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LAX 2-CATEGORIES AND DIRECTED HOMOTOPY

by Marco GRANDIS

Résumé. La Topologie Algébrique Dirigée est un domaine récent, où un 'espace dirigé' X, par exemple un espace topologique ordonné, a des chemins *dirigés* (généralement non réversibles) et une *catégorie* fondamentale, à la place du groupoïde fondamental classique. En dimension 2, les 2-cubes dirigés de X produisent, de façon naturelle, une *2-catégorie lax*. Cette notion généralise celle de bicatégorie, moyennant des cellules de comparaison qui ne sont pas supposées être inversibles, et pour lesquelles il faut donc choisir une diréction. Le cadre géométrique présent donne un choix différent de ceux qui ont déjà été étudiés.

Introduction

At the roots of higher dimensional categories, after Mac Lane's coherence theorem for monoidal categories ([18], 1963), we have Ehresmann's notion of a *2-category* ([7], 1963; [8], 1965) and its weak version, Bénabou's *bicategory* ([2], 1967), where the unit and associativity laws of the arrow-composition are replaced with invertible comparison cells, like $f \cong f \circ 1_x$, $(h \circ g) \circ f \cong h \circ (g \circ f)$.

A *lax* version, where such comparisons are not assumed to be invertible - and therefore the choice of their direction becomes relevant - has been studied only exceptionally. This is likely due to two facts: (a) relevant examples are not frequent; (b) formally, the choice of orientation is not obvious: choosing to direct unit cells towards longer (or shorter) expressions, how should associativity cells be directed?

Burroni [5] introduced, since 1971, a 'pseudocategory', with the following directions:

(1) $f \to f \circ 1_x, \qquad f \to 1_y \circ f, \qquad (h \circ g) \circ f \to h \circ (g \circ f).$

Borceux [3] mentions a similar notion of 'lax category', in a marginal remark

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after the definition of bicategory (7.7.1). Such approaches, *based on nullary and binary compositions*, will be said to be 'biased', as in Leinster's book [17].

The latter uses the term 'lax bicategory' in a different sense ([17], 3.4), which has the advantage of a clear syntactic criterion for the direction of comparisons: he introduces an 'unbiased' structure, where all multiple compositions $f_n \circ ... \circ f_1$ are assigned and there are comparison cells *from* each iterated composition *to* the corresponding multiple composition, as in the following examples

(2) $(k \circ h \circ g) \circ f \to k \circ h \circ g \circ f, \qquad (1 \circ (h \circ g \circ 1)) \circ f \to h \circ g \circ f.$

Thus, there is no single comparison cell between $(h \circ g) \circ f$ and $h \circ (g \circ f)$, but there is one from each of them to $h \circ g \circ f$. This choice amounts to a lax algebra for the 'free strict 2-category' 2-monad on the 2-category of graphs in **Cat** (cf. [17], 3.4.2).

A recent domain, Directed Algebraic Topology, can yield interesting examples together with a different guideline for the orientation of comparisons. Directed Algebraic Topology studies structures having *privileged directions*, like 'directed spaces' in some sense: ordered topological spaces, 'spaces with distinguished paths', simplicial and cubical sets, etc. Such objects have *directed* paths and homotopies, which cannot be reversed, generally. They can thus model non-reversible phenomena, in various domains; the existing applications deal mostly with the analysis of concurrent processes, in Computer Science (references for these applications can be found in [9, 10, 11, 14]).

Directed spaces can be studied with homology and homotopy theories, modified to keep an account of privileged directions: e.g., *preordered* homology groups [12, 13] and fundamental n-*categories* (in some sense) instead of the classical homology groups and fundamental n-*groupoids* of Algebraic Topology. Thus, Directed Algebraic Topology is more clearly linked with higher dimensional Category Theory, and can also give some geometric intuition to the latter.

However, while there is no problem in defining the *fundamental category* $\uparrow\Pi_1(X)$ (see 1.1), the construction of a fundamental (strict) 2-category is complicated, perhaps non natural [15].

Here we introduce, in Section 2, a fundamental *biased d-lax 2-category* $\uparrow b\Pi_2(X)$, with comparison cells

(3) $1_x \otimes a \to a \to a \otimes 1_y, \qquad a \otimes (b \otimes c) \to (a \otimes b) \otimes c,$

where $a \otimes b$ is the concatenation of two consecutive paths: $a(1) = b(0)$. The direction of these cells (*as suggested by the term* 'd-lax') comes from the fact that, in a directed space, a (directed) comparison homotopy can only go from a concatenation to another which, at each instant $t \in [0, 1]$, has made a longer way than the initial one.

The coherence theorem (2.4) for such a structure $\uparrow b\Pi_2(X)$ says that all diagrams naturally constructed with comparison cells commute. This remains true if we add *higher* associativity comparisons, depending on four consecutive arrows

(4) $a\otimes((b\otimes c)\otimes d) \rightarrow (a\otimes b) \otimes (c\otimes d) \rightarrow (a\otimes(b\otimes c))\otimes d$,

which break Mac Lane's pentagon into 3 commutative triangles (2.5).

Then, in Section 3, we define a fundamental *unbiased d-lax 2-category* $\uparrow u\Pi_2(X)$, where (for instance) we also have comparison cells

(5) $a\otimes(b\otimes c) \rightarrow a\otimes b\otimes c \rightarrow (a\otimes b)\otimes c$.

References to the rich literature on higher categories can be found in two beautiful, recent books, by T. Leinster [17] and E. Cheng - A. Lauda [6].

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1. Directed spaces and homotopy

We briefly review some notions of 'directed spaces', starting with preordered spaces. A *preorder* relation is assumed to be reflexive and transitive; it is called a (partial) *order* if it is also anti-symmetric; using a preorder as the main notion, instead of an order, has strong advantages, as recalled below (1.1).

1.1. Homotopy for preordered spaces. The simplest topological setting where one can study directed paths and directed homotopies is likely the category **pTop** of *preordered topological spaces* and *preorder-preserving continuous mappings*; the latter will be simply called *morphisms* or *maps*, when it is understood we are in this category. (Richer settings will be recalled below).

A (directed) *path* in the preordered space X is a map $a: \uparrow[0, 1] \rightarrow X$, defined on the standard directed interval $\uparrow\mathbf{I} = \uparrow[0, 1]$ (with euclidean topology and natural order). A (directed) *homotopy* $\varphi: f \rightarrow g: X \rightarrow Y$, *from* f *to* g, is a map $\varphi: X\times\uparrow\mathbf{I} \rightarrow Y$ coinciding with f on the lower basis of the *cylinder* $X\times\uparrow\mathbf{I}$, with g on the upper one. Of course, this (directed) cylinder is a product in **pTop**: it is equipped with the product topology *and* with the product preorder, where $(x, t) \prec (x', t')$ if x

$\prec$ x' in X and $t \le t'$ in ↑**I**.

The category **pTop** has all limits and colimits, constructed as in **Top** and equipped with the initial or final preorder for the structural maps. The forgetful functor U: **pTop** → **Top** with values in the category of topological spaces has both a left and a right adjoint, $D \dashv U \dashv C$, where DX (resp. CX) is the space X with the *discrete* order (resp. the *chaotic pre*order). The standard embedding of **Top** in **pTop** will be the *chaotic-preorder* one, so that all (ordinary) paths in X are directed in CX. *Note that the category of* ordered *spaces does not allow for such an embedding*, and would not allow us to view classical Algebraic Topology within the Directed one.

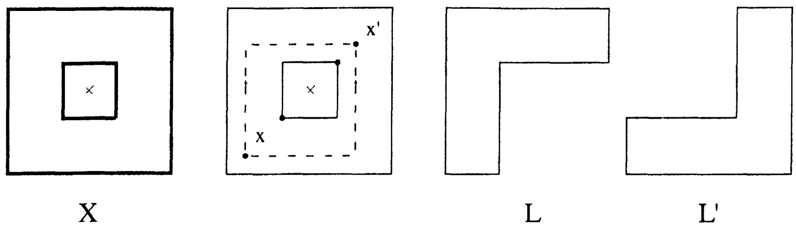
The fundamental category $\uparrow\Pi_1(X)$ of a preordered space X has objects in X, and for arrows, the classes [a]: x → x' of directed paths, up to the equivalence relation *generated* by directed homotopy with fixed endpoints; composition is given by the concatenation of consecutive paths, written as $[a]\otimes[b] = [a\otimes b]$ for [a]: x → x', [b]: x' → x''. $\uparrow\Pi_1(X)$ can be computed by a van Kampen-type theorem, as proved in [11], Thm. 3.6, in a much more general setting ('d-spaces', see 1.4). The obvious functor $\uparrow\Pi_1(X) \to \Pi_1(UX)$ with values in the fundamental groupoid of the underlying space need neither be full (obviously), nor faithful (see 1.2).

A map f: X → Y induces a functor $f_*: \uparrow\Pi_1(X) \to \uparrow\Pi_1(Y)$; a homotopy φ: f → g: X → Y induces a natural transformation $\varphi_*: f_* \to g_*: \uparrow\Pi_1(X) \to \uparrow\Pi_1(Y)$, which generally is *not* invertible. Also because of this, there are crucial differences with the classical fundamental groupoid $\Pi_1(S)$ of a space, for which a model up to homotopy invariance is given by the skeleton: a family of fundamental groups $\pi_1(S, x_i)$, obtained by choosing one point in each path-connected component of S. For instance, if X is *ordered*, the fundamental category has no isomorphisms nor endomorphisms, except the identities. Thus: (a) the category is *skeletal*, and *ordinary equivalence of categories cannot yield any simpler model*; (b) all the fundamental monoids $\uparrow\pi_1(X, x_0) = \uparrow\Pi_1(X)(x_0, x_0)$ are trivial.

1.2. Modelling the fundamental category. An elementary example will give some idea of the information which the fundamental category can give, following the analysis developed in [14]. Let us start from the standard *ordered* square $\uparrow[0, 1]^2$, with the euclidean topology and the product order

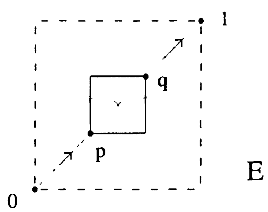
$$(x, y) \le (x', y') \quad \text{if:} \quad x \le x', \ y \le y',$$

and consider a sort of 'square annulus', the (compact) ordered subspace X obtained by taking out the *open* square $]1/3, 2/3[^2$ (marked with a cross)

(1) 

Its directed paths, the continuous order-preserving maps $\uparrow[0, 1] \to X$, move 'rightward and upward' (in the weak sense). The fundamental category $C = \uparrow\Pi_1(X)$ has *some* arrow $x \to x'$ provided that $x \leq x'$ and both points are in L or L' (the closed subspaces represented above). Precisely, there are *two* arrows when $x \leq p = (1/3, 1/3)$ and $x' \geq q = (2/3, 2/3)$ (as in the last figure above), and *one* otherwise. This evident fact can be easily proved with the 'van Kampen' theorem recalled above, using the subspaces L, L' (whose fundamental category is the induced order).

Thus, the whole category C is easy to visualise and 'essentially represented' by the full subcategory E on four vertices 0, p, q, 1 (the central cell does not commute)

(2) 

$$0 \to p \rightrightarrows q \to 1$$

E

which is a *minimal injective model* of C, in a sense made precise in [14].

Finally, notice that - here - the canonical functor $\uparrow\Pi_1(X) \to \Pi_1(UX)$ is faithful and not full. In other cases it can be neither faithful nor full, as it happens for the 3-dimensional analogue, the 'hollow cube' $\uparrow\mathbf{I}^3 \setminus]1/3, 2/3[^3$ ([14], 9.8).

1.3. Other directed structures. More complex directed structures have to be considered, if we want to have *non-reversible* loops and objects like the directed circle $\uparrow\mathbf{S}^1$, or a directed torus $\uparrow\mathbf{S}^1 \times \uparrow\mathbf{S}^1$.

One could extend **pTop** by some *local* notion of ordering - as in the usual geometric models of concurrent processes. The simplest way is perhaps to consider spaces equipped with a relation $\prec$ which is reflexive and locally transitive: every point has some neighbourhood on which the relation is transitive (see [11], 1.4;

stronger properties have been used in the theory of concurrency). But a relevant internal drawback appears, which makes this setting inadequate for directed homotopy and homology: *mapping cones and suspension are lacking* (as well as coequalisers, more generally). Indeed, a locally preordered space cannot have a 'pointlike vortex' (where all neighbourhoods of a point contain some non-reversible loop), whence one cannot realise this way the cone of the directed circle (as proved in detail in [11], 4.6).

1.4. Spaces with distinguished paths. A richer, well-behaved setting has been studied in [11].

A *d-space* is a topological space X equipped with a set dX of (continuous) maps $a: \mathbf{I} \to X$; these maps, called *distinguished paths* or *d-paths*, must contain all constant paths and be closed under concatenation and 'partial increasing reparametrisation' on $\mathbf{I}$: if $a: \mathbf{I} \to X$ is in dX and $h: \mathbf{I} \to \mathbf{I}$ is a continuous order-preserving function, then ah is also distinguished.

A *d-map* $f: X \to Y$ (or *map* of d-spaces) is a continuous mapping between d-spaces which preserves the directed paths: if $a \in dX$, then $fa \in dY$.

The category of d-spaces is written as $d\mathbf{Top}$. It has all limits and colimits, constructed as in $\mathbf{Top}$ and equipped with the initial or final d-structure for the structural maps; for instance a path $\mathbf{I} \to \Pi X_j$ is directed if and only if all its components $\mathbf{I} \to X_j$ are so. The forgetful functor $U: d\mathbf{Top} \to \mathbf{Top}$ preserves thus all limits and colimits; a topological space is generally viewed as a d-space by its *natural* structure, where all (continuous) paths are directed (again, via the right adjoint to U).

Reversing d-paths, by the involution $r(t) = 1 - t$, yields the *reflected*, or *opposite*, d-space $RX = X^{op}$, where $a \in d(X^{op})$ if and only if $a^{op} = ar$ is in dX.

The *standard d-interval* $\uparrow\mathbf{I} = \uparrow[0, 1]$ has directed paths given by the (weakly) increasing maps $\mathbf{I} \to \mathbf{I}$. The *standard directed circle* $\uparrow\mathbf{S}^1 = \uparrow\mathbf{I}/\partial\mathbf{I}$ has the (obvious) quotient d-structure, where paths have to follow a precise orientation. (But note that the directed structure $\uparrow\mathbf{S}^1 \times \uparrow\mathbf{S}^1$ on the torus is not related with an orientation of this surface.)

As in $p\mathbf{Top}$, a (directed) *path* of a d-space X is a map $\uparrow\mathbf{I} \to X$; here, this simply means a distinguished path in the d-structure of X itself. A (directed) *homotopy* $\varphi: f \to g: X \to Y$ is a map $\varphi: X \times \uparrow\mathbf{I} \to Y$ coinciding with f (resp. g) on the lower (resp. upper) basis of the *cylinder* $X \times \uparrow\mathbf{I}$. In particular, a *2-homotopy* $\varphi: a \to b: \uparrow\mathbf{I} \to X$ is a homotopy with fixed endpoints, which means that the mapping $\varphi: \uparrow\mathbf{I} \times \uparrow\mathbf{I} \to X$ induces two constant paths, $\varphi(0, -): a(0) \to b(0)$ and

$\varphi(1,-)$: $a(1) \to b(1)$. The fundamental category $\uparrow\Pi_1(X)$ of a d-space is defined as for preordered spaces (1.1).

(An alternative setting, *inequilogical spaces*, introduced in [13] as a directed version of Dana Scott's equilogical spaces [19, 1], could also be used - but has some disadvantage for concatenation; cf. [13].)

Comparing **pTop** and **dTop**, we have two obvious adjoint functors

(1) $\mathbf{p}$: **dTop** $\rightleftarrows$ **pTop** :$\mathbf{d}$, $\qquad\qquad$ $\mathbf{p} \dashv \mathbf{d}$,

where $\mathbf{d}$ equips a preordered space X with the preorder-preserving maps $\mathbf{I} \to X$, while $\mathbf{p}$ provides a d-space with the *path-preorder* $x \preceq x'$: there exists a d-path from x to x'.

Both functors are faithful, *but* $\mathbf{d}$ *is not an embedding* (nor is $\mathbf{p}$, of course). In fact, for a preordered space $(X, \prec)$, the path-preorder of $\mathbf{pd}(X, \prec) = (X, \preceq)$ can be strictly finer than the original preorder, as it happens for the ordered space considered in 1.2; since $\mathbf{d}(X, \prec) = \mathbf{d}(X, \preceq)$, our claim is proved. Finally, note that the functor $\mathbf{d}$: **pTop** $\to$ **dTop** preserves limits (as a right adjoint) but does not preserve colimits: the coequaliser of the endpoints $\{*\} \rightrightarrows \uparrow\mathbf{I}$ has the chaotic preorder in **pTop** and a non-trivial d-structure $\uparrow\mathbf{S}^1$ in **dTop**. Which is why **dTop** is more interesting.

2. The fundamental biased d-lax 2-category of a directed space

In dimension 2, homotopical analysis of directed spaces leads to a definition of *d-lax* 2-category, with a precise direction for comparison cells.

2.1. The guideline of directed homotopy. We shall work in the setting **dTop** of d-spaces, reviewed above (1.4).

In a d-space, a (directed) homotopy between two iterated concatenations of (the same) paths can only move towards a 'route' which, at each moment, has made a longer way than the initial one, as in the following cases (the homotopy will be made explicit below, in 2.2.5-7)

(1) $a\otimes(b\otimes c) \to (a\otimes b)\otimes c$, $\qquad\qquad$ $1_x\otimes a \to a \to a\otimes 1_{x'}$.

For instance, in the first case, at the instant $t = 1/2$ the second path has already reached the point $x'' = b(1)$, while the first is still in $x' = a(1)$; and the latter can certainly be moved to x'', along b.

(It is interesting to note that Mac Lane's proof of his coherence theorem for monoidal categories follows, *in the associativity part*, a directed approach which agrees with the direction of the associativity homotopy, above: a *directed path*, in the sense of [18], Thm. 3.1, links iterated tensors with decreasing *rank*, and an iterated tensor has rank zero if and only if 'all parentheses start in front'.)

Later, in Section 3, we shall follow an *unbiased* approach, with n-ary concatenations and new comparisons, like the following ones

(2) $a\otimes(b\otimes c) \rightarrow a\otimes b\otimes c \rightarrow (a\otimes b)\otimes c$.

In both approaches, a relevant role will be played by *reparametrisation functions*, i.e. maps $r\colon \uparrow\mathbf{I} \rightarrow \uparrow\mathbf{I}$ which preserve the endpoints; or, equivalently, order-preserving surjective endomappings of the standard interval, necessarily continuous. (One can require them to be piecewise affine, or not.) They have an n-ary concatenation

(3) $(r_1 \otimes ... \otimes r_n)(t) = (i-1)/n + r_i(nt - i + 1)/n$, when $(i-1)/n \leq t \leq i/n$.

The pointwise order $r \leq r'$ produces an *interpolating* directed 2-homotopy, by affine interpolation

(4) $\varphi_0(r, r')\colon r \rightarrow r'\colon \uparrow\mathbf{I} \rightarrow \uparrow\mathbf{I}$, $\varphi_0(r, r')(s, t) = (1-t).r(s) + t.r'(s)$.

(Note: the notation r' has nothing to do with derivatives.) Its class $[\varphi_0(r, r')]$ up to the equivalence relation generated by 3-homotopy (with fixed boundary) is *uniquely determined* by r, r'. In fact, if $\alpha, \beta\colon r \rightarrow r'$ are 2-homotopies, also $(\alpha\vee\beta)(s, t) = \max(\alpha(s, t), \beta(s, t))$ is so, and - plainly - there are 3-homotopies $\alpha \rightarrow \alpha\vee\beta \leftarrow \beta$.

The set of reparametrisation functions, with the pointwise order and the tensor product described above, is an *ordered d-lax monoidal category* (cf. 3.7).

2.2. The construction. Let X be a d-space. The *fundamental biased d-lax 2-category* $\uparrow b\Pi_2(X)$ will have the following objects, arrows, cells, elementary compositions (nullary and binary) and comparisons.

(a) An *object* is a point of X.

(b) An *arrow* $a\colon x \rightarrow y$ is a (directed) path $a\colon \uparrow\mathbf{I} \rightarrow X$ with $a(0) = x$, $a(1) = y$; the *unit-arrow* $1_x\colon x \rightarrow x$ is the constant path at x.

(c) A *cell* $[\alpha]\colon a \rightarrow a'\colon x \rightarrow y$ is a homotopy class of homotopies of paths; more precisely, α is a (directed) *2-homotopy* $a \rightarrow a'$ (with fixed endpoints), which means that the map $\alpha\colon \uparrow\mathbf{I}^2 \rightarrow X$ has the boundary represented below (the thick lines represent constant paths)

$$(1) \qquad \begin{array}{ccc} x & \xrightarrow{\ a\ } & y \\ \| & \alpha & \| \\ x & \xrightarrow[\ a'\ ]{} & y \end{array} \qquad\qquad\qquad \begin{array}{ll} \bullet \longrightarrow s \\ \downarrow t \end{array}$$

and its homotopy class $[\alpha]$ is up to the equivalence relation generated by 3-*homotopies* $\alpha' \to \alpha''$ (with fixed boundary); the *unit-cell* 1_a: $a \to a$ is the class of the trivial 2-homotopy $c_a(s, t) = a(s)$.

(d) The *main composition*, or *upper-level composition*, of $[\alpha]$ with $[\alpha']$: $a' \to a''$: $x \to y$ is defined by the pasting $\alpha \otimes_2 \alpha'$ of any two representatives, with respect to the second variable

$$(2) \quad [\alpha] \otimes_2 [\alpha']: a \to a'': x \to y, \qquad [\alpha] \otimes_2 [\alpha'] = [\alpha \otimes_2 \alpha'];$$

$$(\alpha \otimes_2 \alpha')(s, t) = \begin{cases} \alpha(s, 2t), & 0 \leq t \leq 1/2, \\ \alpha'(s, 2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

(e) The *(lower-level) composition* of a: $x \to y$ with b: $y \to z$ is the standard concatenation $a \otimes b$: $x \to z$ of the paths

$$(3) \quad (a \otimes b)(t) = a(2t) \ \text{ if } \ 0 \leq t \leq 1/2, \qquad (a \otimes b)(t) = b(2t-1) \ \text{ if } \ 1/2 \leq t \leq 1.$$

(f) The *lower-level composition* of $[\alpha]$: $a \to a'$: $x \to y$ with $[\beta]$: $b \to b'$: $y \to z$ is defined by the pasting $\alpha \otimes \beta$ of any two representatives, with respect to the first variable

$$(4) \quad [\alpha] \otimes [\beta]: a \otimes b \to a' \otimes b': x \to z, \qquad [\alpha] \otimes [\beta] = [\alpha \otimes \beta];$$

$$(\alpha \otimes \beta)(s, t) = \begin{cases} \alpha(2s, t), & 0 \leq s \leq 1/2, \\ \beta(2s-1, t), & 1/2 \leq s \leq 1. \end{cases}$$

We shall use abbreviations as: $x \otimes a = 1_x \otimes a = 1 \otimes a$, $a \otimes [\alpha] = 1_a \otimes [\alpha] = 1 \otimes [\alpha]$, $x \otimes [\alpha] = 1_{1_x} \otimes [\alpha]$ (when the domain-arrow of α is degenerate) and so on.

(g) For an arrow a: $x \to y$, the *left-unit* and the *right-unit comparisons* are given by the following 2-homotopies (determined by two 2-homotopies λ_0, ρ_0, which are affine in the second variable)

$$(5)\qquad \begin{array}{ccccc} x & \xrightarrow{1_x} & x & \xrightarrow{a} & y \\ | & & \lambda a & & | \\ x & & \xrightarrow[a]{\qquad} & & y \end{array} \qquad \begin{array}{l} [\lambda a]\colon x \otimes a \to a, \qquad \lambda a = a \circ \lambda_0\colon \uparrow\mathbf{I} \times \uparrow\mathbf{I} \to X, \\ \lambda_0(s, t) = (1-t).r(s) + t.s, \\ r(s) = \max(0, 2s-1), \end{array}$$

$$(6)\qquad \begin{array}{ccccc} x & & \xrightarrow{a} & & y \\ | & & \rho a & & | \\ x & \xrightarrow[a]{} & y & \xrightarrow[1_y]{} & y \end{array} \qquad \begin{array}{l} [\rho a]\colon a \to a \otimes y, \qquad \rho a = a \circ \rho_0\colon \uparrow\mathbf{I} \times \uparrow\mathbf{I} \to X, \\ \rho_0(s, t) = (1-t).s + t.r'(s), \\ r'(s) = \min(2s, 1). \end{array}$$

(h) For three consecutive arrows $a\colon x \to y$, $b\colon y \to z$, $c\colon z \to w$, the *associativity comparison* is expressed as follows (here, the ternary concatenation $a \otimes b \otimes c$ is only used as a shortcut in describing the 2-homotopy):

(7) $[\kappa] = [\kappa(a, b, c)]\colon a \otimes (b \otimes c) \to (a \otimes b) \otimes c, \qquad \kappa = (a \otimes b \otimes c) \circ \kappa_0\colon \uparrow\mathbf{I} \times \uparrow\mathbf{I} \to X,$

$$\begin{array}{ccccccc} x & \xrightarrow{\quad a \quad} & & y & \xrightarrow{b} z & \xrightarrow{c} & w \\ | & & & \kappa(a, b, c) & & & | \\ x & \xrightarrow[a]{} & y & \xrightarrow[b]{} & z & \xrightarrow[c]{\qquad} & w \end{array} \qquad \begin{array}{l} \kappa_0\colon \uparrow\mathbf{I} \times \uparrow\mathbf{I} \to \uparrow\mathbf{I}, \\ \kappa_0(s, t) = (1-t).r(s) + t.r'(s), \end{array}$$

$$r(s) = \begin{cases} 2s/3, & 0 \le s \le 1/2, \\ (4s-1)/3, & 1/2 \le s \le 1, \end{cases} \qquad r'(s) = \begin{cases} 4s/3, & 0 \le s \le 1/2, \\ (2s+1)/3, & 1/2 \le s \le 1. \end{cases}$$

2.3. Definition. Abstracting the previous situation, a *biased d-lax 2-category* **A** will consist of the following data and properties. (Greek letters denote now 2-cells.)

(bdl.0) A set of objects, Ob**A**.

(bdl.1) For any two objects x, y, a category $\mathbf{A}(x, y)$ of *maps* $a\colon x \to y$ and *cells* $\alpha\colon a \to b$, with *main*, or *upper-level*, composition $\alpha \otimes_2 \beta\colon a \to b \to c$ and units $1_a\colon a \to a$.

(bdl.2) For any object x a *lower identity* 1_x; for any triple of objects x, y, z a functor of *lower composition*

(1) $- \otimes -\colon \mathbf{A}(x, y) \times \mathbf{A}(y, z) \to \mathbf{A}(x, z).$

Explicitly, the functorial properties give:

(2) $1_{a\otimes b} = 1_a \otimes 1_b$ (*nullary interchange*),

(3) $(\alpha\otimes\beta)\otimes_2(\alpha'\otimes\beta') = (\alpha\otimes_2\alpha')\otimes(\beta\otimes_2\beta')$ (*binary* or *middle-four interchange*),

(bdl.3) For any map a and for any triple (a, b, c) of consecutive maps, three cells

(4) $\lambda a\colon x\otimes a \to a,$ $\rho a\colon a \to a\otimes y$ (*left* and *right-unit comparison*),

$\kappa(a, b, c)\colon a\otimes(b\otimes c) \to (a\otimes b)\otimes c$ (*associativity comparison*),

forming three natural transformations between the obvious (ordinary) functors $\mathbf{A}(x, y) \to \mathbf{A}(x, y)$ (in the first two cases) and $\mathbf{A}(x, y) \times \mathbf{A}(y, z) \times \mathbf{A}(z, w) \to \mathbf{A}(x, w)$ (in the last).

Explicitly, the naturality properties give the following relations

(5) $(x\otimes\alpha)\otimes_2(\lambda a') = (\lambda a)\otimes_2\alpha$ (*naturality of* λ),

(6) $(\rho a)\otimes_2(\alpha\otimes y) = \alpha\otimes_2(\rho a')$ (*naturality of* ρ),

(7) $(\alpha\otimes(\beta\otimes\gamma))\otimes_2\kappa(a', b', c') = \kappa(a, b, c)\otimes_2((\alpha\otimes\beta)\otimes\gamma)$ (*naturality of* κ).

(bdl.4) (*coherence*) Every diagram (universally) constructed with comparison cells, via $\otimes$- and $\otimes_2$-compositions, commutes.

This last axiom can be made more precise using techniques which will be developed in the next section. One defines the functor of iterated composition along a dichotomic tree τ

(8) $< -; \tau>\colon \mathbf{A}(x_0, x_1) \times \ldots \times \mathbf{A}(x_{n-1}, x_n) \to \mathbf{A}(x_0, x_n),$

$(a_1,\ldots, a_n) \mapsto <a_1,\ldots, a_n; \tau>,$

and requires that, for any two such trees, there be *at most one* natural transformation $< -; \tau> \to < -; \tau'>$ constructed with λ, ρ, α. But the unbiased approach of the ne section will allow for a simpler formulation (3.3).

2.4. Theorem. For a d-space X, the structure $\uparrow b\Pi_2(X)$ constructed above (2.2) is indeed a biased d-lax 2-category, as defined above (2.3).

Proof. The only non obvious part is the proof of the coherence property (bdl.4). This becomes easy introducing generalised comparison cells (which would not make sense in a general biased d-lax 2-category, but will make sense in the unbiased approach of Section 3).

A *generalised comparison cell*

(1) $\varphi(a; r, r') = a[\varphi_0(r, r')]\colon ar \to ar'\colon \uparrow\mathbf{I} \to X,$

is determined by a path $a\colon \uparrow\mathbf{I} \to X$, together with two reparametrisation functions (2.1) $r, r'\colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$, with $r \leq r'$. Recall that the 3-homotopy class $[\varphi_0(r, r')]\colon r \to r'$ (which exists by affine interpolation) is uniquely determined by r, r' (2.1).

The unit and associativity comparisons λa, ρa, $\kappa(a, b, c)$ in 2.2 have been constructed precisely in this way; in the third case, the role of a in (1) is taken by the ternary composite $a{\otimes}b{\otimes}c$.

Now, the set of all generalised comparisons contains the identities of arrows, and is plainly closed under $\otimes$-composition. As to $\otimes_2$-composition, we have

(2) $\quad [\varphi(a; r, r') \otimes_2 \varphi(a; r', r'')] = [\varphi(a; r, r'')]\colon ar \to ar''\colon \uparrow\mathbf{I} \to X.$

This is sufficient to prove the thesis, since in a given diagram 'naturally' constructed with *comparison cells*, each path is a reparametrisation of a *unique* n-ary concatenation $a_1{\otimes}...{\otimes}a_n$, via a *precise* reparametrisation function, which only depends on the type of iterated concatenation of the components $a_1,..., a_n$ (formally, a *tree*, as will be made explicit in Section 3). □

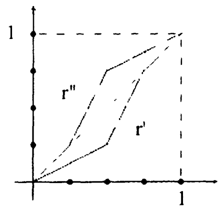
2.5. Higher comparisons. One can *enrich* the fundamental biased d-lax 2-category $\uparrow b\Pi_2(X)$ of a d-space by adding two new *higher associativity comparisons* $\kappa'(a, b, c, d)$ and $\kappa''(a, b, c, d)$, depending on *four* consecutive arrows, so to break Mac Lane's pentagon into 3 commutative triangles

(1)
$$\begin{array}{ccccc} & & (a{\otimes}b) \otimes (c{\otimes}d) & & \\ & \overset{\kappa}{\nearrow} & \kappa' \quad\quad \kappa'' & \overset{\kappa}{\searrow} & \\ a{\otimes}(b{\otimes}(c{\otimes}d)) & & & & ((a{\otimes}b){\otimes}c){\otimes}d \\ & \underset{a\otimes\kappa}{\searrow} & & \underset{\kappa\otimes d}{\nearrow} & \\ & & a{\otimes}((b{\otimes}c){\otimes}d) \xrightarrow[\kappa]{} (a{\otimes}(b{\otimes}c)){\otimes}d & & \end{array}$$

We obtain κ', κ'' as *generalised comparison cells*, in the sense of 2.4.1 (which proves the commutativity of the diagram above). In fact, with respect to the quaternary composite $a{\otimes}b{\otimes}c{\otimes}d$:

- $(a{\otimes}b) \otimes (c{\otimes}d)$ coincides with it, and its reparametrisation function is the identity,
- $a{\otimes}((b{\otimes}c){\otimes}d)$ and $(a{\otimes}(b{\otimes}c)){\otimes}d$ have reparametrisation functions r', r'' such that $r' \leq \mathrm{id} \leq r''$

(2)
$$r'(s) = \begin{cases} s/2 & 0 \leq s \leq 1/2, \\ 2s - 3/4 & 1/2 \leq s \leq 3/4, \\ s & 3/4 \leq s \leq 1, \end{cases} \qquad r''(s) = \begin{cases} s & 0 \leq s \leq 1/4, \\ 2s - 1/4 & 1/4 \leq s \leq 1/2, \\ (s+1)/2 & 1/2 \leq s \leq 1, \end{cases}$$

(3) 

Following 2.4.2, κ' and κ'' are constructed as follows

$$\text{(4)}\quad \kappa'(a, b, c, d) = \varphi(a{\otimes}b{\otimes}c{\otimes}d;\, r', r) = [(a{\otimes}b{\otimes}c{\otimes}d)\circ\kappa'_0]\colon \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to X,$$
$$\kappa''(a, b, c, d) = \varphi(a{\otimes}b{\otimes}c{\otimes}d;\, r, r'') = [(a{\otimes}b{\otimes}c{\otimes}d)\circ\kappa''_0]\colon \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to X,$$
$$\kappa'_0(s, t) = (1 - t).r'(s) + t.s, \qquad\qquad \kappa'_0(s, t) = (1 - t).s + t.r''(s).$$

One can now define an *extended biased d-lax 2-category*, including these higher associativity comparisons in the structure *and* in the axioms (bdl.3-4). Theorem 2.4 still holds in this extended sense, as the new comparisons lie within the generalised ones, dealt with in the proof.

2.6. Basic coherence properties. Taking into account the higher comparisons $\kappa'(a,b,c,d)$, $\kappa''(a,b,c,d)$ (2.4), we can formulate seven 'basic' coherence properties, for *extended* biased d-lax 2-categories. It would be interesting to prove that they are sufficient to ensure that 'all diagrams of comparison cells commute', so that the theory could be formulated as a first-order one.

(a) Given an object x, the cells $\lambda(1_x)\colon 1_x{\otimes}1_x \to 1_x$ and $\rho(1_x)\colon 1_x \to 1_x{\otimes}1_x$ are inverse.

(b)-(d) Given two consecutive arrows a, b, we have:

$$\text{(1)}\quad \rho a \otimes \lambda b = \kappa(a, 1_y, b)\colon a \otimes (1_y \otimes b) \to (a \otimes 1_y) \otimes b,$$
$$\lambda(a \otimes b) = (\lambda a \otimes b) \circ \kappa(1_x, a, b), \qquad\qquad \rho(a \otimes b) = \kappa(a, b, 1_z) \circ (a \otimes \rho b),$$

$$\begin{array}{ccc} 1_x \otimes (a \otimes b) & \xrightarrow{\ \kappa\ } & (1_x \otimes a) \otimes b \\ & {\scriptstyle\lambda(a\otimes b)}\searrow & \downarrow{\scriptstyle\lambda a\otimes b} \\ & & a\otimes b \end{array} \qquad\qquad \begin{array}{ccc} a \otimes (b \otimes 1_z) & \xrightarrow{\ \kappa\ } & (a \otimes b) \otimes 1_z \\ {\scriptstyle a\otimes\rho b}\downarrow & \nearrow{\scriptstyle\rho(a\otimes b)} & \\ a\otimes b & & \end{array}$$

(e)-(g) Given four consecutive arrows a, b, c, d, the comparisons $\kappa, \kappa', \kappa''$ form three commutative triangles (as in 2.5.1).

These properties 'somehow' correspond to Mac Lane's five original coherence axioms for monoidal categories [18]. Here, the comparisons are not invertible and the direction comes from Directed Algebraic Topology; moreover, the original pentagon has been split into three triangles, which might be of help for attacking the coherence problem. Kelly's well-known reduction result [16], showing that - in the classical case - the properties (a), (c), (d) follow from the others should not subsist here, since that proof strongly depends on cancellation of invertible cells (more than Mac Lane's, apparently).

Finally, let us note that, *for* $\mathbf{A} = \uparrow b\Pi_2(X)$, the condition (a) holds in a strict form

(3) $1_x \otimes 1_x = 1_x, \qquad\qquad \lambda 1_x = 1_{1_x} = \rho 1_x.$

2.7. Functoriality. A directed map $f\colon X \to Y$ induces a *strict* 2-functor

(1) $f_*\colon \uparrow b\Pi_2(X) \to \uparrow b\Pi_2(Y),$

$f_*(x) = f(x), \qquad f_*(a\colon x \to x') = (f \circ a\colon fx \to fx'), \qquad f_*[\alpha] = [f \circ \alpha].$

This takes objects, arrows and cells of $\uparrow b\Pi_2(X)$ to similar items of $\uparrow b\Pi_2(Y)$, preserving the whole structure: domains, codomains, units, compositions and comparisons (in the original or in the extended sense of 2.5): $f_*(\lambda^X a) = \lambda^Y f_*(a)$, etc.

A directed homotopy $\alpha\colon f \to g\colon X \to Y$, represented by a directed map $\alpha\colon X \times \uparrow\mathbf{I} \to Y$, induces a *lax natural transformation of 2-functors* (a notion recalled below, in 2.8)

(2) $\alpha_*\colon f_* \to g_*\colon \uparrow b\Pi_2(X) \to \uparrow b\Pi_2(Y), \qquad \alpha_*(x) = \alpha(x, -)\colon f(x) \to g(x),$

$\alpha_*(a\colon x \to x') = [\hat{\alpha}_*(a)]\colon f_*(a) \otimes \alpha(x') \to \alpha(x) \otimes g_*(a)\colon f(x) \to g(x').$

Here $\hat{\alpha}_*(a)$ is the 2-cell associated with the *double homotopy* $\alpha \circ (a \times \uparrow\mathbf{I})\colon \uparrow\mathbf{I} \times \uparrow\mathbf{I} \to Y$, in the usual way, first pasting it with the double cells $g^-(\alpha x)$ and $g^+(\alpha x')$

$$
(3) \qquad
\begin{array}{ccc}
f(x) & \text{———} & f(x) \\
| & g^-(\alpha x) & \downarrow \alpha x \\
f(x) & \text{-}\ \alpha x \to & g(x) \\
fa \downarrow & \alpha(a \times \uparrow\mathbf{I}) & |\ ga \\
f(x') & \text{-}\ \alpha x' \to & g(x') \\
\alpha x' \downarrow & g^+(\alpha x') & | \\
f(x') & \text{———} & g(x')
\end{array}
\qquad
\begin{array}{l}
g^-(\alpha x) = (\alpha x)(\max(s, t)), \\ \\ \\ \\ \\
g^+(\alpha x') = (\alpha x')(\min(s, t)).
\end{array}
$$

('lower and upper connections', a standard tool of cubical homotopical algebra), and then interchanging coordinates

2.8. Lax natural transformations. We end this section recalling the definition of a *lax natural transformation* $\varphi: f \to g: X \to Y$, between strict 2-functors and lax 2-categories (cf. [4]). It assigns

(i) to every object $x \in X$, a map $\varphi x: fx \to gx$ in Y,

(ii) to every map $a: x \to x'$ in X, a *comparison* cell $\varphi a: fa \otimes \varphi x' \to \varphi x \otimes ga$ in Y,

$$(1) \qquad \begin{array}{ccc} fx & \xrightarrow{fa} & fx' \\ \varphi x \downarrow & \swarrow \varphi a & \downarrow \varphi x' \\ gx & \xrightarrow[ga]{} & gx' \end{array}$$

so that the following axioms hold:

(lnt.1) given $x \in X$, $\varphi(1_x) = \lambda(\varphi x) \otimes_2 \rho(\varphi x): 1_{fx} \otimes \varphi x \to \varphi x \otimes 1_{gx}$,

(lnt.2) if $c = a \otimes b: x \to x' \to x''$, then $(fa \otimes \varphi b) \otimes_2 (\varphi a \otimes gb) = \varphi c$,

$$(2) \qquad \begin{array}{ccccc} fx & \xrightarrow{fa} & fx' & \xrightarrow{fb} & fx'' \\ \varphi x \downarrow & \swarrow \varphi a & \varphi x' \downarrow & \swarrow \varphi b & \downarrow \varphi x'' \\ gx & \xrightarrow[ga]{} & gx' & \xrightarrow[gb]{} & gx'' \end{array} \quad = \quad \begin{array}{ccc} fx & \xrightarrow{fc} & fx'' \\ \varphi x \downarrow & \swarrow \varphi c & \downarrow \varphi x'' \\ gx & \xrightarrow[gc]{} & gx'' \end{array}$$

(lnt.3) given a cell $\alpha: a \to b: x \to x'$ in X, then $(f\alpha \otimes \varphi x') \otimes_2 \varphi b = \varphi a \otimes_2 (\varphi x \otimes g\alpha)$,

$$(3) \qquad \begin{array}{ccc} fx & \overset{fa}{\underset{fb}{\genfrac{}{}{0pt}{}{\longrightarrow}{\downarrow f\alpha}}} & fx' \\ \varphi x \downarrow & \swarrow \varphi b & \downarrow \varphi x' \\ gx & \xrightarrow[gb]{} & gx' \end{array} \quad = \quad \begin{array}{ccc} fx & \xrightarrow{fa} & fx' \\ \varphi x \downarrow & \swarrow \varphi a & \downarrow \varphi x' \\ gx & \overset{ga}{\underset{gb}{\genfrac{}{}{0pt}{}{\longrightarrow}{\downarrow g\alpha}}} & gx' \end{array}$$

3. The fundamental unbiased d-lax 2-category

We follow now an unbiased approach, where all n-ary concatenations are assigned.

3.1. Tree concatenation. Again, we begin with the 'geometric case', introducing the fundamental *unbiased d-lax 2-category* $\uparrow u\Pi_2(X)$ of a d-space, and will then abstract the general notion (but we single out from now the main general aspects). The structure has the same items and upper-level composition as the biased one (2.2), with lower-level composition extended as below and comparisons extended as in 3.2.

First, there is an *n-ary lower composition* of (consecutive) maps and cells

(1) $a = a_1 \otimes a_2 \otimes \ldots \otimes a_n, \qquad \alpha = \alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n,$

realised in the obvious way:

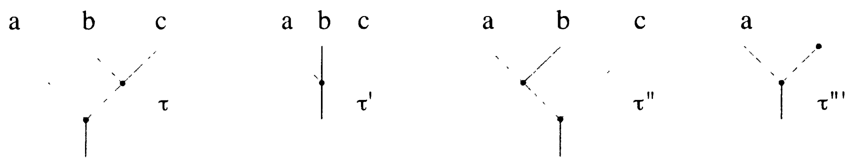
(2) $a(t) = a_i(nt - i + 1), \qquad \text{for } (i-1)/n \leq t \leq i/n,$

$\alpha(t, t') = \alpha_i(nt - i + 1, t'), \qquad \text{for } (i-1)/n \leq t \leq i/n.$

Note that, as a marginal difference with Leinster [17], we make no distinction between an arrow a and its 'unary composition' (a). This choice can be referred to as 'normality' (since it corresponds to *normal* lax algebras, as opposed to the general ones).

Iterating multiple composition, we get a *tree-composition*, which will be described (also in the abstract case) as a finite tree whose n leaves are labelled by a sequence of n consecutive arrows $a_1, \ldots, a_n$, as in the following examples

(3) $f = a \otimes (b \otimes c), \qquad g = a \otimes b \otimes c, \qquad h = (a \otimes b) \otimes c, \qquad k = a \otimes 1,$



We are using the notion of tree defined in [17], 2.3.3, which we shall call a *composition tree*. The last example has two *shoots*, namely one leaf and one *bare shoot* $\uparrow$.

We shall write $<a_1, a_2, \ldots, a_n; \tau>$ to denote the composition of such a labelled tree. We only label leaves: there is no need of labelling bare shoots, since the corresponding identities are determined by the adjacent arrows; *unless all shoots are bare*, in which case one labelling *object* suffices: thus $<x; \tau>$ will denote $1_x \otimes (1_x \otimes 1_x)$, if τ is the 'pruned version' of the first tree above, with all shoots bare. The numbers $l(\tau) \leq s(\tau)$ will denote, respectively, the number of leaves and shoots of a composition tree τ.

(In the biased case, one would only use *dichotomic trees*, with twofold bifurcations; all the examples above are of this type, except the second, which is related with ternary composition.)

3.2. Comparisons. We already know that, in our guideline, comparisons move towards tree-concatenations of the same paths which, at each moment, have made a longer way; for instance, from f to g to h (in 3.1.3), or from $1 \otimes a$ to a and then to $a \otimes 1$.

Let τ be a composition tree, with $n = l(\tau)$ leaves and $m = s(\tau)$ shoots. Its *duration sequence* $h(\tau) = (h_1, \ldots, h_m)$ gives, in the 'geometric case', the duration of the time-interval on which the concatenated path goes along each component (including the constant paths); generally, h_i^{-1} is *defined* as the product of the multiplicities of the nodes which precede the i-th shoot; plainly, $\Sigma\, h_i = 1$.

If there are no bare shoots (i.e., $m = n$), we can use the *cumulative sequence* $k = k(\tau)$, with $k_i = \Sigma_{j \leq i}\, h_j$. Thus, in the previous cases τ, τ', τ'' (3.1.3), we get the sequences

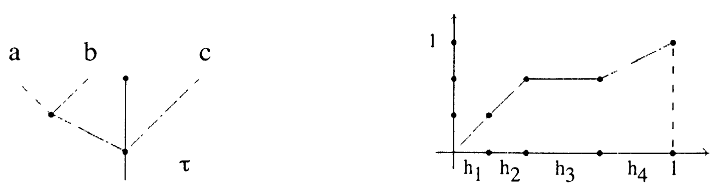
(1) h: (1/2, 1/4, 1/4), (1/3, 1/3, 1/3), (1/4, 1/4, 1/2),

k: (2/4, 3/4, 1), (1/3, 2/3, 1), (1/4, 2/4, 1),

and the fact that $k(\tau) \geq k(\tau') \geq k(\tau'')$ shows that we can construct directed homotopies $f \to g \to h$.

To get also the comparisons $1 \otimes a \to a \to a \otimes 1$ we need a more general criterion, extending the constructions of 2.2.5-7 and already used in the proof of Theorem 2.4. The reparametrisation function (2.1) $r = r(\tau)\colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ is exemplified below, for $(a \otimes b) \otimes 1 \otimes c = \langle a, b, c; \tau\rangle$

(2)



$h_1 = h_2 = 1/6, \quad h_3 = h_4 = 1/3, \qquad r(1/6) = 1/3, \quad r(2/6) = r(4/6) = 2/3.$

For a pruned tree, the reparametrisation function is the identity. Otherwise, the function is affine on each interval $[k_{i-1}, k_i]$ $(i = 1, \ldots, m)$, and increases on the latter of $1/n$ or 0, when the i-th shoot is, respectively, a leaf or bare. (An inductive definition will be given in 3.4a.)

In $\uparrow u\Pi_2(X)$, the reparametrisation function determines the tree-composition as follows

(3) $\langle a_1,..., a_n; \tau\rangle = (a_1 \otimes ... \otimes a_n)\circ r(\tau)\colon \uparrow\mathbf{I} \to X.$

Now, we introduce comparison cells when the labels coincide and the first reparametrisation function is pointwise smaller than the second

(4) $\varphi(a_1,..., a_n; \tau, \tau')\colon \langle a_1,..., a_n; \tau\rangle \to \langle a_1,..., a_n; \tau'\rangle, \qquad r(\tau) \leq r(\tau').$

These will be called *syntactic* comparison cells, since they depend on the *construction* of the two paths from $a_1,..., a_n$ and the trees τ, τ' (rather than on their actual value, cf. 3.7-3.8).

In $\uparrow u\Pi_2(X)$, the homotopy $\varphi(a_1,..., a_n; \tau, \tau')$ is constructed in the obvious way (as in 2.4.1), out of the obvious 2-homotopy $\varphi_0(\tau, \tau') = \varphi_0(r(\tau), r(\tau'))\colon r(\tau) \to r(\tau')$ moving, at each instant $s \in [0, 1]$, along the (directed!) segment, in $\uparrow\mathbf{I}$, from $r(\tau)(s)$ to $r(\tau')(s)$

(5) $\varphi(a_1,..., a_n; \tau, \tau') = (a_1 \otimes ... \otimes a_n)\circ\varphi_0(\tau, \tau')\colon \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to X,$

$\varphi_0(\tau, \tau')\colon \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to \uparrow\mathbf{I}, \qquad \varphi_0(\tau, \tau')(s, t) = (1 - t).r(\tau)(s) + t.r(\tau')(s),$

$$\begin{array}{ccc} r(0) & \xrightarrow{\ r(\tau)\ } & r(1) \\ \Big| & \varphi_0(\tau, \tau') & \Big| \\ r'(0) & \xrightarrow[\ r(\tau')\ ]{} & r'(1) \end{array} \qquad\qquad \begin{array}{ll} \bullet \longrightarrow & s \\ \Big|\ t & \end{array}$$

Note that every pair of iterated compositions *of the same identity* has a comparison, $[x;\tau] \to [x;\tau']$, whatever be the number of shoots of the two pruned trees; indeed, the reparametrisation functions coincide (with the identity); therefore, all such compositions will be isomorphic, in the general case (for a fixed object x, of course). Another case where the reparametrisation functions coincide (with the identity, again) is exemplified by the pair $(a\otimes b)\otimes(c\otimes d)$ and $a\otimes b\otimes c\otimes d$. In our geometric situation, $\uparrow u\Pi_2(X)$, all such pairs of concatenations *strictly coincide.*

3.3. Definition. An *unbiased d-lax 2-category* $\mathbf{A}$ will consist of the following data (and properties).

(udl.0) A set of objects, $\mathrm{Ob}\mathbf{A}$.

(udl.1) For any two objects x, y, a category $\mathbf{A}(x, y)$ of *maps* $a\colon x \to y$ and *cells*

α: a $\rightarrow$ b, with *main*, or *upper-level*, composition $\alpha \otimes_2 \beta$: a $\rightarrow$ b $\rightarrow$ c and units 1_a: a $\rightarrow$ a.

(udl.2) For any sequence of objects $x_0, ..., x_n$, a functor of *lower n-ary composition*

$$\text{(1)} \quad \mathbf{A}(x_0, x_1) \times ... \times \mathbf{A}(x_{n-1}, x_n) \rightarrow \mathbf{A}(x_0, x_n), \qquad (a_1, ..., a_n) \mapsto a_1 \otimes ... \otimes a_n,$$

which, for $n = 0$, reduces to a *lower identity* 1_{x_0}. This defines also the *tree-composition* $<a_1, ..., a_n; \tau>$ of a sequence of n consecutive maps, *along* a tree with n leaves $(n > 0)$; for a tree with no leaves, we have a tree-composition of identities $<x; \tau>$.

(udl.3) For every pair of trees τ, τ' with n leaves and reparametrisation functions $r(\tau) \leq r(\tau')$, a natural transformation (*syntactic comparison*) of ordinary functors in n variables

$$\text{(2)} \quad \varphi(\tau, \tau'): < -; \tau,> \rightarrow < -; \tau'>: \mathbf{A}(x_0, x_1) \times ... \times \mathbf{A}(x_{n-1}, x_n) \rightarrow \mathbf{A}(x_0, x_n),$$
$$\varphi(a_1, ..., a_n; \tau, \tau'): <a_1, ..., a_n; \tau> \rightarrow <a_1, ..., a_n; \tau'>: a_1(x_0) \rightarrow a_n(x_n),$$

whose general component is a cell between two tree-compositions of the same sequence of maps.

(udl.4) (*coherence*) Every diagram (universally) constructed with comparison cells, via $\otimes$- and $\otimes_2$-compositions, commutes.

More explicitly, the last axiom means that:

$$\text{(3)} \quad r(\tau) \leq r(\tau') \leq r(\tau'') \quad \Rightarrow \quad \varphi(\tau', \tau'') \circ \varphi(\tau, \tau') = \varphi(\tau, \tau''),$$
$$r(\sigma_1) \leq r(\tau_1), ..., r(\sigma_n) \leq r(\tau_n) \quad \Rightarrow$$
$$\varphi(\sigma_1, \tau_1) \otimes ... \otimes \varphi(\sigma_n, \tau_n) = \varphi((\sigma_1, ..., \sigma_n), (\tau_1, ..., \tau_n)),$$

(note that, in the second case, $r((\sigma_1, ..., \sigma_n)) \leq r((\tau_1, ..., \tau_n))$.

3.4. Remarks. (a) The reparametrisation function of a tree can be inductively defined as follows:

(i) $r(|) = \text{id}: \uparrow\mathbf{I} \rightarrow \uparrow\mathbf{I}$,

(ii) given a tree $(\tau_1, ..., \tau_p)$, where $n_i \geq 0$ is the number of leaves of τ_i $(i = 1, ..., p)$, $k_i = \Sigma_{j \leq i}\, n_j$ is the cumulative sequence and $n = k_p = \Sigma\, n_i$, the function $r(\tau_1, ..., \tau_p)$ takes values:

$$\text{(1)} \quad r(\tau_1, ..., \tau_p)(t) = k_{i-1}/n + (n_i/n)\, r(\tau_i)(pt - i + 1), \qquad \text{for } t \in [(i-1)/p, i/p],$$

or is the identity when $n = 0$.

(b) The concatenation trees form a preordered set, with respect to the pointwise order of their reparametrisation functions. And a 'preordered d-lax monoidal category' (as defined in 3.7), with

(2) $\tau_1 \otimes ... \otimes \tau_p = (\tau_1,..., \tau_p)$.

We already remarked that the mapping $\tau \mapsto r(\tau)$ is not injective (end of 3.2).

(c) An unbiased d-lax 2-category **A** contains an associated biased one, b**A**, restricting its multiple concatenations to the nullary and binary ones, and restricting the comparison transformations to the required ones: λ, ρ, κ. (For the extended case of 2.5, one should also include the transformations κ', κ''.)

3.5. Theorem. $\uparrow u\Pi_2(X)$ is an unbiased d-lax 2-category. The associated biased d-lax 2-category (3.4c) is $\uparrow b\Pi_2(X)$.

Proof. As in the biased case (2.4), we only have to prove the coherence axiom. The previous proof can be easily extended. Or one can give a more constructive argument, verifying 3.3.3, in the same way. The last assertion is obvious. □

3.6. Functoriality. Also here, as in Section 2, a directed map $f: X \to Y$ induces a *strict* 2-functor

(1) $f_*: \uparrow u\Pi_2(X) \to \uparrow u\Pi_2(Y)$,

$f_*(x) = f(x), \qquad f_*(a: x \to x') = (f \circ a: fx \to fx'), \qquad f_*[\alpha] = [f\alpha]$.

A directed homotopy $\alpha: f \to g: X \to Y$, represented by a directed map $\alpha: X \times \uparrow\mathbf{I} \to Y$, induces a lax natural transformation of 2-functors (with $\hat{\alpha}_*(a)$ defined as in 2.7.3)

(2) $\alpha_*: f_* \to g_*: \uparrow u\Pi_2(X) \to \uparrow u\Pi_2(Y)$,

$\alpha_*(x) = \alpha(x, -): f(x) \to g(x), \qquad \alpha_*(a: x \to x') = [\hat{\alpha}_*(a)]$.

3.7. Preordered d-lax 2-categories. We end with a simpler d-lax structure, where comparisons have an absolute, non-syntactic character. Examples have already appeared, in the d-lax *monoidal* case: the set of reparametrisation functions (2.1) and the set of concatenation trees (3.4b). The simplicity of the structure comes from the fact that, here, coherence is automatic.

A *preordered biased d-lax 2-category* **A** consists of the following data (and properties).

(pdl.0) A set of objects, Ob**A**.

(pdl.1) For any two objects x, y, a set $\mathbf{A}(x, y)$ of *maps* $a: x \to y$, with a preorder relation $a \prec b$.

(pdl.2) For any object x, a *lower identity* 1_x. For any triple of objects x, y, z a preorder-preserving mapping of *composition*

(1) $-\otimes-: \mathbf{A}(x, y) \times \mathbf{A}(y, z) \to \mathbf{A}(x, z)$.

(pdl.3) Given a map $a: x \to x'$ and three consecutive maps a, b, c one has:

(2) $1_x \otimes a \prec a \prec a \otimes 1_{x'}$, $\qquad a\otimes(b\otimes c) \prec a\otimes b\otimes c \prec (a\otimes b)\otimes c$.

In the extended case one should add the following condition, in (pdl.3):

(3) $a\otimes(b\otimes(c\otimes d)) \prec a\otimes((b\otimes c)\otimes d) \prec (a\otimes b)\otimes(c\otimes d) \prec (a\otimes(b\otimes c))\otimes d \prec ((a\otimes b)\otimes c)\otimes d$.

The *unbiased* case can be defined similarly.

3.8. An absolute approach? The preorder considered above gives a sort of absolute comparison cells $a \prec b$, which only depend on the actual values of the arrows a, b rather than on their being produced by comparable tree-concatenations, as in the previous syntactic approaches. *This suggests an interesting problem*: can one define, for a d-space, an 'absolute' d-lax fundamental 2-category? Of course, in contrast with the previous situation (3.7), the present comparisons would not give all the cells of the structure, in general.

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