \mathsf{Set} \\
\mathsf{Set} &
\end{tikzcd}
$$

and ask the question: do we have

$$
\begin{aligned}
(\lim\nolimits_W F)(C) &= \lim\nolimits_{\mathsf{ev}_C(W)} \mathsf{ev}_C(F) \\
(\operatorname{colim}\nolimits_W F)(C) &= \operatorname{colim}\nolimits_{\mathsf{ev}_C(W)} \mathsf{ev}_C(F)\ ?
\end{aligned}
$$

The answer is *yes* for colimits, but *no* for limits.

Now a filtered colimit in Set is one which is computed over a filtered category. In terms of weighted colimits over Set, this means thus a colimit indexed by a weight

$W\colon \mathcal{D} \longrightarrow \mathsf{Set}$ whose category $\mathsf{Elts}(W)$ is filtered. This is precisely requiring that W is flat, that is, its Kan extension along the Yoneda embedding preserves finite limits.

A $\widehat{\mathcal{C}}$-weight $W\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$ is finite when $\mathcal{D}$ has finitely many objects, while each $\mathcal{D}(D, D')$ and each $W(D)$ are finitely presentable presheaves. A finite $\widehat{\mathcal{C}}$-limit is one indexed by such a finite weight W. A flat $\widehat{\mathcal{C}}$-weight is one whose Kan extension along the $\widehat{\mathcal{C}}$-Yoneda embedding preserves finite $\widehat{\mathcal{C}}$-limits. And finally a filtered $\widehat{\mathcal{C}}$-colimit is one indexed by a flat $\widehat{\mathcal{C}}$-weight.

Since weighted $\widehat{\mathcal{C}}$-limits in $\widehat{\mathcal{C}}$ are not computed pointwise, the notion of flat $\widehat{\mathcal{C}}$-weight has no reason *a priori* to reduce to being pointwise flat over Set. As a consequence, the notion of filtered weighted $\widehat{\mathcal{C}}$-colimit has no reason *a priori* to reduce pointwise to the ordinary notion of filtered colimit in Set. Nevertheless we show that these unexpected properties hold in the case of a locally finitely presentable base $\widehat{\mathcal{C}}$.

To prove this, given $\widehat{\mathcal{C}}$-functors

$$W\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}, \quad F\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$$

we investigate first the form of $(\lim_W F)(C)$, which we know not to be in general $\lim_{\mathsf{ev}_C(W)} \mathsf{ev}_C(F)$. We express nevertheless $(\lim_W F)(C)$ as the conical limit of some adequate functor

$$F_C\colon \mathcal{D}_W(C) \longrightarrow \mathsf{Set}$$

on some adequate category $\mathcal{D}_W(C)$. When W is a finite $\widehat{\mathcal{C}}$-weight, the category $\mathcal{D}_W(C)$ is generally not finite, not even finitely presentable nor finitely generated, not even when $\widehat{\mathcal{C}}$ is a locally finitely presentable base. But in this last case, we prove the existence of a finite subdiagram $\mathcal{F}_W(C) \subseteq \mathcal{D}_W(C)$ such that the limit of F_C can be equivalently computed just on $\mathcal{F}_W(C)$.

It follows easily from that analysis that a $\widehat{\mathcal{C}}$-weight which is pointwise flat over Set is in fact a flat $\widehat{\mathcal{C}}$-weight. The converse is rather immediate. Therefore filtered $\widehat{\mathcal{C}}$-colimits in $\widehat{\mathcal{C}}$ are precisely those which are pointwise filtered in Set.

We assume a reasonable familiarity with enriched category theory, as presented in [8] and chapter 6 of [3].

We thank Panegis Karazeris for fruitful discussions concerning our Proposition 2 and our Example 5.

1 Finitely presentable bases of presheaves

We fix once for all a small category $\mathcal{C}$ and consider the corresponding topos $\widehat{\mathcal{C}} = [\mathcal{C}^{op}, \mathrm{Set}]$ of presheaves. We view $\widehat{\mathcal{C}}$ as a symmetric monoidal closed category via its cartesian closed structure.

In [7] the theory of locally presentable categories, and in [4] the theory of accessible categories, are generalized to the enriched context. In the finitely presentable and finitely accessible cases, this is possible when

1. the base category $\mathcal{V}$ is locally finitely presentable;
2. the unit $I \in \mathcal{V}$ is finitely presentable;
3. if $A, B \in \mathcal{V}$ are finitely presentable, so is $A \otimes B$.

Such a symmetric monoidal closed category $\mathcal{V}$ is called a *locally finitely presentable base*.

Lemma 1 *Let $\mathcal{C}$ be a small category such that:*

1. *the terminal presheaf $\mathbf{1} \in \widehat{\mathcal{C}}$ is finitely presentable in $\widehat{\mathcal{C}}$;*
2. *the product of two representable functors is finitely presentable in $\widehat{\mathcal{C}}$.*

Then $\widehat{\mathcal{C}}$ is a locally finitely presentable base.

Proof The finite products of representable functors constitute a strongly generating family, stable under binary products, and constituted of finitely presentable objects. One concludes by Proposition 1.3 in [4]. □

Let us recall that the small category $\mathcal{C}$ admits *finite weak multilimits* when for every finite diagram in $\mathcal{C}$, one can find finitely many cones $\Gamma_1, \ldots, \Gamma_n$ on this diagram such that every cone Γ on the diagram factors in at least one way through at least one of the selected cones Γ_i. When the factorization is unique through a unique Γ_i, we recapture the notion of *finite multilimit*. On the other hand when $n = 1$, we get the notion of *finite weak limit*.

The following result can be infered from the results in [6]; we give here a direct proof, for the comfort of the reader.

Proposition 2 *When the small category $\mathcal{C}$ admits finite weak multilimits, the topos $\widehat{\mathcal{C}}$ is a locally finitely presentable base.*

Proof Consider first two objects $C, D \in \mathcal{C}$ and a finite weak multiproduct of them

$$\left(C \xleftarrow{p_i^C} P_i \xrightarrow{p_i^D} D\right)_{i=1,\dots,n}.$$

By definition of a finite weak multiproduct, the morphism

$$\gamma\colon \coprod_{t=1}^{n} \mathcal{C}(-,P_t) \twoheadrightarrow \mathcal{C}(-,C) \times \mathcal{C}(-,D)$$

of composition with p_t^C and p_t^D is surjective in each component. This proves already that $\mathcal{C}(-,C) \times \mathcal{C}(-,D)$ is finitely generated.

Consider next, for each pair (i,j) of indices, the finite weak multiple limit of the following diagram

$$\begin{tikzcd} P_i \arrow[r, "p_i^C"] \arrow[drr, "p_i^D"'] & & C \\ P_j \arrow[r, "p_j^D"'] \arrow[urr, "p_j^C"] & & D \end{tikzcd} \qquad \text{Diagram } (i,j)$$

which reduces to giving morphisms

$$P_i \xleftarrow{\lambda_k^{i,j}} L_k^{i,j} \xrightarrow{\mu_k^{i,j}} P_j$$

with the adequate property. Compositions with $\lambda_k^{i,j}$ and $\mu_k^{i,j}$ yield two morphisms

$$\mathcal{C}(-,L_k^{i,j}) \rightrightarrows \coprod_{t=1}^{n} \mathcal{C}(-,P_t)$$

identified by γ. This yields further two morphisms λ and μ

$$\coprod_{i,j,k} \mathcal{C}(-,L_k^{i,j}) \underset{\mu}{\overset{\lambda}{\rightrightarrows}} \coprod_{t} \mathcal{C}(-,P_t) \xrightarrow{\ \gamma\ } \mathcal{C}(-,C) \times \mathcal{C}(-,D)$$

such that $\gamma \circ \lambda = \gamma \circ \mu$. Therefore we get a factorization δ of (λ,μ) through the kernel pair of γ. If we can prove that δ is surjective, we shall have $\gamma = \text{Coker}\,(\lambda,\mu)$ and thus $\mathcal{C}(-,C) \times \mathcal{C}(-,D)$ will be finitely presentable.

Choose thus $(u\colon X \longrightarrow P_i, v\colon X \longrightarrow P_j)$ in the kernel pair of γ. This means

$$p_i^C \circ u = p_j^C \circ v, \quad p_i^D \circ u = p_j^D \circ v.$$

The pair (u, v) is a cone on Diagram (i, j) above, thus there exist an index k and a factorization w making the following diagram commutative

$$\begin{tikzcd} & & P_i \\ X \arrow[urr, "u"] \arrow[r, "w"] \arrow[drr, "v"'] & L_k^{i,j} \arrow[ur, "\lambda_k^{i,j}"'] \arrow[dr, "\mu_k^{i,j}"] & \\ & & P_j \end{tikzcd}$$

This proves that $(u, v) = \delta_X(w)$.

It remains to prove that the terminal presheaf is finitely presentable. The finite multilimit of the empty diagram consists in finitely many objects $P_1, \ldots, P_n$, in such a way that for every object $C \in \mathcal{C}$, there exists at least an arrow to at least one P_i. This means that the morphism

$$\coprod_{i=1}^{n} \mathcal{C}(-, P_i) \longrightarrow\!\!\!\!\!\rightarrow \mathbf{1}$$

is surjective , thus $\mathbf{1}$ is already finitely generated. The kernel pair of this morphism is simply

$$\left(\coprod_{i=1}^{n} \mathcal{C}(-, P_i)\right) \times \left(\coprod_{j=1}^{n} \mathcal{C}(-, P_j)\right)$$

and since $\widehat{\mathcal{C}}$ is cartesian closed, this is also

$$\coprod_{i,j=1}^{n} \mathcal{C}(-, P_i) \times \mathcal{C}(-, P_j).$$

By the first part of the proof, this is a finitely presentable object, proving eventually that $\mathbf{1}$ itself is finitely presentable. $\square$

Example 3 *When $\mathcal{C}$ is a category with finite limits, the topos $\widehat{\mathcal{C}}$ of presheaves is a finitely presentable base.*

Proof This is a very special instance of proposition 2. □

Example 4 *When $\mathcal{C}$ is a finite category, the topos $\widehat{\mathcal{C}}$ of presheaves is a finitely presentable base.*

Proof Given a finite diagram in $\mathcal{C}$, the set of all cones on it is trivially a finite weak multilimit. □

Our following example is the one which originally motivated our study; a different (and unpublished) proof of it has been presented by P. Karazeris at the 83rd PSSL in Glasgow (2006).

Example 5 *The topos of simplicial sets is a finitely presentable base.*

Proof Write Pos for the category of posets. The category Δ is the full subcategory of Pos whose objects are the non-empty finite ordered chains

$$[n] = \{0 < 1 < \ldots < n-1 < n\}.$$

The topos of simplicial sets is that of presheaves on Δ.

Given a finite diagram $\mathcal{D}$ in Δ, let L be its limit in Pos. If this limit is empty, the empty family of cones is a finite weak multilimit of $\mathcal{D}$ in Δ. Otherwise every cone on $\mathcal{D}$ in Δ factors through L, thus through a non-empty finite subchain of L. The limit cone in Pos restricted to all the non-empty finite subchains of L is then a finite weak multilimit of $\mathcal{D}$ in Δ. □

Counterexample 6 *When $\mathcal{C}$ is an infinite discrete category, the topos $\widehat{\mathcal{C}}$ is not a locally finitely presentable base.*

Proof Each representable functor $\mathcal{C}(-, C)$ takes the value 1 at C and the empty set elsewhere. Every finite colimit of representable functors has thus only finitely many non empty components. Therefore the terminal presheaf $\mathbf{1}$ is not finitely presentable. □

2 Pointwise weighted colimits

The results of this section hold for an arbitrary small category $\mathcal{C}$, even if we shall only need them in the case of a locally finitely presentable base $\widehat{\mathcal{C}}$.

We investigate here the form of weighted limits and weighted colimits in category theory enriched in $\widehat{\mathcal{C}}$. Let us recall that given two enriched functors defined on a small enriched category $\mathcal{D}$

$$W\colon D \longrightarrow \widehat{\mathcal{C}}, \quad F\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}},$$

the limit of F weighted by W consists in an object $L \in \widehat{\mathcal{C}}$ together with $\widehat{\mathcal{C}}$-natural isomorphisms

$$\mathsf{Nat}\Big(W, \widehat{\mathcal{C}}(P, F(-))\Big) \cong \widehat{\mathcal{C}}(P, L)$$

where Nat indicates the object of $\widehat{\mathcal{C}}$-natural transformations and P runs through $\widehat{\mathcal{C}}$. Weighted colimits are defined dually, using a contravariant weight W:

$$\mathsf{Nat}\Big(W, \widehat{\mathcal{C}}(F(-), P)\Big) \cong \widehat{\mathcal{C}}(L, P).$$

For every object $C \in \mathcal{C}$ we have an evaluation functor

$$\mathsf{ev}_C\colon \widehat{\mathcal{C}} \longrightarrow \mathsf{Set}, \quad P \mapsto P(C).$$

This is a morphism of symmetric monoidal closed categories, which is strict as a monoidal functor (it preserves binary products). This induces in particular a corresponding 2-functor

$$\mathsf{ev}_C\colon \widehat{\mathcal{C}}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Cat}$$

mapping a small $\widehat{\mathcal{C}}$-category $\mathcal{D}$ on the ordinary category with the same objects and whose sets of morphisms are given by

$$\mathsf{ev}_C(\mathcal{D})(X, Y) = \mathcal{D}(X, Y)(C).$$

Definition 7 *Given a small $\widehat{\mathcal{C}}$-category $\mathcal{D}$ and two $\widehat{\mathcal{C}}$-functors*

$$W\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}, \quad F\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$$

we say that the weighted $\widehat{\mathcal{C}}$-limit $\mathsf{lim}_W F$ *is* computed pointwise *when for every object $C \in \mathcal{C}$*

$$(\mathsf{lim}_W F)(C) = \mathsf{lim}_{\mathsf{ev}_C(W)} \mathsf{ev}_C(F).$$

An analogous definition holds for weighted $\widehat{\mathcal{C}}$-colimits.

Let us observe at once that:

Proposition 8 *The weighted $\widehat{\mathcal{C}}$-colimits of $\widehat{\mathcal{C}}$-valued $\widehat{\mathcal{C}}$-enriched functors are computed pointwise.*

Proof We consider the situation of definition 7, with W contravariant. The weighted colimit can be computed as a coend

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{colim}_W F = \int^{D\in\mathcal{D}} W(D)\times F(D) & \\
W(D)\times F(D) \arrow[ur] & & W(D')\times F(D') \arrow[ul] \\
 & W(D')\times F(D)\times\mathcal{D}(D,D') \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

that is, as an ordinary conical colimit involving the "tensors" $W(D)\times F(D)$. But all these ingredients are computed pointwise as in Set. □

The readers familiar with internal limits and colimits in the topos $\widehat{\mathcal{C}}$ will not be amazed to meet an opposite conclusion in the case of limits.

Counterexample 9 *The weighted $\widehat{\mathcal{C}}$-limit of an $\widehat{\mathcal{C}}$-valued $\widehat{\mathcal{C}}$-enriched functor is generally not computed pointwise.*

Proof First let us observe why the argument in Proposition 8 does not transpose to the case of weighted limits. In the limit case one would have to consider the end

$$\lim_W F = \int_{D\in\mathcal{D}} \widehat{\mathcal{C}}\big(W(D),F(D)\big).$$

This end can be expressed as an ordinary conical limit of "cotensors". The conical limit is of course computed pointwise as in Set, but the "cotensor" $\widehat{\mathcal{C}}\big(W(D),F(D)\big)$ is the exponentiation in the topos $\widehat{\mathcal{C}}$, which is by no means computed pointwise as in Set.

To construct an explicit counterexample, simply take for $\mathcal{D}$ the "unit" $\widehat{\mathcal{C}}$-category, that is the category with a single object D and $\mathcal{D}(D,D)$ the terminal presheaf. In that case W and F reduce to two objects of $\widehat{\mathcal{C}}$ and the limit reduces to the cotensor $[W,F]$, that is, the exponentiation F^W in the topos $\widehat{\mathcal{C}}$. On the other hand, for every object $C \in \mathcal{C}$, $\mathsf{ev}_C(\mathcal{D})$ is the terminal category and $\mathsf{ev}_C(W)$, $\mathsf{ev}_C(F)$ reduce to giving the two sets $W(C)$, $F(C)$. The corresponding limit $\lim_{\mathsf{ev}_C(W)} \mathsf{ev}_C(F)$ is thus simply the exponentiation $F(C)^{W(C)}$ in Set.

By the Yoneda lemma, for every object $C \in \widehat{\mathcal{C}}$

$$F^W(C) = \mathsf{Nat}\big(\mathcal{C}(-,C), F^W\big) = \mathsf{Nat}\big(\mathcal{C}(-,C) \times W, F\big)$$

where Nat indicates here the sets of natural transformations. It is clear that in general, this is not the same as

$$F(C)^{W(C)} = \mathsf{Set}\big(W(C), F(C)\big). \qquad \square$$

This difficulty in the case of $\widehat{\mathcal{C}}$-limits is somehow the motivation for the present paper.

3 Flat weights

We keep assuming that the topos $\widehat{\mathcal{C}}$ of presheaves is a locally finitely presentable base. We borrow the following definitions from [7] and [4].

Definition 10 *By a finite weighted $\widehat{\mathcal{C}}$-limit is meant a limit weighted by a $\widehat{\mathcal{C}}$-functor $W : \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$ such that:*

1. *$\mathcal{D}$ has finitely many objects;*
2. *for all $D, D' \in \mathcal{D}$, $\mathcal{D}(D,D') \in \widehat{\mathcal{C}}$ is finitely presentable;*
3. *for all $D \in \mathcal{D}$, $W(D) \in \widehat{\mathcal{C}}$ is finitely presentable.*

Definition 11 *A flat $\widehat{\mathcal{C}}$-weight is a contravariant functor $W : \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$ whose $\widehat{\mathcal{C}}$-Kan extension along the $\widehat{\mathcal{C}}$-Yoneda embedding*

$$\mathsf{Lan}_Y W : [\mathcal{D}, \widehat{\mathcal{C}}] \longrightarrow \widehat{\mathcal{C}}$$

preserves finite weighted $\widehat{\mathcal{C}}$-limits. A filtered weighted $\widehat{\mathcal{C}}$-colimit is one whose weight is flat.

It is well-known that in the case of sets (i.e. when $\mathcal{C}$ is the terminal category), our definition 11 coincides with the usual notion of flat functor. This notion, in the case of sets, is particularly handy since it is equivalent to the category of elements of the weight being filtered in the usual sense.

Now since finite weighted $\widehat{\mathcal{C}}$-limits are generally not computed pointwise, being a flat $\widehat{\mathcal{C}}$-weight and thus being a filtered weighted $\widehat{\mathcal{C}}$-colimit has a priori no reason to be a pointwise notion. The main result of this paper will show that when $\widehat{\mathcal{C}}$ is a locally finitely presentable base, $\widehat{\mathcal{C}}$-flatness reduces to pointwise Set-flatness.

Let us at once observe that $\widehat{\mathcal{C}}$-flatness implies pointwise flatness.

Proposition 12 *Suppose that the topos $\widehat{\mathcal{C}}$ of presheaves is a finitely presentable base. Then every flat $\widehat{\mathcal{C}}$-weight is pointwise flat.*

Proof Given a $\widehat{\mathcal{C}}$-flat weight $W: \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$, its Kan extension along the $\widehat{\mathcal{C}}$-Yoneda embedding preserves finite weighted $\widehat{\mathcal{C}}$-limits. Therefore it preserves ordinary finite conical limits, simply because ordinary finite limits can be presented as finite weighted $\widehat{\mathcal{C}}$-limits.

Indeed, write $\overline{\mathcal{F}}$ for the free $\widehat{\mathcal{C}}$-category generated by a finite category $\mathcal{F}$: same objects, with $\overline{\mathcal{F}}(X,Y) = \amalg_{\mathcal{F}(X,Y)}\mathbf{1}$. Given a $\widehat{\mathcal{C}}$-category $\mathcal{A}$, the limit of an ordinary functor $F: \mathcal{F} \longrightarrow \mathcal{A}$ is the weighted limit of the $\widehat{\mathcal{C}}$-factorization $\overline{F}: \overline{\mathcal{F}} \longrightarrow \mathcal{A}$ weighted by the constant $\widehat{\mathcal{C}}$-functor $\Delta_\mathbf{1}$ on the terminal presheaf $\mathbf{1}$. But $\mathbf{1}$ is finitely presentable since $\widehat{\mathcal{C}}$ is a finitely presentable base. Thus $\Delta_\mathbf{1}$ is indeed a finite $\widehat{\mathcal{C}}$-weight.

Now conical finite limits are computed pointwise both in $[\mathcal{D}^{\mathrm{op}}, \widehat{\mathcal{C}}]$ and in $\widehat{\mathcal{C}}$, proving that in Set, $\mathrm{ev}_C(W)$-weighted colimits commute with finite limits. This proves that each $\mathrm{ev}_C(W)$ is flat. □

4 Pointwise form of weighted limits

Once more the results of this section are valid for an arbitrary small category $\mathcal{C}$, even if we shall need them only in the case where the topos $\widehat{\mathcal{C}}$ of presheaves is a finitely presentable base.

We want this time to investigate the pointwise form of a weighted $\widehat{\mathcal{C}}$-limit $\lim_W F$, that is, the value of the set $(\lim_W F)(C)$ for an object $C \in \mathcal{C}$. By counter-example 9 we know already that in general, this is not $\lim_{\mathrm{ev}_C(W)} \mathrm{ev}_C(F)$.

The case of internal limits in a topos of presheaves on a topological space somehow guides the intuition to handle this question. The internal limit of an internal diagram at some level U is constituted of those families which are compatible along the diagram, not only at the level U, but also at every lower level $V \subseteq U$. And at the level V, there can indeed appear in the diagram some morphisms which are not restrictions of morphisms at the level U.

Construction 13 *Given a covariant $\widehat{\mathcal{C}}$-functor $W : \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$ on a small $\widehat{\mathcal{C}}$-category $\mathcal{D}$, we construct a covariant functor*

$$\mathcal{D}_W : \mathcal{C} \longrightarrow \mathsf{Cat}$$

where Cat *indicates the category of small categories.*

Proof Let us fix an object $C_0 \in \mathcal{C}$.

1. The objects of $\mathcal{D}_W(C_0)$ are the quadruples (C, f, D, x) where
   - $f : C \longrightarrow C_0$ in $\mathcal{C}$,
   - $x \in W(D)(C)$ with $D \in \mathcal{D}$.
2. A morphism is a pair

$$(C, f, D, x) \xrightarrow{(\varphi, \delta)} (C', f', D', x')$$

   where

   - $\varphi : C' \longrightarrow C$ in $\mathcal{C}$;

$$\begin{tikzcd}
& C_0 & \\
C \arrow[ur, "f"] & & C' \arrow[ul, "f'"'] \arrow[ll, "\varphi"]
\end{tikzcd}$$

   - $\delta : D \longrightarrow D'$ in $\mathsf{ev}_{C'}(\mathcal{D})$, i.e. $\delta \in \mathcal{D}(D, D')(C')$;
   - $f \circ \varphi = f'$;
   - $x' = W(\delta)(\varphi)(x)$
     where for simplicity we have written $W(\delta)(\phi)$ to indicate the composite

$$W(D)(C) \xrightarrow{W(D)(\varphi)} W(D)(C') \xrightarrow{\mathsf{ev}_{C'}(W)(\delta)} W(D')(C').$$

3. The composition is that induced by the compositions of $\mathcal{C}$ and $\mathcal{D}$: given

$$(C,f,D,x)\xrightarrow{(\varphi,\delta)}(C',f',D',x')\xrightarrow{(\varphi',\delta')}(C'',f'',D'',x'')$$

their composite is

$$(\varphi',\delta')\circ(\varphi,\delta)=\big(\varphi\circ\varphi',\delta'\circ\mathcal{D}(D,D')(\varphi)(\delta)\big)$$

where indeed

$$\mathcal{D}(D,D')(\varphi')\colon\mathcal{D}(D,D')(C')\longrightarrow\mathcal{D}(D,D')(C''),$$
$$\delta\mapsto\mathcal{D}(D,D')(\varphi')(\delta)$$

with now in $\mathrm{ev}_{C''}(\mathcal{D})$

$$D\xrightarrow{\mathcal{D}(D,D')(\varphi')(\delta)}D'\xrightarrow{\quad\delta'\quad}D''.$$

It is routine to observe that we have so defined a category $\mathcal{D}_W(C_0)$.

Consider now a morphism $\gamma\colon C_0\longrightarrow C_1$ in $\mathcal{C}$. We define a functor

$$\mathcal{D}_W(\gamma)\colon\mathcal{D}_W(C_0)\longrightarrow\mathcal{D}_W(C_1)$$

in the following way:

1. $\mathcal{D}_W(\gamma)(C,f,D,x)=(C,\gamma\circ f,D,x)$

$$\begin{tikzcd} C_0 \arrow[rr, "\gamma"] & & C_1 \\ & C \arrow[ul, "f"] \arrow[ur, "\gamma\circ f"'] & \end{tikzcd} \qquad x\in W(D)(C)$$

2. given

$$(C,f,D,x)\xrightarrow{(\varphi,\delta)}(C',f',D',x')$$

in $\mathcal{D}_W(C_0)$, we define

$$\mathcal{D}_W(\gamma)(\varphi,\delta)=(\varphi,\delta)\colon(C,\gamma\circ f,D,x)\longrightarrow(C',\gamma\circ f',D',x')$$

$$\begin{tikzcd}
C_0 \arrow[rr, "\gamma"] & & C_1 \\
C \arrow[u, "f"] \arrow[urr, "\gamma \circ f"'] & & C' \arrow[ll, "\varphi"] \arrow[ull, "f'"] \arrow[u, "\gamma \circ f'"']
\end{tikzcd}$$

Checking the details is routine. □

Construction 14 *Given two covariant $\widehat{\mathcal{C}}$-functors*

$$W\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}, \quad F\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$$

on a small $\widehat{\mathcal{C}}$-category $\mathcal{D}$, we construct for each object $C_0 \in \mathcal{C}$ a covariant functor

$$F_{C_0}\colon \mathcal{D}_W(C_0) \longrightarrow \mathsf{Set}$$

such that for every morphism $\gamma\colon C_0 \longrightarrow C_1$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}_W(C_0) \arrow[rr, "\mathcal{D}_W(\gamma)"] \arrow[dr, "F_{C_0}"'] & & \mathcal{D}_W(C_1) \arrow[dl, "F_{C_1}"] \\
& \mathsf{Set} &
\end{tikzcd}$$

Proof It suffices to define

1. given $(C, f, D, x) \in \mathcal{D}_W(C_0)$

$$F_{C_0}(C, f, D, x) = F(D)(C);$$

2. given $(\varphi, \delta)\colon (C, f, D, x) \longrightarrow (C', f', D', x')$ in $\mathcal{D}_W(C_0)$

$$F_{C_0}(\varphi, \delta)\colon F(D)(C) \longrightarrow F(D')(C')$$

is the composite

$$F(D)(C) \xrightarrow{F(D)(\varphi)} F(D)(C') \xrightarrow{\mathsf{ev}_{C'}(F)(\delta)} F(D')(C').$$

Checking the functoriality is routine. Notice that trivially, given a morphism $\gamma\colon C_0 \longrightarrow C_1$ in $\mathcal{C}$,

$$\begin{aligned}\big(F_{C_1} \circ \mathcal{D}_W(\gamma)\big)(C,f,D,x) &= F_{C_1}(C,\gamma \circ f, D, x)\\ &= F(D)(C)\\ &= F_{C_0}(C,f,D,x).\end{aligned}$$

This proves the result. □

The whole point about these constructions is then:

Proposition 15 *Given two covariant $\widehat{\mathcal{C}}$-functors*

$$W\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}, \quad F\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$$

on a small $\widehat{\mathcal{C}}$-category $\mathcal{D}$, with the notation of Constructions 13 and 14, one has

$$(\mathsf{lim}_W\, F)(C) = \mathsf{lim}\, F_C$$

while given $\gamma\colon C_0 \longrightarrow C_1$ in $\mathcal{C}$

$$(\mathsf{lim}_W\, F)(\gamma)\colon\ \mathsf{lim}\, F_{C_1} \longrightarrow \mathsf{lim}\, F_{C_0}$$

is the canonical factorization through the limits induced by the functor

$$\mathcal{D}_W(\gamma)\colon \mathcal{D}_W(C_0) \longrightarrow \mathcal{D}_W(C_1).$$

Proof Let us put $L(C) = \mathsf{lim}\, F_C$. The commutative triangle $F_{C_1} \circ \mathcal{D}_W(\gamma) = F_{C_0}$ induces at once a factorization $\mathsf{lim}\, F_{C_1} \longrightarrow \mathsf{lim}\, F_{C_0}$ which we define to be $L(\gamma)$.

$$\begin{tikzcd} \mathsf{lim}\, F_{C_1} \arrow[rr, dotted, "L(\gamma)"] \arrow[dr, "p_{(C,\gamma\circ f,D,x)}"'] & & \mathsf{lim}\, F_{C_0} \arrow[dl, "p_{(C,f,D,x)}"] \\ & F(D)(C) & \end{tikzcd}$$

This yields a presheaf $L \in \widehat{\mathcal{C}}$ and we must prove that $L = \mathsf{lim}_W\, F$.

The weighted limit $\mathsf{lim}_W\, F$ is given by the end

$$\mathsf{lim}_W\, f = \textstyle\int_D \big[W(D), F(D)\big].$$

This end is the ordinary conical limit of a diagram constituted of co-spans,

$$\begin{tikzcd}[column sep=small] \big[W(D), F(D)\big] \arrow[dr] & & \big[W(D'), F(D')\big] \arrow[dl] \\ & \big[\mathcal{D}(D, D'), [W(D), F(D')]\big] & \end{tikzcd}$$

one for each pair of objects $D, D' \in \mathcal{D}$, with the two morphisms of the co-span induced respectively by the actions of F and W on the arrows. This ordinary conical limit is computed pointwise in $\widehat{\mathcal{C}}$ and we know that

$$\big[W(D), F(D)\big](C_0) = \mathsf{Nat}\big(\mathcal{C}(-, C_0) \times W(D), F(D)\big)$$

where Nat indicates the set of natural transformations. But such a natural transformation

$$\alpha_D \colon \mathcal{C}(-, C_0) \times W(D) \Longrightarrow F(D)$$

consists in giving, for each object $C \in \mathcal{C}$, each arrow $f \in \mathcal{C}(C, C_0)$ and each element $x \in W(D)(C)$, a corresponding element in $F(D)$. The conical limit of the diagram of co-spans above consists then in giving a compatible family of such natural transformations, that is, a family of elements in the various $F_{C_0}(C, f, D, x)$, for all objects of $\mathcal{D}_W(C_0)$. It remains to observe — and this is routine computation — that the compatibility condition on the natural transformations α_D, together with the naturality of each α_D, is equivalent to the compatibility of the corresponding family of elements along the functor F_{C_0} on $\mathcal{D}_W(C_0)$, that is, to giving an element of $L(C_0) = \lim F_{C_0}$. □

5 Filtered weighted colimits

We restrict again our attention to the case of a finitely presentable base $\widehat{\mathcal{C}}$. We want first to prove that for a finite $\widehat{\mathcal{C}}$-weight W, the diagrams $\mathcal{D}_W(C)$ of Construction 13 are "essentially finite". For that, we first need a lemma.

Lemma 16 *Given a finitely presentable presheaf $P \in \widehat{\mathcal{C}}$,*

1. *there exist finitely many objects $C_1, \ldots, C_n$ in $\mathcal{C}$ and corresponding elements $x_i \in P(C_i)$ such that for each object $C \in \mathcal{C}$ and each element $x \in P(C)$, there exists an index i and an arrow $f \colon C \longrightarrow C_i$ such that $x = P(f)(x_i)$;*

2. *there exist finitely many objects $A_1, \dots, A_m$ and corresponding pairs of arrows*

$$C_{i_{(\alpha,j)}} \xleftarrow{\alpha_j} A_j \xrightarrow{\beta_j} C_{i_{(\beta,j)}}$$

such that

- $P(\alpha_j)(x_{i_{(\alpha,j)}}) = P(\beta_j)(x_{i_{(\beta,j)}})$;
- *with the notation of condition 1, given an arbitrary span in $\mathcal{C}$*

$$C_i \xleftarrow{\varphi} C \xrightarrow{\psi} C_{i'}$$

if $P(\varphi)(x_i) = P(\psi)(x_{i'})$, there exists a commutative "zig-zag" in $\mathcal{C}$

$$\begin{tikzcd}
 & C_{i_1} & C_{i_2} & \cdots & C_{i_{n-2}} & C_{i_{n-1}} & \\
 & & & \cdots & & & \\
C_i = C_{i_0} & A_{j_1} \arrow[l] \arrow[uu] & A_{j_2} \arrow[uul] \arrow[uu] & \cdots & A_{j_{n-1}} \arrow[uu] \arrow[uur] & A_{j_n} \arrow[uu] \arrow[r] & C_{i_n} = C_{i'} \\
 & & & \cdots & & & \\
 & & & C \arrow[uulll, "\varphi"] \arrow[uull] \arrow[uul] \arrow[uur] \arrow[uurr] \arrow[uurrr, "\psi"'] & & &
\end{tikzcd}$$

with each span

$$C_{i_{k-1}} \longleftarrow A_{j_k} \longrightarrow C_{i_k}$$

having the form $(\alpha_{j_k}, \beta_{j_k})$ or $(\beta_{j_k}, \alpha_{j_k})$.

Proof This follows at once from the fact that P is a coequalizer of the form

$$\coprod_{j=1}^{m} \mathcal{C}(-, A_j) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{i=1}^{n} \mathcal{C}(-, C_i) \xrightarrow{\chi} P.$$

This coequalizer, as well as the two finite coproducts, are computed pointwise as in Set. And these coproducts are disjoint. Giving χ is equivalent to give

$$\chi_i \colon \mathcal{C}(-, C_i) \longrightarrow P$$

for each index i, that is, by the Yoneda lemma, some $x_i \in P(C_i)$. The first condition expresses exactly the fact that χ_C is surjective.

Next giving α is equivalent to giving for each index j

$$\mathcal{C}(-,A_j) \xrightarrow{s_j} \coprod_{j=1}^{m} \mathcal{C}(-,A_j) \xrightarrow{\alpha} \coprod_{i=1}^{n} \mathcal{C}(-,C_i).$$

By disjointness of the second coproduct and the Yoneda lemma, this is equivalent to giving some α_j in some $\mathcal{C}(A_j, C_i)$. An analogous argument holds for β. Now $P(\varphi)(x_i) = P(\psi)(x_{i'})$ can be rephrazed as $\chi_{i,C}(\varphi) = \chi_{i',C}(\psi)$, that is as $\chi_C(\varphi) = \chi_C(\psi)$. The zig-zag condition of the statement follows then from the well-known construction of the equivalence relation yielding the coequalizer $\chi_C = \mathsf{Coker}\,(\alpha_C, \beta_C)$ in Set. □

Proposition 17 *When $\widehat{\mathcal{C}}$ is a locally finitely presentable base and the functor $W\colon \mathcal{D} \longrightarrow$ is a finite $\widehat{\mathcal{C}}$-weight, each category $\mathcal{D}_W(C)$ as in Construction 13 is essentially finite, that is, there exists a finite diagram $\mathcal{F}_W(C_0) \subseteq \mathcal{D}_W(C)$ such that the limit of an arbitrary functor $H\colon \mathcal{D}_W(C_0) \longrightarrow \mathsf{Set}$ can be equivalently computed on $\mathcal{F}_W(C_0)$.*

Proof Consider a functor H as in the statement; its limit is the set of compatible families

$$a_{(C,f,D,x)} \in H(C,f,D,x), \quad (C,f,D,x) \in \mathcal{D}_W(C_0).$$

We shall step by step construct $\mathcal{F}_W(C_0) \subseteq \mathcal{D}_W(C_0)$, the expected finite subdiagram.

We use freely the notation of Construction 13, for the fixed object $C_0 \in \mathcal{C}$. We consider the presheaf

$$P = \mathcal{C}(-,C_0) \times \left(\coprod_{D\in\mathcal{D}} W(D) \right) \times \left(\coprod_{D',D''\in\mathcal{D}} \mathcal{D}(D',D'') \right).$$

Since $\mathcal{D}$ has finitely many objects while each $W(D)$ and each $\mathcal{D}(D,D')$ are finitely presentable, the fact that $\widehat{\mathcal{C}}$ is a finitely presentable base implies that P is a finitely presentable presheaf. As in Lemma 16 we have thus a coequalizer

$$\coprod_{j=1}^{m} \mathcal{C}(-,A_j) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{i=1}^{n} \mathcal{C}(-,C_i) \xrightarrow{\chi} P.$$

But since $\widehat{\mathcal{C}}$ is cartesian closed, we have also

$$P = \coprod_{D,D',D''\in\mathcal{D}} \mathcal{C}(-,C_0) \times W(D) \times \mathcal{D}(D',D'').$$

This coproduct is computed as in Set so that by disjointness, each composite

$$\chi_i \colon \mathcal{C}(-,C_i) \xrightarrow{\ s_i\ } \coprod_{i=1}^{n} \mathcal{C}(-,C_i) \xrightarrow{\ \chi\ } P$$

factors through exactly one of the terms of the coproduct. Thus for each index i, there are uniquely determined objects D_i, D_i', D_i'' such that $\chi_i = \chi \circ s_i$ factors through

$$\mathcal{C}(-,C_0) \times W(D_i) \times \mathcal{D}(D_i',D_i'').$$

The identity on C_i is then mapped by χ_i on a triple

$$f \colon C_i \longrightarrow C_0, \quad x_i \in W(D_i)(C_i), \quad \delta_i \in \mathcal{D}(D_i',D_i'')(C_i).$$

Now given an arbitrary object $(C,f,D,x) \in \mathcal{D}_W(C_0)$, we have in particular

$$(f,x,\mathsf{id}_D^C) \in \mathcal{C}(C,C_0) \times W(D)(C) \times \mathcal{D}(D,D)(C)$$

where id_D^C is the identity morphism on D in $\mathsf{ev}_C(\mathcal{D})$. Applying our Lemma 16 we get the existence of an index i and an arrow $\varphi \colon C \longrightarrow C_i$ such that

$$P(\varphi)(f_i,x_i,\delta_i) = (f,x,\mathsf{id}_D^C).$$

But by disjointness of the coproduct above, this forces both triples to be in the same term of the coproduct. This means in particular $D_i = D_i' = D_i'' = D$. We can thus rephrase the equality above as the existence of a morphism

$$(\varphi,\mathsf{id}_D^C) \colon (C_i,f_i,D_i,x_i) \longrightarrow (C,f,D,x)$$

in $\mathcal{D}_W(C_0)$. We first put in our diagram $\mathcal{F}_W(C_0)$ all the finitely many objects (C_i,f_i,D_i,x_i).

Let us switch a moment back to $\lim H$. With the notation above, by compatibility of the family $a_{(C,f,D,x)}$, we have

$$a_{(C,f,D,x)} = H\big(\varphi,\mathsf{id}_D^C\big)\big(a_{(C_i,f_i,D,x_i)}\big).$$

This proves already that

A compatible family

$$a_{(C,f,D,x)} \in H(C,f,D,x), \quad (C,f,D,x) \in \mathcal{D}_W(C_0)$$

is entirely determined by its components

$$a_{(C_i,f_i,D_i,x_i)} \in H(C_i,f_i,D_i,x_i), \quad i = 1,\dots,n$$

This fact will of course be preserved when enlarging further the diagram $\mathcal{F}_W(C_0)$.

Still with the notation of Lemma 16, let us now consider the diagram

$$\begin{tikzcd}
\mathcal{C}(-,A_j) \arrow[r, "s_j"] & \coprod_j \mathcal{C}(-,A_j) \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & & \mathcal{C}(-,C_0) \\
& \coprod_i \mathcal{C}(-,C_i) \arrow[r, two heads, "\chi"] & P \arrow[ur, "p_1"] \arrow[r, "p_2"] \arrow[dr, "p_3"'] & \coprod_D W(D) \\
& & & \coprod_{D',D''} \mathcal{D}(D',D'')
\end{tikzcd}$$

Again by disjointness of the coproducts the composites with p_2 and p_3 factor through a unique term of the coproducts. This determines thus objects $E_j, E'_j, E''_j \in \mathcal{D}$ together with

$$g_j\colon A_j \longrightarrow C_0,\quad y_j \in W(E_j)(A_j),\quad \varepsilon_j \in \mathcal{D}(E'_j, E''_j)(A_j).$$

But again by disjointness of the coproducts, $\alpha \circ s_j$ factors through a unique term $\mathcal{C}(-, C_{i_{(\alpha,j)}})$, so that necessarily $E_j = C_{i_{(\alpha,j)}}$. An analogous argument holds for β, proving eventually that

$$D_{i_{(\alpha,j)}} = E_j = D_{i_{(\beta,j)}}$$

And still an analogous argument shows that

$$D'_{i_{(\alpha,j)}} = E'_j = D'_{i_{(\beta,j)}},\quad D''_{i_{(\alpha,j)}} = E''_j = D''_{i_{(\beta,j)}}.$$

This gives in particular the following situation in $\mathcal{D}_W(C_0)$

$$\begin{tikzcd}
\left(C_{i_{(\alpha,j)}}, f_{i_{(\alpha,j)}}, D_{i_{(\alpha,j)}}, x_{i_{(\alpha,j)}}\right) \arrow[dr, "{(\alpha_j, \mathrm{id}_{E_j})}"'] & & \left(C_{(i_{\beta,j})}, f_{i_{(\beta,j)}}, D_{i_{(\beta,j)}}, x_{i_{(\beta,j)}}\right) \arrow[dl, "{(\beta_j, \mathrm{id}_{E_j})}"] \\
& (A_j, g_j, E_j, y_j) &
\end{tikzcd}$$

Let us add all those finitely many objects and arrows to our diagram $\mathcal{F}_W(C_0)$.

Let us now make a second detour to our limit $\lim H$. With this extended version of $\mathcal{F}_W(C_0)$, we are now already able to prove that:

Given a compatible family of elements

$$a_{(C,f,D,x)} \in H(C,f,D,x), \quad (C,f,D,x) \in \mathcal{F}_W(C_0)$$

there exists a unique way to extend it into a family of elements

$$a_{(C,f,D,x)} \in H(C,f,D,x), \quad (C,f,D,x) \in \mathcal{D}_W(C_0)$$

which is further compatible along all the morphisms of the form

$$(\theta, \mathrm{id}^C_{D'}) \colon (C,f,D,x) \longrightarrow (C',f',D',x').$$

We have already seen that given $(C,f,D,x) \in \mathcal{D}_W(C_0)$ we can find an index i and an arrow $(\varphi, \mathrm{id}^C_D)$ as above; necessarily, to satisfy our intermediate statement, we must define

$$a_{(C,f,D,x)} = H(\varphi, \mathrm{id}^C_D)\big(a_{(C_i,f_i,D_i,x_i)}\big).$$

Let us first make sure that this definition is independent of the choice of the index i and the morphism φ.

Let us thus assume to have

$$(C_i,f_i,D_i,x_i) \xleftarrow{(\varphi,\mathrm{id}^C_D)} (C,f,D,x) \xrightarrow{(\psi,\mathrm{id}^C_D)} (C_{i'},f_{i'},D_{i'},x_{i'}).$$

This can be rephrased as having

$$P(\varphi)(f_i,D_i,x_i,\delta_i) = (f,D,x,\mathrm{id}^C_D) = P(\psi)(f_{i'},D_{i'},x_{i'},\delta_{i'}).$$

Condition 2 in Lemma 16 yields then a corresponding commutative zig-zag in $\mathcal{C}$. Keeping in mind the various equalities like $E_j = C_{i_{(\alpha,j)}}$, and so on, it is easy to observe that this zig-zag can be seen as the "first component" of a corresponding commutative zig-zag in $\mathcal{D}_W(C_0)$. Simply replace

- each object C_k by $(C_k, f_k, D_k, \delta_k)$;
- each object A_l by (A_l, g_l, E_l, y_l);
- the object C by (C,f,D,x);

- each arrow θ by (θ, id).

Of course, the direction of each arrow has now been reversed. All the morphisms corresponding to the upper part of the zig-zag in Lemma 16 are now in our extended diagram $\mathcal{F}_W(C)$; therefore the compatibility of the various

$$a_{(C_k,f_k,D_k,x_k)} \in H(C_k, f_k, D_k, x_k)$$

along this upper zig-zag, together with the commutativity of the bottom part of the diagram, implies that all the $a_{(C_k,f_k,D_k,x_k)}$ are mapped on the same element of $H(C, f, D, x)$. In particular,

$$H(\varphi, \mathrm{id}^C_D)\big(a_{(C_i,f_i,D_i,\delta_i)}\big) = H(\psi, \mathrm{id}^C_D)\big(a_{(C_{i'},f_{i'},D_{i'},\delta_{i'})}\big)$$

as required.

The proof of our intermediate statement follows at once. Since the definition of $a_{(C',f',D',x')}$ is independent of the various choices that one can make, simply choose to define it the composite

$$(C_i, f_i, D_i, x_i) \xrightarrow{(\varphi, \mathrm{id})} (C, f, D, x) \xrightarrow{(\theta, \mathrm{id})} (C', f', D', x');$$

this makes the conclusion trivial.

But this is not yet the end of the story: we must still make sure that the extended family of elements in H is compatible along *all* the morphisms of $\mathcal{D}_W(C_0)$. To achieve this,it is necessary to extend once again our finite diagram $\mathcal{F}_X(C_0)$.

Notice that an arbitrary morphism of $\mathcal{D}_W(C_0)$ can always be factored as

$$\begin{tikzcd}
(C, f, D, x) \arrow[rr, "{(\theta, \delta)}"] \arrow[dr, "{(\theta, \mathrm{id})}"'] & & (C', f', D', x') \\
& (C', f', D, W(D)(\theta)(x)) \arrow[ur, "{(\mathrm{id}, \delta)}"'] &
\end{tikzcd}$$

Since we ensured already the compatibility along (θ, id), it remains to take care of the compatibility along (id, δ).

Consider thus a morphism of $\mathcal{D}_W(C_0)$ of the form

$$(\mathrm{id}_C, \delta)\colon (C, f, D, x) \longrightarrow (C, f, D', x').$$

We have in particular

$$(f, x, \delta) \in \mathcal{C}(C, C_0) \times W(D)(C) \times \mathcal{D}(D, D')(C) \subseteq P(C).$$

Thus we have an index i and a morphism $\varphi\colon C \longrightarrow C_i$ such that

$$P(\varphi)(f_i, x_i, \delta_i) = (f, x, \delta).$$

Once more by disjointness of the coproduct describing P, necessarily,

$$D_i = D, \quad \delta_i \in \mathcal{D}(D, D')(C_i).$$

We obtain in this way the following commutative square in $\mathcal{D}_W(C_0)$:

$$\begin{tikzcd}
(C_i, f_i, D_i, x_i) \arrow[r, "{(\mathsf{id}, \delta_i)}"] \arrow[d, "{(\varphi, \mathsf{id})}"'] & \big(C_i, f_i, D', \mathsf{ev}_{C_i}(W(D'))(\delta_i)(x_i)\big) \arrow[d, "{(\varphi, \mathsf{id})}"] \\
(C, f, D, x) \arrow[r, "{(\mathsf{id}, \delta)}"'] & (C, f, D', x')
\end{tikzcd}$$

We conclude the construction of our finite diagram $\mathcal{F}_W(C_0)$ by adding all the

$$(C_i, f_i, D_i, x_i) \xrightarrow{(\mathsf{id}, \delta_i)} \big(C_i, f_i, D', \mathsf{ev}_{C_i}(W(D'))(\delta_i)(x_i)\big)$$

for all indices i.

It is now obvious to conclude our proof. With our final definition of $\mathcal{F}_W(C_0)$, a family

$$a_{(C,f,D,x)} \in H(C, f, D, x), \quad (C, f, D, x) \in \mathcal{F}_W(C_0)$$

can thus be extended in a unique way in a family

$$a_{(C,f,D,x)} \in H(C, f, D, x), \quad (C, f, D, x) \in \mathcal{D}_W(C_0)$$

which is compatible along all the morphisms (φ, id), as we already known, but also along all the morphism (id, δ_i), since these are now in the diagram $\mathcal{F}_W(C_0)$. By commutativity of the square above, the family is also compatible along the bottom morphism (id, δ). □

Theorem 18 *When the topos $\widehat{\mathcal{C}}$ of presheaves on a small category $\mathcal{C}$ is a locally finitely presentable base, the following conditions are equivalent, given a small $\widehat{\mathcal{C}}$-enriched category $\mathcal{D}$:*

1. *the $\widehat{\mathcal{C}}$-functor $W\colon \mathcal{D} \longrightarrow \widehat{\mathcal{C}}$ is $\widehat{\mathcal{C}}$-flat;*
2. *for every object $C \in \mathcal{C}$, the functor $\mathsf{ev}_C(W)\colon \mathsf{ev}_C(\mathcal{D}) \longrightarrow \mathsf{Set}$ is flat, that is, its category of elements is filtered.*

In other words, the filtered $\widehat{\mathcal{C}}$-colimits $\mathsf{colim}_W F$ are those which are pointwise filtered in Set.

Proof One implication is attested by Proposition 12. The other one follows at once from Propositions 15 and 17. □

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