

# CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

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*Cahiers de topologie et géométrie différentielle catégoriques*, tome 47, n° 1 (2006), p. 29-49

[http://www.numdam.org/item?id=CTGDC\\_2006\\_\\_47\\_1\\_29\\_0](http://www.numdam.org/item?id=CTGDC_2006__47_1_29_0)

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## COMBINATORIAL STACKS AND THE FOUR-COLOR THEOREM

by *Romain ATTAL*

ABSTRACT. We interpret the number of good four-colourings of the faces of a trivalent, spherical polyhedron as the 2-holonomy of the 2-connection of a fibered category,  $\varphi$ , modeled on  $\mathbf{Rep}_f(\mathfrak{sl}_2)$  and defined over the dual triangulation,  $T$ . We also build an  $\mathfrak{sl}_2$ -bundle with connection over  $T$ , that is a global, equivariant section of  $\varphi$ , and we prove that the four-colour theorem is equivalent to the fact that the connection of this  $\mathfrak{sl}_2$ -bundle vanishes nowhere. This geometric interpretation shows the cohomological nature of the four-colour problem.

**Résumé :** Nous interprétons le nombre de bons quadri-coloriages des faces d'un polyèdre sphérique trivalent comme la 2-holonomie de la 2-connexion d'une catégorie fibrée,  $\varphi$ , modélisée sur  $\mathbf{Rep}_f(\mathfrak{sl}_2)$  et définie sur la triangulation duale,  $T$ . Nous construisons au-dessus de  $T$  un  $\mathfrak{sl}_2$ -fibré avec connexion qui est une section globale équivariante de  $\varphi$ , et nous prouvons que le théorème des quatre couleurs équivaut au fait que la connexion de ce  $\mathfrak{sl}_2$ -fibré ne s'annule nulle part. Cette interprétation géométrique montre la nature cohomologique du problème des quatre couleurs.

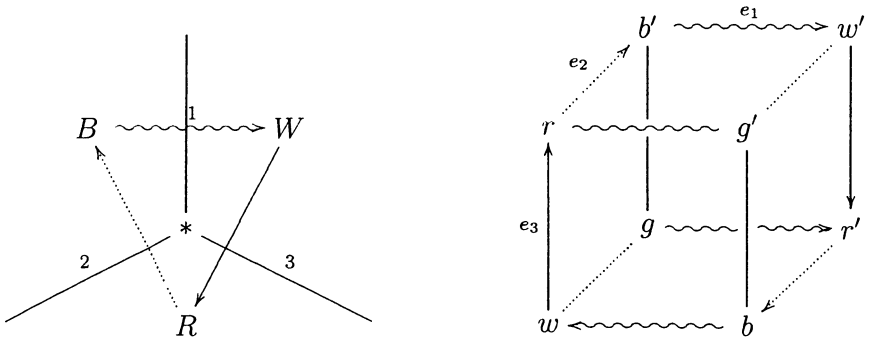
**Keywords :** Map colouring, iterated paths, combinatorial stacks,  $\mathfrak{sl}_2$ .

### 1. Introduction

Let us consider a finite spherical polyhedron,  $P$ , and a palette of four colours,  $\{W, R, G, B\}$ . We will call a good colouring of  $P$  any map which associates one of these colours to each face of  $P$  in such a way that any two adjacent faces carry distinct colours. The four-colour theorem [5, 1, 12] states that such a map exists for any  $P$ . The goal of the present work is to provide a geometric interpretation of this theorem. We obtain here two new results : the number of good colourings of a trivalent, spherical polyhedron

is the 2-holonomy of a 2-connection on a fibered category over the dual triangulation,  $T = P^*$  (Theorem 2) ; the four-colour theorem is equivalent to the existence of a non-vanishing, equivariant global section of this fibered category (Theorem 4).

In order to study the colourability of  $P$ , let us start by making some classical modifications. We first remark that it is sufficient to prove the colourability of trivalent polyhedra. Indeed, by cutting a little disk around each vertex of degree  $> 3$  in  $P$ , one obtains a trivalent polyhedron and each good colouring of the latter provides a good colouring of  $P$  by shrinking this disk to the initial vertex. Henceforth, we will suppose that  $P$  is itself trivalent. Secondly, let us identify our four colours with the pairs of diametrically opposite vertices of a cube :  $W = \{w, w'\}$ ,  $R = \{r, r'\}$ ,  $G = \{g, g'\}$  and  $B = \{b, b'\}$ . Then each good colouring of the three faces which surround a vertex of  $P$  defines an edge-loop in this cube such that the determinant of any triple of successive vectors be  $\pm 1$  :



A map  $(u : T_1 \rightarrow \{e_1, e_2, e_3\})$  satisfying this property will be called a good numbering. Thus, the number of good numberings of the edges of  $T$  is one quarter of the number of good colourings of the faces of  $P$ , as proved by P.G. Tait [13]. We call this integer,  $K_T$ , the chromatic index of  $T$  and the four-colour theorem states that  $K_T \neq 0$  for any finite, spherical triangulation,  $T$ .

Our article is organised as follows. In Section 2, we give a proof of Penrose's formula which expresses  $K_T$  as a partition function. In Section 3, we

define the graph  $\mathcal{P}$  of edge-paths of  $T$ . In Section 4, we collect useful results about representations of  $\mathfrak{sl}_2$ . In Section 5, we construct the chromatic stack,  $\varphi$ , which is a fibered category over  $T$ , endowed with a functorial 1-connection and with a natural 2-connection, and we prove that  $K_T$  is the 2-holonomy of this 2-connection on  $T$ . In Section 6, we define another fibered category,  $\Phi$ , over  $\mathcal{P}$ . By integrating the functorial connection of  $\Phi$  along a 2-path which sweeps each triangle of  $T$  once only, we obtain an equivariant global section of the pull-back of the chromatic stack to a triangulation  $\tilde{T}$  of the disk. This section,  $\zeta$ , is an  $\mathfrak{sl}_2$ -bundle with connection whose holonomy on  $\partial\tilde{T}$  is  $K_T$ . Our construction is an adaptation of Stokes theorem to a case of combinatorial differential forms with values in the tensor category  $\mathbf{A} = \mathbf{Rep}_f(\mathfrak{sl}_2)$  and we can write it symbolically  $K_T = \int_T \varphi = \int_{\partial\tilde{T}} \zeta$ . Since  $K_T$  depends linearly on the value of  $\zeta$  on each inner edge of  $\tilde{T}$ , we obtain this way our second result : the four-colour theorem is equivalent to the fact that  $\zeta$  vanishes nowhere.

## 2. The chromatic index

The idea to translate the four-colour problem in terms of linear algebra is due to Roger Penrose. Let us fix a finite, spherical triangulation  $T = (T_0, T_1, T_2)$ .  $T_0$  is the set of its vertices,  $T_1$  the set of its edges and  $T_2$  the set of its triangles. Following [10], we define the chromatic index of  $T$  as

$$(1) \quad K_T := \sum_u \prod_{[xyz]} i \det(u_{xy}, u_{yz}, u_{zx})$$

In this sum,  $u$  runs over the set of all maps from  $T_1$  to  $\{e_1, e_2, e_3\}$ , the canonical basis of  $\mathbb{R}^3$ , and  $[xyz]$  runs over the set of positively oriented triangles of  $T$ . The integrality of  $K_T$  follows from the fact that, if  $u$  is a good numbering of  $T_1$ , *i.e.* if no determinant vanishes in this product, the number of triangles where  $\det = (+1)$  minus the number of triangles where  $\det = (-1)$  is a multiple of 4, as proves the following lemma.

**Lemma :** *If  $u$  is a good numbering of  $T_1$  and if  $n_+$  (resp.  $n_-$ ) denotes the number of triangles  $[xyz]$  such that  $\det(u_{xy}, u_{yz}, u_{zx}) = (+1)$  (resp.  $(-1)$ ), then  $n_+ \equiv n_- \pmod{4}$ .*

*Proof* : 1) Starting from  $(T, u)$ , we can build another triangulation,  $T'$ , equipped with a good edge numbering,  $u'$ , such that  $n'_+ = 0$ . Indeed, if two adjacent, positively oriented triangles of  $T$ , say  $[xyz]$  and  $[zyw]$ , have  $\det = (+1)$  (positive triangles), then we can flip their common edge  $[yz]$  to  $[xw]$  and obtain a new pair of negative triangles,  $[xyw]$  and  $[wzx]$ , where  $\det = (-1)$ . During this step,  $(n_+ - n_-)$  is reduced by 4. Once all these pairs of neighbour positive triangles have been eliminated this way, the remaining contributions to  $n_+$  are triangles surrounded by three negative neighbours. By adjoining three edges and a trivalent vertex inside each isolated triangle of this kind, we change a positive triangle for three negative ones. Again,  $(n_+ - n_-)$  is reduced by 4, and  $(T', u')$  is reached at the end of this process.

2) Consider all pairs of triangles,  $[xyz]$  and  $[zyw]$ , with  $u'_{yz} = e_1$  on their common edge,  $[yz]$ . Since  $\det(u_{xy}, u_{yz}, u_{zx}) = \det(u_{zy}, u_{yw}, u_{wz}) = (-1)$ , the opposite sides of the rectangle  $[xywz]$  carry the same vector, say  $u'_{xz} = u'_{yw} = e_2$  and  $u'_{xy} = u'_{zw} = e_3$ . Let us join the midpoints of two opposite edges with a simple curve. By repeating this process inside all such pairs of triangles, we obtain two simple closed curves,  $c_2$  and  $c_3$ . If we orient these curves conveniently, their intersection number is equal to  $|u'^{(-1)}(e_1)|$ , the number of edges of  $T'$  marked with  $e_1$ . But, after Jordan's theorem, the intersection number of two simple closed curves in  $S^2$  is even. Therefore,  $|u'^{(-1)}(e_1)|$ , the number of edges mapped to  $e_1$  by  $u'$ , is even. Similarly,  $|u'^{(-1)}(e_2)|$  and  $|u'^{(-1)}(e_3)|$  are also even, as well as the total number of edges of  $T'$  :

$$t'_1 = |u'^{(-1)}(e_1)| + |u'^{(-1)}(e_2)| + |u'^{(-1)}(e_3)| \in 2\mathbb{N}$$

3) Since  $T'$  is a triangulation of a closed surface, we have  $3t'_2 = 2t'_1$ . Since  $t'_1$  is even, we obtain  $t'_2 = n'_- \in 4\mathbb{N}$ . Therefore,  $n_+$  and  $n_-$  are congruent modulo 4 :

$$(2) \quad \boxed{(n_+ - n_-) \in 4\mathbb{Z}}$$

□

**Theorem 1 (R. Penrose)** :  $K_T$  is the number of good numberings of  $T_1$ .

*Proof*: If  $u$  is a bad numbering, then one of the determinants is zero and the corresponding product vanishes. On the other hand, if  $u$  is a good numbering, then the corresponding product is equal to  $i^{(n_+ - n_-)} = 1$ , after the precedent lemma. Therefore, the sum of all these products equals the number of good numberings of  $T_1$ . □

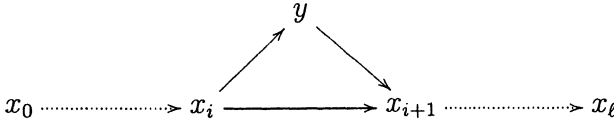
### 3. The graph of edge-paths

Having fixed our triangulation,  $T$ , let us define the graph  $\mathcal{P}$  whose vertices are the edge-paths of  $T$ :

$$\mathcal{P}_0 = \bigcup_{\ell \geq 0} \{ \gamma = (x_0, \dots, x_\ell) : \{x_i, x_{i+1}\} \in T_1 \forall i \}$$

and whose edges, called the 2-edges of  $T$ , are the pairs of paths, with the same source and the same target, which bound a single triangle of  $T$ :

$$\mathcal{P}_1 = \{ \{ (x_0, \dots, x_\ell), (x_0, \dots, x_i, y, x_{i+1}, \dots, x_\ell) \} : \{x_i y x_{i+1}\} \in T_2 \}$$



The oriented 2-edges are the corresponding ordered pairs. A 2-path in  $T$  is an edge-path in  $\mathcal{P}$ , *i.e.* a family  $\Gamma = (\gamma_0, \dots, \gamma_n)$  such that  $\{\gamma_i, \gamma_{i+1}\} \in \mathcal{P}_1$  for  $i = 0, \dots, n - 1$ . They form the set  $\mathcal{P}_2$ :

$$\mathcal{P}_2 = \bigcup_{n \geq 0} \{ \Gamma = (\gamma_0, \dots, \gamma_n) : \{\gamma_i, \gamma_{i+1}\} \in \mathcal{P}_1 \text{ for } i = 0, \dots, n - 1 \}$$

For each 2-path  $\Gamma = (\gamma_0, \dots, \gamma_n)$ , there is a 2-path  $\tilde{\Gamma}$  going backward in time :

$$\tilde{\Gamma} = (\gamma_n, \dots, \gamma_0)$$

The 0-source (resp. 0-target) of  $\Gamma$  is the common source (resp. target) of the  $\gamma_i$ 's. The 1-source of  $\Gamma$  is  $\gamma_0$  and its 1-target is  $\gamma_n$ . The oriented 2-cells of  $T$  are its smallest 2-paths. They have the form  $((xz), (xyz))$  or  $((xyz), (xz))$ , for some triangle  $\{xyz\}$ .

#### 4. Representations of $\mathfrak{sl}_2$

As we have seen above, Penrose's formula involves the determinants of triples of basis vectors of  $\mathbb{R}^3$ . If we endow  $\mathbb{R}^3$  with its canonical euclidian structure and with the corresponding cross-product, we obtain a Lie algebra isomorphic to  $\mathfrak{so}_3$ . Since we will use complex coefficients and Schur's lemma, valid only for representations over an algebraically closed field, we will work with its complexification,  $V = \mathfrak{sl}_2$ . We will note  $I = \text{Id}_V$ ,  $V^\ell = V^{\otimes \ell}$  and  $I^\ell = \text{Id}_{V^\ell}$ , where  $V^\ell$  carries the representation

$$\begin{aligned} \rho_\ell &: V \longrightarrow \text{End}(V^\ell) \\ x &\longmapsto \rho_\ell(x) = \sum_{k=1}^{\ell} I^{k-1} \otimes \text{ad}_x \otimes I^{\ell-k} \end{aligned}$$

Let  $\mathbf{A} = \mathbf{Rep}_f(\mathfrak{sl}_2)$ , the category of finite dimensional representations of  $\mathfrak{sl}_2$  over complex vector spaces. If  $M$  and  $M'$  are two  $V$ -modules, carrying, respectively, the representations  $R$  and  $R'$ , we will often identify  $M$  with  $M \otimes -$ , the endofunctor of  $\mathbf{A}$ , and write  $M'M$  for  $M' \otimes M$ . For each  $j \in \frac{1}{2}\mathbb{N}$ , let  $(R_j : V \rightarrow \text{End}(V_j))$  be a representative of the isomorphism class of representations of spin  $j$  and dimension  $2j + 1$ . For example, we can choose  $V_0 = \mathbb{C}$ ,  $V_{1/2} = \mathbb{C}^2$  and  $V_1 = V$ . After Schur's lemma, the irreducible representations are orthonormal for the bifunctor  $\text{hom}_{\mathbf{A}}$  :

$$(3) \quad \boxed{\text{hom}_{\mathbf{A}}(R_j, R_k) \simeq \delta_{jk} R_0}$$

The intertwining number between two representations  $R$  and  $R'$  is defined as the dimension of the space  $\text{hom}_{\mathbf{A}}(R, R')$  :

$$c(R, R') = \dim_{\mathbb{C}} (\text{hom}_{\mathbf{A}}(R, R'))$$

After Clebsch-Gordan's rule, we have

$$\begin{aligned} V^2 &\simeq V_0 \oplus V_1 \oplus V_2 \\ c(V, V^2) &= c(V^2, V) = 1 \end{aligned}$$

The projectors onto the isotypic components of  $V^2$ , of spin 0, 1 and 2, respectively map  $u \otimes v$  to

$$\begin{aligned} T(u \otimes v) &= (u \cdot v) e_a \otimes e_a \\ A(u \otimes v) &= \frac{1}{2}(u_a v_b - u_b v_a) e_a \otimes e_b \\ S(u \otimes v) &= \frac{1}{2}(u_a v_b + u_b v_a) e_a \otimes e_b - (u_a v_a) e_a \otimes e_a \end{aligned}$$

The line  $L = \text{hom}_{\mathbf{A}}(V, V^2)$  is spanned by the map  $F$  defined by

$$F(e_a) = i e_{a-1} \wedge e_{a+1}$$

and the line  $\tilde{L} = \text{hom}_{\mathbf{A}}(V^2, V)$  is spanned by the bracket, noted  $\tilde{F}$  :

$$\tilde{F}(e_a \otimes e_b) = [e_a, e_b] = i \varepsilon_{abc} e_c$$

All these morphisms of representations satisfy the relations



$$\begin{aligned}
 \tilde{F}F &= 2I \\
 F\tilde{F} &= A \\
 T + A + S &= I^2 \\
 (\tilde{F} \otimes I)(I \otimes F) &= (I \otimes \tilde{F})(F \otimes I) \\
 &= T + 2A - 2S \\
 F &= (\tilde{F} \otimes I)(I \otimes F)F \\
 \tilde{F} &= \tilde{F}(I \otimes \tilde{F})(F \otimes I)
 \end{aligned}$$

### 5. The chromatic stack, $\varphi$

The notion of combinatorial stack appeared in [6] and we used it in [2] to give a construction of non-abelian  $G$ -gerbes over a simplicial complex. Dually, we can also use coefficients in a category of representation. Thus, we define the chromatic stack,  $\varphi$ , as a 2-functor which represents the simplicial homotopy groupoid  $\Pi_1(\mathcal{P})$  into the 2-category of  $\mathbf{A}$ -modules.  $\varphi$  is generated by pasting the following data :

$$\begin{aligned}
 \varphi_x &= \mathbf{A} \\
 \varphi_{xy} &= (V \otimes - : \varphi_y \rightarrow \varphi_x) \\
 \varphi_{(x_0, \dots, x_\ell)} &= (V^\ell \otimes - : \varphi_{x_\ell} \rightarrow \varphi_{x_0}) \\
 \varphi_\sigma &= F \quad \text{if } \sigma = ((xyz), (xz)) \\
 &= \tilde{F} \quad \text{if } \sigma = ((xz), (xyz)) \\
 \varphi_{\gamma\gamma'} &= (I^k \otimes \varphi_\sigma \otimes I^{\ell-k-1} : \varphi_{\gamma'} \rightarrow \varphi_\gamma) \\
 \varphi_{(\gamma_0, \dots, \gamma_n)} &= (\varphi_{\gamma_0\gamma_1} \circ \dots \circ \varphi_{\gamma_{n-1}\gamma_n} : \varphi_{\gamma_n} \rightarrow \varphi_{\gamma_0})
 \end{aligned}$$

The 1-connection of  $\varphi$  is the family of functors  $(\varphi_\gamma)_{\gamma \in \mathcal{P}_0}$ . The 2-connection of  $\varphi$  is the family of natural transformations  $(\varphi_\Gamma)_{\Gamma \in \mathcal{P}_2}$ . In order to compute the chromatic index, we choose a 2-loop,  $\Gamma = (\gamma_0, \dots, \gamma_n)$ , based at  $(a, b) = \gamma_0 = \gamma_n$ , and sweeping each triangle of  $T$  once only. To each path  $\gamma_p = (a, x_{p1}, \dots, x_{p\ell_p-1}, b)$ , of length  $|\gamma_p| = \ell_p$ ,  $\varphi$  associates a copy of  $V^{\ell_p}$ . For each  $p \in \{2, \dots, n\}$ , the loop  $\gamma_p$  differs from  $\gamma_{p-1}$  either by the insertion of

a vertex  $y \in T_0$  between  $x_{p-1,k_p}$  and  $x_{p-1,k_p+1}$  or by the deletion of  $x_{p-1,k_p}$ , where  $x_{p-1,k_p-1}$  and  $x_{p-1,k_p+1}$  are supposed to be adjacent. Each such move is represented by a linear map of the form

$$\varphi_{\gamma_{p-1}\gamma_p} = F_{k_p\ell_p} = (I_{V^{k_p-1}} \otimes F \otimes I_{V^{\ell_p-k_p}} : V^{\ell_p} \longrightarrow V^{\ell_p+1})$$

if  $\ell_{p-1} = \ell_p + 1$  and

$$\varphi_{\gamma_{p-1}\gamma_p} = \widetilde{F}_{k_p\ell_p} = (I_{V^{k_p-1}} \otimes \widetilde{F} \otimes I_{V^{\ell_p-k_p-1}} : V^{\ell_p} \longrightarrow V^{\ell_p-1})$$

if  $\ell_{p-1} = \ell_p - 1$ .

Since Penrose's formula looks like the partition function of a statistical model, it is natural to express  $K_T$  as the trace of a product of transfer matrices which represent linear maps between tensor powers of  $V$ . This approach will give us an efficient way to compute it, because the bad numberings are eliminated progressively during the sweeping process. Geometrically, the construction of the chromatic stack allows us to reinterpret  $K_T$  as a 2-holonomy, which is the categorical analogue of a holonomy in a fiber bundle.

**Definition :** *The 2-holonomy of  $\varphi$  on a 2-loop  $\Gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_0)$  based at  $\gamma_0$ , is the natural transformation*

$$\varphi_\Gamma = \varphi_{\gamma_0\gamma_1} \circ \dots \circ \varphi_{\gamma_{n-1}\gamma_0} : \varphi_{\gamma_0} \longrightarrow \varphi_{\gamma_0}$$

When  $\gamma_0 = (a)$ ,  $\varphi_\Gamma$  is an endomorphism of  $\varphi_{\gamma_0} = \text{Id}_A$  so that  $\varphi_\Gamma$  defines canonically a complex number. Moreover, after the following theorem, which illustrates the pasting lemma [11] in the 2-category of  $A$ -modules, the trace of  $\varphi_\Gamma \in \text{End}(\varphi_{\gamma_0})$  depends only on  $T$  and not on the 2-path  $\Gamma$ .

**Theorem 2 :** *If  $\Gamma$  is a 2-loop which sweeps each triangle of  $T$  once only, then the trace of the 2-holonomy of  $\varphi$  along  $\Gamma$ , evaluated in the representation associated to the base path of  $\Gamma$ , is the chromatic index of  $T$  :*

$$(4) \quad \boxed{\text{tr}_{\varphi_{\gamma_0}}(\varphi_\Gamma) = K_T}$$

*Proof:* Let  $\Gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_0)$  be a 2-loop. Let  $p \in \{0, \dots, n-1\}$  and suppose that  $\gamma_{p+1}$  is obtained from  $\gamma_p$  by inserting  $y$  between  $x_j$  and  $x_{j+1}$ , with  $x_j \neq y \neq x_{j+1} \neq x_j$ :

$$\begin{aligned}\gamma_p &= (x_0, \dots, x_\ell) \\ \gamma_{p+1} &= (x_0, \dots, x_j, y, x_{j+1}, \dots, x_\ell)\end{aligned}$$

Then the 2-arrow  $\varphi_{\gamma_p \gamma_{p+1}}$  is the intertwiner

$$\varphi_{\gamma_p \gamma_{p+1}} = I_{\varphi_{x_0 x_1}} \otimes \dots \otimes I_{\varphi_{x_{j-1} x_j}} \otimes \tilde{F} \otimes I_{\varphi_{x_{j+1}, j+2}} \otimes \dots \otimes I_{\varphi_{x_{\ell-1} x_\ell}} = \tilde{F}_{\ell k}$$

which is represented by the matrix  $M_p$  whose entries are given by

$$M_{p,ab} = \delta_{a_0 b_0} \dots \delta_{a_{j-1} b_{j-1}} (i \varepsilon_{a_j b_j b_{j+1}}) \delta_{a_{j+1} b_{j+2}} \dots \delta_{a_{\ell-1} b_\ell}$$

If  $\gamma_{q+1}$  is obtained from  $\gamma_q$  by deleting a vertex between  $y_k$  and  $y_{k+1}$ , then  $\varphi_{\gamma_q \gamma_{q+1}}$  is the intertwiner going backwards

$$\varphi_{\gamma_q \gamma_{q+1}} = I_{\varphi_{y_0 y_1}} \otimes \dots \otimes I_{\varphi_{y_{k-1} y_k}} \otimes F \otimes I_{\varphi_{y_{k+1}, k+2}} \otimes \dots \otimes I_{\varphi_{y_{\ell-1} y_\ell}} = F_{\ell+1, l}$$

and is represented by the matrix whose entries are

$$M_{q,ab} = \delta_{a_0 b_0} \dots \delta_{a_{k-1} b_{k-1}} ((-i) \varepsilon_{a_k b_{k+1} b_k}) \delta_{a_{k+2} b_{k+1}} \dots \delta_{a_\ell b_{\ell-1}}$$

Now, let  $a^p = (a_1^p, \dots, a_{\ell_p}^p)$  be a generic multi-index for the basis vectors of the representation  $\varphi_{\gamma_p}$ , with  $a_j^p \in \{1, 2, 3\}$  for  $j = 1, \dots, \ell_p$ . The number  $\text{tr}_{\varphi_{\gamma_0}}(\varphi_\Gamma)$  is the trace of the product of these matrices:

$$\text{tr}_{\varphi_{\gamma_0}}(\varphi_\Gamma) = \sum_a \prod_{p=0}^{n-1} M_{p, a^p a^{p+1}}$$

In this sum,  $a$  runs over the set of families  $(a^0, \dots, a^n)$  of multi-indices  $a^p = (a_1^p, \dots, a_{\ell_p}^p)$  with  $a_i^p \in \{1, 2, 3\}$ . To each edge of  $T$  are associated as many indices as there are paths  $\gamma_p$  which contain it. Let  $N_{xy}$  be the number of indices associated to  $(xy)$ . Among them,  $(N_{xy} - 2)$  indices are constrained by the  $\delta$ 's to be equal. Similarly, the two  $\varepsilon$ 's associated to the two triangles which contain  $(xy)$  force the two remaining indices to take the same value. Since the  $\delta$ 's are sandwiched between these two  $\varepsilon$ 's, these two indices are in fact equal and there is one and only one free index  $a_{xy}$  associated to each edge  $(xy)$ . The various factors of the product are equal to one except for the  $\varepsilon$ 's which can be indexed by the positively oriented triangles of  $T$ . Therefore, the precedent formula becomes

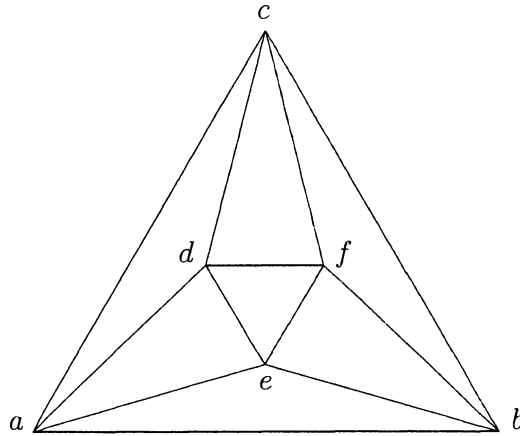
$$\begin{aligned} \text{tr}_{\varphi_{\gamma_0}}(\varphi_{\Gamma}) &= \sum_{(xy) \in T_1} \sum_{a_{xy} \in \{1,2,3\}} \left( \prod_{[xyz] \in T_2} i \varepsilon_{a_{xy} a_{yz} a_{zx}} \right) \\ &= \sum_u \prod_{[xyz]} i \det(u_{xy}, u_{yz}, u_{zx}) \end{aligned}$$

where  $u$  describes the set of all maps from  $T_1$  to  $\{e_1, e_2, e_3\}$  and the triangles  $[xyz]$  all have the same orientation.

□

Initially,  $K_T$  is defined as a sum of  $3^{t_1}$  terms and most of them vanish. By working in the tensor algebra,  $T(V)$ , the bad bumberings are eliminated during the sweeping process and the computation is much quicker if we use formula (4). Moreover, this method provides explicitly all good numberings.

**Example :** Let us apply the relation (4) to the computation of the chromatic index of the octahedron.



We sweep this triangulation with the 2-path

$$\Gamma = ((ab), (aeb), (adeb), (adefb), (adfb), (adcfb), (acfb), (acb), (ab))$$

For simplicity, we will write  $\mathbf{a}_1 \cdots \mathbf{a}_\ell$  for  $e_{a_1} \otimes \cdots \otimes e_{a_\ell}$  with  $\mathbf{a}_i \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ .  
The successive images of  $\mathbf{1}$  via the maps  $\varphi_{\gamma\gamma'}$  are :

$$\begin{aligned}
 \mathbf{1} &\mapsto i(\mathbf{23} - \mathbf{32}) \\
 &\mapsto i^2(\mathbf{313} - \mathbf{133} - \mathbf{122} + \mathbf{212}) \\
 &\mapsto i^3(\mathbf{3112} - \mathbf{3121} - \mathbf{1312} + \mathbf{1321} \\
 &\quad - \mathbf{1231} + \mathbf{1213} + \mathbf{2131} - \mathbf{2113}) \\
 &\mapsto i^4(-\mathbf{331} - \mathbf{122} - \mathbf{111} - \mathbf{111} - \mathbf{133} - \mathbf{221}) \\
 &\mapsto i^5(-\mathbf{3121} + \mathbf{3211} - \mathbf{1312} + \mathbf{1132} - \mathbf{1231} + \mathbf{1321} \\
 &\quad - \mathbf{1231} + \mathbf{1321} - \mathbf{1123} + \mathbf{1213} - \mathbf{2311} + \mathbf{2131}) \\
 &\mapsto i^6(-\mathbf{221} - \mathbf{111} + \mathbf{212} - \mathbf{331} - \mathbf{221} \\
 &\quad - \mathbf{331} - \mathbf{221} + \mathbf{313} - \mathbf{111} - \mathbf{331}) \\
 &\mapsto i^7(\mathbf{23} + \mathbf{23} - \mathbf{32} + \mathbf{23} - \mathbf{32} + \mathbf{23} - \mathbf{32} - \mathbf{32}) \\
 &\mapsto i^8(\mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{1}) = 4 \cdot \mathbf{1}
 \end{aligned}$$

Consequently,  $K_{octa.} = 3! \cdot 4 = 24$  and there exist  $4 \cdot 24 = 96$  good colourings of the dual cube. We have made 64 operations instead of  $3^{12} = 531441$ . It would be interesting to evaluate the complexity of this method for generic triangulations. Using the same method, one can compute the chromatic index of the icosahedron and one finds  $K_{ico.} = 60$ , proving this way that there exist 240 good colourings of the faces of the dual dodecahedron.

## 6. A global section of $\varphi$

$\varphi$  induces over  $\mathcal{P}$  another fibered category,  $\Phi$ , defined as follows. To each path  $\alpha = (a_0, \dots, a_\ell)$ , we associate the category  $\Phi_\alpha$  whose objects are the sections of  $\varphi$  over  $\alpha$ . These are the families of  $V$ -modules,  $\zeta_{a_i} \in \text{Ob}(\varphi_{a_i})$ , connected by intertwiners :

$$\zeta = \left( \zeta_{a_{i-1}} \begin{array}{c} \xleftarrow{\zeta_{a_{i-1}a_i}} \\ \xrightarrow{\zeta_{a_i a_{i-1}}} \end{array} V\zeta_{a_i} \right)_{1 \leq i \leq \ell} \in \text{Ob}(\Phi_\alpha)$$

If  $\zeta, \omega \in \text{Ob}(\Phi_\alpha)$ , then  $\text{hom}_{\Phi_\alpha}(\zeta, \omega)$  is the vector space whose elements are the families  $(u_i : \zeta_{a_i} \rightarrow \omega_{a_i})_{0 \leq i \leq \ell}$  of linear maps such that the following diagrams commute :

$$\begin{array}{ccc}
 \zeta_{a_{i-1}} & \xrightarrow{\zeta_{a_i a_{i-1}}} & V\zeta_{a_i} \\
 \downarrow u_{i-1} & & \downarrow I \otimes u_i \\
 \omega_{a_{i-1}} & \xrightarrow{\omega_{a_i a_{i-1}}} & V\omega_{a_i}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \zeta_{a_{i-1}} & \xleftarrow{\zeta_{a_{i-1} a_i}} & V\zeta_{a_i} \\
 \downarrow u_{i-1} & & \downarrow I \otimes u_i \\
 \omega_{a_{i-1}} & \xleftarrow{\omega_{a_{i-1} a_i}} & V\omega_{a_i}
 \end{array}$$

that is to say :

$$\begin{aligned}
 \omega_{a_i a_{i-1}} \circ u_{i-1} &= (I \otimes u_i) \circ \zeta_{a_i a_{i-1}} \\
 u_{i-1} \circ \zeta_{a_{i-1} a_i} &= \omega_{a_{i-1} a_i} \circ (I \otimes u_i)
 \end{aligned}$$

**Definition :** Let  $\alpha = (a_0, \dots, a_\ell)$  be a path of length  $\ell$  and let  $\zeta \in \text{Ob}(\Phi_\alpha)$  be a section of  $\varphi$  over  $\alpha$ . The direct transport operator of  $\zeta$  along  $\alpha$  is the morphism

$$\begin{aligned}
 T_\alpha(\zeta) &= (I^{\ell-1} \otimes \zeta_{a_\ell a_{\ell-1}}) \circ (I^{\ell-2} \otimes \zeta_{a_{\ell-1} a_{\ell-2}}) \circ \dots \circ (I \otimes \zeta_{a_2 a_1}) \circ \zeta_{a_1 a_0} \\
 &: \zeta_{a_0} \longrightarrow V^\ell \zeta_{a_\ell}
 \end{aligned}$$

and the inverse transport operator of  $\zeta$  is the morphism

$$\begin{aligned}
 \bar{T}_\alpha(\zeta) &= \zeta_{a_0 a_1} \circ (I \otimes \zeta_{a_1 a_2}) \circ \dots \circ (I^{\ell-2} \otimes \zeta_{a_{\ell-2} a_{\ell-1}}) \circ (I^{\ell-1} \otimes \zeta_{a_{\ell-1} a_\ell}) \\
 &: V^\ell \zeta_{a_\ell} \longrightarrow \zeta_{a_0}
 \end{aligned}$$

Let us note that  $T_{\bar{\alpha}} \neq \bar{T}_\alpha$ .  $\Phi|_{\mathcal{P}_1}$  is generated by its restriction to the oriented 2-cells of  $T$ . If  $\sigma = ((xz), (xyz))$  and if  $\zeta \in \text{Ob}(\Phi_{(xyz)})$ , then we define  $\xi = \Phi_\sigma(\zeta) \in \text{Ob}(\Phi_{(xz)})$  by

$$\begin{aligned}
 \xi_x &= \zeta_x \\
 \xi_z &= \zeta_z \\
 \xi_{zx} &= (\tilde{F} \otimes I_{\zeta_z}) \circ (I \otimes \zeta_{zy}) \circ \zeta_{yx} \\
 \xi_{xz} &= \zeta_{xy} \circ (I \otimes \zeta_{yz}) \circ (F \otimes I_{\zeta_z})
 \end{aligned}$$

Similarly, if  $\xi \in \text{Ob}(\Phi_{(xz)})$ , we define  $\zeta = \Phi_{\bar{\sigma}}(\xi) \in \text{Ob}(\Phi_{(xyz)})$  by

$$\begin{aligned}
 \zeta_x &= \xi_x \\
 \zeta_y &= V\xi_z \\
 \zeta_z &= \xi_z \\
 \zeta_{yx} &= (F \otimes I_{\xi_z}) \circ \xi_{zx} \\
 \zeta_{xy} &= \xi_{xz} \circ (\tilde{F} \otimes I_{\xi_z}) \\
 \zeta_{yz} &= I \otimes I_{\xi_z} = \zeta_{zy}
 \end{aligned}$$

If  $u \in \text{hom}_{\Phi_{(xyz)}}(\zeta, \omega)$ , then we have the commutative diagrams

$$\begin{array}{ccccccc}
 \zeta_x & \xrightarrow{\zeta_{yx}} & V\zeta_y & \xrightarrow{I \otimes \zeta_{zy}} & V^2\zeta_z & \xrightarrow{\tilde{F} \otimes I_{\zeta_z}} & V\zeta_z \\
 \downarrow u_x & & \downarrow I \otimes u_y & & \downarrow I^2 \otimes u_z & & \downarrow I \otimes u_z \\
 \omega_x & \xrightarrow{\omega_{yx}} & V\omega_y & \xrightarrow{I \otimes \omega_{zy}} & V^2\omega_z & \xrightarrow{\tilde{F} \otimes I_{\omega_z}} & V\omega_z
 \end{array}$$



$$\begin{array}{ccccccc}
 \zeta_x & \xleftarrow{\zeta_{xy}} & V\zeta_y & \xleftarrow{I \otimes \zeta_{yz}} & V^2\zeta_z & \xleftarrow{F \otimes I_{\zeta_z}} & V\zeta_z \\
 \downarrow u_x & & \downarrow I \otimes u_y & & \downarrow I^2 \otimes u_z & & \downarrow I \otimes u_z \\
 \omega_x & \xleftarrow{\omega_{xy}} & V\omega_y & \xleftarrow{I \otimes \omega_{yz}} & V^2\omega_z & \xleftarrow{F \otimes I_{\omega_z}} & V\omega_z
 \end{array}$$

and we can define the action of  $\Phi_\sigma$  and of  $\Phi_{\bar{\sigma}}$  on the arrows by

$$\begin{aligned}
 \Phi_\sigma(u_x, u_y, u_z) &= (u_x, u_z) \\
 \Phi_{\bar{\sigma}}(v_x, v_z) &= (v_x, I \otimes v_z, v_z)
 \end{aligned}$$

These functors satisfy the relations :

$$\begin{aligned}
 \Phi_\sigma \Phi_{\bar{\sigma}}(\xi_x, \xi_{zx}, \xi_z) &= (\xi_x, 2\xi_{zx}, \xi_z) \\
 \Phi_{\bar{\sigma}} \Phi_\sigma(\zeta_x, \zeta_{yx}, \zeta_y, \zeta_{zy}, \zeta_z) &= (\zeta_x, \zeta_{yx} \circ (A \otimes \zeta_{zy}), V\zeta_z, I \otimes I_{\zeta_z}, \zeta_z)
 \end{aligned}$$

If  $(\alpha, \beta) \in \mathcal{P}_1$  is a generic 2-edge, then  $\Phi_{\alpha\beta}$  acts locally as above without modifying the other entries.

For  $p = 0, \dots, n$ , let  $\Gamma_p = (\gamma_p, \dots, \gamma_n)$  be the partial 2-path made of the last  $(n - p + 1)$  entries of  $\Gamma$  and let

$$\Phi_{\Gamma_p} = \Phi_{\gamma_p \gamma_{p+1}} \circ \dots \circ \Phi_{\gamma_{n-1} \gamma_n} \quad : \quad \Phi_{\gamma_n} \longrightarrow \Phi_{\gamma_p}$$

be the functor which maps the sections of  $\varphi$  over  $\gamma_n$  to sections over  $\gamma_p$ . For example, we can choose  $\gamma_n = (ab)$ . Let us apply  $\Phi_{\Gamma_p}$  to the section  $\zeta^n \in \text{Ob}(\Phi_{(ab)})$  defined by

$$\begin{aligned} \zeta_a^n &= \zeta_b^n = V \\ \zeta_{ba}^n &= F \\ \zeta_{ab}^n &= \tilde{F} \end{aligned}$$

$$\zeta^n = \left( V \begin{array}{c} \xleftarrow{\tilde{F}} \\ \xrightarrow{F} \end{array} V^2 \right)$$

**Theorem 3 :**  $\Phi_\Gamma$  multiplies the arrows of  $\zeta^n$  by  $K_T$  :

$$\Phi_\Gamma(\zeta^n) = \left( V \begin{array}{c} \xleftarrow{K_T \tilde{F}} \\ \xrightarrow{K_T F} \end{array} V^2 \right) \in \text{Ob}(\Phi_{(ab)})$$

*Proof:* The inverse transport operator of  $\zeta^p := \Phi_{\Gamma_p}(\zeta^n) \in \text{Ob}(\Phi_{\gamma_p})$  is

$$\bar{T}_{\gamma_p}(\zeta^p) = \bar{T}_{\gamma_p}(\Phi_{\gamma_p \gamma_{p+1}} \circ \dots \circ \Phi_{\gamma_{n-1} \gamma_n}(\zeta^n)) : V^{\ell_p+1} \longrightarrow V$$

By a decreasing induction on  $p$ , we have :

$$\bar{T}_{\gamma_p}(\zeta^p) = \zeta_{ab}^n \circ (\varphi_{\Gamma_p} \otimes I) = \tilde{F} \circ (\varphi_{\Gamma_p} \otimes I)$$

$$V^{\ell_p+1} \begin{array}{c} \xrightarrow{\varphi_{\Gamma_p} \otimes I} \\ \xrightarrow{\bar{T}_{\gamma_p}(\zeta^p)} \end{array} V^2 \begin{array}{c} \xrightarrow{\zeta_{ab}^n} \\ \xrightarrow{\tilde{F}} \end{array} V$$

For  $p = 0$  :

$$\zeta_{ab}^0 = \overline{T}_{\gamma_0}(\zeta^0) = \widetilde{F} \circ (\varphi_{\Gamma} \otimes I) = K_T \widetilde{F}$$

Similarly, by using the direct transport operator, we obtain

$$\begin{array}{ccccc} V & \xrightarrow{\zeta_{ba}^n} & V^2 & \xrightarrow{\varphi_{\widetilde{\Gamma}_p} \otimes I} & V^{\ell_p+1} \\ & \searrow & & \nearrow & \\ & & & & \\ & \xrightarrow{T_{\gamma_p}(\zeta^p)} & & & \end{array}$$

$$\zeta_{ba}^0 = T_{\gamma_0}(\zeta^0) = (\varphi_{\widetilde{\Gamma}} \otimes I) \circ F = K_T F$$

□

Once  $\Gamma = (\gamma_0, \dots, \gamma_n)$  has been chosen, the sweeping process constructs a  $V$ -module,  $\zeta_x = V^{n_x}$ , for each  $x \in T_0$ , and a morphism,  $\zeta_{xy}$ , for each oriented edge of  $T$ . Each integer  $n_x$  depends only on the partial 2-path  $\Gamma_p$  which reaches  $x$  first and not on the paths  $\gamma_q$  with  $q < p$ . Similarly for each arrow,  $\zeta_{xy}$ . Therefore, we obtain a global section,  $\zeta$ , of  $\varphi$  over  $T$ . More precisely, if we lift  $T$  to a triangulation  $\widetilde{T}$  of the disk  $D^2$  such that  $\widetilde{T}|_{\partial D^2}$  be a pair of arcs both projected onto the base edge  $(ab)$ , then  $\zeta$  is a global section of the pull-back of  $\varphi$  to  $\widetilde{T}$ .

If  $\zeta_{xy} = 0$  for some edge  $(xy)$ , then the transport operator along a path  $\gamma_p$  containing  $(xy)$  vanishes, as well as the subsequent transport operators and, at the end, we obtain  $K_T = 0$ . Conversely, if  $K_T = 0$ , then there exists an edge (at least the last one) where  $\zeta$  vanishes. Consequently, we have obtained a geometric interpretation of the four-colour theorem in terms of sections of  $\varphi$  :

**Theorem 4 :**  $4CT \iff (\zeta_{xy} \neq 0 \quad \forall (xy)).$

**Example :** Let us construct  $\zeta$  on the octahedron. Let us start from  $\zeta_a = \zeta_b = V$  and  $\zeta_{ab}^n = \tilde{F}$  :

$$\begin{aligned}
 \zeta_e &= V^2 \\
 \zeta_{ae} &= \tilde{F} \circ (\tilde{F} \otimes I) \\
 \zeta_{eb} &= I^2 \\
 \zeta_d &= V^3 \\
 \zeta_{ad} &= \tilde{F} \circ (\tilde{F} \otimes I) \circ (\tilde{F} \otimes I^2) \\
 \zeta_{de} &= I^3 \\
 \zeta_f &= V^2 \\
 \zeta_{ef} &= \tilde{F} \otimes I \\
 \zeta_{fb} &= I^2 \\
 \zeta_{df} &= (I \otimes \tilde{F} \otimes I) \circ (F \otimes I^2) \\
 \zeta_c &= V^3 \\
 \zeta_{dc} &= (I \otimes \tilde{F} \otimes I) \circ (F \otimes I^2) \circ (\tilde{F} \otimes I^3) \\
 \zeta_{cf} &= I^3 \\
 \zeta_{ac} &= \tilde{F} \circ (\tilde{F} \otimes I) \circ (\tilde{F} \otimes I^2) \circ (I^2 \otimes \tilde{F} \otimes I) \\
 &\quad \circ (I \otimes F \otimes I^2) \circ (I \otimes \tilde{F} \otimes I^3) \circ (F \otimes I^3) \\
 \zeta_{cb} &= F \otimes I \\
 \zeta_{ab}^0 &= \tilde{F} \circ (\tilde{F} \otimes I) \circ (\tilde{F} \otimes I^2) \circ (I^2 \otimes \tilde{F} \otimes I) \circ (I \otimes F \otimes I^2) \\
 &\quad \circ (I \otimes \tilde{F} \otimes I^3) \circ (F \otimes I^3) \circ (I \otimes F \otimes I) \circ (F \otimes I) \\
 &= \tilde{F} \circ (\varphi_\Gamma \otimes I) \\
 &= K_T \tilde{F}
 \end{aligned}$$

## 7. Conclusion and perspectives

The classical approaches to the four-colour problem study the local form of a planar map to prove its global colourability. This suggests the existence of a cohomological interpretation of this property. In the present work, we have constructed a global section of a fibered category modeled on

$\text{Rep}_f(\mathfrak{sl}_2)$  and proved that the validity of the four-colour theorem is equivalent to the fact that this section does not vanish. We hope that the present approach will be a first step toward an algebraic proof and the understanding of the four-colour theorem.

**Acknowledgements :** I wish to thank Olivier Babelon, Michel Bauer, Marc Bellon, Daniel Bennequin, Luc Frappat, Georges Girardi, Vincent Maillot, Stéphane Peigné, Raphaël Rouquier, Raymond Stora and Frank Thuillier for helpful discussions at various stages of this work. I also thank Marco Mackaay and Roger Picken who gave me the opportunity to present preliminary constructions during the workshop they organized in Lisbon (*Categories and Higher Order Geometry*, July 23<sup>rd</sup> & 24<sup>th</sup>, 2003). I wish to thank all the members of the L.A.P.T.H. (Laboratoire d'Annecy-le-Vieux de Physique Théorique) for their kind hospitality.

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