MARIO GRANDIS
ROBERT PARE

Lax Kan extensions for double categories (on weak double categories, part IV)

Cahiers de topologie et géométrie différentielle catégoriques, tome 48, n° 3 (2007), p. 163-199

<http://www.numdam.org/item?id=CTGDC_2007__48_3_163_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLVIII-3 (2007)

LAX KAN EXTENSIONS FOR DOUBLE CATEGORIES * (ON WEAK DOUBLE CATEGORIES, PART IV)

by Mario GRANDIS and Robert PARE

Résumé. Les extensions de Kan à droite pour les catégories doubles (faibles) généralisent les limites doubles et d'autres constructions, appelées 'vertical companion' et 'vertical adjoint', que nous avons étudiées dans des articles précédents. Nous prouvons ici que ces cas particuliers sont suffisants pour construire toutes les extensions de Kan à droite ponctuelles, le long de foncteurs lax doubles satisfaisant une condition 'de Conduché'. Les catégories doubles 'basées sur les profoncteurs' sont complètes, dans le sens qu'elles admettent toutes ces constructions, tandis que la catégorie double des carrés commutatifs d'une catégorie complète ne l'est pas, en général.

Introduction

This is a sequel to three papers on the general theory of weak (or pseudo) double categories, '*Limits in double categories*' [GP1], '*Adjoint for double categories*' [GP2], and '*Kan extensions in double categories*' [GP3], which will be referred to as Part I, II and III, respectively.

In Part I, it was proved that, in a pseudo double category $\mathbb{A}$, all (small) double limits can be constructed from (small) products, equalisers and *tabulators*, the latter being the double limit of a vertical arrow. Part II deals with the natural notion of adjunction for weak double categories, a *colax/lax adjunction* $G \dashv R$, where G is a *colax* double functor, while R is *lax*; this is also viewed as an internal Kan extension *in* the strict double category $\mathbb{D}\mathrm{bl}$ of weak double categories, lax double functors (as horizontal arrows), colax double functors (as vertical arrows) and suitable cells - as recalled here, in 1.2.

Finally, Part III introduces internal Kan extensions, *in* a weak double category $\mathbb{D}$, and begins to consider Kan extensions *for* weak double categories, choosing $\mathbb{D}$ to be a double category of weak double categories: namely, the 'settings' $\mathbb{D}\mathrm{bl}$, $\mathbb{D}\mathrm{bl}_u$,

(*) Work partially supported by MIUR, INDAM (Italy) and NSERC (Canada).

$\mathrm{Lx}\mathbb{D}\mathrm{bl}_\mathrm{u}$ listed in its Section 6.

The first case yields *colax* right Kan extensions along *lax* double functors. Applying a general result of Part III (Thm. 2.4) which relates absolute Kan extensions in a double category with orthogonal adjunctions therein, we have shown that absolute colax right Kan extensions of the identity along lax double functors amount to the colax/lax adjunctions of Part II, mentioned above.

The second setting is a restriction of $\mathbb{D}\mathrm{bl}$ to the double subcategory $\mathbb{D}\mathrm{bl}_\mathrm{u}$ where the vertical arrows are *unitary* colax double functors. It yields *unitary colax* right Kan extensions along lax double functors; the interest of the restriction was shown by some examples, in Section 7 of Part III.

Here we focus on the third setting, *unitary lax* right Kan extensions along *lax* double functors, based on another double category of double categories, $\mathrm{Lx}\mathbb{D}\mathrm{bl}_\mathrm{u}$, where all arrows are lax double functors, but the vertical ones are required to be unitary. This notion goes well with limits and completeness; most of the present paper is devoted to studying *pointwise* extensions *of this kind.*

Explicitly, the *unitary lax* right Kan extension of a *lax* double functor $S\colon \mathbb{I} \to \mathbb{A}$ along a *lax* double functor $R\colon \mathbb{I} \to \mathbb{J}$ (2.1)

(1)
$$\begin{tikzcd} \mathbb{I} \arrow[r, "R"] \arrow[d, equal] & \mathbb{J} \arrow[r, equal] \arrow[d, "G"] \arrow[dl, phantom, "\swarrow\varepsilon"] & \mathbb{J} \arrow[d, "G'"] \arrow[dl, phantom, "\swarrow\tau"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd}$$

is a *unitary lax* double functor G equipped with a horizontal transformation τ: $GR \to S$ such that any similar pair (G', τ') factors through τ, by a unique horizontal transformation $\tau\colon G' \to G$.

After a first section on the diagrammatic properties of comma double categories, Section 2 begins the study of pointwise extensions of this type. The Reduction Theorem 2.3 shows that the pointwise property needs only to be checked on the three *vertical ordinals* $\underline{1}$, $\underline{2}$, $\underline{3}$ (2.2): the *singleton*, the *vertical arrow* and the *vertical composite*; furthermore, the latter can also be omitted, in some important cases (Thm. 2.4).

Then, Sections 3 and 4 show that pointwise unitary lax right Kan extensions in a pseudo double category $\mathbb{A}$ (the codomain of the extension) extend *lax functorial double limits*, *orthogonal companions* and *orthogonal adjoints* in $\mathbb{A}$ (defined in the previous parts and recalled in 3.1, 4.1). Conversely, our main result (Theorem 5.2), in the last section, proves that all pointwise unitary lax right Kan extensions in $\mathbb{A}$, along any small lax double functor R which satisfies a suitable Conduché condition, can be constructed from these elementary cases. It would be interesting to prove a similar result for the colax case.

We say that the pseudo double category $\mathbb{A}$ is *complete* (5.3) when all the above pointwise extensions exist; or, equivalently, when $\mathbb{A}$ has lax functorial limits, vertical companions and vertical adjoints. This is the case for all of our profunctor-based examples of Part I, related to the pseudo double category $\mathbb{C}\mathbf{at}$ of small categories, functors and profunctors (see 1.1). Finite completeness is a first-order property (5.3). On the other hand, a double category $\mathbb{Q}\mathbf{A}$ of quintets on a 2-complete 2-category $\mathbf{A}$ has all double limits and vertical companions, but generally lacks vertical adjoints. Thus, $\mathbb{Q}\mathbf{Cat}$ is not complete: *the important, natural double structure of categories is* $\mathbb{C}\mathbf{at}$.

We end with an example showing the role of the Conduché condition (5.4).

Double categories where introduced by C. Ehresmann [E1, E2]. Other contributions on double categories, weak or strict, are referred to in the previous Parts. Many more recent references can be found in the paper [DPP].

Size aspects (for double categories of double categories, for instance) can be easily settled working with suitable universes. A reference I.2 or I.2.3, relates to Part I, namely its Section 2 or Subsection 2.3. Similarly for Parts II and III.

1. Diagrammatic lemmas for double commas

We show that double commas $F{\Downarrow\!\Downarrow}R$, constructed in Part II, are comma objects *in* $\mathbb{D}\mathrm{bl}$, with respect to a general definition by universal properties (III.3.2). We end with some diagrammatic lemmas for them.

1.1. Terminology. For double categories, we use the same terminology and notation as in the previous Parts.

The composite of two horizontal arrows $f\colon A \to A'$, $g\colon A' \to A''$ is written gf, while for vertical arrows $u\colon A \twoheadrightarrow B$, $v\colon B \twoheadrightarrow C$ we write $u\otimes v$ or $v\bullet u$, or just vu (note the switch). The boundary of a double cell α, consisting of two horizontal arrows and two vertical ones, as in the left diagram

$$(1) \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v"] \\ B \arrow[r, "g"'] & B' \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "v"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

is displayed as $\alpha\colon (u \, {}^{f}_{g} \, v)$ or $\alpha\colon u \to v$; it is a *special* cell when the horizontal arrows are identities (as at the right). The horizontal and vertical compositions of cells are written as $(\alpha \mid \beta)$ and $(\frac{\beta}{\gamma})$; or also $\gamma \,|\gamma$ and $\gamma \otimes\otimes$ The symbols 1_A, 1_u (resp. $1^\bullet_A$, $1^\bullet_f$) denote horizontal (resp. vertical) identities.

We generally work with *pseudo double categories* (I.7.1), also called weak double categories, where the horizontal structure behaves categorically, while the composition of vertical arrows is associative up to *comparison cells* $(u\otimes v)\otimes w \to u\otimes(v\otimes w)$; these are *special isocells* – horizontally invertible. But we always assume that vertical identities behave *strictly*, a useful simplification, easy to obtain.

The expression 'profunctor-based examples' will refer to the following pseudo double categories, treated in Part I: $\mathbb{C}$at (formed of categories, functors and profunctors, I.3.1), $\mathbb{S}$et (sets, mappings and spans, I.3.2), $\mathbb{P}$os (preordered sets, monotone mappings and order ideals, I.3.3), $\mathbb{M}$tr (generalised metric spaces, weak contractions and metric profunctors, I.3.3), $\mathbb{R}$el (sets, mappings and relations, I.3.4), $\mathbb{R}$ng (unitary rings, homomorphisms and bimodules, I.5.3). In $\mathbb{C}$at, a profunctor $u\colon A \to\!\!\!\!\to B$ is defined as a functor $u\colon A^{op}\times B \to$ **Set**.

A 2-category **A** has an associated (Ehresmann's) double category of quintets $\mathbb{Q}$**A**, where a double cell $\otimes \, (u \, {}^{f}_{g} \, v)$ is defined as a 2-cell $\otimes\colon vf \to gu$ of **A** (see I.1.3). On the other hand, a weak double category $\mathbb{A}$ contains a *bicategory* **V**$\mathbb{A}$ of vertical arrows and special cells, as well as (*because* of unitarity) a *2-category* **H**$\mathbb{A}$ of horizontal arrows and 'vertically special' cells (I.1.9).

Now, a *lax* double functor $R\colon \mathbb{A} \to \mathbb{X}$ between pseudo double categories (II.2.1) preserves the horizontal structure in the strict sense, and the vertical one up to *laxity comparisons*, which are special cells (the *identity* and *composition* comparison)

(2) $\quad R[A]\colon 1^\times_{RA} \to R(1^\times_A)\colon RA \to\!\!\!\!\to RA, \qquad R[u, v]\colon Ru\otimes Rv \to R(u\otimes v)\colon RA \to\!\!\!\!\to RC,$

for A and $u\otimes v\colon A \to\!\!\!\!\to B \to\!\!\!\!\to C$ in $\mathbb{A}$; all this has to satisfy naturality and coherence axioms. (To remember the direction of these cells, one can think of a vertical monad in $\mathbb{A}$ as a lax double functor $\underline{1} \to \mathbb{A}$, defined on the singleton double category.)

This lax R is *unitary* if, for every A in $\mathbb{A}$, the special cell $R[A]\colon 1^\times_{RA} \to R1^\times_A$

is an identity; then, by coherence, also the following cells are (for $u: B \to\!\!\!\!\!\to A$ and $v: A \to\!\!\!\!\!\to C$)

(3) $R[u, 1^\bullet_A]: Ru \otimes R1^\bullet_A \to Ru, \qquad R[1^\bullet_A, v]: R1^\bullet_A \otimes Rv \to Rv.$

As a consequence, which will be relevant here, a unitary lax double functor defined on a weak double category where all vertical compositions are trivial (i.e., all consecutive pairs of vertical arrows contain a vertical identity) is necessarily strict.

By horizontal duality, a *colax* double functor $F: \mathbb{A} \to \mathbb{X}$ has comparison cells in the opposite direction

(4) $[FA]: F(1^\bullet_A) \to 1^\bullet_{FA}, \qquad F[u, v]: F(u \otimes v) \to Fu \otimes Fv.$

A *pseudo* (resp. *strict*) double functor is a lax one, whose comparison cells are horizontally invertible (resp. identities); or, equivalently, a colax one satisfying the same condition. A pseudo double functor can always be made unitary.

A lax or colax double functor $\mathbb{I} \to \mathbb{A}$ is said to be *small* if $\mathbb{I}$ is.

1.2. A double category of double categories. Lax and colax double functors do not compose well. But they can be organised in a *strict* double category $\mathbb{D}\mathrm{bl}$, introduced in II.2.2 and also recalled in III.1.4. Here, we will briefly sketch its definition.

Its objects are pseudo double categories $\mathbb{A}, \mathbb{B}, \ldots$; its horizontal arrows are the *lax* double functors $R, S\ldots$ between them; its vertical arrows are *colax* double functors $F, G\ldots$ (II.2.1). A cell α, as in the left diagram below, is - loosely speaking - a 'horizontal transformation' $\alpha: GR \to SF$ (as stressed by the arrow we are placing in the square)

(1)
$$\begin{tikzcd} \mathbb{A} \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathbb{B} \arrow[d, "G"] \\ \mathbb{C} \arrow[r, "S"'] & \mathbb{D} \end{tikzcd} \qquad \begin{tikzcd} GRA \arrow[r, "\alpha A"] \arrow[d, "GRu"'] \arrow[dr, phantom, "\alpha u"] & SFA \arrow[d, "SFu"] \\ GRA' \arrow[r, "\alpha A'"'] & SFA' \end{tikzcd}$$

More precisely, since these composites GR, SF are neither lax nor colax (just morphisms of double graphs, respecting the horizontal structure), the cell α consists of:

- the lax double functors R, S; the colax double functors F, G;

- maps $\alpha A: GR(A) \to SF(A)$ and cells αu in $\mathbb{D}$ (for A and $u: A \to\!\!\!\!\!\to A'$ in $\mathbb{A}$), as in the right diagram above, satisfying two *naturality conditions* (c0), (c1) and two *coherence conditions* (c2), (c3) based on the comparison cells of the four 'functors' (see Part II or III).

We denote by $\mathbb{D}\mathrm{bl}_u$ the cell-wise full double subcategory of $\mathbb{D}\mathrm{bl}$ where the *vertical* arrows are *unitary* colax double functors, while the horizontal ones are general. Similarly, $\mathbb{D}\mathrm{bl}_p$ has *unitary pseudo* double functors as vertical arrows.

1.3. Double commas. We recall now another main tool inherited from Part II, where one can find the (non obvious) proof of the coherence properties (in II.2.5).

A *colax* double functor F and a *lax* double functor R with the same codomain have a *comma* pseudo double category $F{\Downarrow}R$, forming a cell ω in $\mathbb{D}\mathrm{bl}$

$$(1) \qquad \begin{tikzcd} F{\Downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] & \mathbb{A} \arrow[d, "F"] \arrow[dl, phantom, "\swarrow\omega"] \\ \mathbb{X} \arrow[r, "R"'] & \mathbb{C} \end{tikzcd}$$

whose universal properties will be examined below, in 1.5; the projections P and Q are *strict* double functors.

First, an object of $F{\Downarrow}R$ is a triple $(A, X; c\colon FA \to RX)$. Second, a horizontal map $(a, x)\colon (A,X;c) \to (A', X'; c')$ comes from a commutative square of $\mathbb{C}$, as in the left diagram below

$$(2) \qquad \begin{tikzcd} FA \arrow[r, "c"] \arrow[d, "Fa"'] \arrow[dr, phantom, "="] & RX \arrow[d, "Rx"] \\ FA' \arrow[r, "c'"'] & RX' \end{tikzcd} \qquad\qquad \begin{tikzcd} FA \arrow[r, "c"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\gamma"] & RX \arrow[d, "Rv"] \\ FB \arrow[r, "d"'] & RY \end{tikzcd}$$

Composition is obvious. Third, a vertical arrow $(u, v; \gamma)\colon (A, X; c) \twoheadrightarrow (B, Y; d)$ comes from a cell $\gamma\colon (Fu \,{}^{c}_{d}\, Rv)$ in $\mathbb{C}$, as in the right diagram above. The composition of vertical arrows is displayed below

$$(3) \qquad \begin{tikzcd} \bullet \arrow[r, equal] \arrow[dd, "F(u\otimes u')"'] & \bullet \arrow[r, dashed, "c"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\gamma"] & \bullet \arrow[r, equal] \arrow[d, "Rv"] & \bullet \arrow[dd, "R(v\otimes v')"] \\ {} \arrow[r, phantom, "F{[u, u']}"] & \bullet \arrow[r, dashed, "d"] \arrow[d, "Fu'"'] \arrow[dr, phantom, "\delta"] & \bullet \arrow[d, "Rv'"] \arrow[r, phantom, "R{[v, v']}"] & {} \\ \bullet \arrow[r, equal] & \bullet \arrow[r, dashed, "e"] & \bullet \arrow[r, equal] & \bullet \end{tikzcd}$$

It is constructed with the *colaxity* cell of F and the *laxity* cell of R (note the 'necessity' of this direction of comparisons).

Fourth, a cell (α, ξ) is a pair of cells $\xi\colon (u \,{}^{a}_{b}\, u')$, $\xi\colon (v \,{}^{x}_{y}\, v')$ in $\mathbb{A}$ and $\mathbb{X}$, respectively, such that $F\xi$ and $R\xi$ are coherent with γ, γ' in $\mathbb{C}$

$$(4)\qquad \begin{array}{ccc} (A, X; c) & \xrightarrow{(a,\, x)} & (A', X'; c') \\ {\scriptstyle (u,\, v;\, \gamma)}\downarrow & (\gamma\, ,\gamma) & \downarrow{\scriptstyle (u',\, v';\, \gamma')} \\ (B, Y; d) & \xrightarrow[(b,\, y)]{} & (B, Y; d') \end{array} \qquad\qquad (F\gamma \mid \gamma') = (\gamma \mid R\gamma).$$

Their horizontal and vertical compositions are obvious.

The associativity isocell for three consecutive vertical arrows $(u, v; \gamma)$, $(u', v'; \delta)$, $(u'', v''; \varepsilon)$ *is* the pair $(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))$ of associativity isocells of $\mathbb{A}, \mathbb{X}$ for the triples $\mathbf{u} = (u, u', u'')$, $\mathbf{v} = (v, v', v'')$

$$(5)\quad (\varepsilon\mathbf{u}, \varepsilon\mathbf{v})\colon ((u, v; \varepsilon)\otimes(u', v'; \otimes))\otimes(u'', v''; \otimes) \to (u, v; \otimes)\otimes((u', v'; \otimes)\otimes(u'', v''; \otimes)).$$

Finally, P and Q are projections and the components of ω on objects and vertical arrows are:

$$(6)\quad \omega(A, X; c) = c\colon FA \to RX, \qquad\qquad \omega(u, v; \omega) = \omega\ (Fu \,{}^{\omega(A,\, X;\, c)}_{\omega(B,\, Y;\, d)}\, Rv).$$

1.4. Cells and commutative cells. Let us come back to examining the cells of $\mathbb{D}$bl. The set of cells ω with a specified boundary, as in the left diagram below, can be denoted as $[F \,{}^{R}_{S}\, G]$; but we shall also write $[GR, SF]$, following the previous abuse of notation $\omega\colon GR \to SF$ (in 1.2)

$$(1)\qquad \begin{array}{ccc} \mathbb{A} & \xrightarrow{R} & \mathbb{B} \\ {\scriptstyle F}\downarrow & \swarrow\omega & \downarrow{\scriptstyle G} \\ \mathbb{C} & \xrightarrow[S]{} & \mathbb{D} \end{array} \qquad\qquad\qquad \begin{array}{ccc} \mathbb{A} & \xrightarrow{R} & \mathbb{B} \\ {\scriptstyle F}\downarrow & \lambda /\!/ & \downarrow{\scriptstyle G} \\ \mathbb{C} & \xrightarrow[S]{} & \mathbb{D} \end{array}$$

Note that, *when* F, G *are pseudo*, their colaxity cells have horizontal inverses which are laxity cells. Thus, the composites GR, SF are *lax* and the set [GR, SF] *coincides with the set of ordinary horizontal transformations* $\lambda\colon GR \to SF$ (as evident from the coherence conditions (c2)-(c3), in 1.2).

If, moreover, it happens that GR = SF (including the laxity cells!), the horizontal identity $1\colon GR \to SF$ yields a cell λ, represented in the right diagram above. It will be called a *commutative cell*, and written as λ: (GR = SF). Similar facts hold *when* R, S *are pseudo*, and hence the composites GR, SF are *colax*.

All this agrees with a general definition of commutative cells in a weak double category, given in III.3.1 (and based on orthogonal companions, see 4.1). In fact, a pseudo double functor $\mathbb{A} \to \mathbb{B}$ is a *strong arrow* in $\mathbb{D}$bl, with a 'horizontal version' $\mathbb{A} \to \mathbb{B}$ (with invertible *laxity* cells) and a 'vertical version' $\mathbb{A} \twoheadrightarrow \mathbb{B}$ (with invertible *colaxity* cells).

Composing a cell α with a commutative one can often be reduced to a 'whiskering operation' with a lax or colax double functor, with an evident meaning on computing components. There are four cases

(2)
$$\begin{tikzcd}
 & \bullet \arrow[r, "RM"] \arrow[d, "M"'] \arrow[dr, phantom, "\mu /\!/"] & \bullet \arrow[d, equal] & \\
\bullet \arrow[r, "L"] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha /\!/"] & \mathbb{A} \arrow[r, dashed, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha \swarrow"] & \mathbb{B} \arrow[r, "L'G"] \arrow[d, "G"] \arrow[dr, phantom, "\alpha' /\!/"] & \bullet \arrow[d, "1"] \\
\mathbb{C} \arrow[r, "FL"'] & \mathbb{C} \arrow[r, dashed, "S"'] \arrow[d, equal] \arrow[dr, phantom, "\mu' /\!/"] & \mathbb{D} \arrow[r, "L'"'] \arrow[d, "M'"] & \bullet \\
 & \bullet \arrow[r, "M'S"'] & \bullet &
\end{tikzcd}$$

(a) *If* F *is pseudo*, one can define the whiskering αL with any lax double functor L of codomain $\mathbb{A}$, as αL = (α | α), where α: (FL = FL). On components, the notation αL is fully justified, because (αL)(X) = α(LX).

(b) *If* G *is pseudo*, one can define the whiskering L'α, with any lax double functor L' of domain $\mathbb{D}$, as L'α = (α | α'). On components, (L'α)(A) = L'(αA).

(c) Similarly, if R (resp. S) is pseudo one can define the whiskering αM = $\mu\otimes\otimes$ (resp. M'$\otimes$ = $\otimes\otimes\mu'$), with a colax double functor M (resp. M').

1.5. Universal properties of comma squares. Comma squares satisfy the three universal properties of comma objects in the double category $\mathbb{D}$bl (III.3.2). Moreover, restricting all vertical arrows, they also work in $\mathbb{D}\mathrm{bl}_u$ and $\mathbb{D}\mathrm{bl}_p$ (1.2).

The first two properties have already been proved, in Thm. II.2.6. The horizontal universal property says that

$$(1)\qquad \begin{tikzcd} \mathbb{U} \arrow[rr, "S"] \arrow[d, "I"'] & & \mathbb{A} \arrow[d, "F"] \\ \mathbb{U} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "R"'] \arrow[ur, phantom, "\swarrow\alpha"] & \mathbb{C} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{U} \arrow[r, dashed, "L"] \arrow[rr, bend left, "S"] \arrow[d, "I"'] \arrow[dr, phantom, "\lambda /\!/"] & F\Downarrow R \arrow[r, dashed, "P"] \arrow[d, "Q"] \arrow[dr, phantom, "\swarrow\omega"] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{U} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "R"'] & \mathbb{C} \end{tikzcd}$$

(a) for every pair of *lax* double functors S, T and every cell $\omega\colon (1\ {}^{\ S}_{RT}\ F)$ (in $\mathbb{D}$bl) there is a unique lax double functor $L\colon \mathbb{U} \to F\Downarrow R$ such that $S = PL$, $T = QL$ and $\omega = \omega L = (\omega \mid \omega)$ where $\omega\colon (QL = T)$ is the commutative cell defined by a horizontal identity of lax double functors (and the whiskering ωL is defined in 1.4). Moreover, L is unitary (or pseudo, or strict) if and only if both S and T are.

Similarly, we have a vertical universal property (proved in II.2.6(b)), displayed in the left diagram below:

(b) for every pair of *colax* double functors G, H and every cell $\beta\colon (G\ {}^{1}_{R}\ FH)$ there is a unique colax double functor M such that $G = QM$, $H = PM$ and $\beta = \beta M = \mu\otimes\otimes$ where $\mu\colon (H = PM)$ is a commutative cell. Moreover, M is pseudo if and only if both G and H are.

$$(2)\qquad \begin{tikzcd} \mathbb{V} \arrow[r, equal] \arrow[d, "M"] \arrow[dd, bend right=60, "G"'] \arrow[dr, phantom, "\mu /\!/"] & \mathbb{V} \arrow[d, "H"] \\ F\Downarrow R \arrow[r, dashed, "P"] \arrow[d, "Q"] \arrow[dr, phantom, "\otimes \swarrow"] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{X} \arrow[r, "R"'] & \mathbb{C} \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{W} \arrow[rr, "P'"] \arrow[dr, "M"] \arrow[dd, "Q'"'] & & \mathbb{A} \arrow[d, equal] \arrow[dl, phantom, "\mu /\!/"] \\ & F\Downarrow R \arrow[r, dashed, "P"] \arrow[d, "Q"] \arrow[dl, phantom, "\otimes /\!/"] \arrow[dr, phantom, "\otimes \swarrow"] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{X} \arrow[r, equal] & \mathbb{X} \arrow[r, "R"'] & \mathbb{C} \end{tikzcd}$$

These first two properties reduce, for pseudo double functors, to a *symmetric universal property*, (displayed in the right diagram above) which determines the solution up to isomorphism in a clearer way

(c) for every pair of double functors P', Q' and every cell $\otimes\colon FP' \to RQ'$ (in $\mathbb{D}$bl) there is a unique double functor L such that $P' = PL$, $Q' = QL$ and $\otimes = \otimes L = (\otimes \mid \otimes) = \mu\otimes\otimes$, for the commutative cells $\otimes$ $(QL = Q')$ and $\mu\colon (P' = PL)$.

Finally, there is also a (quite strong) global universal property, III.3.2(c), which will not be used here (but replaced with direct computations). Its verification in $\mathbb{D}$bl is easy, as in III.3.4 for $\mathbb{Q}$**Cat**.

1.6. Pasting Lemma. *In* $\mathbb{D}$bl, *consider the pasting of two double commas, displayed in the left diagram, and the comma* $\otimes''$ *of the vertical composite,* *displayed in the middle diagram*

(1)
$$
\begin{tikzcd}
F'\downdownarrows P \arrow[r, "P'"] \arrow[d, "Q'"'] \arrow[dr, phantom, "\swarrow \omega'"] & \mathbb{B} \arrow[d, "F'"] \\
F\downdownarrows R \arrow[r, dashed, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\swarrow \omega"] & \mathbb{A} \arrow[d, "F"] \\
\mathbb{X} \arrow[r, "R"'] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
(FF')\downdownarrows R \arrow[r, "P''"] \arrow[d, "Q''"'] \arrow[dr, phantom, "\swarrow \omega''"] & \mathbb{B} \arrow[d, "FF'"] \\
\mathbb{X} \arrow[r, "R"'] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[dd, "1"'] & F'\downdownarrows P \arrow[r, "P'"] \arrow[d, "M"] \arrow[dr, phantom, "\mu"] & \mathbb{B} \arrow[d, equal] \\
 & (FF')\downdownarrows R \arrow[r, dashed, "P''"] \arrow[d, "L"] \arrow[dl, phantom, "\swarrow \varphi"] \arrow[dr, phantom, "\lambda"] & \mathbb{B} \arrow[d, equal] \\
\bullet \arrow[r, equal] & F'\downdownarrows P \arrow[r, "P'"'] & \mathbb{B}
\end{tikzcd}
$$

Then the double functor P" *(a projection of the comma of the composite) is a* vertical deformation retract *of* P' *(a projection of the iterated comma), in the strong sense of III.1.6: there are arrows* M, L *as above, commutative cells* λ, μ *and a comparison cell* λ: LM $\to$ $1^\bullet$, *as displayed in the right diagram above, satisfying*:

(2) $ML = 1, \qquad \lambda L = 1_L, \qquad M\lambda = 1_M.$

Proof. This lemma is a particular case of a general Pasting Theorem, holding for internal comma objects in a double category $\mathbb{D}$ (III.5.2). In the present case, i.e. for $\mathbb{D} = \mathbb{D}\mathrm{bl}$, we prefer to give a direct, constructive proof, which is much simpler and shorter than the abstract one.

First, consider the natural comparison M: $F'\downdownarrows P \to (FF')\downdownarrows R$, from the pasting of commas $\lambda'\otimes\otimes$ to the comma $\otimes''$ of the composite FF', by the symmetric universal property of $\otimes''$ (1.5(c))

(3) $P''M = P', \qquad Q''M = QQ', \qquad \otimes''M = \otimes'\otimes\otimes.$

This forms a commutative cell $\bullet = 1$: $P' \to P''M$. To define the second comparison L, we begin by constructing a *colax* N: $(FF')\downdownarrows R \twoheadrightarrow F\downdownarrows R$, using the vertical universal property of $F\downdownarrows R$ (1.5(b))

(4) $PN = F'P'', \qquad QN = Q'', \qquad \otimes N = \otimes'',$

and then a colax L: $(FF')\downdownarrows R \twoheadrightarrow F'\downdownarrows P$, by the vertical universal property of $F'\downdownarrows P$

(5) $P'L = P'', \qquad Q'L = N, \qquad \otimes' L = 1\colon F'P'' \to PN.$

The equality ML = 1 is then detected by the universal property of $(FF')\downdownarrows R$

(6) $P''ML = P'L = P'', \qquad Q''ML = QQ'L = QN = Q'',$

$\otimes''ML = (\otimes'\otimes\otimes)L = (\otimes'L)\otimes\otimes = 1_{PN}\otimes\otimes = \otimes N = \otimes''.$

It can also be proved by direct computation. On objects, we have:

(7) $M(B, A, X; c: FA \to RX; a: F'B \to A) = (B, X; c.Fa: FF'B \to RX)$,
$L(B, X; c: FF'B \to RX) = (B, F'B, X; c: FF'B \to RX; 1: F'B \to F'B)$.
$ML(B, X; c: FF'B \to RX) = (B, X; c.F1)$.

Finally, we have a cell φ with $\varphi L = 1$, $M\varphi = 1$ (in the abstract case, one has to use the *global* universal property of comma objects to construct φ, and again to verify the previous equations)

(8) $\varphi: LM \to 1$,
$\varphi(B, A, X; c, a) = (1_B, a, 1_X): (B, F'B, X; c.Fa, 1_{FB}) \to (B, A, X; c, a)$,
$\varphi L(B, X; c: FF'B \to RX) = \varphi(B, F'B, X; c: FF'B \to RX; 1_{F'B})$
$= (1_B, 1_{FB}, 1_X) = id(L(B, X, c))$,
$M\varphi(B, A, X; c, a) = M(1_B, a, 1_X) = (1_B, 1_X) = id(M(B, A, X; c, a))$. □

1.7. Special Pasting Lemma. *In* $\mathbb{D}$bl, *consider the pasting, in the left diagram, of a comma* ω *and a commutative cell* λ *whose boundary is formed of four* strict *double functors. Then* λ⊗⊗ *is a comma square if and only if* ⊗ *is a pullback (in the category of weak double categories and strict double functors)*

(1)
$$\begin{tikzcd}
\mathbb{Y} \arrow[r, "P'"] \arrow[d, "G"'] \arrow[dr, phantom, "⊗ /\!/"] & \mathbb{B} \arrow[d, "F'"] \\
F\Downarrow R \arrow[r, dashed, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\alpha \swarrow"] & \mathbb{A} \arrow[d, "F"] \\
\mathbb{X} \arrow[r, "R"'] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[rr, bend left, "P''"] \arrow[ddr, bend right, "H"'] \arrow[dr, phantom, "\swarrow\alpha"] & & \\
& \mathbb{Y} \arrow[r, "P'"] \arrow[d, "QG"'] \arrow[dr, phantom, "\alpha\alpha\alpha \swarrow"] & \mathbb{B} \arrow[d, "FF'"] \\
& \mathbb{X} \arrow[r, "R"'] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[rr, bend left, "P''"] \arrow[ddr, bend right, "H"'] \arrow[dr, phantom, "/\!/"] & & \\
& \mathbb{Y} \arrow[r, "P'"] \arrow[d, "G"'] \arrow[dr, phantom, "\alpha /\!/"] & \mathbb{B} \arrow[d, "F'"] \\
& F\Downarrow R \arrow[r, "P"'] & \mathbb{A}
\end{tikzcd}$$

Proof. First, if α is a pullback, given a cell $\alpha: FF'P'' \to RH$ as in the middle diagram above (with strict P'', H), by the symmetric universal property of α there is one double functor $K: \mathbb{Z} \to F\Downarrow R$ such that

(2) $QK = H$, $\qquad PK = F'P''$, $\qquad \alpha K = \alpha$.

Then, by the universal property of α there is one double functor $K': \mathbb{Z} \to \mathbb{Y}$ such that

(3) $P'K' = P''$, $\qquad GK' = K$.

This K' is also the unique double functor satisfying:

(4) $P'K' = P''$, $\qquad QGK' = H$, $\qquad (\alpha\alpha\alpha)K = \alpha$.

Conversely, suppose that $\lambda\otimes\otimes$ is a comma square, and take a commutative cell $\otimes$: (F'P" = PH), as in the right diagram (1). Then, there is a unique double functor $K: \mathbb{Z} \to \mathbb{Y}$ such that

(5) $P'K = P''$, $\qquad QGK = QH$, $\qquad (\otimes\otimes\otimes)K = \otimes\otimes\otimes: FF'P'' \to RQH$,

and then GK = H, as detected by the original comma square $\otimes$

(6) $P(GK) = F'P'K = F'P'' = PH$, $\qquad QGK = QH$,

$\otimes(GK) = (\otimes\otimes\otimes)K = \otimes\otimes\otimes = \otimes H$.

We have thus found a double functor K such that P'K = P" and GK = H; its uniqueness is similarly proved. □

1.8. Pullback Lemma. *Let us suppose that* F, R *are strict double functors.*

(i) *The double comma* $F{\Downarrow}R$ *is linked to the corresponding pullback* $\mathbb{A}\times_{\mathbb{C}}\mathbb{X}$ *(with projections* P', Q'*) by a comparison strict double functor* L *and a commutative cell* $\otimes$ (P' = PL)

$$
(1) \qquad
\begin{tikzcd}
\mathbb{A}\times_{\mathbb{C}}\mathbb{X} \arrow[r, "P'"] \arrow[d, "L"'] \arrow[dr, phantom, "\otimes"] & \mathbb{A} \arrow[d, equal] \\
F{\Downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\otimes"] & \mathbb{A} \arrow[d, "F"] \\
\mathbb{X} \arrow[r, "R"'] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{array}{l}
PL = P', \quad QL = Q', \\
\otimes L = 1: FP' \to RQ', \\
L(A, X) = (A, X, 1: FA = RX).
\end{array}
$$

(ii) *if* $\mathbb{C}$ *is* horizontally discrete (*i.e., all horizontal maps and all cells are horizontal identities), then* L *is an isomorphism:* $F{\Downarrow}R$ *'coincides' with the pullback* $\mathbb{A}\times_{\mathbb{C}}\mathbb{X}$, *so that* P' *is a vertical deformation retract of* P *(III.1.6) and conversely.*

Proof. Obvious. □

2. Pointwise unitary lax right Kan extensions and their models

We focus now on *unitary lax* right Kan extensions, according to the third setting for external Kan extensions considered in III.6.

2.1. Unitary lax right Kan extensions. Let us recall some definitions on the 'kind' of Kan extensions which will be studied here (as introduced in III.6.4).

The general setting will be $\mathrm{Lx}\mathbb{D}\mathrm{bl}_\mathrm{u}$, the double category of weak double categories, lax double functors (horizontally) and *unitary* lax double functors (vertically); a cell $\alpha \colon GR \to SF$, as in the left diagram below, simply is a horizontal transformation between the composed lax double functors GR, SF

(1)
$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathbb{B} \arrow[d, "G"] \\
\mathbb{C} \arrow[r, "S"'] & \mathbb{D}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{I} \arrow[r, "R"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\varepsilon"] & \mathbb{J} \arrow[r, equal] \arrow[d, "G"] \arrow[dr, phantom, "\swarrow\tau"] & \mathbb{J} \arrow[d, "G'"] \\
\mathbb{I} \arrow[r, "S"'] & \mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}
$$

Working in this setting, the *unitary lax* right Kan extension of a *lax* double functor $S \colon \mathbb{I} \to \mathbb{A}$ along a *lax* double functor $R \colon \mathbb{I} \to \mathbb{J}$, as in the right diagram above, is a *unitary lax* double functor G equipped with a cell τ, such that any similar data (G', τ') factors through τ, by a unique special cell, consisting of a horizontal transformation $\tau \colon G' \to G$. We write $G = \mathrm{Ran}_R(S)$, with a notation adapted to the present case. (According to the general definitions of Part III, we should rather write $\mathrm{Ran}_{R,S}(\mathrm{id}\mathbb{I})$, and speak of an extension *of* $\mathrm{id}\mathbb{I}$ *along* R, S).

Now, $\mathrm{Lx}\mathbb{D}\mathrm{bl}_\mathrm{u}$ does not have all comma objects, but its double subcategory $\mathrm{Lx}\mathbb{D}\mathrm{bl}_\mathrm{p} = \mathbb{D}\mathrm{bl}_\mathrm{p}$ (with *unitary pseudo* double functors as vertical arrows) plainly does: just the double commas $F{\downdownarrows}R$ of the previous section, where F is so restricted (as already noted in 1.5). And the diagrammatic lemmas of the previous section also hold with this restriction, for all vertical arrows.

Therefore, speaking of *pointwise* Kan extensions in the present setting we will always mean *pointwise on unitary pseudo double functors,* as defined in III.4.1(b): G is the pointwise *unitary lax* right Kan extension of S along R (both lax) if for every *unitary pseudo* double functor $H \colon \mathbb{J}' \twoheadrightarrow \mathbb{J}$, we have $GH = \mathrm{Ran}_P(SQ)$, via $\omega\otimes\otimes$ (where $\otimes$ is the comma cell of $H{\downdownarrows}R$)

(2)
$$
\begin{tikzcd}
H{\downdownarrows}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\swarrow\otimes"] & \mathbb{J}' \arrow[d, "H"] \\
\mathbb{I} \arrow[r, "R"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\otimes"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{I} \arrow[r, "S"'] & \mathbb{A}
\end{tikzcd}
$$

Then, G *is* indeed a Kan extension, as proved in III.4.2.

This framework is adequate for studying *pointwise extensions and their* *relationship with limits*, and will be the main object of the rest of this paper.

2.2. Vertical models. Three elementary (strict) double categories, the *vertical models* $\underline{1}$, $\underline{2}$, $\underline{3}$, will play a relevant role - in detecting objects, vertical arrows and their composition. The *formal object* $\underline{1}$, or *singleton*, is the trivial double category on one object 0. The *formal vertical arrow* $\underline{2}$ has one vertical arrow $0 \nrightarrow 1$ (and is otherwise trivial). The *formal vertical composite* $\underline{3}$ is the double category spanned by two vertical arrows $0 \nrightarrow 1 \nrightarrow 2$.

A strict double functor $\mathbf{a}\colon \underline{1} \to \mathbb{A}$ amounts to an object $A = \mathbf{a}(0)$ of $\mathbb{A}$; however, it is often better to distinguish between $\mathbf{a}$ and A (as we show at the end of this subsection). Similarly, a strict $\mathbf{u}\colon \underline{2} \to \mathbb{A}$ amounts to a vertical arrow u and a strict $W\colon \underline{3} \to \mathbb{A}$ to a vertical composition $w = u \otimes v$ (of vertical arrows).

Our models are linked by some canonical double functors, which will also be useful to detect the structure of double categories

$$(1) \qquad \underline{1} \;\overset{\rightrightarrows}{\leftarrow}\; \underline{2} \;\overset{\to}{\rightrightarrows}\; \underline{3}$$

namely, two *faces* $\partial_0, \partial_1\colon \underline{1} \to \underline{2}$, the *degeneracy* $e\colon \underline{2} \to \underline{1}$, and three embeddings $\underline{2} \to \underline{3}$. In particular

$$(2) \quad c\colon \underline{2} \to \underline{3}, \qquad\qquad c(0 \to 1) = 0 \to 2,$$

will be called the *precomposition*; in fact, on a strict $W\colon \underline{3} \to \mathbb{A}$ as above, it gives $Wc = \mathbf{w}\colon \underline{2} \to \mathbb{A}$.

On the other hand, a *lax* double functor $T\colon \underline{1} \to \mathbb{A}$ amounts to a *vertical monad* in $\mathbb{A}$

$$(3) \quad \begin{aligned} A &= T(0), & \qquad t &= T1_0^\bullet\colon A \nrightarrow A,\\ \eta &= T[0]\colon 1_A^\bullet \to t, & \mu &= T[1_0^\mu, 1_0^\mu]\colon t\eta t \to t, \end{aligned}$$

formed of an object A, a vertical endoarrow t and two special cells η, μ satisfying the usual axioms, because of the coherence conditions on T. A colax double functor $\underline{1} \to \mathbb{A}$ gives a vertical comonad.

Coming back to the need to distinguish the double functor $\mathbf{a}$ from the object $A = \mathbf{a}(0)$, consider the following compositions

$$(4) \qquad \mathbb{I} \xrightarrow{R} \underline{1} \xrightarrow{\mathbf{a}} \mathbb{A} \xrightarrow{S} \mathbb{B}$$

At the left hand, the composite $\mathbf{a}R\colon \mathbb{I} \to \mathbb{A}$ is a constant double functor, and can be written as AR without ambiguity. But, at the right, the lax double functor S:

$\mathbb{A} \to \mathbb{B}$ yields a vertical monad **Sa** in $\mathbb{B}$, which cannot be reduced to its support, the object $S(A)$.

2.3. Reduction Theorem I [Models for pointwise extensions]. *Let* $\varepsilon\colon GR \to S$ *be a cell in the double category* $\mathrm{Lx}\mathbb{D}\mathrm{bl}_u$, *as in 2.1.1.*

Then $G\colon \mathbb{J} \twoheadrightarrow \mathbb{A}$ *is the pointwise unitary lax Ran via* ε *if (and only if) it satisfies the pointwise property for strict double functors defined on the* models $\mathbb{K} = \underline{1}, \underline{2}, \underline{3}$ *considered above (2.2). In other words, if and only it satisfies the following condition, with* $\underline{n} = \underline{1}, \underline{2}, \underline{3}$

(a) for every strict *double functor* $J\colon \underline{n} \twoheadrightarrow \mathbb{J}$, *we have* $GJ = \mathrm{Ran}_{P_J}(SQ_J)$ *via the vertical pasting* $\omega_J \otimes\otimes$ *in* $\mathbb{D}$ *(where* $\otimes_J$ *is the cell of the double comma* $J{\Downarrow\!\Downarrow}R$).

Note. This theorem also holds in the unitary colax case, as defined in III.6.3.

Proof. Assume that G satisfies the condition (a). Let us fix a unitary pseudo double functor $H\colon \mathbb{K} \twoheadrightarrow \mathbb{J}$, a unitary lax $G'\colon \mathbb{K} \twoheadrightarrow \mathbb{A}$ and a cell $\alpha\colon G'P \to SQ$. We have to prove that there is precisely one cell $\tau\colon G' \to GH$ such that $(\tau\tau\tau \mid \tau) = \tau$. The main argument below shows the existence, while the simpler argument for uniqueness is given in brackets.

(A) *Definition on objects.* Take first an object k in $\mathbb{K}$, as a strict double functor $\mathbf{k}\colon \underline{1} \to \mathbb{K}$, and complete the following diagram with the upper pullback λ_k, so that $\lambda_{\mathbf{k}} = \lambda_k\lambda\,\lambda$ provides $\mathbf{Hk}{\Downarrow\!\Downarrow}R$ (Special Pasting Lemma, 1.7)

$$(1)\qquad \begin{tikzcd}
\mathbf{Hk}{\Downarrow\!\Downarrow}R \arrow[r, "P_k"] \arrow[d, "Q_k"'] \arrow[dr, phantom, "\lambda_k"] & \underline{1} \arrow[r, equal] \arrow[d, "\mathbf{k}"] & \underline{1} \arrow[d, "\mathbf{k}"] \\
H{\Downarrow\!\Downarrow}R \arrow[r, dashed, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\lambda\lambda\lambda"] & \mathbb{K} \arrow[d, "GH"] \arrow[r, phantom, "\lambda\mathbf{k}"] & \mathbb{K} \arrow[d, "G'"] \\
\mathbb{X} \arrow[r, "S"'] & \mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}$$

Applying the hypothesis on $J = \mathbf{Hk}$, we get $G.\mathbf{Hk} = \mathrm{Ran}_{P_J}(SQQ_k)$ via $\lambda_k\lambda\,\lambda\lambda\,\lambda$, so that the cell $\lambda Q_k = \lambda_k\lambda\,\lambda\colon G'\mathbf{k}P_k \to SQQ_k$ factors through $\lambda_k\lambda\,\lambda\lambda\,\lambda$ via a unique cell $\lambda\mathbf{k}$, forming a horizontal transformation of unitary lax double functors

(2) $\lambda\mathbf{k}\colon G'\mathbf{k} \to G\mathbf{Hk}\colon \underline{1} \to \mathbb{A}$, $\qquad (\lambda_k\lambda\,\lambda\lambda\,\lambda \mid \lambda\mathbf{k}) = \lambda_k\lambda\,\lambda$.

This defines, in $\mathbb{A}$, a horizontal morphism $\lambda k = (\lambda\mathbf{k})(0)\colon G'(k) \to GH(k)$ and a cell $\lambda 1^\bullet_k$

$$(3)\qquad \begin{tikzcd} G'k \arrow[r, "\tau k"] \arrow[d, dotted, "G'(1^\bullet_k)"'] \arrow[dr, phantom, "\tau 1^\bullet_k"] & GHk \arrow[d, dotted, "GH(1^\bullet_k)"] \\ G'k \arrow[r, "\tau k"'] & GHk \end{tikzcd} \qquad\qquad \tau 1^\bullet_k = (\tau\mathbf{k})(1^\bullet_0).$$

The latter coincides with $1^\bullet_{\tau k}$, by coherence with the laxity cells of the (unitary!) vertical functors GH and G', on the object k

(4) $(G'[k] \mid \tau 1^\bullet_k) = (1^\bullet_{\tau k} \mid (GH)[k])$.

(On the other hand, if we are given a cell $\tau\colon G' \to GH$ such that $(\omega\otimes\varepsilon \mid \varepsilon) = \alpha$, then the composite $\lambda\mathbf{k}$ is uniquely determined as above: $(\lambda_k\lambda\,\lambda\lambda\lambda \mid \lambda\mathbf{k}) = \lambda_k\lambda\,(\lambda\lambda\lambda \mid \lambda) = \lambda_k\lambda\,\lambda$.)

(B) *Naturality on objects.* Now, given a horizontal map $f\colon k \to k'$ in $\mathbb{K}$, let us verify that

(5) $\lambda k'.G'f = GHf.\lambda k\colon G'k \to GHk'$,

viewing f as a horizontal transformation $f\colon \mathbf{k} \to \mathbf{k}'\colon \underline{1} \to \mathbb{K}$. First, we link the pullbacks λ_k and $\lambda_{k'}$ with the double functor $f^*\colon \mathbf{Hk'}{\downdownarrows}R \to \mathbf{Hk}{\downdownarrows}R$ such that

(6) $P_k f^* = P_{k'}$, $\qquad Q_k f^* = Q_{k'}$, $\qquad (\lambda_k f^* = (\lambda_{k'} \mid f))$.

This gives a commutative cell $\lambda_f\colon P_{k'} \to P_k f^*$ such that $\lambda_f\lambda\,\lambda_k = \lambda_k f^* = (\lambda_{k'} \mid f)$. Now, it suffices to cancel $(\lambda_{k'}\lambda\,\lambda\lambda\lambda \mid -)$ in the first terms of the following equations

(7) $(\lambda_{k'}\lambda\,\lambda\lambda\lambda \mid \lambda\mathbf{k}' \mid G'f) = (\lambda_{k'}\lambda\,\lambda \mid G'f) = (\lambda_{k'} \mid f)\lambda\,\lambda$,

$\quad (\lambda_{k'}\lambda\,\lambda\lambda\lambda \mid GHf \mid \lambda\mathbf{k}) = ((\lambda_{k'} \mid f)\lambda\,\lambda\lambda\lambda \mid \lambda\mathbf{k}) = (\lambda_f\lambda\,\lambda_k\lambda\,\lambda\lambda\lambda \mid \lambda\mathbf{k})$

$\qquad = \lambda_f\lambda\,(\lambda_k\lambda\,\lambda\lambda\lambda \mid \lambda\mathbf{k}) = \lambda_f\lambda\,\lambda_k\lambda\,\lambda = (\lambda_{k'} \mid f)\lambda\,\lambda$,

$$\begin{tikzcd} \mathbf{Hk'}{\downdownarrows}R \arrow[r, "P_{k'}"] \arrow[d, dotted, "f^*"'] \arrow[dr, phantom, "\lambda_f /\!/"] & \underline{1} \arrow[r, equal] \arrow[d, dotted] \arrow[dr, phantom, "/\!/"] & \underline{1} \arrow[d, dotted] \\ \mathbf{Hk}{\downdownarrows}R \arrow[r, "P_k"] \arrow[d, dotted, "Q_k"'] \arrow[dr, phantom, "\lambda_k /\!/"] & \underline{1} \arrow[r, equal] \arrow[d, dotted, "\mathbf{k}"] \arrow[dr, phantom, "\lambda\mathbf{k} \swarrow"] & \underline{1} \arrow[d, dotted, "\mathbf{k}"] \\ H{\downdownarrows}R \arrow[r, "P"] \arrow[d, dotted, "Q"'] \arrow[dr, phantom, "\lambda\lambda\lambda \swarrow"] & \mathbb{K} \arrow[d, dotted, "GH"] & \mathbb{K} \arrow[d, dotted, "G'"] \\ \mathbb{X} \arrow[r, "S"] & \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbf{Hk'}{\downdownarrows}R \arrow[r] \arrow[d, dotted, "Q_{k'}"'] \arrow[dr, phantom, "\lambda_{k'} /\!/"] & \underline{1} \arrow[r, equal] \arrow[d, dotted, "\mathbf{k}'"] \arrow[dr, phantom, "f \swarrow"] & \underline{1} \arrow[d, dotted, "\mathbf{k}"] \\ \mathbf{Hk}{\downdownarrows}R \arrow[r, "P"] \arrow[d, dotted, "Q"'] & \mathbb{K} \arrow[r, equal] \arrow[dr, phantom, "\lambda \swarrow"] & \mathbb{K} \arrow[d, dotted, "G'"] \\ \mathbb{X} \arrow[r, "S"] & \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd}$$

(C) *Definition and naturality on vertical arrows.* We have to define the component of τ on a vertical arrow $u\colon k \to\!\!\!\!\!\!\bullet\;\, k'$ in $\mathbb{K}$, which will be viewed as a strict double functor $\mathbf{u}\colon \underline{2} \to \mathbb{K}$. Proceeding as above on the diagram below, we find a unique cell $\tau\mathbf{u}$ between lax double functors (and again, if τ is given, the composite $\tau\mathbf{u}$ is necessarily so determined)

(8) $\tau\mathbf{u}\colon G'\mathbf{u} \to GH\mathbf{u},$ $\qquad\qquad (\lambda_u \otimes \omega\omega\varepsilon \mid \varepsilon\mathbf{u}) = \varepsilon\,_u\varepsilon\,\alpha,$

$$\begin{tikzcd}
H\mathbf{u}{\downdownarrows}R \arrow[r, "P_u"] \arrow[d, "Q_u"'] \arrow[dr, phantom, "\alpha_u /\!/"] & \underline{2} \arrow[r, equal] \arrow[d, "\mathbf{u}"] & \underline{2} \arrow[d, "\mathbf{u}"] \\
H{\downdownarrows}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\alpha\alpha\alpha \swarrow"] & \mathbb{K} \arrow[d, "GH"] \arrow[r, phantom, "\alpha\mathbf{u} \swarrow"] & \mathbb{K} \arrow[d, "G'"] \\
\mathbb{X} \arrow[r, "S"] & \mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}$$

This defines a cell $\alpha u = (\alpha\mathbf{u})(0 \to 1)$ in $\mathbb{A}$, whose naturality is proved as above, in point (B)

$$(9) \qquad \begin{tikzcd}
G'k \arrow[r] \arrow[d, "G'u"'] \arrow[dr, phantom, "\alpha u"] & GHk \arrow[d, "GHu"] \\
G'k' \arrow[r] & GHk'
\end{tikzcd}$$

(D) *Consistence of definitions.* First, we verify that the upper horizontal map of αu, in (9), *does* coincide with αk (similarly, the second coincides with $\alpha k'$). This is proved by cancelling $(\alpha_k\alpha\,\alpha\alpha\alpha \mid -)$ in the following equation, where α is the commutative cell produced by the first face $\partial_0\colon \underline{1} \to \underline{2}$, so that $\partial\partial\,\partial_u = \partial_k$

(10) $(\partial_k\partial\,\partial\,\partial\,\partial \mid (\partial\mathbf{u})\partial_0) = (\partial\partial\,\partial_u\partial\,\partial\,\partial\,\partial \mid (\partial\mathbf{u})\partial_0) = \partial\partial\,(\partial_u\partial\,\partial\,\partial\,\partial \mid \partial\mathbf{u})$

$= \partial\partial\,\partial_u\partial\,\partial = \partial_k\partial\,\partial = (\partial_k\partial\,\partial\,\partial\,\partial \mid \partial\mathbf{k}),$

$$(11) \qquad \begin{tikzcd}
H\mathbf{k}{\downdownarrows}R \arrow[r, "P_k"] \arrow[d] \arrow[dr, phantom, "\partial /\!/"] & \underline{1} \arrow[r, equal] \arrow[d, "\partial_0"] \arrow[dr, phantom, "/\!/"] & \underline{1} \arrow[d, "\partial_0"] \\
H\mathbf{u}{\downdownarrows}R \arrow[r, "P_u"] \arrow[d, "Q_u"'] \arrow[dr, phantom, "\partial_u /\!/"] & \underline{2} \arrow[r, equal] \arrow[d, "\mathbf{u}"] & \underline{2} \arrow[d, "\mathbf{u}"] \\
H{\downdownarrows}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\partial\,\partial\,\partial \swarrow"] & \mathbb{K} \arrow[d, "GH"] \arrow[r, phantom, "\partial\mathbf{u} \swarrow"] & \mathbb{K} \arrow[d, "G'"] \\
\mathbb{X} \arrow[r, "S"] & \mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}$$

Second, we show that, for a vertical identity $u = 1^\bullet_k$, the present definition of τu coincides with $\tau 1^\bullet_k = (\tau\mathbf{k})(1^\bullet_0)$, as found above, in (3). This is proved by cancelling $(\lambda_u \otimes \omega\omega\varepsilon \mid -)$ in the equation (12), where ε is the commutative cell produced by the degeneracy $e\colon \underline{2} \to \underline{1}$, so that $\varepsilon\varepsilon\, \varepsilon_k = \varepsilon_u$

(12) $\quad (\varepsilon_u\varepsilon\ \varepsilon\ \varepsilon\ \varepsilon \mid \varepsilon\mathbf{u}) \;=\; \varepsilon_u\varepsilon\ \alpha \;=\; \alpha\alpha\,\alpha_k\alpha\,\alpha \;=\; \alpha\alpha(\alpha_k\alpha\,\alpha\alpha\,\alpha \mid \alpha\mathbf{k})$

$\quad = (\alpha\alpha\,\alpha_k\alpha\,\alpha\alpha\,\alpha \mid (\alpha\mathbf{k})e),$

$$\begin{tikzcd}
Hu\Downarrow\Downarrow R \arrow[r, "P_u"] \arrow[d] & \underline{2} \arrow[r, equal] \arrow[d, "e"] \arrow[dl, phantom, "\alpha" near start] \arrow[dr, phantom, "//"] & \underline{2} \arrow[d, "e"] \\
Hk\Downarrow\Downarrow R \arrow[r, "P_k"] \arrow[d, "Q_k"'] & \underline{1} \arrow[r, equal] \arrow[d, "k"] \arrow[dl, phantom, "\alpha_k" near start] & \underline{1} \arrow[d, "k"] \\
H\Downarrow\Downarrow R \arrow[r, "P"] \arrow[d, "Q"'] & \mathbb{K} \arrow[d, "GH"] \arrow[dl, phantom, "\alpha\alpha\alpha" near start] & \mathbb{K} \arrow[d, "G'"] \arrow[l, phantom, "\alpha\mathbf{k}"] \\
\mathbb{X} \arrow[r, "S"] & \mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}$$

(E) *Coherence*. Taking into account the last result, the condition (c2) on $\alpha\colon G' \to GH$ has already been verified in (4), and we are left with proving (c3). Take a vertical composite $w = u\alpha v$ in $\mathbb{K}$, amounting to a strict $K\colon \underline{3} \to \mathbb{K}$; we have to prove that

(13) $\quad (G'[u, v] \mid \alpha w) \;=\; (\alpha u \alpha\, \alpha v \mid (GH)[u, v]).$

By the pointwise condition on $\underline{3}$, $GHK = \mathrm{Ran}_{P_K}(SQQ_K)$, via the cell $\alpha_K = \alpha_K\alpha\,\alpha$ (see the diagram (15)). This provides one horizontal transformation of lax double functors $\alpha K\colon G'K \to GHK$ such that $(\alpha_K\alpha\,\alpha\alpha\,\alpha \mid \alpha K) = \alpha_K\alpha\,(\alpha K)$. By precomposing with the three embeddings $\underline{2} \to \underline{3}$, we can show (see below) that

(14) $\quad \alpha K(0 \to 1) = \alpha u, \qquad \alpha K(1 \to 2) = \alpha v, \qquad \alpha K(0 \to 2) = \alpha w,$

so that the coherence condition (c3) on the cell αK, taking into account the strictness of K, gives precisely the thesis, equation (13).

As to (14), its last equation, for instance, is proved by cancelling $(\alpha_w\alpha\,\alpha\alpha\,\alpha \mid -)$ in the equation below, where $c\colon \underline{2} \to \underline{3}$ is the *precomposition* (2.2.2) and α is the commutative cell produced by the former, so that $\alpha K(0 \to 2) = (\alpha K)c$ and $\alpha\alpha\,\alpha_K = \alpha_w$

(15) $\quad (\alpha_w\alpha\,\alpha\alpha\,\alpha \mid \alpha\mathbf{w}) \;=\; \alpha_w\alpha\,\alpha \;=\; \alpha\alpha\,\alpha_K\alpha\,\alpha \;=\; \alpha\alpha(\alpha_K\alpha\,\alpha\alpha\,\alpha \mid \alpha K)$

$\quad = (\alpha\alpha\,\alpha_K\alpha\,\alpha\alpha\,\alpha \mid (\alpha K)c) \;=\; (\alpha_w\alpha\,\alpha\alpha\,\alpha \mid (\alpha K)c),$

$$\begin{tikzcd}
\mathrm{Hw}{\Downarrow\Downarrow}\mathrm{R} \arrow[r, "P_w"] \arrow[d] \arrow[dr, phantom, "\lambda"] & \underline{2} \arrow[r, equal] \arrow[d, "c"] \arrow[dr, phantom, "//"] & \underline{2} \arrow[d, "c"] \\
\mathrm{HK}{\Downarrow\Downarrow}\mathrm{R} \arrow[r, "P_K"] \arrow[d, "Q_K"'] \arrow[dr, phantom, "\lambda_K"] & \underline{3} \arrow[r, equal] \arrow[d, "K"] & \underline{3} \arrow[d, "K"] \\
\mathrm{H}{\Downarrow\Downarrow}\mathrm{R} \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\omega\otimes\varepsilon"] & \mathbb{K} \arrow[d, "GH"] \arrow[dr, phantom, "\tau K"] & \mathbb{K} \arrow[d, "G'"] \\
\mathbb{X} \arrow[r, "S"] & \mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}$$

□

2.4. Reduction Theorem II [The vertically trivial case]. *Let again* ε: GR → S *be a cell in* $\mathrm{Lx}\mathbb{D}\mathrm{bl}_u$, *as in 2.1.1. If in* $\mathbb{J}$ *all the vertical compositions* $u\varepsilon\, v$ *are trivial (i.e.,* u *or* v *is an identity) and all its special cells are horizontal identities, then in condition 2.3(a) it is sufficient to take* $\underline{n} = \underline{1}, \underline{2}$.

Note 1. If R is an ordinary functor, viewed as a double functor between the associated 'horizontal' double categories, the problem we are considering amounts to the one which led to formulating the Conduché condition for R (cf. 5.1).

Note 2. Taking $\underline{n} = \underline{1}$ in condition 2.3(a) is not sufficient, even when all vertical arrows are trivial and all cells are horizontal identities, as is the case with $\mathbb{J} = \underline{1}$ (cf. Section 3).

Proof. It suffices to prove that, in the present case, point (E) of the previous proof (i.e., verifying condition (c3) on ε) can be proved by means of the pointwise properties on $\underline{1}, \underline{2}$.

Take a vertical composition $w = u\varepsilon\, v$: $k \twoheadrightarrow k' \twoheadrightarrow k''$ in $\mathbb{K}$. One at least of Hu and Hv is a vertical identity; let us choose $Hv = 1^{\bullet}_{Hk'}$; moreover, the special cell H[u, v]: Hw → $Hu\varepsilon\, Hv$ = Hu is a horizontal identity, which implies that Hw = Hu; since G is *unitary* lax, also the following comparison cell is trivial

(1) $(GH)[u, v] = G[Hu, 1^{\bullet}_{Hk'}] = 1_{GHu}$.

Our thesis amounts thus to proving that

(2) $(G'[u, v] \mid \varepsilon w) = \varepsilon u\, \varepsilon\, \varepsilon v$.

Coming back to the definition of GHu and εu in Point (C) of the previous proof, via a right Kan extension along P_u: **Hu**${\Downarrow\Downarrow}$R → $\underline{2}$, we have a universal cone $\pi = \pi_u\pi\,\pi\pi\,\pi$: GH**u**$P_u$ → SQQ_u indexed on **Hu**${\Downarrow\Downarrow}$R

(3) $$\begin{tikzcd} GHk \arrow[r, "\pi(i,h)"] \arrow[d, "GHu"'] \arrow[dr, phantom, "\pi(t,j)"] & Si \arrow[d, "St"] \\ GHk' \arrow[r, "\pi(i',h')"'] & Si' \end{tikzcd}$$

$t: i \nrightarrow i'$ in $\mathbb{I}$,

$j: (Hu \,{}^{h}_{h'}\, Rt)$ in $\mathbb{J}$,

(4) $$\begin{tikzcd} G'k \arrow[r] \arrow[d, "G'u"'] \arrow[dr, phantom, "\tau u"] & GHk \arrow[r] \arrow[d, "GHu"] \arrow[dr, phantom, "\tau(t,j)"] & Si \arrow[d, "St"] \\ G'k' \arrow[r] & GHk' \arrow[r] & Si' \end{tikzcd} = \begin{tikzcd} G'k \arrow[r] \arrow[d, "G'u"'] \arrow[dr, phantom, "\alpha(u,t,j)"] & Si \arrow[d, "St"] \\ G'k' \arrow[r] & Si' \end{tikzcd}$$

Equation (2) comes from cancelling the projections $\alpha(t, j)$ in the following

(5) $(\alpha u \otimes \otimes v \mid \otimes(t, j)) = \otimes(u, t, j) \otimes \otimes(v, 1^{\bullet}, 1^{\bullet}) = (G'[u, v] \mid \otimes(w, t, j))$
$= (G'[u,v] \mid \otimes w \mid \otimes(t, j))$,

where the first equality follows form the coherence of $\otimes$ with the laxity cells of G':

(6) $$\begin{tikzcd} G'k \arrow[r] \arrow[d, "G'u"'] \arrow[dr, phantom, "\otimes u"] & GHk \arrow[r] \arrow[d, "GHu"'] \arrow[dd, bend left, "GHw"] \arrow[ddr, phantom, "\otimes(t,j)"] & Si \arrow[dd, "St"] \\ G'k' \arrow[r] \arrow[d, "G'v"'] \arrow[dr, phantom, "\otimes v"] & GHk' \arrow[d, "1^{\bullet}"'] & \\ G'k'' \arrow[r] & GHk' \arrow[r] & Si' \end{tikzcd} =$$

$$\begin{tikzcd} G'k \arrow[r] \arrow[d, "G'u"'] \arrow[dr, phantom, "\otimes u"] & GHk \arrow[r] \arrow[d, "GHu"] \arrow[dr, phantom, "\otimes(t,j)"] & Si \arrow[d, "St"] \\ G'k' \arrow[r] \arrow[d, "G'v"'] \arrow[dr, phantom, "\otimes v"] & GHk' \arrow[r] \arrow[d, "1^{\bullet}"] \arrow[dr, phantom, "1^{\bullet}"] & Si' \arrow[d, "1^{\bullet}"] \\ G'k'' \arrow[r] & GHk' \arrow[r] & Si' \end{tikzcd} = \begin{tikzcd} G'k \arrow[r] \arrow[d, "G'u"'] \arrow[dr, phantom, "\otimes(u,t,j)"] & Si \arrow[d, "St"] \\ G'k' \arrow[r] \arrow[d, "G'v"'] \arrow[dr, phantom, "\otimes(v,1^{\bullet},1^{\bullet})"] & Si' \arrow[d, "1^{\bullet}"] \\ G'k'' \arrow[r] & Si' \end{tikzcd}$$ □

3. Right Kan extensions on the singleton

Pointwise unitary lax right Kan extensions on the singleton $\underline{1}$ amount to double limits, as studied in Part I.

3.1. Reviewing double limits. In this section we study right Kan extensions on $\mathbb{J} = \underline{1}$. Note that a comma with a double functor $\mathbb{K} \rightarrow \underline{1}$ is the same as the corresponding pullback, i.e. the product $-\times\mathbb{K}$ (a trivial fact, and an instance of the Pullback Lemma,1.8(ii)). Recall also that a unitary lax double functor $A\colon \underline{1} \to \mathbb{A}$ amounts to an object of $\mathbb{A}$ (and is strict).

Let us begin by rewriting the definition of the double limit of a lax double functor $S\colon \mathbb{I} \to \mathbb{A}$ (I.4), with the present terminology.

A horizontal cone for S (as defined in I.4.1), is a pair (A, x), where A is an object of $\mathbb{A}$ and $x\colon AR \to S$ is a horizontal transformation defined on the double functor AR, constant at A (and a cell in $\mathrm{Lx}\mathbb{D}\mathrm{bl}_u$)

(1)
$$\begin{tikzcd} \mathbb{I} \arrow[r, "R"] \arrow[d, "1"'] & \underline{1} \arrow[d, "A"] \arrow[dl, phantom, "\swarrow x"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \end{tikzcd}$$

Explicitly, the horizontal transformation x amounts to the following data (a), (b), subject to the axioms (hc.0-3) (where (hc.0), in fact, follows from (hc.1)):

(a) horizontal maps $xi\colon A \to Si$, for i in $\mathbb{I}$,

(b) cells $xu\colon (1^\bullet_A \; {}^{xi}_{xj} \; Su)$, for $u\colon i \twoheadrightarrow j$ in $\mathbb{I}$,

(hc.0) $Sf.xi = xi'$, for $f\colon i \to i'$ in $\mathbb{I}$,

(hc.1) $(xu \mid Sa) = xv$, for $a\colon (u \; {}^{f}_{g} \; v)$ in $\mathbb{I}$,

(hc.2) $(x(1^\bullet_i) \mid S\varphi i) = (1^\bullet_{xi} \mid S[i])\colon (1^\bullet_A \; {}^{xi}_{xi} \; 1^\bullet_{Si})$, for i in $\mathbb{I}$,

(hc.3) $(x(u\otimes v) \mid S\otimes(u, v)) = (xu\otimes xv \mid S[u, v])$, for u, v vertical in $\mathbb{I}$.

The cone (A, x) is said to be the *1-dimensional double limit* of S (I.4.2) if:

(dl.1) for every A' in $\mathbb{A}$, the mapping $[A', A] \to [A'R, S]$, $t \mapsto (x \mid t)$ is bijective; in other words, for every cone $(A', x'\colon A'R \to S)$ there is precisely one horizontal map $t\colon A' \to A$ in $\mathbb{A}$ such that $(x \mid t) = x'$.

(2)
$$\begin{tikzcd} \mathbb{I} \arrow[r, "R"] \arrow[d, "1"'] & \underline{1} \arrow[r, equal] \arrow[d, "A"] \arrow[dl, phantom, "\swarrow x"] & \underline{1} \arrow[d, "A'"] \arrow[dl, phantom, "\swarrow t"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{I} \arrow[r, "R"] \arrow[d, "1"'] & \underline{1} \arrow[d, "A'"] \arrow[dl, phantom, "\swarrow x'"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \end{tikzcd}$$

Furthermore, (A, x) is a *double limit* (in the full 2-dimensional sense) if it satisfies the following stronger property (written in the present notation) (dl.2) for every vertical arrow $v\colon A' \twoheadrightarrow A''$ in $\mathbb{A}$, the mapping $[v, AH] \to [vP, SQ]$, $\tau \mapsto (\lambda\otimes x \mid \otimes)$ (as in the following diagram) is bijective

$$(3) \qquad \begin{tikzcd}
\underline{2}\times\mathbb{I} \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\otimes"] & \underline{2} \arrow[r, equal] \arrow[d, "H"] & \underline{2} \arrow[dd, "v"] \\
\mathbb{I} \arrow[r, "R"] \arrow[d, "1"'] \arrow[dr, phantom, "x"] & \underline{1} \arrow[d, "A"] \arrow[r, phantom, "\otimes"] & \\
\mathbb{I} \arrow[r, "S"'] & \mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}$$

In other words, we are saying that $AH = \mathrm{Ran}_P(SQ)$. Applying the second Reduction Theorem (2.4), this is clearly equivalent to the complete pointwise condition for $\mathrm{Ran}_R(S)$.

We have thus proved the following characterisation.

3.2. Theorem [Double limits as Kan extensions]. *Given a lax double functor* $S\colon \mathbb{I} \to \mathbb{A}$, *its 1-dimensional double limit* (A, x) *amounts to the unitary lax right Kan extension* $\mathrm{Ran}_R(S)$ *along the projection* $R\colon \mathbb{I} \to \underline{1}$. *The cone* (A, x) *is the (2-dimensional) double limit if and only if this extension is pointwise.*

Proof. Already given above. □

3.3. Theorem [The construction theorem for double limits, Part I]. *The double limit of a lax double functor* $S\colon \mathbb{I} \to \mathbb{A}$ *defined on a small pseudo double category* $\mathbb{I}$ *can be constructed with the 'basic' double limits in* $\mathbb{A}$, *considered in Part I:* small double products, double equalisers *(of horizontal arrows) and* tabulators *(the latter being the double limit of a vertical arrow).*

Proof. See I.5.5-5.7. □

3.4. Lemma [Computing pointwise extensions]. *Let* $G = \mathrm{Ran}_R(S)$ *be a general pointwise unitary lax right Kan extension, via* $x\colon GR \to S$ *(as in 2.1.1). Then, its 1-dimensional horizontal entries, for an object* j *and an arrow* $f\colon j \to j'$ *in* $\mathbb{J}$, *can be computed as the following double limits (2-dimensional)*

(1) $G(j) = \lim(SQ_j)$, $\qquad SQ_j\colon (j{\Downarrow\Downarrow}R) \to \mathbb{I} \to \mathbb{A}$,

$p(i, h) = x(i)\circ Gh\colon G(j) \to G(Ri) \to Si$ $\qquad (h\colon j \to Ri$ in $\mathbb{J})$,

(2) $G(f)\colon G(j) \to G(j')$,

$\quad p(i, h')\circ G(f) \;=\; p(i, h'f)\colon\; G(j) \to Si' \qquad\qquad (h'\colon j' \to Ri \text{ in } \mathbb{J}).$

Proof. For j in $\mathbb{J}$, apply the pointwise property along $H = j\colon \underline{1} \to \mathbb{J}$, as in the following diagram

$$
(3)\qquad
\begin{tikzcd}
j{\Downarrow}R \arrow[r, "P"] \arrow[d, "Q_j"'] \arrow[dr, phantom, "\swarrow\omega"] & \underline{1} \arrow[d, "j"] \\
\mathbb{I} \arrow[r, dashed, "R"] \arrow[d, "1"'] \arrow[dr, phantom, "\swarrow x"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{I} \arrow[r, "S"'] & \mathbb{A}
\end{tikzcd}
\qquad
\begin{array}{l}
Q_j(i, h\colon j \to Ri) \;=\; i,\\
\omega(i, h) \;=\; h\colon j \to Ri,\\
(\omega\otimes x)(i, h) \;=\; xi\circ Gh\colon\; Gj \to GRi \to Si.
\end{array}
$$

By Pointwise Stability (III.5.3), $G(j) = \mathrm{Ran}_P(SQ_j)$ is still a *pointwise* extension, so that, by Thm. 3.2, $G(j)$ is the 2-dimensional double limit of SQ_j, via the pasted cell $p = \otimes\otimes x$. It follows that, on a horizontal arrow $f\colon j \to j'$ in $\mathbb{J}$, $G(f)$ is determined as above, for $h'\colon j' \to Ri$

(4) $p(i, h')\circ G(f) \;=\; x(i)\circ G(h'f) \;=\; p(i, h'f).$ □

4. Kan extensions on the vertical arrow

Pointwise unitary lax right Kan extensions for $\mathbb{J} = \underline{2}$ provide: orthogonal companions, orthogonal adjoints (4.3) and limits of 'extended' vertical transformations (4.4). Conversely, they can be constructed from these instances (4.6).

4.1. Orthogonal companions and adjoints. Let us recall a few notions, from II.1.2-1.3. First, the horizontal morphism $f\colon A \to B$ and the vertical morphism $u\colon A \twoheadrightarrow B$ (in the pseudo double category $\mathbb{A}$) are made *orthogonal companions* by assigning a pair (η, ε) of cells as below, called the *unit* and *counit*, satisfying the identities $(\varepsilon \mid \varepsilon) = 1^\bullet_f$, $\varepsilon\,\varepsilon\,\varepsilon = 1_u$

$$
(1)\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\varepsilon"] & A \arrow[d, "u"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, "1"] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

Given f, this is equivalent (*by unitarity* of $\mathbb{A}$, see 1.1) to saying that the pair

(u, ε) satisfies the following horizontal universal property:

(a) for every cell ε': $(u' \,{}^{f}_{g}\, B)$ there is a unique cell λ: $(u' \,{}^{A}_{g}\, u)$ such that $\lambda' = (\lambda \mid \lambda)$

$$
(2) \qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lambda'"] & B \arrow[d, "1"] \\
A' \arrow[r, "g"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "u'"'] \arrow[dr, phantom, "\lambda"] & A \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\lambda"] & B \arrow[d, "1"] \\
A' \arrow[r, "g"'] & B \arrow[r, equal] & B
\end{tikzcd}
$$

(There are of course dual characterizations, see II.1(b).) Therefore, if f has a vertical companion, this is determined up to a unique special isocell, *and will often be written* f_*. Companions compose in the obvious (covariant) way: if $g: B \to C$ also has a companion, then $g_* f_*: A \twoheadrightarrow C$ is companion to $gf: A \to C$. Companionship is preserved by *unitary* lax or colax double functors.

We say that $\mathbb{A}$ *has vertical companions* if every horizontal arrow has a vertical companion. All our profunctor-based pseudo double categories (cf. Introduction) have vertical companions, given by the obvious embedding of horizontal arrows into the vertical ones. For instance, in $\mathbb{C}$at, the vertical companion of a functor $f: A \to B$ is the associated profunctor $f_*: A \twoheadrightarrow B$, $f_*(a, b) = B(f(a), b)$. Secondly, transforming companionship by vertical (or horizontal) duality, the arrows $f: A \to B$ and $v: B \twoheadrightarrow A$ are made *orthogonal adjoints* by a pair (α, β) of cells as below

$$
(3) \qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, "v"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "v"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, "1"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

with $(\beta \mid \beta) = 1^\bullet_f$ and $\beta \otimes \otimes = 1_v$. Then, f is the *horizontal adjoint* and v the *vertical* one. Again, given f, these relations can be described by universal properties for $(v, \otimes)$ or $(v, \otimes)$ (cf. II.1.3).

The vertical adjoint of f is determined up to a special isocell and will often be written f^*: vertical adjoints compose, contravariantly, letting $(gf)^* = f^* g^*$.

$\mathbb{A}$ is said *to have vertical (orthogonal) adjoints* if every horizontal arrow has a vertical adjoint. All of our profunctor-based examples satisfy this condition. For instance, in $\mathbb{C}$at, the vertical adjoint to a functor $f: A \to B$ is the associated profunctor $f^*: B \twoheadrightarrow A$, $f^*(b, a) = B(b, f(a))$; in $\mathbb{R}$el, the vertical adjoint of a function $f: A \to B$ is the opposite relation $f^\#: B \twoheadrightarrow A$, with $\otimes: 1 \leq f^\# f$, $\leq: f f^\# \leq 1$.

On the other hand, a double category of quintets $\mathbb{Q}\mathbf{A}$, generally, does not have (all) vertical adjoints, since our data amount to an adjunction in the 2-category $\mathbf{A}$, $v \dashv f$, with $\alpha\colon vf \to 1$, $\beta\colon 1 \to fv$.

A more general condition will also be of interest: $\mathbb{A}$ is *horizontally invariant* (II.1.5) if every horizontal isomorphism in $\mathbb{A}$ has a vertical companion (or, equivalently, a vertical adjoint).

4.2. Complements. We want to recall a procedure, called *orthogonal flipping* (II.1.6), which is made possible by the existence, in our pseudo double category $\mathbb{D}$, of vertical companions or adjoints.

If the horizontal map $f\colon A \to B$ has a vertical companion $f_*\colon A \twoheadrightarrow B$, there is a bijective correspondence between cells φ_1 and cells ψ_1, as below, whose boundaries are obtained by 'flipping' f to f_* or vice versa

$$
(1)\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & A \arrow[r, "f"] & B \arrow[d] \\
\bullet \arrow[rr] \arrow[urr, phantom, "\psi_1"] & & \bullet
\end{tikzcd}
\quad
\begin{tikzcd}
\bullet \arrow[rr] \arrow[d] & & \bullet \arrow[d] \\
A \arrow[r, "f"'] \arrow[urr, phantom, "\psi_2"] & B \arrow[r] & \bullet
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r] & \bullet \arrow[d] \\
\bullet \arrow[rr] \arrow[urr, phantom, "\psi_3"] & & \bullet
\end{tikzcd}
\quad
\begin{tikzcd}
\bullet \arrow[rr] \arrow[d] & & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[urr, phantom, "\psi_4"] & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

$$
(2)\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[dd] & A \arrow[d, "f_*"] \\
\arrow[r, phantom, "\psi_1"] & B \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\quad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[dd] \\
A \arrow[d, "f_*"'] \arrow[r, phantom, "\psi_2"] & \\
B \arrow[r] & \bullet
\end{tikzcd}
\quad
\begin{tikzcd}
B \arrow[r] \arrow[d, "f^*"'] & \bullet \arrow[dd] \\
A \arrow[d] \arrow[r, phantom, "\psi_3"] & \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\quad
\begin{tikzcd}
\bullet \arrow[r] \arrow[dd] & \bullet \arrow[d] \\
\arrow[r, phantom, "\psi_4"] & B \arrow[d, "f^*"] \\
\bullet \arrow[r] & A
\end{tikzcd}
$$

By horizontal and vertical duality, the previous statement has three other forms, which establish a bijective correspondence between cells φ_i and ψ_i as above (in the last two cases, flipping f to its vertical adjoint f^*). Starting from a given cell, and applying the flipping process to various arrows, successively, one can *often* show that the final result does not depend on the order of such steps (cf. II.1.6 and III.3).

Here, vertical companions and adjoints will be viewed as Kan extensions, based on the (strict) double category $\mathbb{L}$ represented below, together with its horizontal and vertical opposites

$$
(3)\qquad
\underset{\mathbb{L}}{
\begin{tikzcd}
0 \arrow[d] & \\
1' \arrow[r] & 1
\end{tikzcd}}
\qquad
\underset{\mathbb{L}^{h}}{
\begin{tikzcd}
& 0 \arrow[d] \\
1 \arrow[r] & 1'
\end{tikzcd}}
\qquad
\underset{\mathbb{L}^{v}}{
\begin{tikzcd}
0' \arrow[r] \arrow[d] & 0 \\
1 &
\end{tikzcd}}
\qquad
\underset{\mathbb{L}^{hv}}{
\begin{tikzcd}
0 \arrow[r] & 0' \arrow[d] \\
& 1
\end{tikzcd}}
$$

Note that a unitary lax (or colax) double functor defined on one of these is necessarily strict (1.1).

4.3. Theorem [Companions and adjoints as Kan extensions]. *Let* f: A → B *be a horizontal map of* $\mathbb{A}$. *For each of the double categories* $\mathbb{L}^i$ *listed above (4.2.3), we have an obvious projection* R: $\mathbb{L}^i \to \underline{2}$ *and a strict double functor* S: $\mathbb{L}^i \to \mathbb{A}$ *sending the horizontal arrow to* f *and the vertical arrow to the appropriate vertical identity. Then*:

(a) if the vertical companion of f exists, it is the pointwise unitary lax Ran of S: $\mathbb{L}^{hv} \to \mathbb{A}$ *along* R;

(b) if the vertical adjoint of f *exists, it is the pointwise unitary lax Ran of* S: $\mathbb{L}^h \to \mathbb{A}$ *along* R.

Conversely, provided that **A** *is* horizontally invariant *(4.1)*:

(a') if the pointwise unitary lax Ran of S: $\mathbb{L}^{hv} \to \mathbb{A}$ *along* R *exists, it can be realised as the vertical companion of* f;

(b') if the pointwise unitary lax Ran of S: $\mathbb{L}^h \to \mathbb{A}$ *along* R *exists, it can be realised as the vertical adjoint of* f.

Proof. It is sufficient to prove (a) and (a'). But let us begin by showing the role of horizontal invariance, for (a'). Take the double category **A** formed of a horizontal isomorphism f: A → B between two distinct objects (plus the identities). Then the right Kan extension G: $\underline{2} \to \mathbb{A}$ can only be realised as the vertical identity of A or B, and cannot provide a vertical companion to f - which does not exist.

(a) Let f: A → B have a vertical companion f_*. Then the associated functor S: $\mathbb{L}^{hv} \to \mathbb{A}$ has a right Kan extension G: $\underline{2} \to \mathbb{A}$ along R: $\mathbb{L}^{hv} \to \underline{2}$

(1)
$$\begin{tikzcd} \mathbb{L}^{hv} \arrow[r, "R"] \arrow[dr, "S"'] & \underline{2} \arrow[d, "G"] \arrow[dl, phantom, "\downarrow\alpha"] \\ & \mathbb{A} \end{tikzcd}$$

which sends the vertical arrow to f_*: A ⇸ B, with cell $\alpha = (1_A, \varepsilon)$: GR → S: $\mathbb{L}^{hv} \to \mathbb{A}$ produced by the counit of companionship $\varepsilon\colon (f_* \; {f \atop B} \; B)$, as in the left-hand diagram (a commutative cell of $\mathbb{A}$, in the sense of III.3.1)

(2)
$$
\begin{tikzcd}
 & A \arrow[r, "f"] & B \arrow[d, "1"] \\
A \arrow[ur, "1"] \arrow[r, equal] & A \arrow[ur, "f"] \arrow[d, "f_*"'] \arrow[r, phantom, "\varepsilon"] & B \\
 & B \arrow[ur, "1"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & A \arrow[r, "f"] & B \arrow[d, "1"] \\
A' \arrow[ur, "h"] \arrow[r, equal] & A' \arrow[ur, "fh"] \arrow[d, "u"'] \arrow[r, phantom, "\varepsilon'"] & B \\
 & B' \arrow[ur, "k"'] &
\end{tikzcd}
$$

In fact, a Kan cone (h, ε'): $G'R \to S$ (as above, at the right hand) factors uniquely through G, by $h: A' \to A$ and the cell $(u \,{}^{h}_{k}\, f_*)$ obtained by flipping f to f_* in ε' (4.2).

As to the pointwise property, the second Reduction Theorem (2.4) shows that we only need to consider the stability of the extension with respect to the faces ∂_0, ∂_1: $\underline{1} \to \underline{2}$. But this simply means that the values of G on the objects $0, 1$ of $\underline{2}$ coincide with the limit of the restrictions of S to the pullback of $R: \mathbb{L}^{hv} \to \underline{2}$ along ∂_0 or ∂_1, respectively, which is true.

(a') Conversely, assume that **A** is horizontally invariant. Following backwards the previous argument, the right Kan extension $G: \underline{2} \to \mathbb{A}$ yields a vertical arrow $u: A' \nrightarrow B'$ and a universal cone (h, ∂'): $GR \to S$, as in the right-hand diagram (2). Moreover, the pointwise condition says that h, k are horizontal isomorphisms.

Now, another characterisation of horizontal invariance (again in II.1.5) says that there exists a vertical arrow $f_*: A \nrightarrow B$ and a horizontally invertible cell λ

(3)
$$
\begin{tikzcd}
A' \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\lambda"] & A \arrow[d, "f_*"] \\
B' \arrow[r, "k"'] & B
\end{tikzcd}
$$

Using λ, we can modify the right Kan extension as in the left diagram (2); and f_* is a vertical companion of f. □

4.4. Limits of lax vertical transformations. A *lax vertical transformation* $V: S_0 \nrightarrow S_1: \mathbb{I} \to \mathbb{A}$ will be a lax double functor $V: \underline{2}\times\mathbb{I} \to \mathbb{A}$, where $S_t = V.\lambda_t\times\mathbb{I}$: $\mathbb{I} \to \underline{2}\times\mathbb{I} \to \mathbb{X}$ $(t = 0, 1)$; this extends the strong vertical transformations of lax double functors used in Part I.

The limit $v: \underline{2} \to \mathbb{A}$ of $V: S_0 \nrightarrow S_1$ will be the pointwise unitary lax right Kan extension of V along the projection $R: \underline{2}\times\mathbb{I} \to \underline{2}$. By the second Reduction

Theorem (2.4), this amounts to a unitary lax right Kan extension stable under comma with the injections $\partial_t: \underline{1} \to \underline{2}$.

Thus, $v: A_0 \twoheadrightarrow A_1$ is a vertical arrow in $\mathbb{A}$, equipped with a universal family of cells

(1) $x(e, i): (v \; {}^{x(0,i)}_{x(1,i)} \; S(e,i))$,

and the stability condition means that, for $t = 0, 1$:

(a) A_t is the 1-dimensional double limit of $S_t: \mathbb{I} \to \mathbb{A}$, with cone $x(t, i): A_t \to S_t(i)$.

Extending I.4.4, we will say that $\mathbb{A}$ has a *lax functorial choice* L of $\mathbb{I}$-limits if we can choose:

- the 1-dimensional double limit $L(S)$ of every lax double functor $S: \mathbb{I} \to \mathbb{A}$,

- the limit $L(V): L(S_0) \twoheadrightarrow L(S_1)$, for every lax vertical transformation $V: S_0 \twoheadrightarrow S_1: \mathbb{I} \to \mathbb{A}$, *so that vertical identities are preserved* $(L1^\bullet_S = 1^\bullet_{LS})$.

If this holds, all double limits indexed by $\mathbb{I}$ are 2-dimensional, i.e. pointwise extensions. In fact, the last condition amounts to requiring that the limit $L(S)$ of a lax double functor be stable under the projection $\underline{2} \to \underline{1}$, i.e. 2-dimensional (by the second Reduction Theorem, 2.4).

If $\mathbb{A}$ is *horizontally invariant* (as is always the case if it has vertical companions or vertical adjoints, see 4.1), then one can modify the limit of $S_t: \mathbb{I} \to \mathbb{A}$ up to horizontal isomorphism (as in I.4.6).

4.5. Proposition (I.5.5). *If $\mathbb{A}$ has a lax functorial choice of products, equalisers and tabulators then it has a lax functorial choice of $\mathbb{I}$-limits, for every small $\mathbb{I}$.* □

4.6. Theorem [The construction of pointwise Kan extensions on the vertical arrow]. *The following conditions for a pseudo double category $\mathbb{A}$ are equivalent:*

(i) $\mathbb{A}$ is horizontally invariant and has all pointwise unitary lax right Kan extensions of lax double functors $S: \mathbb{I} \to \mathbb{A}$, along every small $R: \mathbb{I} \to \underline{2}$ *(which means that $\mathbb{I}$ is small);*

(ii) $\mathbb{A}$ has vertical companions, vertical adjoints and a lax functorial choice of double $\mathbb{I}$-limits, for all small weak double categories $\mathbb{I}$.

When these conditions hold, $\mathbb{A}$ has also all pointwise unitary lax right Kan extensions along every small $R: \mathbb{I} \to \underline{1}$.

Proof. It will be sufficient to prove that (ii) implies (i), since the converse follows immediately from the previous results, 4.3 and 4.4, and the last assertion from Section 3. We write $v: 0 \twoheadrightarrow 1$ the non-trivial arrow of $\underline{2}$.

For $t = 0, 1$, let $\mathbb{I}_t$ be the weak double subcategory of $\mathbb{I}$ which R projects to t and its identities, and let S_t be the restriction of S to this substructure. Define

(1) $G(t) = \lim (S_t: \mathbb{I}_t \to \mathbb{A})$ $\qquad\qquad (t = 0, 1)$,

with universal cones $pi: G(0) \to Si$, $qi': G(1) \to Si'$ (for i in $\mathbb{I}_0$ and i' in $\mathbb{I}_1$).

Further, let $\mathbb{I}_v$ be the following *vertically discrete* double category. An object is a vertical arrow $u: i \twoheadrightarrow i'$ in $\mathbb{I}$ such that $R(u) = v$, and a horizontal arrow is an $\mathbb{I}$-cell $a: u \to u'$ such that $R(a) = 1_v$,; vertical arrows and cells are trivial. Take now the (strict) double category $\underline{2}\times\mathbb{I}_v$, where a cell is either a vertical identity or of the following type

$$\text{(2)}\qquad \begin{array}{ccc} (0, u) & \xrightarrow{(0, a)} & (0, u') \\ {\scriptstyle (v, u)}\downarrow & {\scriptstyle (1_v, a)} & \downarrow{\scriptstyle (v, u')} \\ (1, u) & \xrightarrow[(1, a)]{} & (1, u') \end{array} \qquad\qquad (a: u \to u' \text{ in } \mathbb{I}).$$

(The formal vertical identity $1^\bullet_u$ is written as u, and similarly for a.)

We have a commutative diagram, at the left, with strict double functors P (first projection) and Q

$$\text{(3)}\qquad \begin{array}{ccc} \underline{2}\bullet\mathbb{I}_v & & \\ {\scriptstyle Q}\downarrow & \searrow{\scriptstyle P} & \\ \mathbb{I} & \xrightarrow[R]{} & \underline{2} \end{array} \qquad\qquad \begin{array}{ccccc} \mathbb{I}_v & & & & \\ {\scriptstyle J_t}\downarrow & \searrow{\scriptstyle Q_t} & & & \\ \underline{2}\bullet\mathbb{I}_v & \xrightarrow[P_t]{} & \mathbb{I}_t & \longrightarrow & \{t\} \end{array}$$

(4) $Q(v, u) = u,$ $\qquad\qquad Q(1_v, a) = a,$

so that $Q(0, u)$ is the domain of u (and belongs to $R^{-1}(0)$), while $Q(0, a)$ is the vertical domain of the cell a. Then, we form the right diagram above (with $t = 0$ or 1), letting J_t be the embedding as t-basis of the cylinder and defining P_t by vertical domain (D_0) and codomain (D_1)

(5) $P_t(v, u) = D_t(u),$ $\qquad\qquad P_t(1_v, a) = D_t(a).$

Now, $SQ: \underline{2}\bullet\mathbb{I}_v \to \mathbb{A}$ is a vertical transformation between the lax double functors $SQJ_t: \mathbb{I}_v \to \mathbb{A}$. Its limit is a vertical arrow $w: A \twoheadrightarrow B$, with a universal cone consisting of cells in $\mathbb{A}$

$$(6) \qquad \begin{array}{ccc} A & \xrightarrow{pu} & SD_0(u) \\ {\scriptstyle w}\downarrow & \pi u & \downarrow{\scriptstyle Su} \\ B & \xrightarrow[qu]{} & SD_1(u) \end{array}$$

whose vertices are linked with $G(0)$, $G(1)$ by canonical horizontal maps f, g

$$(7) \qquad \begin{array}{ccccccc} G(0) & & & \xrightarrow{\quad pi \quad} & & & \\ \vdots & \dashrightarrow^{f} & A & \xrightarrow{pu} & Si & & pu\circ f = pi, \\ {\scriptstyle G(v)}\,\vdots & \gamma & \downarrow{\scriptstyle w} & \gamma u & \downarrow{\scriptstyle Su} & & \\ \downarrow & \dashrightarrow_{g} & B & \xrightarrow{qu} & Si' & & qu\circ g = qi. \\ G(1) & & & \xrightarrow[\quad qi \quad]{} & & & \end{array}$$

The vertical companion of f and the vertical adjoint of g yield a vertical arrow $G(v)$, together with a cell γ

(8) $G(v) \ = \ f_* \otimes (w \otimes g^*)\colon \ G(0) \to G(1),$ $\qquad\qquad \otimes\,(Gv \ {}^{f}_{g}\ w),$

associated to the horizontal identity $1\colon G(v) \to f_* \otimes (w \otimes g^*)$ (flipping back g^* and then f_*, cf. 4.2). The cell $\otimes$ is determined by the fact that each $(\otimes | \otimes u)$ is *the* cell associated to the composite $\lambda\lambda\,(\mu\lambda\,\lambda u)$, where the cells

$$\lambda\colon\ (f_* \ {}^{pi}_{pu}\ 1^{\bullet}_{Si}), \qquad\qquad \bullet\colon\ (g^* \ {}^{qu}_{qi'}\ 1^{\bullet}_{Si'}),$$

are obtained by flipping, in $\mathbb{A}$, the identities $pu\circ f = pi$, $qu\circ g = qi$ of (7).

Finally, we have defined a strict double functor $G\colon \underline{2} \to \mathbb{A}$, with a universal cone

(9) $\varepsilon u \ = \ (\varepsilon \mid \varepsilon u)\colon G(v) \to Su \qquad\qquad (R(u) = v).$ $\qquad\qquad \square$

5. The construction of lax right Kan extensions

Pointwise unitary lax right Kan extensions with values in the weak double category $\mathbb{A}$, along a small lax double functor satisfying a suitable Conduché property, can be constructed from small double limits, vertical companions and vertical adjoints in $\mathbb{A}$ (Thm. 5.2). We say that $\mathbb{A}$ is *complete* when all these exist.

5.1. Reflection properties. Let $R\colon \mathbb{I} \to \mathbb{J}$ be a lax double functor between pseudo double categories. As in I.1.3, we write $hor_1\mathbb{I}$ the 1-category of *vertical arrows* (as objects) and *cells* of $\mathbb{I}$ (as morphisms), with horizontal composition, and $R_1 =$

$\mathrm{hor}_1 R\colon \mathrm{hor}_1\mathbb{I} \to \mathrm{hor}_1\mathbb{J}$ the induced functor.

We say that R *satisfies the right Conduché condition* if

(i) Every cell $b\colon v'\otimes v'' \to Ru$ in $\mathbb{J}$ (with u in $\mathbb{I}$) can be factored as below

$$
(1)\qquad
\begin{tikzcd}
j_0 \arrow[r, "f"] \arrow[d, "v'"'] & Ri_0 \arrow[dd, "Ru"] \\
j_1 \arrow[d, "v''"'] \arrow[r, phantom, "b"] & \\
j_2 \arrow[r, "g"'] & Ri_2
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
j_0 \arrow[r, "f'"] \arrow[d, "v'"'] \arrow[dr, phantom, "b'"] & \bullet \arrow[r, equal] \arrow[d, "Ru'"] & \bullet \arrow[r, "Rh"] \arrow[dd, "R(u'\otimes u'')"] \arrow[ddr, phantom, "Ra"] & Ri_0 \arrow[dd, "Ru"] \\
j_1 \arrow[r] \arrow[d, "v''"'] \arrow[dr, phantom, "b''"] & \bullet \arrow[d, "Ru''"] \arrow[r, phantom, "{R[u',u'']}"] & & \\
j_2 \arrow[r, "g'"'] & \bullet \arrow[r, equal] & \bullet \arrow[r, "Rk"'] & Ri_2
\end{tikzcd}
$$

by means of two objects (u', b') in $v'{\downarrow}R_1$ and (u'', b'') in $v''{\downarrow}R_1$, of the laxity cell $R[u',u'']$ and of a cell $a\colon u'{\downarrow}u'' \to u$ in $\mathbb{I}$.

(ii) This factorisation is unique up to the equivalence relation generated by the existence of a *morphism* between two factorisations

(2) $(a', a'')\colon (u', b', u'', b'', a) \to (\bar{u}', \bar{b}', \bar{u}'', \bar{b}'', \bar{a})$,

which obviously consists of two cells $a'\colon u' \to \bar{u}'$, $a''\colon u'' \to \bar{u}''$ of $\mathbb{I}$ coherent with the other data:

(3) $(b' \mid Ra') = \bar{b}'$, $\qquad (b'' \mid Ra'') = \bar{b}''$, $\qquad (a' - a'' \mid \bar{a}) = a$,

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "u'"'] \arrow[dr, phantom, "a'"] & \bullet \arrow[r] \arrow[d, "\bar{u}'"] \arrow[ddr, phantom, "\bar{a}"] & i_0 \arrow[dd, "u"] \\
\bullet \arrow[r] \arrow[d, "u''"'] \arrow[dr, phantom, "a''"] & \bullet \arrow[d, "\bar{u}''"] & \\
\bullet \arrow[r] & \bullet \arrow[r] & i_2
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "u'"'] \arrow[ddr, phantom, "a"] & i_0 \arrow[dd, "u"] \\
\bullet \arrow[d, "u''"'] & \\
\bullet \arrow[r] & i_2
\end{tikzcd}
$$

(Coherence with the laxity cells of R necessarily holds, cf. II.2.1(ii).)

The horizontal dual, of interest for left Kan extensions and colimits, will be called the *left* Conduché condition.

5.2. Main Theorem [The construction of pointwise lax Kan extensions]. *The following conditions for a horizontally invariant (4.1) pseudo double category* $\mathbb{A}$ *are equivalent:*

(i) $\mathbb{A}$ *has all pointwise unitary lax right Kan extensions of lax double functor* S:

$\mathbb{I} \to \mathbb{A}$, *along every small lax* $R: \mathbb{I} \to \mathbb{J}$ *which satisfies the right Conduché condition (5.1);*

(ii) $\mathbb{A}$ *satisfies the same condition for* $\mathbb{J} = \underline{1}, \underline{2}$;

(iii) $\mathbb{A}$ *satisfies the same condition for* $\mathbb{J} = \underline{2}$;

(iv) $\mathbb{A}$ *has vertical companions and vertical adjoints (hence, it is horizontally invariant) and has lax functorial double limits.*

By horizontal duality, the existence of all pointwise unitary colax left *Kan extensions in* $\mathbb{A}$ *(horizontally invariant), along every small colax double functor which satisfies the* left *Conduché condition, amounts to the existence of colax functorial double colimits, vertical companions and vertical adjoints.*

Proof. It will be sufficient to prove that (ii) implies (i), since the converse is obvious, and the equivalence of (ii), (iii), (iv) has been proved in 4.6. (For the horizontally dual case, see III.6).

(A) First, the 1-dimensional horizontal part of $G: \mathbb{J} \to \mathbb{A}$ is defined in the usual way (3.4.1-2; and just needs the existence of 1-dimensional double limits in $\mathbb{A}$). For an object j and a horizontal map $f: j \to j'$ in $\mathbb{J}$, we let $G(j) = \mathrm{Ran}_{P_j}(SQ_j)$ via $p_j: G(j).P_j \to SQ_j$, and define $G(f)$ accordingly:

(1) $SQ_j: (j{\Downarrow}R) \to \mathbb{I} \to \mathbb{A}$, $\qquad G(j) = \mathrm{Ran}_{P_j}(SQ_j)$,

$G(j) = \lim SQ_j$, $\qquad p_j(i, h): G(j) \to Si \quad (i \text{ in } \mathbb{I};\ h: j \to Ri \text{ in } \mathbb{J})$,

$G(f): G(j) \to G(j')$, $\quad p_{j'}(i, h').G(f) = p_j(i, h'f): G(j) \to Si' \quad (h': j' \to Ri \text{ in } \mathbb{J})$.

This also gives the value of ε on the object i:

(2) $\varepsilon i = p_{Ri}(i, 1_{Ri}): GRi \to Si$.

(B) Similarly, to define G on a vertical arrow $v: j \twoheadrightarrow j'$, viewed as a double functor $v: \underline{2} \to \mathbb{J}$, we use the right Kan extension $Gv = \mathrm{Ran}_{P_v}(SQ_v)$ via $\pi_v: Gv.P_v \to SQ_v$

(3) $SQ_v: (v{\Downarrow}R) \to \mathbb{I} \to \mathbb{A}$, $\qquad G(v) = \mathrm{Ran}_{P_v}(SQ_v)$,

$\pi_v(u, b): (Gv \ {}^{p_j(i,h)}_{p_{j'}(i',h')}\ Su) \qquad (u: i \twoheadrightarrow i' \text{ in } \mathbb{I};\ b: (v \ {}^{h}_{h'}\ Ru) \text{ in } \mathbb{J})$,

(3)
$$\begin{array}{ccc} G(j) & \xrightarrow{p_j(i,h)} & Si \\ {\scriptstyle G(v)}\downarrow & \pi_v(u,b) & \downarrow{\scriptstyle Su} \\ G(j') & \xrightarrow[p_{j'}(i',h')]{} & Si' \end{array} \qquad \varepsilon u = \pi_{Ru}(u, 1_{Ru}): GRu \to Su.$$

In particular, $G(1^\bullet_j) = 1^\bullet_{Gj}$ by the 2-dimensional universal property of $G(j) = \lim SQ_j$ (3.1).

(C) Take now a cell $c\colon (v \,{}^{f}_{g}\, v')$ in $\mathbb{J}$, and define $G(c)$ by means of the universal property of $G(v')$

$$
(4) \qquad
\begin{tikzcd}
\bullet \arrow[r, "Gf"] \arrow[d, "Gv"', "\bullet" marking] \arrow[dr, phantom, "Gc"] & \bullet \arrow[r, "p_j(i,h)"] \arrow[d, "Gv'"', "\bullet" marking] \arrow[dr, phantom, "\pi_{v'}(u{,}b')"] & \bullet \arrow[d, "Su", "\bullet" marking] \\
\bullet \arrow[r, "Gg"'] & \bullet \arrow[r, "p_{j'}(i'{,}h')"'] & \bullet
\end{tikzcd}
\quad = \quad \pi_v(u, (c \mid b'))
$$

$(b'\colon (v' \,{}^{h'}_{k'}\, Ru)$ in $\mathbb{J})$.

Its horizontal arrows are indeed as claimed above, as detected by the projections $p_j(i, h')$, $p_{j'}(i, k')$. Plainly, G preserves horizontal composition.

(D) Here begins the crucial point of the proof. We already know that G is unitary. To make it a *lax* double functor, let us start from a vertical composition $v = v' \otimes v''\colon j_0 \twoheadrightarrow j_1 \twoheadrightarrow j_2$ in $\mathbb{J}$. We want to define the laxity cell

(5) $\gamma = G[v', v'']\colon Gv' \gamma Gv'' \to Gv.$

Using the universal property of Gv, this amounts to defining a horizontal transformation ρ

$$
(6) \qquad
\begin{tikzcd}
v{\downarrow\downarrow}R \arrow[r, "P_v"] \arrow[d, "Q_v"', "\bullet" marking] \arrow[dr, phantom, "\rho_v \swarrow"] & \underline{2} \arrow[r, equal] \arrow[d, "Gv", "\bullet" marking] \arrow[dr, phantom, "\rho \swarrow"] & \underline{2} \arrow[d, "Gv'\rho Gv''", "\bullet" marking] \\
\mathbb{I} \arrow[r, "S"'] & \mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
v{\downarrow\downarrow}R \arrow[r, "P_v"] \arrow[d, "Q_v"', "\bullet" marking] \arrow[dr, phantom, "\rho \swarrow"] & \underline{2} \arrow[d, "Gv'\rho Gv''", "\bullet" marking] \\
\mathbb{I} \arrow[r, "S"'] & \mathbb{A}
\end{tikzcd}
$$

on the objects and vertical arrows of $v{\downarrow\downarrow}R$. On objects, we use the previous projections, of (1)

(7) $\rho(i_t, h_t\colon j_t \to Ri_t) = p_{j_t}(i_t, h_t)\colon G(j_t) \to Si_t$ $\qquad (t = 0, 2).$

Vertical arrows belong to three types, corresponding to the three vertical arrows of $\underline{2}$, namely the vertical identities of j_0, j_2 and $v\colon j_0 \twoheadrightarrow j_2$. For the first type, a vertical arrow $(u_0, b_0)\colon (i_0, h_0) \twoheadrightarrow (i_0, h'_0)$ (with $b_0\colon 1^\bullet \to Ru_0$) also belongs to $v'{\downarrow\downarrow}R$, and we can take a projection of (3)

(8) $\rho(u_0, b_0) = \rho_{v'}(u_0, b_0)\colon G1^\bullet \to Su_0.$

Similarly for the second type. For the third, we need a new procedure because we want a cell starting from $Gv'\rho Gv''$, rather than from Gv.

Take then a vertical arrow $(u, b)\colon (i_0, h_0) \twoheadrightarrow (i_2, h_2)$, where $b\colon v \to Ru$ is a cell in $\mathbb{J}$. By the Conduché condition on R, we can factor it as below

$$
(9)\qquad
\begin{tikzcd}
j_0 \arrow[r, "h_0"] \arrow[d, "v'"'] & Ri_0 \arrow[dd, "Ru"] \\
j_1 \arrow[d, "v''"'] \arrow[r, phantom, "b"] & \\
j_2 \arrow[r, "h_2"'] & Ri_2
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
j_0 \arrow[r] \arrow[d, "v'"'] \arrow[dr, phantom, "b'"] & \bullet \arrow[r, equal] \arrow[d, "Ru'"] & \bullet \arrow[r] \arrow[dd, "R(u'\otimes u'')"] \arrow[dr, phantom, "Ra"] & Ri_0 \arrow[dd, "Ru"] \\
j_1 \arrow[r] \arrow[d, "v''"'] \arrow[dr, phantom, "b''"] & \bullet \arrow[d, "Ru''"] \arrow[r, phantom, "{R[u',u'']}"] & & \\
j_2 \arrow[r] & \bullet \arrow[r, equal] & \bullet \arrow[r] & Ri_2
\end{tikzcd}
$$

and we define $\rho(u, b)$ by the following pasting (using the cells π_v of (3))

$$
(10)\qquad
\begin{tikzcd}
Gj_0 \arrow[r] \arrow[d, "Gv'"'] \arrow[dr, phantom, "{\pi_v(u', b')}"] & \bullet \arrow[r, equal] \arrow[d, "Su'"] & \bullet \arrow[r] \arrow[dd, "S(u'\pi u'')"] \arrow[dr, phantom, "Sa"] & Si_0 \arrow[dd, "Su"] \\
Gj_1 \arrow[r] \arrow[d, "Gv''"'] \arrow[dr, phantom, "{\pi_v(u'', b'')}"] & \bullet \arrow[d, "Su''"] \arrow[r, phantom, "{S[u',u'']}"] & & \\
Gj_2 \arrow[r] & \bullet \arrow[r, equal] & \bullet \arrow[r] & Si_2
\end{tikzcd}
$$

To see that this is well defined, take a morphism of factorisations (5.1.2)

(11) $\quad (a', a'')\colon (u', b', u'', b'', a) \to (\bar{u}', \bar{b}', \bar{u}'', \bar{b}'', \bar{a})$,

and recall that $(a' - a'' \mid \bar{a}) = a$, $(b' \mid Ra') = \bar{b}'$, $(b'' \mid Ra'') = \bar{b}''$. Therefore:

(12) $\quad (-_v(u', b') - -_v(u'', b'') \mid S[u', u''] \mid Sa) =$

$\quad (-_v(u', b') - -_v(u'', b'') \mid S[u', u''] \mid S(a' - a'') \mid S\bar{a}) =$

$\quad (-_v(u', b') - -_v(u'', b'') \mid (Sa') - (Sa'') \mid S[\bar{u}', \bar{u}''] \mid S\bar{a}) =$

$\quad (-_v((\bar{u}', \bar{b}') - -_v(\bar{u}'', \bar{b}'') \mid S[\bar{u}', \bar{u}''] \mid S\bar{a})$.

(E) One verifies now that $-$ is indeed a horizontal transformation, and that the laxity cells of G are coherent. These computations will not be written down.

(F) Finally, G is indeed the pointwise unitary lax right Kan extension of S along R.

We know that it suffices to prove the pointwise property, by III.4.2. Moreover, by the Reduction Theorem 2.3, it suffices to verify this property for (strict) double functors defined on the three vertical models; and actually on $\underline{3}$, since on $\underline{1}$ and $\underline{2}$ we already know that it holds, by the previous construction.

Now, a double functor $V: \underline{3} \to \mathbb{J}$ amounts to a vertical composition $v' \otimes v'' = v$, as considered in point (D). Let us be given a unitary lax double functor $G': \underline{3} \to \mathbb{A}$, and a horizontal transformation α

$$(13) \qquad \begin{tikzcd} V\Downarrow R \arrow[r, "Pv"] \arrow[d, "Qv"'] \arrow[dr, phantom, "\varepsilon_V \swarrow"] & \underline{3} \arrow[r, equal] \arrow[d, "GV"] \arrow[dr, phantom, "\tau \swarrow"] & \underline{3} \arrow[d, "G'"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd} \quad = \quad \begin{tikzcd} v\Downarrow R \arrow[r, "Pv"] \arrow[d, "Qv"'] \arrow[dr, phantom, "\tau \swarrow"] & \underline{3} \arrow[d, "G'"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \end{tikzcd}$$

We want to show that it factors through the cell $\tau_V = \omega_V \omega \omega$ by a unique horizontal transformation ω: $G' \to G$. Note that G' amounts to a cell γ': $u' \gamma\, u'' \to u$ in $\mathbb{A}$ (namely, its laxity cell $G'[0 \twoheadrightarrow 1, 1 \twoheadrightarrow 2]$ for the only non-trivial vertical composition in $\underline{3}$).

Now, since G is pointwise on $\underline{2}$, we have a uniquely determined triple of $\mathbb{A}$-cells, coherent with the general data

(14) $\quad t' = \gamma(0 \twoheadrightarrow 1): u' \to Gv', \qquad\qquad t'' = \gamma(1 \twoheadrightarrow 2): u'' \to Gv'',$

$\qquad t = \gamma(0 \twoheadrightarrow 2): u \to Gv,$

and we have only to check that

(15) $\quad (\, t' \gamma\, t'' \mid G[v', v'']) = (\gamma' \mid t): u' \gamma\, u'' \to Gv.$

In fact, pasting both terms with the (cancellable) cell π_v, we get the following results (and one should not confuse cells in $\mathbb{A}$, used in (14), with the corresponding cells in $\mathrm{Lx}\mathbb{D}\mathrm{bl}$, used below)

$$(16) \qquad \begin{tikzcd} v\Downarrow R \arrow[r, "P_v"] \arrow[d, "Q_v"'] \arrow[dr, phantom, "\pi_v \swarrow"] & \underline{2} \arrow[r, equal] \arrow[d, "Gv"] \arrow[dr, phantom, "\pi \swarrow"] & \underline{2} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "t'\pi\, t'' \swarrow"] & \underline{2} \arrow[d, "u'\pi\, u''"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \arrow[r, equal] & \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd} \quad \begin{aligned} &= (\rho \mid t'\rho\, t'') \\ &= \rho(0 \twoheadrightarrow 1)\rho\, \rho(1 \twoheadrightarrow 2), \end{aligned}$$

$$(17) \qquad \begin{tikzcd} v\Downarrow R \arrow[r, "P_v"] \arrow[d, "Q_v"'] \arrow[dr, phantom, "\rho_v \swarrow"] & \underline{2} \arrow[r, equal] \arrow[d, "Gv"] \arrow[dr, phantom, "t \swarrow"] & \underline{2} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\rho' \swarrow"] & \underline{2} \arrow[d, "u'\rho\, u''"] \\ \mathbb{I} \arrow[r, "S"'] & \mathbb{A} \arrow[r, equal] & \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd} \quad = (\rho(0 \twoheadrightarrow 2) \mid \rho')$$

which coincide, by coherence of ρ. □

5.3. Completeness. We say that the pseudo double category $\mathbb{A}$ is *complete* if it is horizontally invariant and satisfies the equivalent conditions of the previous theorem.

Because of Parts I, II, we already know that this holds for all the profunctor-based examples of Part I, whose archetype is the pseudo double category $\mathbb{C}\mathbf{at}$ (of small categories, functors and profunctors) (see 1.1). On the other hand, a double category $\mathbb{Q}\mathbf{A}$ of quintets on a 2-complete 2-category $\mathbf{A}$ has all double limits and vertical companions, but generally lacks vertical adjoints and is not complete, in the present sense.

In the previous statement, the properties (i) - (iv) are also equivalent for *finite* double categories $\mathbb{I}, \mathbb{J}$, and *finite* lax functorial double limits in $\mathbb{A}$ (with the same proof); we say then that $\mathbb{A}$ is *finitely complete*. By the construction theorem of double limits (I.5.5), finite completeness of double categories is a first-order property, and amounts to having: vertical companions, vertical adjoints, a double terminal, lax functorial binary products, lax functorial equalisers, lax functorial tabulators.

5.4. A pointwise unitary colax extension. We end with an example showing a case where R does not satisfy the Conduché condition, but there is a solution in the alternative setting $\mathbb{D} = \mathbb{D}\mathrm{bl}_u$ (cf. III.6.3). Take $R: \underline{2} \to \underline{3}$ the strict double functor which takes the vertical arrow $0 \twoheadrightarrow 1$ to $0 \twoheadrightarrow 2$ (and does not 'lift' the vertical factorisation $(0 \twoheadrightarrow 2) = (0 \twoheadrightarrow 1) \otimes (1 \twoheadrightarrow 2)$)

$$(1) \qquad \begin{tikzcd} \underline{2} \arrow[r, "R"] \arrow[d, "1"'] & \underline{3} \arrow[d, "G"] \arrow[dl, phantom, "\swarrow \varepsilon"] \\ \underline{2} \arrow[r, "w"'] & \mathbb{A} \end{tikzcd}$$

Take a strict double functor $w: \underline{2} \to \mathbb{A}$, which amounts to a vertical arrow $w: A \twoheadrightarrow B$ in $\mathbb{A}$. The pointwise *unitary colax* Kan extension of w along R is a unitary colax double functor $G: \underline{3} \twoheadrightarrow \mathbb{A}$, consisting of a cell $\gamma: w \to u\gamma\, v$, universal in the obvious sense, yielding a universal 'colax decomposition' of w (if it exists in $\mathbb{A}$).

If $\mathbb{A}$ has a terminal object T, vertical companions and adjoints, the Kan extension can be constructed as the colax double functor $G = (G, \gamma)$ displayed in the left diagram below, using the vertical arrows $t_*: A \twoheadrightarrow T$ and $t^*: T \twoheadrightarrow B$

$$
(2) \qquad
\begin{tikzcd}
 & A \arrow[dd, "t_*"] & \\
A \arrow[ur, equal] \arrow[dd, "w"'] \arrow[rr, "1"] & & A \arrow[dd, "w"] \\
 {} \arrow[r, phantom, "\gamma \to"] & T \arrow[dd, "t^*"] & \\
B \arrow[dr, equal] \arrow[rr, "1"] & & B \\
 & B &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & X \arrow[rr, dashed] \arrow[dd] & & A \arrow[dd, "t_*"] \\
X \arrow[ur, equal] \arrow[dd, "w'"'] \arrow[rr, "f"] & & A \arrow[ur, equal] \arrow[dd] & \\
 {} \arrow[r, phantom, "\gamma' \to"] & Z \arrow[rr, dashed] \arrow[dd] & & T \arrow[dd, "t^*"] \\
Y \arrow[dr, equal] \arrow[rr, "g"] & & B \arrow[dr, equal] \arrow[r, phantom, "\gamma \to"] & {} \\
 & Y \arrow[rr, dashed] & & B
\end{tikzcd}
$$

Universality is plain, from the right diagram above.

References

[DPP] R.J.MacG. Dawson - R. Paré - D.A. Pronk, *Paths in double categories*, Th. Appl. Categ. **16** (2006), 460-521.

[E1] C. Ehresmann, *Catégories structurées*, Ann. Sci. Ecole Norm. Sup. **80** (1963), 349-425.

[E2] C. Ehresmann, *Catégories et structures*, Dunod, Paris 1965.

[G2] M. Grandis, *The shape of a category up to directed homotopy*, Theory Appl. Categ. 15 (2005/06) (CT2004), No. 4, 95-146.

[GP1] M. Grandis - R. Paré, *Limits in double categories*, Cah. Topol. Géom. Différ. Catég. **40** (1999), 162-220.

[GP2] M. Grandis - R. Paré, *Adjoint for double categories*, Cah. Topol. Géom. Différ. Catég. **45** (2004), 193-240.

[GP3] M. Grandis - R. Paré, *Kan extensions in double categories (On weak double categories, III)*, Dip. Mat. Univ. Genova, Preprint **553** (2007).

Dipartimento di Matematica
Università di Genova
via Dodecaneso 35
16146 Genova, Italy
grandis@dima.unige.it

Department of Mathematics and Statistics
Dalhousie University
Halifax NS
Canada B3H 3J5
pare@mathstat.dal.ca