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THE MALTSEV OPERATION ON EXTENSIONS

Dedicated to Jiří Adámek on the occasion of his sixtieth birthday

by Dominique BOURN

Abstract

Etant donnée une catégorie pointée protomodulaire $\mathbb{C}$ possèdant des classifiants d'extensions scindées, nous donnons une description explicite de l'opération de Mal'tsev associée à l'action simplement transitive sur les extensions $Ext_{\phi}(Y,K)$ déterminée par le théorème de Schreier-Mac Lane. Elle apparait naturellement comme une somme fibrée le long d'une opération de Mal'tsev interne.

Introduction

Following Schreier [18] and Mac Lane [16], we know that any extension of groups:

$$1 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y \longrightarrow 1$$

determines, via conjugation in the group X, a group homomorphism $\phi : Y \to AutK/IntK$, called the *abstract kernel* of this extension, which allows to recover the whole set $Ext_{\phi}(Y,K)$ of isomorphic classes of extensions with this abstract kernel ϕ. For that, there was shown that there is a simply transitive action of the abelian group $Ext_{\bar{\phi}}(Y,ZK)$ on this set $Ext_{\phi}(Y,K)$, where ZK denotes the center of K and the ZK-module structure $\bar{\phi}$ is given by the restriction $\bar{\phi}(y)$ of the automorphism $\phi(y)$ to ZK.

Thanks to the recent introduction of the notion of *split extension classifier* (see [6] and [7]), the previous result on extensions was fully extended (see [11]) to any pointed protomodular category $\mathbb{C}$ with split extension classifiers, provided it is exact (in the sense of Tierney (see [1] I.3.1)). The proof of this result intensively needed the use of internal profunctors [2] and a re-interpretation of the classical Schreier-Mac Lane extension theorem in terms of action of profunctors.

On the other hand, any simply transitive action of a group G on a set X is equivalent to an associative Mal'tsev operation on the set X, namely, a ternary operation $p : X \times X \times X \to X$ such that $p(x, y, y) = x, p(x, x, y) = y$ [17] and

$$p(p(x, y, z), u, v) = p(x, y, p(z, u, v)) \quad \text{(associativity)}$$

provided the direction of this Mal'tsev operation p is the group G in question [9], see Section 1 below.

The modest aim of this article is to re-translate the profunctorial results of [11] in terms of extensions and to make a diagrammatic explicit description of the Mal'tsev operation associated with the simply transitive action on extensions with abstract kernel ϕ recalled above. The pleasant aspect of this description is that it is actually a pushout along an internal Mal'tsev operation.

1 Mal'tsev operation and group action

Let X be a set endowed with an associative Mal'tsev operation p. Recall from [9]:

Definition 1.1. *The Chasles relation R_p associated with the Mal'tsev law p is the relation on the set $X \times X$ defined by:*

$$(x, t)R_p(y, z) \quad \Leftrightarrow \quad t = p(x, y, z)$$

When p is associative, R_p is an equivalence relation. The quotient $(X \times X)/R_p$ is called the direction of the associative Mal'tsev operation and denoted by $d_p(X)$. The terminology is clearly inspired by the affine situation. We denote by $\overrightarrow{xy}$ the equivalence class of the pair (x, y).

Then $d_p(X)$ is canonically endowed with a group structure. It is abelian if and only if the Mal'tsev law p is commutative, namely, if and only if we have $p(x, y, z) = p(z, y, x)$. In any case this group has a simply transitive left action on the set X. We get classically:

$$\overrightarrow{xx} = 1 \ , \ \overrightarrow{xy}.\overrightarrow{yz} = \overrightarrow{xz} \text{ and } \overrightarrow{xy}.x = y$$

Conversely any simply transitive action of a group G on a set X produces an associative Matsev operation with direction G given by:

$$p : X \times X \times X \to X \quad (x, y, z) \mapsto g.z \quad \text{with} \quad g.y = x$$

Clearly all the previous observations can be transfered to any finitely complete category $\mathbb{C}$, provided it is exact and the object X has global support. Let us denote by $AM_g(\mathbb{C})$ the category whose objects are the pairs (X, p) where X is an object in $\mathbb{C}$ having global support and p is an associative Mal'tsev operation on X, and whose morphisms are the maps preserving the Mal'tsev operations. Let us denote by $d : AM_g(\mathbb{C}) \to Gp(\mathbb{C})$ the direction functor, where $Gp(\mathbb{C})$ is the category of internal groups in $\mathbb{C}$. The main results in [9] were the following:

Theorem 1.2. *When $\mathbb{C}$ is finitely complete and exact, the functor d is a quasi-cofibration (the existence of cocartesian maps is assured only up to isomorphism). Moreover the functor d reflecting the isomorphisms, any map in $AM_g(\mathbb{C})$ is cocartesian, and any fibre is a groupoid.*

and:

Corollary 1.3. *Given any abelian group A in $\mathbb{C}$, the fibre above A with respect to the direction functor d is canonically endowed with a closed symmetric monoidal structure. Any change of base functor along a homomorphism between abelian groups produces a monoidal functor.*

Proof. The proof is strikingly simple: the tensor and unit functors on the fibre above A are given by the cocartesian maps above the following maps in $Gp(\mathbb{C})$:

$$1 \rightarrowtail^{0} A \xleftarrow{+} A \times A$$

□

Taking into consideration the examples of the following section, the tensor product in the fibres was called *Baer sum*. It gives an abelian group structure to the set of connected components of the fibre above A.

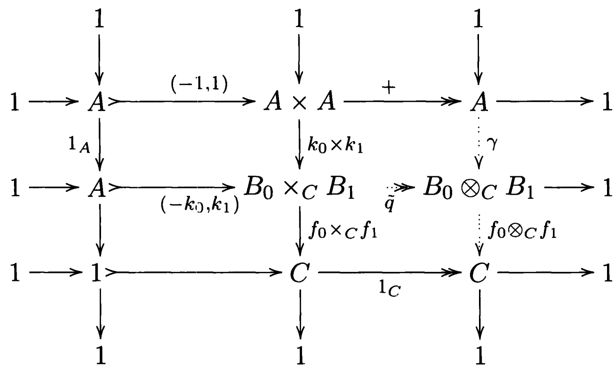
2 Extensions: the classical cases

2.1 The additive case

We know that in any abelian (i.e. additive+exact) category $\mathbb{A}$, there is an abelian group structure on the set $Ext(C, A)$ of all isomorphic classes of extensions between A and C. Starting from two extensions ($i \in \{0, 1\}$):

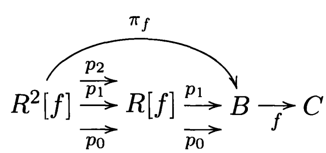
$$1 \longrightarrow A \overset{k_i}{\rightarrowtail} B_i \overset{f_i}{\twoheadrightarrow} C \longrightarrow 1$$

the group operation is given by the following 3×3 Baer sum construction:



where $f_0 \times_C f_1$ is the pullback of f_1 along f_0. In other words, the sum of two extensions is obtained by the pushout along the internal sum $A \times A \xrightarrow{+} A$ of the middle pullback extension.

Actually this is a particular case of the situation described in the first section: in an additive category, any map $f : B \to C$, considered as an objet in the slice category $\mathbb{A}/C$, is canonically endowed with an associative and commutative Mal'tsev operation π_f:



where the previous diagram is made of the iterated kernel relations and follows the simplicial notations, and where we have $\pi_f(xRyRz) = x - y + z$. This object f has a global support in $\mathbb{A}/C$ if and only if its is a regular epimorphism. The direction of this Mal'tsev operation is then the split epimorphism $p_C : A \times C \to C$, where A is the kernel of f. And the abelian group structure on $Ext(C, A)$ is precisely the one given on the set of connected components of the fibre above the internal abelian group $p_C : A \times C \to C$ in the slice category $\mathbb{A}/C$ with respect to the direction functor $d : AM_g(\mathbb{A}/C) \to Gp(\mathbb{A}/C)$. We are, here, in a very specific situation, since the Mal'tsev operation, being canonical, gives rise to a natural transformation, namely, since the category $\mathbb{A}/C$ is Naturally Mal'tsev in the sense of [15].

2.2 Extensions with abelian kernel in the protomodular case

Suppose now $\mathbb{C}$ is only an exact pointed protomodular category. Let be given any exact sequence:

$$1 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y \longrightarrow 1$$

Suppose moreover the map f, seen as an object in the protomodular slice category $\mathbb{C}/Y$, is abelian, namely, suppose this object f is endowed with a (necessarily unique) internal Mal'tsev operation π:

$$\begin{array}{c} \overset{\pi}{\frown} \\ R^2[f] \underset{p_0}{\overset{p_2,\ p_1}{\rightrightarrows}} R[f] \underset{p_0}{\overset{p_1}{\rightrightarrows}} X \xrightarrow[f]{} Y \end{array}$$

Then the Mal'tsev operation π is necesarily associative and conmmutative, the kernel K of f is necessarily an abelian object in $\mathbb{C}$, and the inclusion $K \overset{k}{\rightarrowtail} X \overset{s_0}{\rightarrowtail} R[f]$, where the monomorphism s_0 is given by the reflexivity of the equivalence relation $R[f]$, is necessarily normal. Actually f is abelian if and only if the monomorphism $s_0.k$ is normal, see [12]. Recall [10]:

Definition 2.1. *The direction of this extension is defined as the right hand side split extension given by the following diagram :*

$$\begin{array}{ccccc} & & R[f] & \overset{q_f}{\twoheadrightarrow} & d_X(Y) \\ & \nearrow^{s_0.k} & p_0 \downarrow \uparrow \downarrow p_1 & & d \downarrow \uparrow s \\ K & \underset{k}{\rightarrowtail} & X & \underset{f}{\twoheadrightarrow} & Y \end{array}$$

where q_f is the cokernel of the normal monomorphism $s_0.k$.

The following 3×3 diagram shows that the kernel of the direction d is the abelian object K:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K & \xrightarrow{s_0} & K \times K & \xrightarrow{(-1_K,1_K)} & K & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle 1_K} & & \downarrow{\scriptstyle (k,k)} & & \vdots & & \\
1 & \longrightarrow & K & \xrightarrow{s_0.k} & R[f] & \xrightarrow{q_f} & d_Y(X) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle f.p_0} & & \downarrow{\scriptstyle d} & & \\
1 & \longrightarrow & 1 & \longrightarrow & Y & \xrightarrow[1_Y]{} & Y & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

When $\mathbb{C}$ is the category Gp of groups, then $d_Y(X)$ is nothing other than the classical semi-direct product of the groups Y and K under the action of the group Y on the abelian group K induced by the surjective homomorphism f. In any exact pointed protomodular category $\mathbb{C}$ the previous definition of the direction of an extension with abelian kernel relation coincides with the internalisation in the category $\mathbb{C}/Y$ of the direction construction described in Section 1 in respect with the associative Mal'tsev operation π associated with the abelian object f in $\mathbb{C}/Y$.

Then the set $Ext_d(Y, K)$ of all isomorphic classes of extensions with abelian kernel relation and direction d is still endowed with an abelian group structure. Starting from two extensions ($i \in \{0, 1\}$) with direction d:

$$1 \longrightarrow K \rightarrowtail^{k_i} X_i \xrightarrow{f_i} Y \longrightarrow 1$$

the group operation is given by the following 3×3 Baer sum construction, see [10] Section 3 and Proposition 2.6:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K & \xrightarrow{(-1,1)} & K \times K & \xrightarrow{+} & K & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle 1_K} & & \downarrow{\scriptstyle k_0 \times k_1} & & \vdots{\scriptstyle \gamma} & & \\
1 & \longrightarrow & K & \xrightarrow[(-k_0,k_1)]{} & X_0 \times_Y X_1 & \xrightarrow[\bar{q}]{} & X_0 \otimes_Y X_1 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle f_0 \times_Y f_1} & & \vdots{\scriptstyle f_0 \otimes_Y f_1} & & \\
1 & \longrightarrow & 1 & \longrightarrow & Y & \xrightarrow[1_Y]{} & Y & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

The main difficult point being to show, when the directions of the two extensions are the same, that the monomorphism $(-k_0, k_1)$ is a normal monomorphism. Again the sum of two extensions is obtained by the pushout along the internal sum $K \times K \xrightarrow{+} K$ of the middle pullback extension. And again the abelian group structure on $Ext_d(Y, K)$ is precisely the one given in Section 1 on the set of connected components of the fibre above the internal abelian group $d : d_Y(X) \to Y$ in the slice category $\mathbb{C}/Y$ with respect to the direction functor $AM_g(\mathbb{C}/Y) \to Gp(\mathbb{C}/Y)$.

2.3 Centrality in the protomodular case

When the equivalence relation $R[f]$ is not only abelian but also central, the internal Mal'tsev operation π can be decomposed in a way we shall need later on. Let us briefly recall that, given any pair (R, S) of reflexive (and hence equivalence) relations on an object X in a protomodular category $\mathbb{C}$, it is said to commute when there is a (necessarily unique) partial Mal'tsev operation (see [12] and also [13]):

$$\pi : R \times_X S \to X \ ; \ (xRySz) \mapsto \pi(xRySz)$$

satisfying $\pi(xRxSz) = z$ and $\pi(xRySy) = x$. We classically denote this situation by $[R, S] = 0$. Accordingly the equivalence R is abelian when $[R, R] = 0$ and central when $[R, \nabla_X] = 0$, where ∇_X is the indiscrete equivalence relation on X. Recall also that, when $\mathbb{C}$ is pointed, the *normalization* of an equivalence relation R is the subobject $n = d_1.k : I_R \rightarrowtail X$ given by the following left hand side pullback:

$$\begin{array}{ccccc} I_R & \overset{k}{\rightarrowtail} & R & \overset{d_1}{\longrightarrow} & X \\ \downarrow & & \downarrow{\scriptstyle d_0} & & \\ 0 & \underset{\alpha_X}{\rightarrowtail} & X & & \end{array}$$

Suppose R is central, then we denote by $+ : I_R \times X \to X$ the map defined by $a + z = \pi(aR0, z)$ and call it the canonical action of the central subobject I_R on X. Moreover the factorization $(+, p_X) : I_R \times X \to R$ is an isomorphism. This defines a map $d : R \to I_R$ which, when $\mathbb{C}$ is the category Gp of groups, is given by $d(x, y) = x.y^{-1}$. Any central relation R is abelian, and, thanks to the previous isomorphism, its Mal'tsev operation $\pi : R^2 \to X$ can be canonically decomposed:

$$R^2 \overset{(d.p_0, p_2)}{\longrightarrow} I_R \times X \xrightarrow{+} X; \quad \pi(xRyRz) = d(x, y) + z$$

3 Non-abelian extensions

We are now interested in extensions whose kernels are no longer abelian, and for that we shall need one more tool.

3.1 Split extension classifier

We shall suppose here that $\mathbb{C}$ is only a pointed protomodular category. Recall the following definition from [6] and [7]:

Definition 3.1. *An object X of the pointed protomodular category $\mathbb{C}$ is said to have a split extension classifier when there is a split extension:*

$$X \xrightarrow{\gamma} D_1X \underset{s_0}{\overset{d_0}{\rightleftarrows}} DX$$

which is universal in the sense that any other split extension:

$$X \xrightarrow{k} H \underset{s}{\overset{f}{\rightleftarrows}} G$$

determines a unique pair of morphisms (χ, χ_1) such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{k} & H & \underset{s}{\overset{f}{\rightleftarrows}} & G \\ \downarrow{1_X} & & \downarrow{\chi_1} & & \downarrow{\chi} \\ X & \xrightarrow[\gamma]{} & D_1X & \underset{s_0}{\overset{d_0}{\rightleftarrows}} & DX \end{array}$$

The category $\mathbb{C}$ will be said to be action representative when such a universal split extension exists for any object X.

Of course, the category $\mathbb{C}$ being protomodular and 1_X being an isomorphism, the right hand side commutative square is necessarily a pullback. Moreover the map χ_1 is uniquely determined by χ, since the pair (k, s) is jointly strongly epic (this is actually one of the equivalent definition of protomodularity). This map χ will be called the *classifying map* of the split extension.

Examples 1) A pointed protomodular category $\mathbb{C}$ is additive if and only if it is trivially action representative (see [7] and [5]), in the sense that, for any object X, the split extension classifier does exist and is:

$$X \xrightarrow{1_X} X \underset{\alpha_X}{\overset{\tau_X}{\rightleftarrows}} 0$$

2) The protomodular category Gp of groups is clearly action representative, with $DX = AutX$ the group of automorphisms of X.
3) The protomodular category $R\text{-}Lie$ of R-Lie algebras is also clearly action representative with $DX = DerX$, the R-Lie algebra of derivations of X.
4) The dual of the category of pointed objects of certain (bi-Heyting) toposes $\mathbb{E}$ is action representative, see [4].

Obviously, each splitting of a given extension determines a different classifying map. Certainly the classifying map of the following split extension must be $0 : X \to DX$:

$$X \xrightarrow{r_X=(0,1)} X \times X \underset{l_X=(1,0)}{\overset{p_0}{\rightleftarrows}} X$$

By $j_X : X \to DX$ we shall take to mean the classifying map associated with another splitting:

$$X \xrightarrow{r_X} X \times X \underset{s_0}{\overset{p_0}{\rightleftarrows}} X$$

where, following the simplicial notations, s_0 is the diagonal. It makes the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{r_X} & X \times X & \underset{s_0}{\overset{p_0}{\rightleftarrows}} & X \\ \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \tilde{j}_X} & & \downarrow{\scriptstyle j_X} \\ X & \xrightarrow[\gamma]{} & D_1X & \underset{s_0}{\overset{d_0}{\rightleftarrows}} & DX \end{array}$$

When $\mathbb{C} = Gp$, the map j_X is precisely the classical group homomorphism $X \to AutX$.

Actually the split extension classifier of the object X underlies an internal groupoid structure, see [5]. Here we briefly recall its construction:

Theorem 3.2. *Let $\mathbb{C}$ be a pointed protomodular category and X any object with split extension classifier. Then this split extension classifier underlies a structure of groupoid $\underline{D}_1X$ such that $d_1.\gamma = j_X$. We shall call it the* action groupoid *of the object X.*

Proof. Consider the following split extension, with $R[d_0]$ the kernel relation of $d_0 : D_1X \to DX$, and s_1 (according to the simplicial notations) the unique map such that $p_0.s_1 = s_0.d_0$ and $p_1.s_1 = 1_{D_1X}$:

$$X \xrightarrow{s_1.\gamma} R[d_0] \underset{s_0}{\overset{d_0}{\rightleftarrows}} D_1X$$

It determines a unique pair (d_1, δ_2) of arrows making the following commutative square a pullback:

$$\begin{array}{ccc} R[d_0] & \xrightarrow{\delta_2} & D_1X \\ {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \\ D_1X & \xrightarrow[d_1]{} & DX \end{array}$$

Since any protomodular category $\mathbb{C}$ is Mal'tsev, this is sufficient to produce the following groupoid $\underline{D}_1X$:

$$R[d_0] \overset{\delta_2}{\underset{d_0}{\overset{d_1}{\rightrightarrows}}} D_1X \underset{d_0}{\overset{d_1}{\underset{s_0}{\leftrightarrows}}} DX$$

Moreover, we have $d_1.\gamma = j_X$ since these two maps clearly classify the same split extension. □

Example 1) When $\mathbb{C} = \mathbb{A}$ is additive, the action groupoid associated with any object X is nothing other than the canonical (internal) abelian group structure on X, namely:

$$X \times X \overset{+}{\underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} X \underset{\tau_X}{\overset{\alpha_X}{\leftrightarrows}} 0$$

2) When the category $\mathbb{C}$ is not additive, the split extension classifier still has a strong connexion with additivity, see [5]: the kernel relation $R[j_X]$ is the centre of X, i.e. the greatest central equivalence relation on X. As a consequence, the object j_X is abelian in $\mathbb{C}/DX$, it is endowed with a (associative and commutative) Mal'tsev operation $\pi_X : R^2[j_X] \to X$, and the kernel $\kappa_X : ZX \rightarrowtail X$ of j_X is the centre of X.

The previous observation provides a characterization of abelian objects in $\mathbb{C}$, again see [5]. Indeed, the following conditions are equivalent:
1) the object X is abelian;
2) the map j_X is a zero map (or equivalently $R[j_X] = \nabla_X$);
3) $d_0 = d_1$; in other words, the groupoid $\underline{D}_1X$ is totally disconnected.

3.2 The universal property of the action groupoid

Let us recall now the universal property of the action groupoid [5]. An internal groupoid $\underline{Z}_1$ in $\mathbb{C}$ will be presented (see [8]) as a reflexive graph $Z_1 \rightrightarrows Z_0$ endowed with an operation ζ_2:

$$\begin{array}{ccccccc} & R(\zeta_2) & & \zeta_2 & & & \\ R^2[z_0] & \overset{p_2,\ p_1}{\underset{p_0}{\rightrightarrows}} & R[z_0] & \overset{p_1}{\underset{p_0}{\rightrightarrows}} & Z_1 & \overset{z_1,\ s_0 \,(\leftarrow)}{\underset{z_0}{\rightleftarrows}} & Z_0 \end{array}$$

making the previous diagram satisfy all the simplicial identities (including the ones involving the degeneracies). By $R[z_0]$ we denote the kernel equivalence relation of the map z_0. In the set theoretical context, this operation ζ_2 associates the composite $\psi.\phi^{-1}$ with any pair (ϕ, ψ) of arrows with same domain.

Suppose its *normalization* $n = z_1.k_0$ has the object $X = Ker z_0$ as domain:

$$\begin{array}{ccccc} X & \overset{k_0}{\rightarrowtail} & Z_1 & \overset{z_1}{\longrightarrow} & Z_0 \\ \downarrow & & \downarrow{\scriptstyle z_0} & & \\ 0 & \underset{\alpha Z_0}{\rightarrowtail} & Z_0 & & \end{array}$$

Then the universal property of the action groupoid $\underline{D}_1X$ is that there is a unique internal functor $\underline{\chi}_1 = (\chi_0, \chi_1)$ such that $\chi_1.k_0 = \gamma$:

$$\begin{array}{ccccc} R[z_0] & \overset{\zeta_2,\ z_1}{\underset{z_0}{\rightrightarrows}} & Z_1 & \overset{z_1,\ s_0\,(\leftarrow)}{\underset{z_0}{\rightleftarrows}} & Z_0 \\ {\scriptstyle R(\chi_1)}\downarrow & & \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_0} \\ R[d_0] & \overset{\delta_2,\ d_1}{\underset{d_0}{\rightrightarrows}} & D_1X & \overset{d_1,\ s_0\,(\leftarrow)}{\underset{d_0}{\rightleftarrows}} & DX \end{array}$$

Clearly this functor is a discrete fibration. Now suppose we have a map $f : X \to Y$ with kernel $k : K \rightarrowtail X$. Then the previous universal property determines a functor $(\check{k}, \check{k}_1) : R[f] \to \underline{D}_1K$:

$$\begin{array}{ccccc} R^2[f] & \overset{p_2,\ p_1}{\underset{p_0}{\rightrightarrows}} & R[f] & \overset{p_1,\ s_0\,(\leftarrow)}{\underset{p_0}{\rightleftarrows}} & X \\ {\scriptstyle R(\check{k}_1)}\downarrow & & \downarrow{\scriptstyle \check{k}_1} & & \downarrow{\scriptstyle \check{k}} \\ R[d_0] & \overset{\delta_2,\ d_1}{\underset{d_0}{\rightrightarrows}} & D_1K & \overset{d_1,\ s_0\,(\leftarrow)}{\underset{d_0}{\rightleftarrows}} & DK \end{array}$$

We can check that $\check{k}.k = j_K$ since the two maps classify the same split extension. When $\mathbb{C} = Gp$, this map $\check{k}$ is the classical homomorphism $X \to AutK$ associated with the normal subgroup $k : K \rightarrowtail X$.

From the previous characterization of abelian objects, we get the following characterization of extensions with abelian kernel relation which is all the more interesting since it does not hold in any pointed protomodular category:

Proposition 3.3. *Suppose the pointed protomodular category $\mathbb{C}$ is action representative. Then a map $f : X \to Y$ has its kernel relation $R[f]$ abelian (or equivalently the map f, seen as an object in the slice category $\mathbb{C}/Y$, is abelian) if and only if its kernel object K is abelian. The kernel relation $R[f]$ is central if and only if its classifying map $\check{k}$ is* 0.

3.3 Extensions

We shall suppose from now on that the action representative category $\mathbb{C}$ is moreover exact [1]. Consider any extension whose kernel is no longer abelian:

$$1 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y \longrightarrow 1$$

The following diagram induces a factorization ϕ, we shall call the *abstract direction* of the extension, where $q_K : DK \twoheadrightarrow Q_K$ is the coequalizer of the pair $(d_0, d_1) : D_1K \rightrightarrows DK$, or equivalently the cokernel of $j_K : K \to DK$:

$$\begin{array}{ccc} R[f] & \xrightarrow{\check{k}_1} & D_1K \\ d_0 \downarrow\downarrow d_1 & & d_0 \downarrow\downarrow d_1 \\ X & \xrightarrow[\check{k}]{} & DK \\ f \downarrow & & \downarrow q_K \\ Y & \underset{\phi}{\dashrightarrow} & Q_K \end{array}$$

We shall denote by $Ext_\phi(Y, K)$ the set of isomorphic classes of extensions with abstract direction ϕ.

When the kernel $K = A$ is abelian, we have $d_0 = d_1$ and $Q_A = DA$. Then the abstract direction $\phi : Y \to DA$ is such that $\phi.f = \check{k}$. This map ϕ is, in turn, the classifying map of a split extension with abelian kernel A:

$$A \overset{k_\phi}{\rightarrowtail} E\phi \underset{s_\phi}{\overset{e_\phi}{\rightleftarrows}} Y$$

The following diagram made of pullbacks

$$\begin{array}{ccccccc}
 & & & \xrightarrow{\qquad\quad \check{k}_1 \quad\qquad} & & & \\
 & & R[f] & \overset{\tilde{f}}{\twoheadrightarrow} & E\phi & \xrightarrow{\tilde{\varphi}} & D_1(A) \\
 & & {\scriptstyle f_0}\downarrow\uparrow{\scriptstyle s_0} & & {\scriptstyle e_\phi}\downarrow\uparrow{\scriptstyle s_\phi} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \\
A & \underset{k}{\rightarrowtail} & X & \overset{f}{\twoheadrightarrow} & Y & \xrightarrow{\varphi} & D(A) \\
 & & & \xrightarrow[\qquad\quad \check{k} \quad\qquad]{} & & &
\end{array}$$

shows us that the map $\tilde{f}$ is necessarily the cokernel of $s_0.k : A \to R[f]$ and that consequently $E\phi = d_Y(X)$, i.e. the direction of the extension with abelian kernel, as defined in Section 2.2.

Suppose K is no longer abelian. At the moment, it is quite natural to introduce the following lower pullback:

$$\begin{array}{ccc}
D_1\phi & \xrightarrow{d_{1\phi}} & D_1K \\
{\scriptstyle d_0}\downarrow\ \downarrow{\scriptstyle d_1} & & {\scriptstyle d_0}\downarrow\ \downarrow{\scriptstyle d_1} \\
D\phi & \xrightarrow{d_\phi} & DK \\
{\scriptstyle q_\phi}\downarrow & & \downarrow{\scriptstyle q} \\
Y & \xrightarrow[\phi]{} & Q_K
\end{array}$$

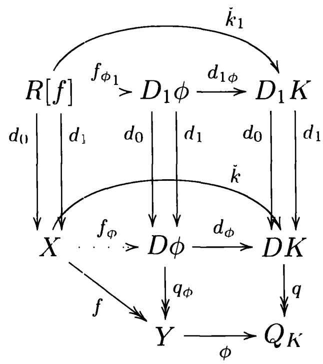
which determines a groupoid $\underline{D}_1\phi$ and a discrete fibration $\underline{d}_{1\phi} : \underline{D}_1\phi \to \underline{D}_1K$. The groupoid $\underline{D}_1\phi$ is the *categorical expression of the obstruction to extensions*, see [11].

This pullback produces a unique factorization j_ϕ:

$$\begin{array}{ccc}
 & & K \\
 & \swarrow^{j_\phi} & \downarrow{\scriptstyle j_K} \\
D\phi & \xrightarrow[d_\phi]{} & DK \\
{\scriptstyle q_\phi}\downarrow & & \downarrow{\scriptstyle q_K} \\
Y & \xrightarrow[\phi]{} & Q_K
\end{array}$$

such that $d_\phi.j_\phi = j_K$ and $q_\phi.j_\phi = 0$. The kernel relation $R[j_\phi]$ of j_ϕ is necessarily $R[j_K]$, and consequently the kernel of j_ϕ is the kernel $\kappa_K : ZK \rightarrowtail K$ of j_K. The map j_K and j_ϕ having the same regular epimorphic part, the map q_ϕ is the cokernel of j_ϕ.

Consider now the following diagram:



The functor $\underline{d}_{1\phi} : \underline{D}_1\phi \to \underline{D}_1K$ is a discrete fibration by construction. Let us denote by f_ϕ and f_{ϕ_1} the factorizations induced by $\check{k}$ and $\check{k}_1$. They make $\underline{f}_{\phi 1} : R[f] \to \underline{D}_1\phi$ a discrete fibration. Since we have $f = q_\phi . f_\phi$, we have clearly $\pi_0(\underline{f}_{\phi 1}) = 1_Y$. Notice also that, thanks to the universal property of the pullback, we get $f_\phi . k = j_\phi$. When $K = A$ is abelian, then $q_\phi = 1_Y$ and $f_\phi = f$.

We need now a last structural ingredient. What is rather awkward and uncomfortable is that the objects DX (although having a universal property) do not give rise to any functorial process. However we have some very specific constructions. For instance, there is a comparison morphism $\zeta_X : DX \to DZX$ such that $\zeta_X . j_X = 0$, see Proposition 2.2 in [11]. When $\mathbb{C} = Gp$ is the category of groups, this comparison morphism is given by the restriction of any automorphism on the group X to its centre ZX. Since q_X is the cokernel of j_X, there is a unique factorization $\xi_X : Q_X \to DZX$ such that $\xi_X . q_X = \zeta_X$. From this, we get:

Proposition 3.4. *The map f_ϕ underlies an extension with abelian kernel relation whose abstract direction is $\zeta_K . d_\phi$:*

$$1 \longrightarrow ZK \xrightarrow{k.\kappa_K} \bar{X} \xrightarrow{f_\phi} D\phi \longrightarrow 1$$

A profunctorial interpretation of this proposition leads to an intrinsic Schreier-Mac Lane extension theorem:

Theorem 3.5. *Suppose $\mathbb{C}$ is an exact action representative category. When we have $Ext_\phi(Y, K) \neq \emptyset$, there is on the set $Ext_\phi(Y, K)$ a canonical simply transitive action of the abelian group $Ext_{\xi_K . \phi}(Y, ZK)$.*

This result holds a fortiori in any semi-abelian [14] action representative category.

3.4 The action on extensions

Let be given any extension with abstract direction ϕ:

$$1 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y \longrightarrow 1$$

and any extension with abstract kernel $\xi_K.\phi$:

$$1 \longrightarrow ZK \overset{n}{\rightarrowtail} T \overset{t}{\twoheadrightarrow} Y \longrightarrow 1$$

The action of latter extension on the former one asserted by the theorem is given by the following 3×3 diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & ZK & \overset{(-1,\kappa_K)}{\rightarrowtail} & ZK \times K & \overset{+}{\twoheadrightarrow} & K & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle 1_{ZK}} & & \downarrow{\scriptstyle n\times k} & & \vdots{\scriptstyle \gamma} & & \\
1 & \longrightarrow & ZK & \underset{(-n,k.\kappa_K)}{\rightarrowtail} & T \times_Y X & \underset{\tilde{q}}{\twoheadrightarrow} & T \otimes_Y X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle t\times_Y f} & & \vdots{\scriptstyle t\otimes_Y f} & & \\
1 & \longrightarrow & 1 & \rightarrowtail & Y & \underset{1_Y}{\twoheadrightarrow} & Y & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where $ZK \times K \overset{+}{\rightarrow} K$ is the natural action of the centre. Notice moreover that the upper right hand side square is a pullback, and, since the two horizontal edges are regular epimorphisms, it is also a pushout.

Proof. According to Theorem 3.3 in [11], the result of the action is given by the composition $(f_\phi, f) \otimes (t,t)$ of the two profunctors (f_ϕ, f) and (t,t) associated with the two extensions:

$$\begin{array}{ccccccc}
 & & & T \times_Y X & & & \\
 & & \swarrow & & \searrow & & \\
 & T & & & & X & \\
{\scriptstyle t}\swarrow & & \searrow{\scriptstyle t} & & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f_\phi} \\
Y & & & Y & & & D\phi
\end{array}$$

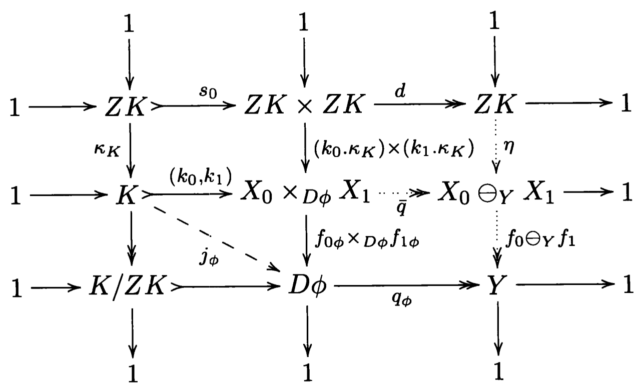
namely, by a quotient of the pullback $T \times_Y X$ which is precisely described by the 3×3 construction. □

3.5 Simply transitive aspect of this action

This action being simply transitive, any pair of extensions ($i \in \{0,1\}$) with abstract direction ϕ:

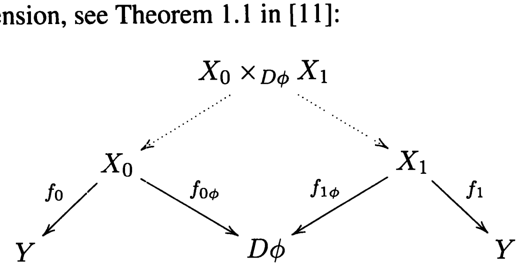
$$1 \longrightarrow K \overset{k_i}{\rightarrowtail} X_i \overset{f_i}{\twoheadrightarrow} Y \longrightarrow 1$$

must determine a unique element in the abelian group $Ext_{\mathcal{E}_K.\phi}(Y, ZK)$. It is given by the following 3×3 diagram:



where $ZK \times ZK \xrightarrow{d} ZK$ is defined by $d(x,y) = x.y^{-1}$ and the lower left hand side corner of the diagram is the canonical decomposition of $j_\phi : K \twoheadrightarrow K/ZK \rightarrowtail D\phi$. The commutativity of the lower left hand side square is asserted by the equality $f_{i\phi}.k_i = j_\phi$ recalled above.

Proof. According to Theorem 3.3 in [11], the result of the action is given by the composition $(f_{1\phi}, f_1) \otimes (f_{0\phi}, f_0)^*$, where $(f_{1\phi}, f_1)$ is the profunctor associated with the second extension, while $(f_{0\phi}, f_0)^*$ is the inverse of the profunctor associated with the first extension, see Theorem 1.1 in [11]:



namely, by a quotient of the pullback $X_0 \times_{D\phi} X_1$ which is precisely described by the 3×3 construction. □

3.6 The Mal'tsev operation on extensions

Section 1 insures that the previous simply transitive action gives the set $Ext_\phi(Y, K)$ an associative and commutative Mal'tsev operation. So consider any triple ($i \in \{0, 1, 2\}$) of extensions with abstract direction ϕ:

$$1 \longrightarrow K \overset{k_i}{\rightarrowtail} X_i \overset{f_i}{\twoheadrightarrow} Y \longrightarrow 1$$

First, take the limit of the following lower diagram:

$$\begin{array}{ccccc} & & X_0 \times_{D\phi} X_1 \times_{D\phi} X_2 & & \\ & \swarrow^{p_0} & \downarrow p_1 & {}^{p_2}\searrow & \\ X_0 & & X_1 & & X_2 \\ & {}_{f_{0\phi}}\searrow & \downarrow f_{1\phi} & \swarrow_{f_{2\phi}} & \\ & & D\phi & & \end{array}$$

and set $h = f_{i\phi}.p_i$ for any $i \in \{0, 1, 2\}$. Then there is a unique factorization $k : R^2[j_K] \rightarrowtail X_0 \times_{D\phi} X_1 \times_{D\phi} X_2$ making the following diagram commute, for any $i \in \{0, 1, 2\}$ (recall that $R[j_K] = R[j_\phi]$ and $f_i.k_i = j_\phi$):

$$\begin{array}{ccc} R^2[j_K] & \overset{k}{\rightarrowtail} & X_0 \times_{D\phi} X_1 \times_{D\phi} X_2 \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p_i} \\ K & \overset{k_i}{\rightarrowtail} & X_i \\ & {}_{j_\phi}\searrow \quad D\phi \quad \swarrow_{f_i} & \end{array}$$

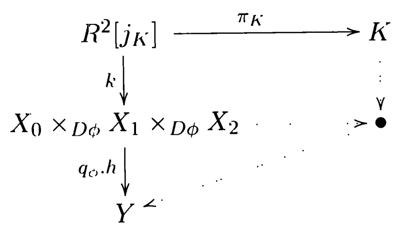
It is straightforward to check that the map k is the kernel of the map

$$X_0 \times_{D\phi} X_1 \times_{D\phi} X_2 \xrightarrow{h} D\phi \xrightarrow{q_\phi} Y$$

namely, that the following sequence is exact:

$$1 \longrightarrow R^2[j_K] \overset{k}{\rightarrowtail} X_0 \times_{D\phi} X_1 \times_{D\phi} X_2 \overset{q_\phi.h}{\twoheadrightarrow} Y \longrightarrow 1$$

Then the value of the Mal'tsev operation on the triple of extensions with abstract direction ϕ is given by the following pushout of the vertical extension along the internal Mal'tsev operation π_K (see Section 3.1, Example 2) :



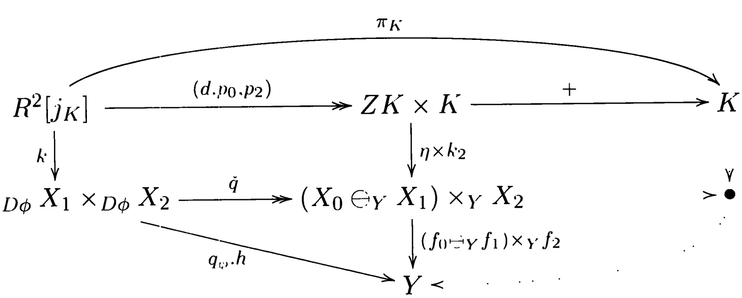
Proof. According to Section 1, the value of the Mal'tsev operation on the triple of extensions is the result of the action of the extension with abelian kernel:

$$1 \longrightarrow ZK \stackrel{\eta}{\rightarrowtail} X_0 \ominus_Y X_1 \stackrel{f_0 \ominus_Y f_1}{\twoheadrightarrow} Y \longrightarrow 1$$

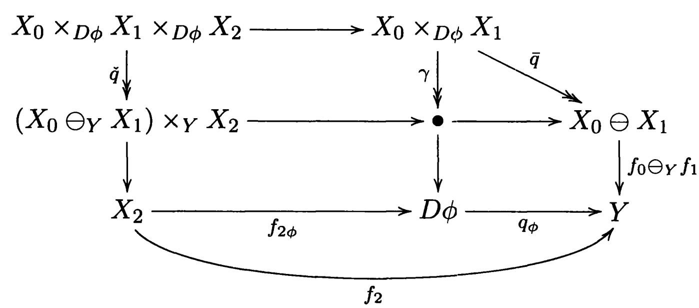
on the following one:

$$1 \longrightarrow K \stackrel{k_2}{\rightarrowtail} X_2 \stackrel{f_2}{\twoheadrightarrow} Y \longrightarrow 1$$

Namely, the pushout along the canonical action $ZK \times K \xrightarrow{+} K$ of the following middle extension:



We noticed that the upper right hand square is a pushout. There is also a natural comparison morphism $\check{q}$. The same reasons apply to make the associated upper left hand side square a pushout. Indeed, the map $\check{q}$ is the canonical factorization induced by the quotient map $\bar{q} : X_0 \times_{D\phi} X_1 \twoheadrightarrow X_0 \ominus X_1$. Let us consider the following diagram:



where the left hand side vertical rectangle is a pullback in the same way as the lower right hand side square. Since $\mathbb{C}$ is protomodular and exact, and since the vertical right hand side quadrangle is a pushout by the 3×3 construction, the factorization γ is a regular epimorphism. The upper left hand side square is a pullback. Accordingly the map $\check{q}$ is a regular epimorphism. On the other hand, the restriction of $\check{q}$ to the kernels is precisely the factorization $(d.p_0, p_2)$ defined in Section 2.3 whose composition with the canonical action $ZK \times K \xrightarrow{+} K$ was showed to be the internal Mal'tsev operation π_K. □

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