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A CATEGORICAL PROOF OF THE EQUIVALENCE OF LOCAL COMPACTNESS AND EXPONENTIABILITY IN LOCALE THEORY

by Christopher F. TOWNSEND

RESUME. Un résultat bien connu de la théorie des locales affirme qu'un locale est localement compact si, et seulement si, il est exponentiable. Un résultat récent de Vickers et Townsend représente un homomorphisme (préservant les bornes supérieures des parties majorées) entre ouverts de locales en terme de transformations naturelles. Ici ce théorème de représentation est utilisé pour donner une preuve catégorique du fait qu'un locale est localement compact si, et seulement si, il est exponentiable.

1. INTRODUCTION

Given topological spaces X and Y the function space Y^X, with its compact-open topology, is known to be well behaved provided X is locally compact. Being 'well behaved' can be given a precise categorical meaning: X is an exponentiable object in the category of topological spaces; that is the endofunctor $(\_) \times X : \mathbf{Top} \to \mathbf{Top}$ has a right adjoint. For T_0 spaces the converse holds and so exponentiability almost characterizes local compactness. The situation is nicer in locale theory since any locale is exponentiable if and only if it is locally compact. This was originally shown by Hyland, [1], and Theorem C4.1.9 in [3] and VII 4.10 in [2] give textbook accounts of the result. The purpose of this paper is to re-prove the result for locales using a representation theorem for directed join preserving maps between the lattices of opens of locales. This representation theorem, [4], is in terms of natural transformations allowing the main proof to be entirely categorical.

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In locale theory local compactness can be expressed as a lattice theoretic condition on the lattice of opens of a locale. The condition is that the lattice of opens must be a continuous lattice so providing a fundamental link between a topological notion (local compactness) and a lattice theoretic one (continuity). A continuous lattice is, equivalently, a retract via directed join preserving maps of an algebraic directed complete partial order (dcpo). Since we have a representation theorem for the directed join preserving maps, the only work needed for this paper is to ensure that certain algebraic dcpos are exponentiable, a fact that is widely known and, we shall see, easy to prove. Once this is done the relationship between general exponentiabilty and local compactness will become immediate via a categorical argument.

2. Notation and Background Results

A *frame* is a complete poset such that finite meets distribute over arbitrary joins. A frame homomorphism is a map between frames that is required to preserve arbitrary joins and finite meets. The category **Fr** is defined. The notation ΩX is used for a frame where X is the *corresponding locale*. This comes from the definition of the category of locales:

$$\mathbf{Loc} \equiv \mathbf{Fr}^{op}.$$

See [2] and C1 of [3] for an account of the theory of frames and locales. The theory of frames is suitably algebraic in that the free frame on a set of generators subject to a set of frame equations can always be constructed. For example the Sierpiński locale $\mathbb{S}$ can be defined by

$$\Omega\mathbb{S} \equiv Fr\langle\{*\}\rangle;$$

that is, the free frame on the singleton set subject to no relations. Regular monomorphisms in **Loc** are exactly regular epimorphisms in **Fr** and since this category is suitably algebraic, these are exactly frame surjections. Note, given this observation, that $\mathbb{S}$ is injective in **Loc** with respect to all regular monomorphisms. This is clear given that locale maps $X \to \mathbb{S}$ are in bijection with ΩX.

Locale products exist and are given by suplattice tensor,

$$\Omega(X \times Y) = \Omega X \otimes_{\mathbf{Sup}} \Omega Y,$$

where suplattice homomorphisms are those maps which preserve arbitrary joins.

A result in [4] shows that directed join preserving maps between frames (say from ΩX to ΩY) are in bijection with natural transformations

$$\mathbf{Loc}(\_\times X, \mathbb{S}) \dot{\rightarrow} \mathbf{Loc}(\_\times Y, \mathbb{S})$$

where $\mathbf{Loc}(\_\times X, \mathbb{S})$ and $\mathbf{Loc}(\_\times Y, \mathbb{S}) : \mathbf{Loc}^{op} \rightarrow \mathbf{Set}$ are presheaves on **Loc**. It is this representation theorem for dcpo homomorphisms that is key to the main result.

2.1. **Locally Compact Locales.** Let ΩX be the frame of opens of a locale, then X is locally compact if for any $v \in \Omega X$

$$v = \bigvee^{\uparrow}\{u \mid u \ll v\},$$

where $u \ll v$ if for every directed cover $v \leq \bigvee^{\uparrow} S$ there is $s \in S$ such that $u \leq s$. Equivalently X is locally compact if and only if ΩX is a continuous poset and this leads to various characterizations of locally compact:

Lemma 1. *Given a locale X the following are equivalent,*

(i) X is locally compact,

(ii) ΩX is a retract, via dcpo homomorphisms, of $idl(L)$ for some poset L,

(iii) ΩX is a retract, via dcpo homomorphisms, of $idl(L)$ for some meet semilattice L,

(iv) ΩX is a retract, via dcpo homomorphisms, of $\mathcal{D}L$ for some meet semilattice L, and

(v) ΩX is a retract, via dcpo homomorphisms, of $\mathcal{U}L$ for some join semilattice L.

Of course, $\mathcal{D}L$ is the set of lower closed subsets of L and $\mathcal{U}L$ is the set of upper closed subsets of L.

Proof. (i)$\Longleftrightarrow$(ii) is a well known characterization of continuous poset, e.g. Theorem VII 2.3 [2]. Since L can always be taken to be ΩX in this, (i)$\Longleftrightarrow$(iii is clear. (iii)$\Longleftrightarrow$(iv) is clear since $\mathcal{D}L$ is the free frame on the meet semilattice L and so is order isomorphic to $idl(F_{\vee}L)$ where $F_{\vee}L$ is the free join semilattice on the poset L. (iv)$\Longleftrightarrow$(v) is clear since $\mathcal{U}(L) = \mathcal{D}(L^{op})$ for any poset L. $\square$

2.2. **Exponentiation of** $Idl(L)$. For any poset L we can define the ideal completion locale $Idl(L)$ by

$$\Omega Idl(L) \equiv \mathcal{U}(L).$$

It is called the ideal completion locale since its points ($\mathbf{Loc}(1, Idl(L))$) are in order isomorphism with $idl(L)$, the set of ideals of L. It is well known that for any locales X and Y, monotone maps $L \to \mathbf{Loc}(X, Y)$ are in order isomorphism with

$$\mathbf{Loc}(X \times Idl(L), Y),$$

from which it is easy to establish that,

Theorem 1. *For any poset L, $Idl(L)$ is exponentiable in* **Loc**.

Proof. For any locales X and Y we must establish a natural bijection $\mathbf{Loc}(X : Idl(L), Y) \cong \mathbf{Loc}(X, W)$ for some locale W. Let us set

$$\Omega W = \mathbf{Fr}\langle (l, b) \in L \times \Omega X \mid (\_,\_) \text{ qua poset in first, qua frame in second} \rangle$$

so that locale maps $X \to W$ are in bijection with maps $\phi : L \times \Omega Y \to \Omega X$ that are monotone in the first coordinate and preserve frame homomorphisms in the second. Now $\mathbf{Loc}(X \times Idl(L), Y)$ is naturally isomorphic to maps

$$\phi : L \times \Omega Y \to \Omega X$$

which are monotone in the first component and are frame homomorphisms in the second; but these maps are in bijection with $\mathbf{Loc}(X, W)$ by construction of W and so $W \cong Y^{Idl(L)}$. □

Recalling that regular monomorphisms in **Loc** correspond to frame surjections it is therefore clear that any locale Y can be embedded in $\mathbb{S}^{Idl(L)}$ for some join semilattice L by taking $L = \Omega Y$.

3. Main Result

Theorem 2. *If X is a locale then the following are equivalent,*
(i) X is locally compact,
(ii) $\mathbb{S}^X$ exists, and
(iii) X is exponentiable.

Proof. (ii)$\Longrightarrow$(iii). We have just established that for any Y, there is an equalizer diagram, $Y \hookrightarrow \mathbb{S}^{Idl(L)} \rightrightarrows Z$, since an embedding is a regular monomorphism (i.e. occurs as an equalizer). But Z itself must embed in $\mathbb{S}^{Idl(L^Z)}$, and so in fact there is an equalizer diagram

$$Y \hookrightarrow \mathbb{S}^{Idl(L)} \rightrightarrows \mathbb{S}^{Idl(L^Z)}$$

in **Loc**. But if $\mathbb{S}^X$ exists then certainly $\mathbb{S}^{Idl(L)\times X}$ must exist as $Idl(L)$ is exponentiable. Therefore Y^X can be defined as the equalizer of a diagram

$$\mathbb{S}^{Idl(L)\times X} \rightrightarrows \mathbb{S}^{Idl(L^Z)\times X}.$$

(iii)$\Longrightarrow$(ii) is trivial.

(ii)$\Longrightarrow$(i). We have noted above that X is locally compact if and only if there is a join semilattice L such that ΩX is a retract, via dcpo homomorphisms of $\mathcal{U}(L) \cong \Omega Idl(L)$. Since dcpo homomorphisms between frames correspond to natural transformations between presheaves we have that X is locally compact if there exists a join semilattice L such that $\mathbf{Loc}(\_\times X, \mathbb{S})$ is a retract, in $[\mathbf{Loc}^{op}, \mathbf{Set}]$, of the presheaf $\mathbf{Loc}(\_\times Idl(L), \mathbb{S})$. Say $Y = \mathbb{S}^X$, which exists by assuming (ii), then we have that $\mathbf{Loc}(\_\times X, \mathbb{S})$ is naturally isomorphic to $\mathbf{Loc}(\_, Y)$. Let $Y \hookrightarrow \mathbb{S}^{Idl(L)}$ be the embedding used as above where, certainly, it can be assumed that L is a join semilattice. Since $\mathbb{S}$ is injective with respect to regular monomorphisms, and this property is inherited by any power of $\mathbb{S}$, $\mathbb{S}^{Idl(L)}$ is also injective and so the identity $Y \to Y$ in **Loc** factors via $\mathbb{S}^{Idl(L)}$. By applying the Yoneda embedding to this retract diagram and noting that $\mathbf{Loc}(\_\times Idl(L), \mathbb{S})$ is naturally isomorphic to $\mathbf{Loc}(\_, \mathbb{S}^{Idl(L)})$ since $Idl(L)$ is exponentiable, we have that $\mathbf{Loc}(\_\times X, \mathbb{S})$ is a retract of $\mathbf{Loc}(\_\times Idl(L), \mathbb{S})$ in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ as required to show that X is locally compact.

(i)$\Longrightarrow$(ii). Say X is locally compact. Then there exists a join semilattice L such that $\mathbf{Loc}(\_\times X, \mathbb{S})$ is a retract, in $[\mathbf{Loc}^{op}, \mathbf{Set}]$, of the presheaf $\mathbf{Loc}(\_\times Idl(L), \mathbb{S})$. That is, there are natural transformations

$$\alpha_i : \mathbf{Loc}(\_\times X, \mathbb{S}) \to \mathbf{Loc}(\_\times Idl(L), \mathbb{S})$$

and

$$\alpha_q : \mathbf{Loc}(\_\times Idl(L), \mathbb{S}) \to \mathbf{Loc}(\_\times X, \mathbb{S})$$

with $\alpha_q \alpha_i = Id$. Notice that this forces

$$\alpha_i : \mathbf{Loc}(\_\times X, \mathbb{S}) \to \mathbf{Loc}(\_\times Idl(L), \mathbb{S})$$

to be the equalizer in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ of the diagram

$$\mathbf{Loc}(\_\times Idl(L), \mathbb{S}) \rightrightarrows \mathbf{Loc}(\_\times Idl(L), \mathbb{S})$$

where the top map is the identity and the bottom arrow is the (idempotent) $\alpha_i\alpha_q$. But $Idl(L)$ is exponentiable and so $\mathbf{Loc}(\_\times Idl(L),\mathbb{S}) \cong \mathbf{Loc}(\_,\mathbb{S}^{Idl(L)})$ i.e. is representable. It follows by Yoneda's lemma that $\alpha_i\alpha_q = \mathbf{Loc}(\_,a)$ for some locale map $a : Idl(L) \to Idl(L)$ which must be idempotent. Equalizers exist in **Loc** and so define $E \hookrightarrow \mathbb{S}^{Idl(L)}$ to be the equalizer of

$$\mathbb{S}^{Idl(L)} \underset{a}{\overset{Id}{\rightrightarrows}} \mathbb{S}^{Idl(L)}.$$

Finally since the Yoneda embedding preserves equalizers we have that $\mathbf{Loc}(\_$, is naturally isomorphic to $\mathbf{Loc}(\_\times X,\mathbb{S})$ which is sufficient to prove $E \cong \mathbb{S}^X$. □

4. Categorical Summary

Given the relationship between dcpo homomorphisms and natural transformations it becomes a categorical triviality that a locale is locally compact if and only if it is exponentiable. Because dcpo homomorphisms correspond to natural transformations it follows that local compactness, almost by definition, exhibits $\mathbf{Loc}(\_\times X,\mathbb{S})$ as a retract of a representable functor, $\mathbf{Loc}(\_,\mathbb{S}^{Idl(L)})$. Since the retract of a representable functor is always representable (provided equalizers exist) this forces $\mathbf{Loc}(\_\times X,\mathbb{S})$ to be representable and therefore $\mathbb{S}^X$ exists. If every object Y can be embedded in $\mathbb{S}^{Idl(L)}$ for an exponentiable object $Idl(L)$ then the existence of $\mathbb{S}^X$ is sufficient to prove the existence of Y^X for all Y.

On the other hand, if $\mathbb{S}^X$ exists and can be embedded in $\mathbb{S}^{Idl(L)}$ then, using injectivity of $\mathbb{S}$, this embedding is a retraction and applying the Yoneda embedding to the retract diagram proves local compactness.

In summary the main result can be re-worked entirely categorically. Say we are given a cartesian category $\mathcal{C}$ with an injective object $\mathbb{S}$ and a collection of exponentiable objects $Idl(L)$ such that every object Y embeds in $\mathbb{S}^{Idl(L)}$ for some $Idl(L)$. Then provided we take as our definition that an object X is locally compact if and only if $\mathcal{C}(\_\times X,\mathbb{S})$ is a retract of $\mathcal{C}(\_\times Idl(L),\mathbb{S})$ in the presheaf category $[\mathcal{C}^{op},\mathbf{Set}]$, it follows that an object X is exponentiable if and only if it is locally compact; i.e. Hyland's result.

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