CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

SUSAN NIEFIELD

Homotopic pullbacks, Lax pullbacks, and exponentiability

Cahiers de topologie et géométrie différentielle catégoriques, tome 47, n° 1 (2006), p. 50-80

<http://www.numdam.org/item?id=CTGDC_2006__47_1_50_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLVII-1 (2006)

HOMOTOPIC PULLBACKS, LAX PULLBACKS, AND EXPONENTIABILITY

by Susan NIEFIELD

RESUME. Cet article propose une approche unifiée des produits fibrés homotopiques et autres produits fibrés généralisés, et étudie la notion d'exponentiabilité correspondante.

1 Introduction

This work began as a study of exponentiable maps when pullbacks are replaced by homotopy pullbacks in the category **Top** of topological spaces and continuous maps.

Recall that a continuous map $q: Y \longrightarrow B$ is called *exponentiable* if the functor $- \times_B Y: \mathbf{Top}/B \longrightarrow \mathbf{Top}/B$ has a right adjoint, where $\mathbf{Top}/B$ is the *slice* or *fiber category* of spaces and commutative triangles over B.

The homotopy pullback of $p: X \longrightarrow B$ and $q: Y \longrightarrow B$ can be described as the space with points (x, σ, y), where $x \in X$, $y \in Y$, and σ is any path from px to qy in B. This is the fibered space

$$X \times_B B^I \times_B Y$$

where I is the unit interval and the projections $B^I \longrightarrow B$ are given by evaluation at 0 when B^I is on the right of $\times_B$ and evaluation at 1 when B^I is on the left. These spaces can be used to capture some of the structure of B not detected by ordinary pullbacks. Although two maps p_1 and p_2 are homotopic and thus intuitively give similar information, the ordinary pullbacks along them can be very different, for example: let B be a path-connected space and $q: Y \longrightarrow B$ the inclusion of a proper subspace. Let $y \in Y$, $x \in B \setminus Y$ and p_x, p_y the two functions with singleton domain picking out these elements. Any path from x to y gives a homotopy from p_x to p_y, but the pullback of q along p_x is empty while that along p_y is a singleton. Thus, deforming the point along the path gives a jump of homotopy type in the pullback. This is avoided if one replaces the pullback by the homotopy pullback.

Note that the square

$$\begin{array}{ccc} X \times_B B^I \times_B Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & \xrightarrow{H} & \downarrow{\scriptstyle q} \\ X & \xrightarrow[p]{} & B \end{array}$$

where H is the usual homotopy from $p\pi_X$ to $q\pi_Y$, is called the *standard homotopy pullback* of $p: X \longrightarrow B$ and $q: Y \longrightarrow B$ in [2] and [15]. A homotopy pullback, in the sense of [16], is any square of this form in which $X \times_B B^I \times_B Y$ is replaced by any space that is homotopy equivalent to $X \times_B B^I \times_B Y$.

Given $q: Y \longrightarrow B$, one can ask if the functor $\mathbf{Top}/B \longrightarrow \mathbf{Top}$ defined by

$$p: X \longrightarrow B \quad \mapsto \quad X \times_B B^I \times_B Y$$

has a right adjoint, in particular, if there is a relationship between the "homotopy exponentiability" (called I-exponentiability, here) and the ordinary exponentiability of a map.

It soon became apparent that $X \times_B B^I \times_B Y$ could be described by a universal property in a general 1-categorical setting which applied to many other known constructions (including lax pullbacks and pseudo-pullbacks of categories, posets, locales, and Grothendieck toposes), and the adjunction results were completely general. In fact, the results in **Cat** turned out to be more interesting, in some sense, than those in **Top** since there are not many I-exponentiable space over B for many "nice" spaces B (see Theorem 7.1 and Corollary 7.6).

Although "up to homotopy" versions of "homotopy exponentiability" may be of interest as well, we study this simpler setting here leaving the higher-dimensional structure for future consideration.

The paper proceeds as follows. We begin, in the first two sections, with the introduction of the notions of E-cells and E-pullbacks, generalizing homotopies and homotopy pullbacks, where E is an object in a category with finite limits. In Section 4, we consider E-exponentiablity and establish its relationship to ordinary exponentiability in this general setting, which we then apply to the categories in question, in the five sections that remain.

The author would like to thank the editor for many useful comments, including the homotopy pullback example described above.

2 E-cells

Suppose $\mathcal{E}$ is a category with finite products.

Definition 2.1 *Given an object E of $\mathcal{E}$, an E-cell $\vec{p}: X \Rightarrow B$ is a morphism $p: X \times E \longrightarrow B$ of $\mathcal{E}$. The objects X and B are called the domain and codomain of $\vec{p}$.*

When $\mathcal{E} = \mathbf{Top}$, and E is the unit interval $I = [0, 1]$, an I-cell $\vec{p}: X \Rightarrow B$ is just a homotopy $p: X \times I \longrightarrow B$. Another choice of E in **Top** is the Sierpinski space $\mathbf{2} = \{0, 1\}$, with $\{0\}$ open but not $\{1\}$. In this case, a **2**-cell $\vec{p}: X \Rightarrow B$ is given by a pair $f, g: X \longrightarrow B$ of continuous maps such that $fx \leq gx$ in the specialization order on B (in the sense of [6]), for all $x \in X$.

For $\mathcal{E} = \mathbf{Cat}$, the category of small categories, taking E as the category $\mathbf{2} = \{0, 1\}$ with one non-trivial morphism $0 \longrightarrow 1$, it is not difficult to show that a **2**-cell $\vec{p}: \mathbf{X} \Rightarrow \mathbf{B}$ is a natural transformation from $p(-, 0)$ to $p(-, 1)$. If E is taken to be the category $\mathbf{Iso} = \{0, 1\}$ with two non-trivial morphisms $\alpha: 0 \longrightarrow 1$ and $\beta: 1 \longrightarrow 0$ such that $\alpha^{-1} = \beta$, then **Iso**-cells in **Cat** are the natural isomorphisms.

Considering **2** to be an object of the category **Poset** of partially-ordered sets and order-preserving maps, as in the case of **Top**, a **2**-cell $\vec{p}: X \Rightarrow B$ is given by a pair $f, g: X \longrightarrow B$ of order-preserving maps such that $fx \leq gx$, for all $x \in X$. Another choice for E in **Poset** is given by

Here, an E-cell $\vec{p}: X \Rightarrow B$ is given by order-preserving maps $p, f, g: X \longrightarrow B$ such that px is an upper bound of both fx and gx, for all $x \in X$.

For $\mathcal{E} = \mathbf{Loc}$, the category of locales and locale morphisms (in the sense of Isbell [10] or Johnstone [12]), $\mathcal{O}(I)$-cells are homotopies of morphisms, where $\mathcal{O}(I)$ is the locale of open sets of I. One could also consider E to be the Sierpinski locale, which can be described as $\mathcal{O}(\mathbf{2})$, where $\mathbf{2}$ is the Sierpinski space, or as the locale $\mathcal{O}(\mathbf{1})^2$ obtained by glueing along the identity map $\mathcal{O}(\mathbf{1}) \longrightarrow \mathcal{O}(\mathbf{1})$ (cf., [21]). Then the E-cells $\vec{p}: X \Rightarrow B$ are morphisms $p: X \times \mathcal{O}(\mathbf{1})^2 \longrightarrow B$. But, $X \times \mathcal{O}(\mathbf{1})^2 \cong X^2$, where X^2 is the locale obtained by glueing along the identity map $X \longrightarrow X$, and it follows that an E-cell $\vec{p}: X \Rightarrow B$ is given by a pair of morphisms $f, g: X \longrightarrow B$ such that $f \leq g$ in the usual order on morphism of **Loc**, i.e., the usual 2-cell when **Loc** is consider as a 2-category.

More generally, when $\mathcal{E}$ is the category **GTop** of Grothendieck toposes and geometric morphisms, the $Sh(I)$-cells are generalized homotopies. As in the case of locales, since $Sh(\mathbf{2}) \sim \mathbf{Sets}^2$, which is isomorphic to the topos obtained by glueing along the identity $\mathbf{Sets} \longrightarrow \mathbf{Sets}$ and $\mathcal{X} \times \mathbf{Sets}^2 \simeq \mathcal{X}^2$ (c.f. [11]), a $Sh(\mathbf{2})$-cell $\vec{p}: \mathcal{X} \Rightarrow \mathcal{B}$ consists of a pair $f, g: \mathcal{X} \longrightarrow \mathcal{B}$ of geometric morphisms together with a natural transformation $f^* \longrightarrow g^*$.

Although E-cells do not necessarily compose (for example, there are clearly non-composable $\bigwedge$-cells in **Poset**), morphisms compose with E-cells. In particular, the composites

$$X' \xrightarrow{f} X \overset{\vec{p}}{\Rightarrow} B \quad \text{and} \quad X \overset{\vec{p}}{\Rightarrow} B \xrightarrow{g} B'$$

are the E-cells corresponding to

$$X' \times E \xrightarrow{f \times id} X \times E \xrightarrow{p} B \quad \text{and} \quad X \times E \xrightarrow{p} B \xrightarrow{g} B'$$

Note that even when E-cells do compose, composition is not necessarily associative, as in the homotopy example in **Top**. Thus, using E-cells to define a relaxed version of the slice category $\mathcal{E}/B$ does not necessarily yield a category, unless further structure is put on E to ensure that composition of E-cells is defined and associative, and such an addition would eliminate the homotopy example in **Top**. However, we can consider a generalized slice category in which the E-cells themselves are the objects.

Let $\mathcal{E}\Downarrow B$ denote the category whose objects are E-cells with codomain B, and morphisms $(X \overset{\vec{p}}{\Rightarrow} B) \longrightarrow (X' \overset{\vec{p}'}{\Rightarrow} B)$ are morphisms $f: X \longrightarrow X'$ of $\mathcal{E}$ such that $\vec{p}'f = \vec{p}$.

Recall that an *exponential* B^E is said to exist in $\mathcal{E}$ if there is an object B^E and a morphism $\varepsilon: B^E \times E \longrightarrow B$ such that for every $f: X \times E \longrightarrow B$, there exists a unique $\hat{f}\colon X \longrightarrow B^E$, called the *exponential transpose of* f, making the following diagram commute

$$\begin{array}{ccc} X \times E & \xrightarrow{\hat{f} \times id} & B^E \times E \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \varepsilon} & \\ & B & \end{array}$$

The object E is said to be *exponentiable* if B^E exists, for all B, (or equivalently, if the functor $- \times E\colon \mathcal{E} \longrightarrow \mathcal{E}$ has a right adjoint). When $\mathcal{E}$ has finite limits, a morphism $E \longrightarrow B$ is said to be *exponentiable* in $\mathcal{E}$ if it is exponentiable as an object of $\mathcal{E}/B$. The category $\mathcal{E}$ is called *cartesian closed* (respectively, *locally cartesian closed*) if every object (respectively, morphism) is exponentiable.

All of the objects E considered in the examples above are known to be exponentiable. Both **Cat** and **Poset** are cartesian closed. Although **Top** is not cartesian closed, the exponentials B^I and $B^{\mathbf{2}}$ are given by the usual function spaces with the compact open-topology, and there are appropriate generalizations to **Loc** [9] and **GTop** [13], as well.

Proposition 2.2 *Suppose $\mathcal{E}$ has finite products. Then $\mathcal{E}{\Downarrow}B$ has a terminal object if and only if B^E exists in $\mathcal{E}$. Moreover, $\vec{\varepsilon}\colon B^E \Rightarrow B$ is the terminal object of $\mathcal{E}{\Downarrow}B$, and composition with $\vec{\varepsilon}$ induces an isomorphism $\mathcal{E}/B^E \cong \mathcal{E}{\Downarrow}B$.*

Proof. An E-cell $\vec{\varepsilon}\colon T \longrightarrow B$ is a terminal object of $\mathcal{E}{\Downarrow}B$ if and only if for all $\vec{p}\colon X \Rightarrow B$, there is a unique morphism $f\colon X \longrightarrow T$ of $\mathcal{E}$ such that $\vec{\varepsilon}f = \vec{p}$, or equivalently, for all $p\colon X \times E \longrightarrow B$, there is a unique morphism $f\colon X \longrightarrow T$ in $\mathcal{E}$ such that the following diagram commutes

$$\begin{array}{ccc} X \times E & \xrightarrow{f \times id} & T \times E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \varepsilon} & \\ & B & \end{array}$$

Since this says that T is an exponential object B^E in $\mathcal{E}$, it follows that $\mathcal{E}{\Downarrow}B$ has a terminal object if and only if B^E exists in $\mathcal{E}$, and composition with $\vec{\varepsilon}$ induces the desired isomorphism. □

3 E-Pullbacks

Suppose that $\mathcal{E}$ has finite limits, and fix two morphisms $s, t: 1 \longrightarrow E$. An E-cell $\vec{p}: X \Rightarrow B$ gives rise to a pair

$$p_s: X \xrightarrow{\langle id,s \rangle} X \times E \xrightarrow{p} B \quad \text{and} \quad p_t: X \xrightarrow{\langle id,t \rangle} X \times E \xrightarrow{p} B$$

of morphisms of $\mathcal{E}$ called the *source* and *target* of $\vec{p}$. Thus, $\vec{p}$ can be viewed as an E-cell from p_s to p_t and written $\vec{p}: p_s \longrightarrow p_t$. Moreover, the assignments $\vec{p} \mapsto p_s$ and $\vec{p} \mapsto p_t$ define two forgetful functors

$$\mathcal{E} \Downarrow B \underset{\Sigma_t}{\overset{\Sigma_s}{\rightrightarrows}} \mathcal{E}/B$$

Note that, by definition of B^E, $p_s = \varepsilon_s \hat{p}$ and $p_t = \varepsilon_t \hat{p}$, where $\hat{p}: X \longrightarrow B^E$ is the exponential transpose of p.

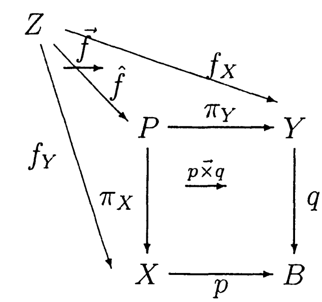
The examples in the previous section all have natural choices for s and t. For instance, taking $s = 0$ and $t = 1$ in **Top**, an I-cell $\vec{p}: X \longrightarrow B$ is a homotopy from p_0 to p_1. The other examples have similar interpretations.

Definition 3.1 *An E-pullback (relative to $s, t: 1 \longrightarrow E$) of $p: X \longrightarrow B$ and $q: Y \longrightarrow B$ is a universal E-cell*

$$\begin{array}{ccc} P & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & \xrightarrow{p \vec{\times} q} & \downarrow{\scriptstyle q} \\ X & \xrightarrow[p]{} & B \end{array}$$

i.e., an object P together with morphisms $\pi_X: P \longrightarrow X$ and $\pi_Y: P \longrightarrow Y$ and an E-cell $p \vec{\times} q: p\pi_X \longrightarrow q\pi_Y$ such that given any object Z with morphisms

$f_X: Z \longrightarrow X$ and $f_Y: Z \longrightarrow Y$, and an E-cell $\vec{f}: pf_X \longrightarrow qf_Y$,



there exists a unique morphism $\hat{f}: Z \longrightarrow P$ with $\pi_X \hat{f} = f_X$, $\pi_Y \hat{f} = f_Y$, and $(p\vec{\times}q)\hat{f} = \vec{f}$.

Using the universal property of E-pullbacks or the description below, one obtains the following examples. If $\mathcal{E} = \mathbf{Top}$, then I-pullbacks are homotopy pullbacks. When $\mathcal{E} = \mathbf{Cat}$, the **Iso**-pullbacks are pseudo-pullbacks. Also, **2**-pullbacks in **Cat** and **Poset**, as well as $\mathcal{O}(\mathbf{2})$-pullbacks in **Loc**, are lax pullbacks. For $\mathcal{E} = \mathbf{GTop}$, the $Sh(\mathbf{2})$-pullbacks are lax pullbacks (in the sense of [17]) and $Sh(I)$-pullbacks are generalized homotopy pullbacks.

For the following description of E-pullbacks and throughout the remainder of this paper, when B^E appears as a factor in a pullback over B, it will be written on the right as in $X \times_B B^E$ when $\varepsilon_s: B^E \longrightarrow B$, and on the left as in $B^E \times_B X$ when $\varepsilon_t: B^E \longrightarrow B$.

Proposition 3.2 *The following are equivalent (relative to $s, t: 1 \longrightarrow E$) for a finitely complete category $\mathcal{E}$:*

(a) $\mathcal{E}$ has E-pullbacks over B

(b) The E-pullback of $id_B \vec{\times} id_B$ exists in $\mathcal{E}$

(c) B^E exists in $\mathcal{E}$

(d) $\mathcal{E}\Downarrow B$ has finite products

(e) $\mathcal{E}\Downarrow B$ has a terminal object

(f) $\Sigma_s: \mathcal{E}\Downarrow B \longrightarrow \mathcal{E}/B$ has a right adjoint (denoted by s^*)

(g) $\Sigma_t: \mathcal{E}\Downarrow B \longrightarrow \mathcal{E}/B$ has a right adjoint (denoted by t^*)

Moreover, the E-pullback $p\vec{\times}q$ is given by the product $s^*(p) \times t^*(q)$ in $\mathcal{E}\Downarrow B$, or equivalently, the square

$$
\begin{array}{ccc}
X \times_B B^E \times_B Y & \xrightarrow{\pi_Y} & Y \\
{\scriptstyle \pi_X}\downarrow & \overset{p\vec{\times}q}{\Rightarrow} & \downarrow{\scriptstyle q} \\
X & \xrightarrow[p]{} & B
\end{array}
$$

where $p\vec{\times}q$ corresponds to the pullback diagram

$$
\begin{array}{ccccc}
X \times_B B^E \times_B Y & \rightarrow & B^E \times_B Y & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle q} \\
X \times_B B^E & \longrightarrow & B^E & \xrightarrow[\varepsilon_t]{} & B \\
\downarrow & & \downarrow{\scriptstyle \varepsilon_s} & & \\
X & \xrightarrow[p]{} & B & &
\end{array}
$$

Proof. By Proposition 2.2, (c) through (e) are equivalent, and they imply (f) and (g) since $\mathcal{E}/B^E \cong \mathcal{E}\Downarrow B$. Also, (b), (f), and (g) each imply (e) since $id_B\vec{\times}id_B$, $s^*(id_B)$, and $t^*(id_B)$ can easily be seen to provide a terminal object for $\mathcal{E}\Downarrow B$. Thus, (c) through (g) are equivalent. Since (a)$\Rightarrow$(b) is clear, it suffices to show that these equivalent conditions imply (a) and that the E-pullback $p\vec{\times}q$ is given by the desired diagram.

Given morphisms $p: X \longrightarrow B$ and $q: Y \longrightarrow B$, consider the diagram

$$
\begin{array}{ccc}
X \times_B B^E \times_B Y & \xrightarrow{\pi_Y} & Y \\
{\scriptstyle \pi_X}\downarrow & \overset{p\vec{\times}q}{\Rightarrow} & \downarrow{\scriptstyle q} \\
X & \xrightarrow[p]{} & B
\end{array}
$$

where $p\vec{\times}q: X \times_B B^E \times_B Y \Rightarrow B$ is obtained by composing the projection $X \times_B B^E \times_B Y \longrightarrow B^E$ with $\vec{\varepsilon}: B^E \Rightarrow B$. To show that this square is an

E-pullback, suppose

$$\begin{array}{ccc} Z & \xrightarrow{f_Y} & Y \\ {\scriptstyle f_X}\big\downarrow & \xrightarrow{\vec{f}} & \big\downarrow{\scriptstyle q} \\ X & \xrightarrow[p]{} & B \end{array}$$

is another such square. Then $\Sigma_s(\vec{f}) = pf_X$ and $\Sigma_t(\vec{f}) = qf_Y$, and so

$$f_X\colon \Sigma_s(\vec{f}) \longrightarrow p \quad \text{and} \quad f_Y\colon \Sigma_t(\vec{f}) \longrightarrow q$$

are morphisms in $\mathcal{E}/B$. Applying the adjunctions, we get

$$\hat{f}_X\colon \vec{f} \longrightarrow s^*(p) \quad \text{and} \quad \hat{f}_Y\colon \vec{f} \longrightarrow t^*(q)$$

and hence, $\hat{f}\colon \vec{f} \longrightarrow s^*(p) \times t^*(q)$ with $\pi_1\hat{f} = \hat{f}_X$ and $\pi_2\hat{f} = \hat{f}_Y$ in $\mathcal{E}\Downarrow B$. Using the isomorphism $\mathcal{E}/B^E \cong \mathcal{E}\Downarrow B$, we know that $s^*(p) \times t^*(q)$ is the E-cell

$$p\vec{\times}q\colon X \times_B B^E \times_B Y \Rightarrow B$$

given above, and $(p\vec{\times}q)\hat{f} = \vec{f}$. It remains to show that $\pi_X\hat{f} = f_X$ and $\pi_Y\hat{f} = f_X$. Applying the functor Σ_s to the equation $\pi_1\hat{f} = \hat{f}_X$ and composing with the counit $\Sigma_s s^*(p) \longrightarrow p$, we get the commutative diagram of objects over B

$$\begin{array}{ccc} Z & \xrightarrow{\hat{f}} & X \times_B B^E \times_B Y \\ & {\scriptstyle f_X}\searrow & \big\downarrow{\scriptstyle \pi_X} \\ & & X \end{array}$$

and so $\pi_X\hat{f} = f_X$. Similarly, $\pi_Y\hat{f} = f_X$. Therefore, E-pullbacks exist over B and are defined as desired in $\mathcal{E}$. □

Note that s and t are not assumed to be distinct, though they are in the examples of interest here. If they happen to coincide, the proofs of the following section (e.g., 4.4 and 4.5) can be greatly simplified.

4 E-Exponentiability

Throughout this section, we assume $\mathcal{E}$ has finite limits and B^E exists so that the equivalent conditions of Proposition 3.2 hold. Then E-pullback along $q: Y \longrightarrow B$ (relative to $s, t: 1 \longrightarrow E$) induces four functors

$$\mathcal{E}/B \xrightarrow{-\vec{\times}q} \mathcal{E}\Downarrow B$$

$$\mathcal{E}/B \xrightarrow{-\vec{\times}q} \mathcal{E}\Downarrow B \xrightarrow{\Sigma_s} \mathcal{E}/B$$

$$\mathcal{E}/B \xrightarrow{-\vec{\times}q} \mathcal{E}\Downarrow B \xrightarrow{\Sigma_t} \mathcal{E}/B$$

$$\mathcal{E}/B \xrightarrow{-\vec{\times}q} \mathcal{E}\Downarrow B \xrightarrow{\Sigma_s} \mathcal{E}/B \xrightarrow{\Sigma_B} \mathcal{E}$$

We will use the following lemma to show that if any one of these E-pullback functors has a right adjoint, then they all do. This lemma was proved in [19], where it was used to show that the pullback functor $q^*: \mathcal{E}/B \longrightarrow \mathcal{E}/Y$ preserves exponentiable morphisms, and that q^* has a right adjoint (usually denoted by Π_q) if and only if q is exponentiable in $\mathcal{E}$.

Lemma 4.1 *Suppose $\mathcal{D}$ is a category with finite limits. Then $F: \mathcal{D} \longrightarrow \mathcal{E}/B$ has a right adjoint if and only if $\Sigma_B F: \mathcal{D} \longrightarrow \mathcal{E}$ has a right adjoint, where $\Sigma_B: \mathcal{E}/B \longrightarrow \mathcal{E}$ is the forgetful functor.*

Proof. If F has a right adjoint, then so does $\Sigma_B F$, since B^* is right adjoint to Σ_B. For the converse, suppose G' is right adjoint to $\Sigma_B F$. Given any object X of $\mathcal{D}$, then FX can be viewed as a morphism $FX: \Sigma_B FX \longrightarrow B$ of $\mathcal{E}$, and hence corresponds, via the adjunction, to a morphism $\sigma_X: X \longrightarrow G'B$ of $\mathcal{D}$. Then it is not difficult to show that the functor $G: \mathcal{E}/B \longrightarrow \mathcal{D}$ defined on objects $q: Y \longrightarrow B$ by the equalizer

$$G(q) \longrightarrow G'Y \underset{\sigma_{G'Y}}{\overset{G'q}{\rightrightarrows}} G'B$$

is right adjoint to F. □

Proposition 4.2 *The following are equivalent for $q: Y \longrightarrow B$ (relative to $s, t: 1 \longrightarrow E$):*

(a) $-\vec{\times} q\colon \mathcal{E}/B \longrightarrow \mathcal{E}\Downarrow B$ *has a right adjoint*

(b) $\Sigma_s(-\vec{\times} q)\colon \mathcal{E}/B \longrightarrow \mathcal{E}/B$ *has a right adjoint*

(c) $\Sigma_t(-\vec{\times} q)\colon \mathcal{E}/B \longrightarrow \mathcal{E}/B$ *has a right adjoint*

(d) $- \times_B B^E \times_B Y\colon \mathcal{E}/B \longrightarrow \mathcal{E}$ *has a right adjoint*

Proof. This follows directly from Lemma 4.1 since $\mathcal{E}\Downarrow B \cong \mathcal{E}/B^E$. □

Definition 4.3 *A morphism $q\colon Y \longrightarrow B$ of $\mathcal{E}$ is E-exponentiable if any, and hence all, of the four E-pullback functors on $\mathcal{E}/B$ have a right adjoint.*

Note that $q\colon Y \longrightarrow B$ is E-exponentiable if and only if for every $p\colon X \longrightarrow B$ and every $\vec{r}\colon Z \Rightarrow B$, there is an object $[q,\vec{r}]$ of $\mathcal{E}/B$ and a bijection $\mathcal{E}\Downarrow B(p\vec{\times} q, \vec{r}) \longrightarrow \mathcal{E}/B(p,[q,\vec{r}])$ which is natural p and $\vec{r}$.

Next, we consider the relationship between E-exponentiability and exponentiability in $\mathcal{E}$.

Theorem 4.4 *The following are equivalent for $s,t\colon 1 \longrightarrow E$:*

(a) Every exponentiable morphism over B is E-exponentiable in $\mathcal{E}$

(b) The identity morphism $id_B\colon B \longrightarrow B$ is E-exponentiable

(c) The morphism $\varepsilon_s\colon B^E \longrightarrow B$ is exponentiable in $\mathcal{E}$

Proof. The proof of (a)⇒(b) is clear.

(b)⇒(c) Suppose id_B is E-exponentiable, and consider $\varepsilon_s^*\colon \mathcal{E}/B \longrightarrow \mathcal{E}/B^E$, which can easily be seen to factor as

$$\begin{array}{ccc} \mathcal{E}/B & \xrightarrow{\varepsilon_s^*} & \mathcal{E}/B^E \\ & {\scriptstyle -\vec{\times} id_B}\searrow \quad \nearrow{\scriptstyle \cong} & \\ & \mathcal{E}\Downarrow B & \end{array}$$

Since $-\vec{\times} id_B$ has a right adjoint, it follows that ε_s^* does as well, and so ε_s is exponentiable in $\mathcal{E}$.

(c)⇒(a) Suppose ε_s and $q\colon Y \longrightarrow B$ are exponentiable in $\mathcal{E}$. To show $-\vec{\times} q$ has a right adjoint, it suffices to show that the morphism $f\colon \mathcal{E}/B \longrightarrow \mathcal{E}/B^E$ given by the composite

$$\mathcal{E}/B \xrightarrow{-\vec{\times} q} \mathcal{E}\Downarrow B \cong \mathcal{E}/B^E$$

has a right adjoint. By Proposition 3.2, we know that f takes the morphism $p: X \longrightarrow B$ to the projection $\pi_{B^E}: X \times_B B^E \times_B Y \longrightarrow B^E$, and so it factors as

$$\begin{array}{ccc} \mathcal{E}/B & \xrightarrow{\ f\ } & \mathcal{E}/B^E \\ & {\scriptstyle \varepsilon_s^*}\searrow \quad \nearrow{\scriptstyle - \times \varepsilon_t^*(q)} & \\ & \mathcal{E}/B^E & \end{array}$$

But, ε_s^* has a right adjoint since ε_s is exponentiable by assumption, and $- \times \varepsilon_t^*(q)$ has a right adjoint since ε_t^* preserves exponentiability. Thus, f has a right adjoint, as desired. □

Note that from the proof of (c)⇒(a), it follows that when $\varepsilon_s: B^E \longrightarrow B$ and $q: Y \longrightarrow B$ are exponentiable in $\mathcal{E}$, then the right adjoint to $-\vec{\times} q$ is given by the composite

$$[q, -]: \mathcal{E}\Downarrow B \cong \mathcal{E}/B^E \xrightarrow{(\ )^{\varepsilon_t^*(q)}} \mathcal{E}/B^E \xrightarrow{\Pi_{\varepsilon_s}} \mathcal{E}/B$$

With an additional assumption on B^E, we get the following converse to (a).

Theorem 4.5 *If the exponential transpose $\hat{\pi}_2: B \longrightarrow B^E$ is exponentiable in $\mathcal{E}$, then E-exponentiable morphisms over B are exponentiable in $\mathcal{E}$.*

Proof. Suppose $\hat{\pi}_2: B \longrightarrow B^E$ is an exponentiable and $q: Y \longrightarrow B$ is E-exponentiable. Then the composite

$$\mathcal{E}/B \xrightarrow{-\vec{\times} q} \mathcal{E}\Downarrow B \cong \mathcal{E}/B^E \xrightarrow{\hat{\pi}_2^*} \mathcal{E}/B$$

has a right adjoint. By Proposition 3.2, this composite is given by pulling back the projection $\pi_{B^E}: X \times_B B^E \times_B Y \longrightarrow B^E$ along $\hat{\pi}_2: B \longrightarrow B^E$. Note that π_{B^E} is given by the commutative square

$$\begin{array}{ccc} X \times_B B^E \times_B Y & \longrightarrow & B^E \times_B Y \\ \downarrow & & \downarrow \\ X \times_B B^E & \longrightarrow & B^E \end{array}$$

For the pullback of the bottom row of this square, consider the pullbacks

$$\begin{array}{ccc} X \times_B B & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \hat{\pi}_2 \\ X \times_B B^E & \longrightarrow & B^E \\ \big\downarrow & & \big\downarrow \varepsilon_s \\ X & \underset{p}{\longrightarrow} & B \end{array}$$

Since the morphism $B \longrightarrow B$ in $X \times_B B$ is $\varepsilon_s \hat{\pi}_2 = id_B$, it follows that $X \times_B B \cong X$ and so the top row is given by $p: X \longrightarrow B$. Similarly, since $\varepsilon_t \hat{\pi}_2 = id_B$, the pullback of the morphism $B^E \times_B Y \longrightarrow B^E$ is $q: Y \longrightarrow B$. Thus, it follows that the functor in question is just $- \times q: \mathcal{E}/B \longrightarrow \mathcal{E}/B$, and so q is exponentiable in $\mathcal{E}$. □

Combining these two theorems gives:

Corollary 4.6 *If $\varepsilon_s: B^E \longrightarrow B$ and $\hat{\pi}_2: B \longrightarrow B^E$ are exponentiable in $\mathcal{E}$, then $q: Y \longrightarrow B$ is E-exponentiable (relative to $s, t: 1 \longrightarrow E$) if and only if it is exponentiable in $\mathcal{E}$.*

5 E-Exponentiability in Cat

In this section, we apply the results of Section 4 to relate 2-exponentiability and **Iso**-exponentiability to ordinary exponentiability in **Cat**.

As noted earlier, 2-pullbacks in **Cat** are lax pullbacks which are constructed by

$$\begin{array}{ccc} (p \downarrow q) & \xrightarrow{\pi_{\mathbf{Y}}} & \mathbf{Y} \\ \pi_{\mathbf{X}} \big\downarrow & \xrightarrow{\sim} & \big\downarrow q \\ \mathbf{X} & \underset{p}{\longrightarrow} & \mathbf{B} \end{array}$$

where $(p \downarrow q)$ is the comma category whose objects are triples

$$(X, Y, pX \xrightarrow{b} qY)$$

and morphisms are compatible pairs of morphisms between the objects of $\mathbf{X}$ and $\mathbf{Y}$. So, q is $\mathbf{2}$-exponentiable if and only if any, and hence all, of the lax pullback functors on $\mathbf{Cat}/\mathbf{B}$ have a right adjoint.

Also, $\mathbf{Iso}$-pullbacks are pseudo-pullbacks which can be constructed by

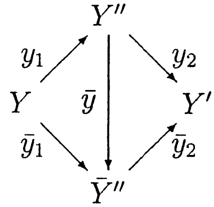
$$\begin{array}{ccc} (p \cong q) & \xrightarrow{\pi_{\mathbf{Y}}} & \mathbf{Y} \\ {\scriptstyle \pi_{\mathbf{X}}}\downarrow & \longrightarrow & \downarrow{\scriptstyle q} \\ \mathbf{X} & \xrightarrow[p]{} & \mathbf{B} \end{array}$$

where $(p \cong q)$ is the full subcategory of $(p \downarrow q)$ consisting of objects (X, Y, b) such that b is an isomorphism. Thus, q is $\mathbf{Iso}$-exponentiable if and only if any (and hence all) of the pseduo-pullback functors on $\mathbf{Cat}/\mathbf{B}$ has a right adjoint.

Recall [4], [5] that a functor $q: \mathbf{Y} \longrightarrow \mathbf{B}$ is exponentiable in $\mathbf{Cat}$ if and only if it has the following factorization lifting property. Given $y: Y \longrightarrow Y'$ in $\mathbf{Y}$ and a factorization $qy = b_2 b_1$ in $\mathbf{B}$, the following diagram can be completed

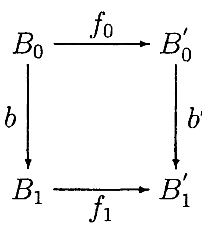
$$\begin{array}{c} \mathbf{Y} \\ \\ {\scriptstyle q}\Bigg\downarrow \\ \\ \mathbf{B} \end{array} \qquad \begin{array}{c} \begin{array}{ccc} Y & \xrightarrow{y} & Y' \\ {\scriptstyle y_1}\searrow & & \nearrow{\scriptstyle y_2} \\ & Y'' & \end{array} \\ \\ \begin{array}{ccc} qY & \xrightarrow{qy} & qY' \\ {\scriptstyle b_1}\searrow & & \nearrow{\scriptstyle b_2} \\ & B'' & \end{array} \end{array}$$

i.e., there exists a factorization $y = y_2 y_1$ in $\mathbf{Y}$ such that $qy_1 = b_1$ and $qy_2 = b_2$. Furthermore, it is required that any two such pairs are equivalent via the equivalence relation generated by $(y_1, y_2) \sim (\bar{y}_1, \bar{y}_2)$ if there is a commutative diagram



such that $q\bar{y} = \mathrm{id}_{B''}$.

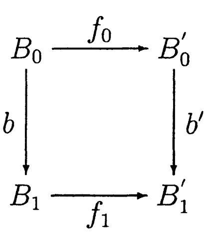
In particular, **Cat** is cartesian closed. Objects of $\mathbf{B}^{\mathbf{E}}$ are functors $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ and morphisms are natural transformations. Thus, $\mathbf{B}^{\mathbf{2}}$ can be identified with the category whose objects are morphisms $b\colon B_0 \longrightarrow B_1$ of $\mathbf{B}$ and morphisms are commutative squares



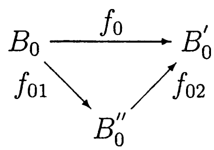
and $\mathbf{B}^{\mathbf{Iso}}$ with the full subcategory consisting of all isomorphisms b. In each case, the 2-cell $\vec{\varepsilon}\colon \mathbf{B}^{\mathbf{E}} \Rightarrow \mathbf{B}$ has the projections $\varepsilon_0(B_0 \xrightarrow{b} B_1) = B_0$ and $\varepsilon_1(B_0 \xrightarrow{b} B_1) = B_1$.

Lemma 5.1 *The functors* $\varepsilon_i\colon \mathbf{B}^{\mathbf{2}} \longrightarrow \mathbf{B}$ *and* $\varepsilon_i\colon \mathbf{B}^{\mathbf{Iso}} \longrightarrow \mathbf{B}$ *are exponentiable in* **Cat**, *for* $i = 0, 1$.

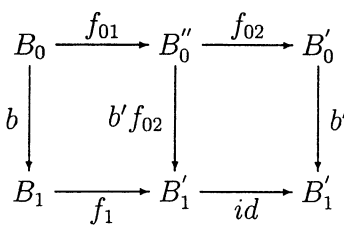
Proof. To show that $\varepsilon_0\colon \mathbf{B}^{\mathbf{2}} \longrightarrow \mathbf{B}$ is exponentiable, suppose $(f_0, f_1)\colon b \longrightarrow b'$ given by



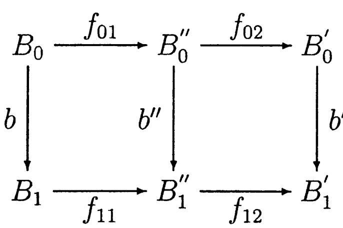
is a morphism of $\mathbf{B}^2$ and



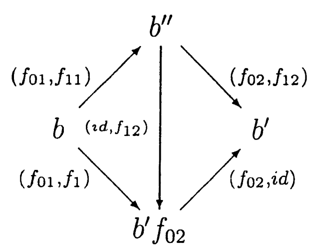
is a factorization of $\varepsilon_0(f_0, f_1)$ in $\mathbf{B}$. Then the following is a factorization of (f_0, f_1) in $\mathbf{B}^2$



Any other factorization of (f_0, f_1) is a commutative diagram of the form



where $f_{12}f_{11} = f_1$, and these two are easily seen to be equivalent via the diagram



Thus, $\varepsilon_0 \colon \mathbf{B}^2 \longrightarrow \mathbf{B}$ is exponentiable, and the proof for ε_1 is similar.

To show that $\varepsilon_0 \colon \mathbf{B}^{\mathbf{Iso}} \longrightarrow \mathbf{B}$ is exponentiable, suppose $(f_0, f_1) \colon b \longrightarrow b'$ is a morphism of $\mathbf{B}^{\mathbf{Iso}}$ given by

$$\begin{array}{ccc} B_0 & \xrightarrow{f_0} & B_0' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b'} \\ B_1 & \xrightarrow[f_1]{} & B_1' \end{array}$$

where b and b' are isomorphisms, and

$$\begin{array}{ccc} B_0 & \xrightarrow{f_0} & B_0' \\ & {\scriptstyle f_{01}}\searrow \quad \nearrow{\scriptstyle f_{02}} & \\ & B_0'' & \end{array}$$

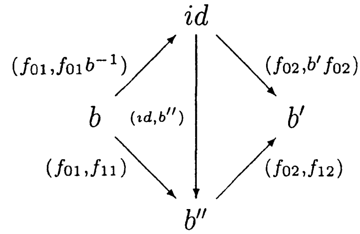
is a factorization of $\varepsilon_0(f_0, f_1)$ in $\mathbf{B}$. Then the following is a factorization of (f_0, f_1) in $\mathbf{B}^{\mathbf{Iso}}$

$$\begin{array}{ccccc} B_0 & \xrightarrow{f_{01}} & B_0'' & \xrightarrow{f_{02}} & B_0' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle b'} \\ B_1 & \xrightarrow[f_{01}b^{-1}]{} & B_0'' & \xrightarrow[b'f_{02}]{} & B_1' \end{array}$$

Any other factorization of (f_0, f_1) is a commutative diagram of the form

$$\begin{array}{ccccc} B_0 & \xrightarrow{f_{01}} & B_0'' & \xrightarrow{f_{02}} & B_0' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b''} & & \downarrow{\scriptstyle b'} \\ B_1 & \xrightarrow[f_{11}]{} & B_1'' & \xrightarrow[f_{12}]{} & B_1' \end{array}$$

where $f_{12}f_{11} = f_1$, and a straightforward calculation shows that these two are equivalent via the diagram



Thus, $\varepsilon_0 \colon \mathbf{B}^{\mathbf{Iso}} \longrightarrow \mathbf{B}$ is exponentiable, and the proof for ε_1 is similar. □

Theorem 5.2 *Every exponentiable morphism is* 2*-exponentiable in* Cat.

Proof. This follows immediately from Theorem 4.4, since $\varepsilon_0 \colon \mathbf{B}^2 \longrightarrow \mathbf{B}$ is exponentiable by Lemma 5.1. □

Note that from the proof of (c)⇒(a) of Theorem 4.4, it follows that if $q \colon \mathbf{Y} \longrightarrow \mathbf{B}$ is exponentiable in **Cat**, then the right adjoint to $-\vec{\times} q$ is given by the composite

$$[q,-] \colon \mathbf{Cat}{\Downarrow}\mathbf{B} \cong \mathbf{Cat}/\mathbf{B}^2 \xrightarrow{(\;)^{\varepsilon_1^*(q)}} \mathbf{Cat}/\mathbf{B}^2 \xrightarrow{\Pi_{\varepsilon_0}} \mathbf{Cat}/\mathbf{B}$$

whose value $[q, \vec{r}]$ at an object $\vec{r} \colon \mathbf{Z} \longrightarrow \mathbf{B}$ can be constructed as follows.

Given an object B of $\mathbf{B}$, let $B \times_{\mathbf{B}} \mathbf{B}^2 \times_{\mathbf{B}} \mathbf{Y}$ denote the category whose objects are pairs (β, Y), where Y is an object of $\mathbf{Y}$ and $\beta \colon B \longrightarrow qY$ in $\mathbf{B}$, and morphisms $(\beta_1, Y_1) \longrightarrow (\beta_2, Y_2)$ are morphisms $y \colon Y_1 \longrightarrow Y_2$ such that $qy\beta_1 = \beta_2$. Then objects in the fiber $[q, \vec{r}]_B$ over B are functors

$$f \colon B \times_{\mathbf{B}} \mathbf{B}^2 \times_{\mathbf{B}} \mathbf{Y} \longrightarrow \mathbf{Z}$$

such that $\vec{r}(\beta, Y) = \beta$, and $r_0 f y = id_B$ and $r_1 f y = qy$, for all $y \colon (\beta_1, Y_1) \longrightarrow (\beta_2, Y_2)$. Morphisms $f \longrightarrow f'$ over $b \colon B \longrightarrow B'$ are families

$$\{\phi_y \colon f(\beta, Y) \longrightarrow f'(\beta', Y')\}$$

indexed by $y: Y \longrightarrow Y'$ such that

$$\begin{array}{ccc} B & \xrightarrow{\ b\ } & B' \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ qY & \xrightarrow[qy]{} & qY' \end{array}$$

commutes, and satisfying $r_0\phi_y = b$ and $r_1\phi_y = qy$. Composition is well-defined by the factorization-lifting property (i.e., exponentiability) of $q: \mathbf{Y} \longrightarrow \mathbf{B}$ and $\varepsilon_0: \mathbf{B^2} \longrightarrow \mathbf{B}$.

Also, note that Theorem 4.5 cannot be applied to prove that 2-exponentiable morphisms over $\mathbf{B}$ are exponentiable when $\mathbf{B}$ is nontrivial, since it can be shown that $\hat{\pi}_2: \mathbf{B} \longrightarrow \mathbf{B^2}$ is exponentiable if and only if $\mathbf{B}$ is discrete.

Lemma 5.3 *The exponential transpose* $\hat{\pi}_2: \mathbf{B} \longrightarrow \mathbf{B^{Iso}}$ *is exponentiable in* **Cat** *if and only if the only isomorphisms of* $\mathbf{B}$ *are identity morphisms.*

Proof. Suppose the only isomorphisms of $\mathbf{B}$ are identities. Then $\hat{\pi}_2: \mathbf{B} \longrightarrow \mathbf{B^{Iso}}$ is easily seen to satisfy the lifting property of exponentiable morphisms in **Cat**. For the converse, suppose $\hat{\pi}_2$ is exponentiable in **Cat**. To show that the only isomorphisms of $\mathbf{B}$ are identities, suppose $b: B \longrightarrow B'$ is an isomorphism in $\mathbf{B}$. Since the factorization

$$\begin{array}{ccccc} B & \xrightarrow{\ id\ } & B & \xrightarrow{\ b\ } & B' \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle id} \\ B & \xrightarrow[b]{} & B' & \xrightarrow[id]{} & B' \end{array}$$

of $\hat{\pi}_2(B \xrightarrow{b} B')$ lifts to one of the form

$$\begin{array}{ccc} B & \xrightarrow{\ b\ } & B' \\ & {\scriptstyle b}\searrow \quad \nearrow{\scriptstyle id} & \\ & B' & \end{array}$$

in $\mathbf{B}$, it follows that $b = \hat{\pi}_2(B') = id_{B'}$, and so the only isomorphisms of $\mathbf{B}$ are identities. □

Theorem 5.4 *If* $q\colon \mathbf{Y} \longrightarrow \mathbf{B}$ *is exponentiable, then it is* Iso*-exponentiable in* Cat. *The converse holds, when the only isomorphisms of* $\mathbf{B}$ *are identity morphisms.*

Proof. The first statement follows from Theorem 4.4, since $\varepsilon_0\colon \mathbf{B}^{\mathbf{Iso}} \longrightarrow \mathbf{B}$ is exponentiable by Lemma 5.1. When the only isomorphisms of $\mathbf{B}$ are identities, $\hat{\pi}_2\colon \mathbf{B} \longrightarrow \mathbf{B}^{\mathbf{Iso}}$ is exponentiable by Lemma 5.3, and so the converse follows from Theorem 4.5. □

By Lemma 5.3, it is necessary to have the assumption in Theorem 5.4, if Theorem 4.5 is to be used in the proof, but it may be possible to find a different method of proof that would allow for a weakening of this assumption.

Finally, as in the case of $\mathbf{E} = \mathbf{2}$, if $q\colon \mathbf{Y} \longrightarrow \mathbf{B}$ are exponentiable in Cat, then the right adjoint to $-\vec{\times} q\colon \mathbf{Cat}/\mathbf{B} \longrightarrow \mathbf{Cat}\Downarrow\mathbf{B}$ is given by the composite

$$[q,-]\colon \mathbf{Cat}\Downarrow\mathbf{B} \cong \mathbf{Cat}/\mathbf{B}^{\mathbf{Iso}} \xrightarrow{(\ )^{\varepsilon_1^*(q)}} \mathbf{Cat}/\mathbf{B}^{\mathbf{Iso}} \xrightarrow{\Pi_{\varepsilon_0}} \mathbf{Cat}/\mathbf{B}$$

and can be constructed as in the case where $\mathbf{E} = \mathbf{2}$.

6 E-Exponentiability in Poset

As in Cat, 2-pullbacks in Poset are just lax pullbacks which are constructed using comma objects, $q\colon Y \longrightarrow B$ is 2-exponentiable if and only if any, and hence all, of the lax pullback functors on $\mathbf{Poset}/B$ have a right adjoint.

Recall [22] that an order-preserving map $q\colon Y \longrightarrow B$ is exponentiable in Poset if and only if it satisfies the following interpolation-lifting property. Given $y \leq y'$ in Y and $qy \leq b \leq qy'$ in B, the following diagram can be completed in Y

$$\begin{array}{ccccccc} Y & \quad & y & \leq & y'' & \leq & y' \\ \downarrow & & | & & | & & | \\ B & & py & \leq & b & \leq & py' \end{array}$$

Note that this is just the exponentiability condition in Cat without the connectivity condition, and so the following analogues of the results from Section 5 are easily established.

Lemma 6.1 *The map* $\varepsilon_i\colon B^{\mathbf{2}} \longrightarrow B$ *is exponentiable in* Poset, *for* $i = 0, 1$.

Theorem 6.2 *Every exponentiable morphism is* 2*-exponentiable in* Poset.

Note that, as in Cat, it can be shown that $\hat{\pi}_2: B \longrightarrow B^2$ is exponentiable if and only if B is discrete, and so Theorem 4.5 cannot be applied to show that 2-exponentiable morphisms over **B** are exponentiable when **B** is nontrivial.

7 E-Exponentiability in Top

As noted before, **I**-pullbacks in **Top** are homotopy pullbacks, given by

$$\begin{array}{ccc} X \times_B B^I \times_B Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & \longrightarrow & \downarrow{\scriptstyle q} \\ X & \xrightarrow[p]{} & B \end{array}$$

where B^I has the compact-open topology, $\varepsilon_0, \varepsilon_1: B^I \longrightarrow B$ are evaluation at 0 and 1, and $X \times_B B^I \times_B Y$ is the space of triples (x, σ, y) such that σ is a path from px to qy in B with the fiber product topology. Thus, q is I-exponentiable if and only if any, and hence all, of the homotopy pullback functors on $\mathbf{Top}/B$ have a right adjoint.

Applying the theorems from Section 4, we get:

Theorem 7.1 *Every exponentiable map over B is I-exponentiable if and only if $id_B: B \longrightarrow B$ is I-exponentiable if and only if the evaluation map $\varepsilon_0: B^I \longrightarrow B$ is exponentiable in* **Top**. *If $\hat{\pi}_2: B \longrightarrow B^I$ is exponentiable, then every I-exponentiable map over B is exponentiable in* **Top**.

Exponentiable maps were characterized in [18] and published in [19]. There it was shown that if Y is locally compact and B is Hausdorff, then every $q: Y \longrightarrow B$ is exponentiable in **Top**. Since ε_0 and ε_1 involve path spaces, we will apply this result, rather than the more general theorem in [19], in order to get examples of spaces B to which Theorem 7.1 applies.

Recall that the compact-open topology on B^I is generated by sets of the form $\langle K, W\rangle = \{\sigma \in B^I | \sigma(K) \subseteq W\}$, where K is compact in I and W is open in B.

Lemma 7.2 *If B is Hausdorff, then so is B^I.*

Proof. Suppose B is Hausdorff, and $\sigma \neq \tau$ in B. Then $\sigma(t) \neq \tau(t)$, for some $t \in I$, and so there are disjoint open neighborhoods U and V of $\sigma(t)$ and $\tau(t)$, respectively, in B. Thus, $\sigma \in \langle\{t\}, U\rangle$ and $\tau \in \langle\{t\}, V\rangle$, and these sets are easily seen to be disjoint since U and V are. □

Proposition 7.3 *If B is a locally compact Hausdorff space, then $\hat{\pi}_2 \colon B \longrightarrow B^I$ is exponentiable.*

Proof. Applying Lemma 7.2, we see that under these conditions $\hat{\pi}_2 \colon B \longrightarrow B^I$ is a locally compact space over a Hausdorff space, and hence exponentiable. □

Combining this proposition with Theorem 7.1, we get:

Corollary 7.4 *If B is a locally compact Hausdorff space, then every I-exponentiable map over B is exponentiable in* **Top**.

However, it turns out that there are not many I-exponentiable maps since B^I is not often locally compact as seen by the following result from [23].

Proposition 7.5 *If B has a non-constant path consisting of closed points, then B^I is not locally compact.*

Corollary 7.6 *If B is a locally compact space with a non-constant path consisting of closed points, then $\varepsilon_0 \colon B^I \longrightarrow B$ is not exponentiable, and so $id_B \colon B \longrightarrow B$ is an exponentiable map which is not I-exponentiable in* **Top**.

Proof. This follows from Theorem 7.1, Proposition 7.5, and the fact that $\varepsilon_0 \colon B^I \longrightarrow B$ exponentiable implies B^I is exponentiable (and hence, locally compact), since the composite of exponentiable maps is exponentiable. □

The theorems of Section 4 can also be applied to **2**-pullbacks in **Top**, where **2** is the Sierpinski space. These results will be used in the subsequent two sections in the consideration of adjoints to the lax pullback functor on locales and toposes. Perhaps there is also a connection to Sorkin's casual set dynamics (c.f., [3], [24]).

We will soon restrict our attention to Alexandrov spaces, i.e., T_0-spaces in which any intersection of open sets is open. The restriction makes sense since this is an interesting class of T_0-spaces, and we will soon see that in the general case, if we impose too much separation, the results obtained become trivial. Of course, any Alexandrov T_1-space is discrete.

Proposition 7.7 *The following are equivalent for a topological space B:*

(a) $\hat{\pi}_2 \colon B \longrightarrow B^{\mathbf{2}}$ is an isomorphism

(b) $\hat{\pi}_2 \colon B \longrightarrow B^{\mathbf{2}}$ is exponentiable in **Top**

(c) B is a T_1-space

(d) Every continuous map $\mathbf{2} \longrightarrow B$ is constant

Proof. The proof of (a)⇒(b) is clear
(b)⇒(c) Note that $\hat{\pi}_2$ is an embedding since $\hat{\pi}_2^{-1}(\langle \mathbf{2}, U\rangle) = U$, for all U open in B. Since exponentiable embeddings are locally closed [19], we know $\hat{\pi}_2(B) = V \cap F$, for some V open and F closed in $B^{\mathbf{2}}$.

Suppose $\hat{\pi}_2$ is exponentiable and B is not T_1. Then there exist $x \neq y$ in B such that $y \in U \Rightarrow x \in U$, for all U open in B, and so the map $\sigma \colon \mathbf{2} \longrightarrow B$, given by $\sigma(0) = x$ and $\sigma(1) = y$, is continuous. Since $\hat{\pi}_2(y) \in V$, there exists W open in by such that $\hat{\pi}_2(y) \in \langle \mathbf{2}, W\rangle \subseteq V$. Since $y \in W$, we know $x \in W$, and it follows that $\sigma \in \langle \mathbf{2}, W\rangle \subseteq V$. Also, it is not difficult to show that $\sigma \in F$, since $\hat{\pi}_2(x) \in F$ and F is closed in $B^{\mathbf{2}}$. Thus, $\sigma \in V \cap F = \hat{\pi}_2(B)$, contradicting that σ is not constant.
(c)⇒(d) since a non-constant maps $\sigma \colon \mathbf{2} \longrightarrow B$ gives rise to $x \neq y$ in B such that $y \in U \Rightarrow x \in U$.
(d)⇒(a) Suppose every continuous map $\mathbf{2} \longrightarrow B$ is constant. Then $\hat{\pi}_2$ is a bijection. Since $\hat{\pi}_2$ is an embedding, as noted in the proof of (b)⇒(c) above, the desired result follows. □

Corollary 7.8 *If B is a T_1-space, then $\mathbf{2}$-pullbacks coincide with ordinary pullbacks, and so $Y \longrightarrow B$ is $\mathbf{2}$-exponentiable if and only if it is exponentiable in* **Top**.

Proof. This follows directly from Proposition 7.7, since $\hat{\pi}_2 \colon B \longrightarrow B^{\mathbf{2}}$ is an isomorphism. □

Note that if B is not a T_1-space, since $\hat{\pi}_2$ is not exponentiable, we cannot apply Theorem 4.5 to conclude that $\mathbf{2}$-exponentiable maps over B are exponentiable. However, we will see that the converse holds, if we restrict to Alexandrov spaces.

Given a poset P, let $P^{\downarrow}$ denote the space whose points are the elements of P and open sets are downward closed subsets. Then it is not difficult to show that B is an Alexandrov space if and only if $B = P^{\downarrow}$, for some poset P (c.f., [1]). In particular, $\mathbf{2} = \mathbf{2}^{\downarrow}$. To apply Theorem 4.4 when B is an Alexandrov space, we would like to show that $\varepsilon_0 \colon B^{\mathbf{2}} \longrightarrow B$ is exponentiable in **Top**. By Lemma 6.1, we know $\varepsilon_0 \colon P^{\mathbf{2}} \longrightarrow P$ is exponentiable in **Poset**. We will see that it is also hereditarily compact (in the sense of [22]), which is precisely what is needed for the exponentiability of $\varepsilon_0^{\downarrow} \colon (P^{\mathbf{2}})^{\downarrow} \longrightarrow P^{\downarrow}$ in **Top** provided that B is a sober space [22]. Thus, we prove the following lemma.

Lemma 7.9 *If P is any poset, then $(P^{\mathbf{2}})^{\downarrow} = (P^{\downarrow})^{\mathbf{2}}$.*

Proof. First, $(P^{\mathbf{2}})^{\downarrow} = (P^{\downarrow})^{\mathbf{2}}$ as sets, since $\sigma \colon \mathbf{2} \longrightarrow P^{\downarrow}$ is continuous if and only if $\sigma \colon \mathbf{2} \longrightarrow P$ is order-preserving. Also, every compact-open subset of $(P^{\downarrow})^{\mathbf{2}}$ is downward closed since the basic opens $\langle 0, W \rangle$ and $\langle 1, W \rangle$ are easily seen to be downward closed. Thus, it remains to show that every downward closed subset H of $P^{\mathbf{2}}$ is in the compact-open topology. But,

$$H = \bigcup_{\sigma \in H} \langle 0, \downarrow \sigma(0) \rangle \cap \langle 1, \downarrow \sigma(1) \rangle$$

and the desired result follows. □

Recall from [22] that $p^{\downarrow} \colon X^{\downarrow} \longrightarrow P^{\downarrow}$ is exponentiable in **Top** if and only if $p \colon X \longrightarrow P$ is exponentiable in **Pos** and *hereditarily compact* (i.e., $\downarrow x \cap p^{-1}(\downarrow b)$ is compact in $X^{\downarrow}$, for all $x \in X$ and $b \leq px$).

Theorem 7.10 *If B is an Alexandrov sober space, then every exponentiable map over B is* **2***-exponentiable in* **Top**.

Proof. Suppose $B = P^{\downarrow}$, for some poset P. In view of Lemma 7.9 and the remarks above, it suffices to show that $\varepsilon_0 \colon P^{\mathbf{2}} \longrightarrow P$ is hereditarily compact. Given $\sigma \in P^{\mathbf{2}}$ and $b \leq \varepsilon_0(\sigma)$, define $\rho \colon \mathbf{2} \longrightarrow P$ by $\rho(0) = b$ and $\rho(1) = \sigma(1)$ Then one can show that $\downarrow \sigma \cap \varepsilon_0^{-1}(\downarrow b) = \downarrow \rho$, which is clearly compact in $(P^{\mathbf{2}})^{\downarrow}$, and the desired result follows. □

8 E-Exponentiability in Loc

There are two generalized pullbacks of interest in **Loc**, namely, homotopy and lax pullbacks. Recall from [9] that a locale is exponentiable if and only if it locally compact, i.e., a continuous lattice (in the sense of Scott [25]). Thus, as noted earlier, $\mathcal{O}(E)$ is exponentiable, when E is the unit interval I or the Sierpinski space $\mathbf{2}$, and the $\mathcal{O}(E)$-pullback is given by

$$\begin{array}{ccc} X \times_B B^{\mathcal{O}(E)} \times_B Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & \longrightarrow & \downarrow{\scriptstyle q} \\ X & \xrightarrow[p]{} & B \end{array}$$

which is the homotopy pullback when $E = I$ and the lax pullback when $E = \mathbf{2}$. Applying the theorems of Section 4, we get:

Theorem 8.1 *Suppose $E = I$ or $E = \mathbf{2}$. Then every exponentiable morphism over B is $\mathcal{O}(E)$-exponentiable if and only if $\varepsilon_0 \colon B^{\mathcal{O}(E)} \longrightarrow B$ is exponentiable in* **Loc**. *If $\hat{\pi}_2 \colon B \longrightarrow B^{\mathcal{O}(E)}$ is exponentiable, then every $\mathcal{O}(E)$-exponentiable map is exponentiable in* **Loc**.

Corollary 8.2 *If B is a locally compact Hausdorff space and $\mathcal{O}(B)^{\mathcal{O}(I)}$ is spatial, then every $\mathcal{O}(I)$-exponentiable morphism over $\mathcal{O}(B)$ is exponentiable in* **Loc**.

Proof. By Proposition 7.3, we know $\hat{\pi}_2 \colon B \longrightarrow B^I$ is exponentiable in **Top**. Since $\mathcal{O}$ preserves exponentiable morphisms over Hausdorff spaces by Theorem 2 of [21], it follows that $\hat{\pi}_2 \colon \mathcal{O}(B) \longrightarrow \mathcal{O}(B)^{\mathcal{O}(I)}$ is exponentiable **Loc**. But, $\mathcal{O}(B^I) \cong \mathcal{O}(B)^{\mathcal{O}(I)}$ since the latter is spatial, and so the desired result follows from Theorem 8.1. □

Corollary 8.3 *If B is a locally compact space with a non-constant path consisting of closed points, then $\varepsilon_0 \colon \mathcal{O}(B)^{\mathcal{O}(I)} \longrightarrow \mathcal{O}(B)$ is not exponentiable, and so $id_{\mathcal{O}(B)} \colon \mathcal{O}(B) \longrightarrow \mathcal{O}(B)$ is not $\mathcal{O}(I)$-exponentiable in* **Loc**.

Proof. Assume $\varepsilon_0 \colon \mathcal{O}(B)^{\mathcal{O}(I)} \longrightarrow \mathcal{O}(B)$ is exponentiable. Then $\mathcal{O}(B)^{\mathcal{O}(I)}$ is locally compact, and hence spatial, since the composite of exponentiable morphisms is exponentiable. But, then $\varepsilon_0: B^I \longrightarrow B$ is exponentiable, and so $\varepsilon_0: \mathcal{O}(B^I) \longrightarrow \mathcal{O}(B)$ is by Theorem 2 of [21], contradicting Corollary 7.6. □

Corollary 8.4 *If B is a T_1-space and $\mathcal{O}(B)^{\mathcal{O}(\mathbf{2})}$ is spatial, then all $\mathcal{O}(\mathbf{2})$-pullbacks coincide with ordinary pullbacks, and so $p: Y \longrightarrow \mathcal{O}(B)$ is $\mathcal{O}(\mathbf{2})$-exponentiable if and only if it is exponentiable in* **Loc**.

Proof. Since $\mathcal{O}(B^\mathbf{2}) \cong \mathcal{O}(B)^{\mathcal{O}(\mathbf{2})}$, the corollary follows directly from from Proposition 7.7. □

Note that if B is not a T_1-space and $\mathcal{O}(B)^{\mathcal{O}(\mathbf{2})}$ is spatial, then

$$\hat{\pi}_2: \mathcal{O}(B)^{\mathcal{O}(\mathbf{2})} \longrightarrow \mathcal{O}(B)$$

is not exponentiable, and so as was the case with **Top**, we cannot apply Theorem 4.5 to conclude that $\mathcal{O}(\mathbf{2})$-exponentiable morphisms over $\mathcal{O}(B)$ are exponentiable. However, we will see that the converse holds, if we restrict to Alexandrov spaces.

Corollary 8.5 *If B is an Alexandrov sober space, then every exponentiable morphism over $\mathcal{O}(B)$ is $\mathcal{O}(\mathbf{2})$-exponentiable in* Loc *if and only if $\mathcal{O}(B)^{\mathcal{O}(\mathbf{2})}$ is spatial.*

Proof. Suppose exponentiable morphisms over $\mathcal{O}(B)$ are $\mathcal{O}(\mathbf{2})$-exponentiable. Then $\varepsilon_0: \mathcal{O}(B)^{\mathcal{O}(\mathbf{2})} \longrightarrow \mathcal{O}(B)$ is exponentiable in **Loc** by Theorem 8.1. Since $\mathcal{O}(B)$ is exponentiable, we know $\mathcal{O}(B)^{\mathcal{O}(\mathbf{2})}$ is exponentiable, and hence spatial, as desired.

Conversely, suppose $\mathcal{O}(B)^{\mathcal{O}(\mathbf{2})}$ is spatial. Then $\mathcal{O}(B^\mathbf{2}) \cong \mathcal{O}(B)^{\mathcal{O}(\mathbf{2})}$. As in Theorem 7.10, we know $\varepsilon_0: B^\mathbf{2} \longrightarrow B$ is exponentiable in **Top** and so $\varepsilon_0: \mathcal{O}(B)^{\mathcal{O}(\mathbf{2})} \longrightarrow \mathcal{O}(B)$ in **Loc** by by Theorem 2 of [21]. Therefore, every exponentiable morphism over $\mathcal{O}(B)$ is $\mathcal{O}(\mathbf{2})$-exponentiable in **Loc** by Theorem 8.1. □

9 E-Exponentiability in GTop

Recall [11] that diagrams in **GTop** are assumed to commute up to coherent isomorphism, and so the functors involved in exponentiability are just pseudo-functors. One can show that the results of the first four sections of this paper hold if we replace all commutative diagrams by ones that commute up to isomorphism, all isomorphisms of objects by equivalences, and all functors by pseudo-functors.

As in the case of **Loc**, there are two generalized pullbacks of interest in **GTop**, namely, homotopy and lax pullbacks. In fact, descent theory for the latter is considered by Moerdijk and Vermeulen [17]. Recall from [13] that a topos is exponentiable if and only if it a continuous category, and a localic topos is exponentiable if and only if the corresponding locale is metastably locally compact. Thus, the sheaf functor Sh does not preserve exponentiability. But, as noted earlier, $Sh(E)$ is exponentiable, when E is the unit interval I or the Sierpinski space **2**, and the $Sh(E)$-pullback is given by

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{B}} \mathcal{B}^{Sh(E)} \times_{\mathcal{B}} \mathcal{Y} & \xrightarrow{\pi_{\mathcal{Y}}} & \mathcal{Y} \\ {\scriptstyle \pi_{\mathcal{X}}}\downarrow & \longrightarrow & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow[p]{} & \mathcal{B} \end{array}$$

which is the homotopy pullback when $E = I$ and the lax pullback when $E = \mathbf{2}$. Applying the theorems of Section 4, we get:

Theorem 9.1 *Suppose $E = I$ or $E = \mathbf{2}$. Then every exponentiable geometric morphism over $\mathcal{B}$ is $Sh(E)$-exponentiable if and only if $\varepsilon_0 \colon \mathcal{B}^{Sh(E)} \longrightarrow \mathcal{B}$ is exponentiable in* **GTop**. *If $\tilde{\pi}_2 \colon \mathcal{B} \longrightarrow \mathcal{B}^{Sh(E)}$ is exponentiable, then every $Sh(E)$-exponentiable map is exponentiable in* **GTop**.

For the remainder of this section, we consider the case where $\mathcal{B}$ is spatial, i.e., $\mathcal{B} = Sh(B)$, for some space topological B. In [21], it was shown that if B and E are any locales and E is exponentiable, then $Sh(B)^{Sh(E)}$ exists in **GTop** and is equivalent to $Sh(B^E)$. Thus, if B and E are spaces and $Sh(B)^{Sh(E)}$ is spatial, then $Sh(B)^{Sh(E)} \simeq Sh(\mathcal{O}(B)^{\mathcal{O}(E)}) \simeq Sh(B^E)$.

Corollary 9.2 *If B is a locally compact Hausdorff space and $Sh(B)^{Sh(I)}$ is spatial, then every $Sh(I)$-exponentiable geometric morphism over $Sh(B)$ is exponentiable in* **GTop**.

Proof. By Proposition 7.3, we know $\hat{\pi}_2: B \longrightarrow B^I$ is exponentiable in **Top**. Since $\hat{\pi}_2$ is an embedding, we know it is locally closed [19], and so $\hat{\pi}_2: Sh(B) \longrightarrow Sh(B^I)$ is locally closed. Since locally closed inclusions are exponentiable in **GTop** [20], the desired result follows from Theorem 9.1. □

Corollary 9.3 *If B is a locally compact space with a non-constant path consisting of closed points, then $\varepsilon_0: Sh(B)^{Sh(I)} \longrightarrow Sh(B)$ is not exponentiable, and so $id_{Sh(B)}: Sh(B) \longrightarrow Sh(B)$ is an exponentiable geometric morphism which is not $Sh(I)$-exponentiable in* **GTop**.

Proof. Assume $\varepsilon_0: Sh(B)^{Sh(I)} \longrightarrow Sh(B)$ is exponentiable. Then so is $Sh(B)^{Sh(I)}$, since a composite of exponentiable geometric morphisms is exponentiable. Since $Sh(B)^{Sh(I)} \simeq Sh(\mathcal{O}(B)^{\mathcal{O}(I)})$, it follows that $\mathcal{O}(B)^{\mathcal{O}(I)}$ is metastably locally compact, and hence, spatial. Then $\mathcal{O}(B)^{\mathcal{O}(I)} \cong \mathcal{O}(B^I)$, making B^I locally compact, contradicting Proposition 7.5, and the desired result follows. □

Corollary 9.4 *If B is a T_1-space and $Sh(B)^{Sh(\mathbf{2})}$ is spatial, then all $Sh(\mathbf{2})$-pullbacks coincide with ordinary pullbacks, and so $p: \mathcal{Y} \longrightarrow Sh(B)$ is $Sh(\mathbf{2})$-exponentiable if and only if it is exponentiable in* **GTop**.

Proof. Since $Sh(B)^{Sh(\mathbf{2})} \simeq Sh(B^{\mathbf{2}})$, the corollary follows directly from from Proposition 7.7. □

Note that if B is not a T_1-space and $Sh(B)^{Sh(\mathbf{2})}$ is spatial, then the geometric morphism $\hat{\pi}_2: Sh(B)^{Sh(\mathbf{2})} \longrightarrow Sh(B)$ is not exponentiable, and so as was the case with **Loc**, we cannot apply Theorem 4.5 to conclude that $Sh(\mathbf{2})$-exponentiable morphisms over $Sh(B)$ are exponentiable. Restricting to Alexandrov spaces, we can prove that if every exponentiable geometric morphism over $Sh(B)$ is $Sh(\mathbf{2})$-exponentiable in **GTop**, then $Sh(B)^{Sh(\mathbf{2})}$ is spatial, but we cannot prove the converse, as we did for **Loc**, since the sheaf functor does not preserve exponentiable morphisms.

10 Conclusion

The next step is to consider the homotopy pullback as a endofunctor on the bicategory whose objects are space over B and morphisms are triangle commuting up to homotopy. Since composition is now only associative up to homotopy, the higher-order structure of **Top** becomes relevant. In particular, homotopy over B and coherence issues must be considered (c.f., the work of Hardie, Kamps, and Porter in [8] and [14]). Another related project is to investigate the relationship of this work to Grothendieck's pursuing stacks which were introduced in a 1983 letter to Quillen and further discussed in [7].

References

[1] P. S. Alexandrov, Über die Metrisation der im kleinen kompakten topologische Räume, Math. Ann. 92 (1924), 294–301.

[2] H. J. Baues, Algebraic Homotopy, Cambridge University Press, 1989.

[3] L. Bombelli, J. Lee, D. Meyer, and R. Sorkin, Space-time as a causal set. Phys. Rev. Lett. 59 (1987), 521–524.

[4] F. Conduché, Au sujet de l'existence d'adjoints à droite aux foncteurs "image réciproque" dans la catégorie des catégories, C. R. Acad. Sci. Paris 275 (1972), A891–894.

[5] J. Giraud, Méthode de la descente, Bull. Math. Soc. France, Memoire 2 (1964).

[6] A. Grothendieck and J. L. Verdier, Theorie des Topos (SGA 4), Springer Lecture Notes in Math. 269–270 (1972), 1–340.

[7] A. Grothendieck, Esquisse d'un programme, London Math. Soc. Lecture Note Ser. 242 (1997), 5–48.

[8] K. A. Hardie, K. H. Kamps, and T. Porter, The coherent homotopy category over a fixed space is a category of fractions, Topology Appl. 40 (1991), 265–274.

[9] J. M. E. Hyland, Function spaces in the category of locales, Springer Lecture Notes in Math. 871 (1981), 264–281.

[10] J. R. Isbell, Atomless parts of spaces, Math. Scand. 31 (1972), 5–32.

[11] P. T. Johnstone, Topos Theory, Academic Press, 1977.

[12] P. T. Johnstone, Stone Spaces, Cambridge University Press, 1982.

[13] P. T. Johnstone and A. Joyal, Continuous categories and exponentiable toposes, J. Pure Appl. Algebra 25 (1982), 255–296.

[14] K. H. Kamps and T. Porter, Abstract homotopy and simple homotopy theory, World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[15] R. W. Kieboom, Notes on homotopy pull-backs, Quaest. Math. 14 (1991), 445–452.

[16] M. Mather, Pull-backs in homotopy, Can. J. Math. 28 (1976), 225–263.

[17] I. Moerdijk and J. J. C. Vermeulen, Proper maps of toposes, Amer. Math. Soc. Memoirs 705 (2000).

[18] S. B. Niefield, Cartesianness, Ph.D. Thesis, Rutgers University, 1978.

[19] S. B. Niefield, Cartesianness: topological spaces, uniform spaces, and affine schemes, J. Pure Appl. Algebra 23 (1982), 147–167.

[20] S. B. Niefield, Cartesian inclusions: locales and toposes, Comm. in Alg. 9(16) (1981), 1639–1671.

[21] S. B. Niefield, Cartesian spaces over T and locales over $\Omega(T)$, Cah. Topol. Géom. Differ. Catég. 23-3 (1982), 257–267.

[22] S. B. Niefield, Exponentiable morphisms: posets, spaces, locales, and Grothendieck toposes, Theory Appl. Categ. 8 (2001), 16–32.

[23] S. B. Niefield, Locally compact path spaces, Appl. Categ. Structures 13 (2005), 65–69.

[24] D. P. Rideout and R. D. Sorkin, Evidence for a continuum limit in causal set dynamics, Phys. Rev. D 63 (2001), 104011.

[25] D. S. Scott, Continuous lattices, Springer Lecture Notes in Math. 274 (1972), 97–137.

Susan Niefield
Union College
Department of Mathematics
Schenectady, NY 12308 U.S.A.
e-mail: niefiels@union.edu