CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

E. GIULI

J. SLAPAL

# Raster convergence with respect to a closure operator

*Cahiers de topologie et géométrie différentielle catégoriques*, tome 46, nº 4 (2005), p. 275-300

<http://www.numdam.org/item?id=CTGDC_2005__46_4_275_0>

© Andrée C. Ehresmann et les auteurs, 2005, tous droits réservés.

L'accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme*
*Numérisation de documents anciens mathématiques*
http://www.numdam.org/

*CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES* *Volume XLVI-4 (2005)*

# RASTER CONVERGENCE WITH RESPECT TO A CLOSURE OPERATOR

*by E. GIULI and J. SLAPAL*

**Résumé.** Nous introduisons et étudions le concept de convergence sur une catégorie concrète $\mathcal{K}$ par rapport à un opérateur $c$ de clôture sur $\mathcal{K}$. Nous commençons en définissant et examinant les voisinages des sous-objets d'un $\mathcal{K}$-objet donné par rapport à $c$. En suite, les voisinages sont utilisés pour l'introduction de la convergence à l'aide de certains filtres généralisés. Quelques propriétés de base sont en suite discutées et la séparation et compacité sont étudiées en plus détail. Nous montrons que la séparation et compacité induites par la convergence se comportent d'une façon analogue à la séparation et compacité des espaces topologiques et ça d'une façon plus décente que la $c$-séparation et la $c$-compacité usuelle.

# 0 Introduction

In this paper, we continue the study of closure operators on categories in the sense of D. Dikranjan and E. Giuli [12]. At present, the theory of closure operators on categories is an important branch of categorical topology and a number of authors have contributed to its development. Categories with closure operators generalize the category **Top** and it was shown by some of these authors that many topological concepts can be naturally extended from topological spaces to objects of categories with closure operators. For example, separation and compactness are extended and studied in [9],[10],[13], connectedness in [3],[4] and [7], quotient maps in [11] and openness in [17]. But, up to now, no attempt has been made to define and study a convergence

The second author gratefully acknowledges financial supports from NATO-CNR (Outreach Fellowship No. 219.33), from Grant Agency of the Czech Republic (Project No. 201/03/0933) and from the Ministry of Education of the Czech Republic (Project No. MSM0021630518).

with respect to a closure operator on a category. And this is what we are concerned with in this paper.

Categories equipped with certain types of convergence are dealt with in [19–23]. In particular, in [22], a general concept of convergence on a category is introduced and discussed and it is shown that the convergence induces, under some natural conditions, a closure operator on the category. The aim of the present paper is just converse - we will show that a closure operator on a category gives rise to a convergence on the category. This convergence will then be investigated. But, instead of using the approach of [19–23], which is based on employing (generalized) nets for expressing the convergence, we rather use the more common approach based on employing (generalized) filters.

Given a concrete category $\mathcal{K}$ with a closure operator $c$, we start with introducing the notion of a neighborhood of a subobject of a $\mathcal{K}$-object. These neighborhoods, which generalize the usual neighborhoods in **Top**, are not closed under finite meets in general. So, when we use them for defining a convergence with respect to $c$ on (objects of) $\mathcal{K}$, we have to employ certain structures that are more general than filters. The structures employed are called rasters and the defined convergence is referred to as the raster convergence (with respect to $c$). The raster convergence is investigated and it is shown that it behaves analogously to the filter convergence in topological spaces. Focus is then put on the study of raster separation and raster compactness with respect to $c$, i.e., separation and compactness induced by the raster convergence with respect to $c$. Results analogous to some classical ones on separation and compactness in topological spaces are proved including the "minimality", "absolute closedness" and "Tychonoff's theorem". It follows that raster separation and raster compactness with respect to $c$ behave better that the usual $c$-separation and $c$-compactness. Relations between the two kinds of separation (respectively, compactness) are also discussed in the paper.

# 1 Preliminaries

For the basic categorical terminology used see [1] and [14].

Throughout the paper, $\mathcal{X}$ denotes a finitely complete category with a proper $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms.

The condition that $(\mathcal{E}, \mathcal{M})$ is proper means that $\mathcal{E}$ is a class of $\mathcal{X}$-epimorphisms and $\mathcal{M}$ is a class of $\mathcal{X}$-monomorphisms (which then contains all extremal $\mathcal{X}$-monomorphisms). It is imposed only for the reason of making formulations of the presented results as brief as possible. For instance, the condition implies that $\mathcal{M}$ contains all points of any $\mathcal{X}$-object $X$, i.e., $\mathcal{X}$-morphisms $1_X \to X$ (by $1_X$ we denote an arbitrary but fixed terminal object of $\mathcal{X}$). But it also results in each of the following two conditions: (1) for every $\mathcal{X}$-object $X$ the diagonal morphism $\delta_X =< \mathrm{id}_X, \mathrm{id}_X >: X \to X \times X$ (where $\mathrm{id}_X : X \to X$ denotes the identity morphism) belongs to $\mathcal{M}$, (2) $g \circ f \in \mathcal{M} \Rightarrow f \in \mathcal{M}$. It will be clear from the context which results remain valid even when $(\mathcal{E}, \mathcal{M})$ is not proper, i.e., when it is supposed only that $\mathcal{M}$ is a class of $\mathcal{X}$-monomorphisms.

Note that $\mathcal{M}$ is closed under products and pullbacks (along arbitrary $\mathcal{X}$-morphisms). We assume that $\mathcal{X}$ has multiple pullbacks of arbitrary large families of $\mathcal{M}$-morphisms with a common codomain. In this case, $\mathcal{M}$ is closed under multiple pullbacks. Given an $\mathcal{X}$-object $X$, each $\mathcal{M}$-morphism with codomain $X$ is called a *subobject* of $X$. We denote by sub$X$ the (possibly large) complete lattice of all isomorphism classes of subobjects of $X$. As usual, we identify isomorphism classes of subobjects of $X$ with their representatives. So, each subobject of $X$ is considered to be an element of sub$X$, and we write $m = n$ instead of $m \simeq n$ for subobjects $m, n$ of $X$. In the same way, by saying that $m$ and $n$ are different, in symbols $m \neq n$, we mean that $m$ and $n$ are not isomorphic. The joins and meets in sub$X$ will be denoted by the usual symbols $\vee$, $\bigvee$ and $\wedge$, $\bigwedge$, respectively. The least element of sub$X$ is denoted by $o_X$. If $\mathrm{id}_X = o_X$, then the $\mathcal{X}$-object $X$ is called *trivial.* Of course, if $\mathcal{E}$ is stable under pullbacks, then an $\mathcal{X}$-object X is non-trivial if and only if there exists an $\mathcal{E}$-morphism $f : X \to Y$ with $Y$ non-trivial (because then $m_1 \neq m_2$ implies $f^{-1}(m_1) \neq f^{-1}(m_2)$ whenever $m_1, m_2 \in$ sub$Y$).

We suppose that, for each $\mathcal{X}$-object $X$, the lattice sub$X$ is *pseudocomplemented.* This means that for each $m \in$ sub$X$ there exists $\overline{m} \in$ sub$X$ such that the equivalence $m \wedge n = o_X \Leftrightarrow n \leq \overline{m}$ is valid whenever $n \in$ sub$X$. The (unique) morphism $\overline{m}$ is called the *pseudocomplement* of $m$. Clearly, given $m, p \in$ sub$X$, we have $m \leq p \Rightarrow \overline{p} \leq \overline{m}$ and $m \leq \overline{\overline{m}}$. It also follows that sub$X$ is a (possibly large) Boolean algebra (with complements given by pseudocomplements) if and only if $\overline{\overline{m}} = m$ for all $m \in$ sub$X$ (cf. [18]) .

Note that, since sub$X$ is pseudocomplemented, any atom $p \in$ sub$X$ has the property $p \leq m$ or $p \leq \overline{m}$ whenever $m \in \mathrm{sub}X$ (because $p \nleq m \Rightarrow p \wedge m = o_X \Rightarrow p \leq \overline{m}$).

In the category $\mathcal{X}$, each morphism $f : X \to Y$ is assumed to fulfill the following two axioms:

1° $f(m) = o_Y \Rightarrow m = o_X$ whenever $m \in \mathrm{sub}X$,

2° $\overline{f^{-1}(n)} \leq f^{-1}(\bar{n})$ whenever $n \in \mathrm{sub}Y$.

Of course, 1° is equivalent to $f^{-1}(o_Y) = o_X$.

We will need the following observations:

**Lemma 1.1** *Let $f : X \to Y$ be an $\mathcal{X}$-morphism, $m \in subX$ and $p \in subY$. If $o_Y < p \leq f(m)$, then $o_X < m \wedge f^{-1}(p)$.*

*Proof.* Let $o_Y < p \leq f(m)$ and suppose that $m \wedge f^{-1}(p) = o_X$. Then $m \leq \overline{f^{-1}(p)}$, hence $f(m) \leq f(\overline{f^{-1}(p)}) \leq f(f^{-1}(\overline{p})) \leq \overline{p}$. Consequently, $p \leq \overline{\overline{p}} \leq \overline{f(m)}$ and thus $p \wedge f(m) = o_L$. But this is a contradiction with $o_L < p \leq f(m)$. □

**Lemma 1.2** *Let $f : X \to Y$ be an $\mathcal{X}$-morphism and $m \in subX$ be an atom. Then $f(m) \in subY$ is an atom too.*

*Proof.* Let $p \in \mathrm{sub}Y$, $0 < p \leq f(m)$. By Lemma 1.1, $o_X < m \wedge f^{-1}(p)$. Thus $m \wedge f^{-1}(p) = m$ which yields $m \leq f^{-1}(p)$. Hence $f(m) \leq f(f^{-1}(p)) \leq p$. Consequently, $f(m) = p$. Therefore $f(m) \in \mathrm{sub}Y$ is an atom. □

**Lemma 1.3** *Let $X = \prod_{i \in I} X_i$ be a product in $\mathcal{X}$ and $p_i \in subX_i$ be an atom for each $i \in I$. If all $p_i$, $i \in I$, have the same domain (up to isomorphisms), then $\langle p_i;\ i \in I\rangle \in subX$ is an atom too.*

*Proof.* Let $Z$ denote the domain of all $p_i$, $i \in I$. As $Z$ is non-trivial, there holds $\langle p_i;\ i \in I\rangle > o_X$. Let $m \in \mathrm{sub}X$, $o_X < m \leq \langle p_i;\ i \in I\rangle$. Then there exists $r \in \mathrm{sub}Z$, $r > o_Z$, with $m = \langle p_i;\ i \in I\rangle \circ r$. Hence $p_i \circ r = p_i(r) > o_X$ for each $i \in I$. Thus, since $p_i \circ r \leq p_i$, we get $p_i \circ r = p_i$ for each $i \in I$. Consequently, $m = \langle p_i;\ i \in I\rangle \circ r = \langle p_i \circ r;\ i \in I\rangle = \langle p_i;\ i \in I\rangle$. Therefore, $\langle p_i;\ i \in I\rangle \in \mathrm{sub}X$ is an atom. □

**Remark 1.4** Let $X$ be an $\mathcal{X}$-object. If $\mathrm{card}(\mathrm{sub}1_{\mathcal{X}}) = 2$, then any point $x$ of $X$ is an atom of sub$X$. Conversely, if every non-terminal $\mathcal{X}$-object $Z$ has the property $\mathrm{card}(\mathrm{sub}Z) \neq 2$, then any atom of sub$X$ is a point of $X$. Thus, for any $\mathcal{X}$-object $X$, the points of $X$ coincide with the atoms of sub$X$ provided that terminal objects of $\mathcal{X}$ are just the $\mathcal{X}$-objects $T$ with $\mathrm{card}(\mathrm{sub}T) = 2$.

Further, we suppose there is given a concrete category $\mathcal{K}$ over $\mathcal{X}$ with the corresponding underlying functor $|\ \ | : \mathcal{K} \to \mathcal{X}$. As usual, we do not distinguish notationally between $\mathcal{K}$-morphisms and their underlying $\mathcal{X}$-morphisms (i.e., we write $f$ instead of $|f|$ whenever $f$ is a $\mathcal{K}$-morphism). Given a $\mathcal{K}$-object $K$, by a *subobject* of $K$ we will always mean a subobject of $|K|$ and correspondingly, we will write briefly sub$K$ and $o_K$ instead of sub$|K|$ and $o_{|K|}$, respectively. This will cause no confusion because only the category $\mathcal{X}$, and not $\mathcal{K}$, is assumed to have a subobject structure. The category $\mathcal{K}$ is also supposed to have finite concrete products, and by a (not necessarily finite) product in $\mathcal{K}$ we always mean a concrete one. The class of all $\mathcal{M}$-embeddings in $\mathcal{K}$, which are called briefly *embeddings*, is denoted by $\mathrm{Emb}_{\mathcal{M}}$.

Recall that a *closure operator* on $\mathcal{K}$ (with respect to $(\mathcal{E}, \mathcal{M})$) is a family of maps $c = (c_K : \mathrm{sub}K \to \mathrm{sub}K)_{K \in \mathcal{K}}$ with the following properties that hold for each $\mathcal{K}$-object $K$ and each $m, p \in \mathrm{sub}K$:

(1) $m \leq c_K(m)$,

(2) $m \leq p \Rightarrow c_K(m) \leq c_K(p)$,

(3) $f(c_K(m)) \leq c_L(f(m))$ for each $\mathcal{K}$-morphism $f : K \to L$.

Note that, for any $\mathcal{X}$-morphism $f : K \to L$ and any $m \in \mathrm{sub}K$, $f(m)$ is the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ m$. In fact, the above defined closure operator $c$ on $\mathcal{K}$ is a so-called *$U$-closure operator* [2] where $U = |\ \ |$. A classical closure operator introduced in [12] is obtained when $\mathcal{K} = \mathcal{X}$ and $|\ \ |$ is the identity functor.

The closure operator $c$ is called

(a) *grounded* if $c_K(o_K) = o_K$ for each $K \in \mathcal{K}$,

(b) *idempotent* if $c_K(c_K(m)) = c_K(m)$ for each $K \in \mathcal{K}$ and each $m \in \mathrm{sub}K$,

(c) *additive* if $c_K(m \vee p) = c_K(m) \vee c_K(p)$ for each $K \in \mathcal{K}$ and each $m, p \in \text{sub}K$,

(d) *hereditary* if, whenever $m : M \to K$ is an embedding in $\mathcal{K}$, $c_M(p) = m^{-1}(c_K(m \circ p))$ for each $p \in \text{sub}M$.

Given a $\mathcal{K}$-object $K$, a subobject $m \in \text{sub}K$ is said to be *c-closed* (respectively, *c-dense*) provided that $c_K(m) = m$ (respectively, $c_K(m) = \text{id}_K$). A $\mathcal{K}$-morphism $f : K \to L$ is called *c-preserving* if $f(c_K(m)) = c_L(f(m))$ whenever $m \in \text{sub}K$. Thus, if $f$ is $c$-preserving, then it maps $c$-closed subobjects to $c$-closed subobjects, and vice versa provided that $c$ is idempotent.

Throughout the paper, we assume there is given a grounded closure operator $c = (c_K)_{K \in \mathcal{K}}$ on $\mathcal{K}$. Given a pair $K$, $L$ of $\mathcal{K}$-objects with $|K| = |L|$, we put $c_K \leq c_L$ provided that $c_K(m) \leq c_L(m)$ for each $m \in \text{sub}K (= \text{sub}L)$.

**Example 1.5** (1) Basic examples of the above introduced category $\mathcal{K}$ with a closure operator are certain topological constructs with $\mathcal{X} = \mathbf{Set}$ where $|\ | : \mathcal{K} \to \mathbf{Set}$ is the forgetful functor and the (surjections, injections)-factorization structure for morphisms is considered in the base category **Set**. A number of such examples are given in [2], [10], [13], [14]. Among them, of course, the most natural one is $\mathcal{K} = \mathbf{Top}$, i.e., the construct of topological spaces and continuous maps, with the Kuratowski closure operator. Further examples can be found among concrete categories over topological constructs (with a singleton fibre of the empty set) which always have the (surjections,embeddings)-factorization structure for morphisms. For instance, let $\mathcal{X} = \mathbf{Top}$, let $\mathcal{K}$ be the category **TopGrp** of topological groups (and continuous homomorphisms), and let $|\ \ | : \mathbf{TopGrp} \to \mathbf{Top}$ be the forgetful functor (that forgets the group structure). A closure operator on **TopGrp** is given by the (classical) Kuratowski closure operator on **Top**. On the other hand, **TopGrp** equipped with the classical Kuratowski closure operator (i.e., the closure operator where $|\ \ | : \mathbf{TopGrp} \to \mathbf{TopGrp}$ is the identity functor) is not an example of $\mathcal{K}$ because **TopGrp** does not fulfill the axiom 1°.
(2) Recall that a projection space is a pair $(X, (\alpha_n)_{n \in \mathbb{N}})$ where $X$ is a set and $(\alpha_n)_{n \in \mathbb{N}} = (\alpha_n : X \to X)_{n \in \mathbb{N}}$ is a sequence of maps such that $\alpha_n \circ \alpha_m = \alpha_{\min(m,n)}$. Given projection spaces $(X, (\alpha_n)_{n \in \mathbb{N}})$ and $(Y, (\beta_n)_{n \in \mathbb{N}})$, a map $g : X \to Y$ is called a projection function of $(X, (\alpha_n)_{n \in \mathbb{N}})$ into $(Y, (\beta_n)_{n \in \mathbb{N}})$ provided that $\beta_n \circ g = g \circ \alpha_n$ for all $n \in \mathbb{N}$. Projection spaces $(X, (\alpha_n)_{n \in \mathbb{N}})$

with $\alpha_n = f$ for all $n \in \mathbb{N}$, where $f : X \to X$ is a map, coincide with idempotent mono-unary algebras. Let $\mathcal{K}$ be the category of projection spaces and projection functions.
(a) Let $\mathcal{X} = \mathcal{K}$, let $|\,| : \mathcal{K} \to \mathcal{K}$ be the identity functor, and consider the (surjections,injections)-factorization structure for morphisms in $\mathcal{K}$. With respect to this factorization structure, there is a closure operator $c = (c_K)_{K \in \mathcal{K}}$ on $\mathcal{K}$ given by $c_K(m) = \{x \in X; \forall n \in \mathbb{N} : \alpha_n(x) \in m(M)\}$ whenever $K = (X, (\alpha_n)_{n\in\mathbb{N}})$ is a projection space and $m : M \to K$ is a subobject of $K$. This closure operator coincides with the closure operator $c_\infty$ from [16]. So, by [16], $c$ is idempotent, additive and hereditary. It can easily be seen that, given a $\mathcal{K}$-object $K$, sub$K$ need not be a Boolean algebra (e.g., let $K = (X, f)$ be the idempotent mono-unary algebra with $X = \{0, 1\}$, $f(0) = 1$ and $f(1) = 1$).
(b) Let $\mathcal{X} = \mathbf{Set}$, let $|\,| : \mathcal{K} \to \mathcal{X}$ be the forgetful functor and consider the (surjections,injections)-factorization structure for morphisms in **Set**. With respect to this factorization structure, there is a closure operator $c = (c_K)_{K \in \mathcal{K}}$ on $\mathcal{K}$ given by $c_K(m) = m(M) \cup \{x \in X; \forall n \in \mathbb{N} : \alpha_n(x) \in m(M)\}$ whenever $K = (X, (\alpha_n)_{n\in\mathbb{N}})$ is a projection space and $m : M \to X$ is a subobject of $K$. Moreover, $c$ is clearly idempotent and hereditary. It is a so-called non-standard closure operator - see [2]. Of course, for those subobjects $m$ of $K$ which coincide with (underlying sets of) subobjects of $K$ in the sense of (a), $c_K(m)$ coincides with (the underlying set of) $c_K(m)$ from (a).
(c) Let the situation be the same as in (b). Then, with respect to the factorization structure considered, there is another closure operator $c = (c_K)_{K\in\mathcal{K}}$ on $\mathcal{K}$ defined as follows: $c_K(m) = m(M) \cup \{\alpha_n(x);\ x \in m(M) \text{ and } n \in \mathbb{N}\}$ whenever $K = (X, (\alpha_n)_{n\in\mathbb{N}})$ is a projection space and $m : M \to X$ is a subobject of $K$. Clearly, this closure operator is not only idempotent and hereditary, but also additive. Therefore, it is more appropriate than the closure operator $c$ from (b). It is also obvious that $c_K$-closed subobjects of a $\mathcal{K}$-object $K$ (i.e., subsets of $K$) coincide with the subobjects of $K$ from (a). In other words, $c$ is a so-called hull operator - see [2].

In what follows, whenever a construct is taken as an example of $\mathcal{K}$ without mentioning the functor $|\,| : \mathcal{K} \to \mathcal{X}$ , we always suppose that $\mathcal{X} = \mathbf{Set}$, $|\,| : \mathcal{K} \to \mathbf{Set}$ is the forgetful functor, and the $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms in the base category **Set** is just the (surjections,injections)-factorization structure. In the case $\mathcal{K} = \mathbf{Top}$ we moreover suppose that $c$ is the Kuratowski closure operator.

**Remark 1.6** Example 1.5(2) demonstrates one of the advantages of $U$-closure operators over the classical closure operators - the former often act on richer domains than the latter (cf. also Example 2.3(2)). Another advantage is that, when studying $U$-closure operators where $U$ is an underlying functor, we can use concepts like an embedding, a fibre, etc. to describe behavior of these operators (cf. 4.10–4.12).

# 2 Neighborhoods and rasters

**Definition 2.1** Let $K$ be a $\mathcal{K}$-object and $m \in \underline{\text{sub}K}$. A *$c_K$-neighborhood* of $m$ is any subobject $n \in \text{sub}K$ such that $m \leq \overline{c_K(\overline{n})}$.

We denote by $\mathcal{N}_{c_K}(m)$ the class of all $c_K$-neighborhoods of $m$. If it is clear which $\mathcal{K}$-object $K$ is considered, we will call $c_K$-neighborhoods briefly *neighborhoods* and instead of $\mathcal{N}_{c_K}(m)$, will write briefly $\mathcal{N}(m)$.

**Proposition 2.2** *Let $K$ be a $\mathcal{K}$-object and $m, p \in subK$. Then*

*(1)* $id_K \in \mathcal{N}(m)$,

*(2)* $\mathcal{N}(o_K) = subK$,

*(3)* $n \in \mathcal{N}(m) \Leftrightarrow m \wedge c_K(\overline{n}) = o_K$

*(4)* $n \in \mathcal{N}(m)$ *implies* $m \leq n$ *provided that* $m$ *is an atom of* $subK$ *or* $\overline{\overline{n}} = n$,

*(5)* $n \in \mathcal{N}(m)$ *and* $p \geq n$ *imply* $p \in \mathcal{N}(m)$,

*(6)* $p \leq m \Rightarrow \mathcal{N}(m) \subseteq \mathcal{N}(p)$,

*(7) if* $m > o_K$ *and* $n_1, n_2, ..., n_k \in \mathcal{N}(m)$ $(k \in \mathbb{N})$, *then* $m \wedge n_1 \wedge n_2 \wedge ... \wedge n_k > o_K$,

*(8) if* $m > o_K$ *and* $n_1, n_2, ..., n_k \in \mathcal{N}(m)$ $(k \in \mathbb{N})$, *then* $n_1 \wedge n_2 \wedge ... \wedge n_k > o_K$,

*(9) if* $n_1, n_2 \in \mathcal{N}(m)$, *then* $n_1 \wedge n_2 \in \mathcal{N}(m)$ *provided that* $c$ *is additive and* $subK$ *is a Boolean algebra.*

*Proof.* Let $n \in \mathcal{N}(m)$ and $m \wedge n = o_K$. Then $m \leq \overline{n} \leq c_K(\overline{n})$. Thus, since $m \leq \overline{c_K(\overline{n})}$, we have $m \leq c_K(\overline{n}) \wedge \overline{c_k(\overline{n})} = o_K$. Hence (7) is valid for $k = 1$. Suppose that it is valid for some $k \in \mathbb{N}$ and let $n_1, n_2, ..., n_k, n_{k+1} \in \mathcal{N}(m)$. Then $m \wedge n_1 \wedge n_2 \wedge ... \wedge n_k > o_K$ and $n_{k+1} \in \mathcal{N}(m \wedge n_1 \wedge n_2 \wedge ... \wedge n_k)$, thus $m \wedge n_1 \wedge n_2 \wedge ... \wedge n_k \wedge n_{k+1} > o_k$ (because (7) is valid for $k = 1$). This proves (7), and (8) is an immediate consequence of (7). The other conditions are obvious. □

**Example 2.3** (1) Of course, if $\mathcal{K} =$ **Top**, the above defined neighborhoods coincide with the usual neighborhoods (of sets).
(2) In Example 1.5(2), let $K \in \mathcal{K}$ be the projection space $K = (\mathbb{N}, (\alpha_n)_{n \in \mathbb{N}})$ where, for each $n, p \in \mathbb{N}$, $\alpha_n(p) = \min(n, p)$.
If $c$ is the closure operator on $\mathcal{K}$ given in the part (a) of the Example, then $n > o_K \Rightarrow n \in \mathcal{N}(m)$ whenever $m, n$ are subobjects of $K$ (because the subobjects of $K$ are $c$-closed and coincide with the subsets of $\mathbb{N}$ having the form $\{x \in \mathbb{N};\ x < n\}$ where $n \in \mathbb{N} \cup \infty$, so that $n > o_K \Rightarrow \overline{n} = o_k$ for each subobject $n$ of $K$).
On the other hand, if $c$ is the closure operator on $\mathcal{K}$ given in the part (b), then $c_K$ coincides with the discrete topology on $\mathbb{N}$. Therefore, we have $n \in \mathcal{N}(m) \Leftrightarrow m \leq n$ whenever $m, n$ are subobjects of $K$.
Finally, let $c$ be the closure operator on $\mathcal{K}$ from the part (c) of Example 1.5(2). Let $m : M \to \mathbb{N}$ be an arbitrary subobject of $K$ with $m > o_K$. Then one can easily see that $c_K(m) = \mathbb{N}$ if $m(M)$ is infinite, and $c_K(m) = \{1, 2, ..., \max m(M)\}$ if $m(M)$ is finite. Consequently, we have $\mathcal{N}(m) = \{N \subseteq \mathbb{N};\ x \in N \text{ for each } x \in \mathbb{N} \text{ with } x \geq \min m(M)\}$. Thus, if $x \in \mathbb{N}$ is a point, then $\{y \in \mathbb{N};\ y \geq x\}$ is the smallest neighborhood of $x$. It follows that $(\mathbb{N}, c_K)$ is nothing but the so-called right topology on (the linearly ordered set) $\mathbb{N}$.

**Proposition 2.4** *Let $f : K \to L$ be a $\mathcal{K}$-morphism, $m \in subK$ and $n \in \mathcal{N}(f(m))$. Then $f^{-1}(n) \in \mathcal{N}(m)$.*

*Proof.* Assume $f^{-1}(n) \notin \mathcal{N}(m)$. Then $m \not\leq \overline{c_K(\overline{f^{-1}(n)})}$ and therefore, $m \wedge c_K(\overline{f^{-1}(n)}) > o_K$. Consequently, $f(m) \wedge f(c_K(\overline{f^{-1}(n)})) > o_L$. But $f(c_K(\overline{f^{-1}(n)})) \leq c_L(f(\overline{f^{-1}(n)})) \leq c_L(f(f^{-1}(\overline{n}))) \leq c_L(\overline{n})$. Hence $f(m) \wedge c_L(\overline{n}) > o_L$, which yields $f(m) \not\leq \overline{c_L(\overline{n})}$. But this is a contradiction with $n \in \mathcal{N}(f(m))$. □

**Proposition 2.5** *Let $K$ be a $\mathcal{K}$-object and $m, p \in subK$, $m > o_K$. If $m \leq c_K(p)$, then $n \wedge p > o_K$ for each $n \in \mathcal{N}(m)$, and vice versa provided that $subK$ is a Boolean algebra and $m$ is an atom of $subK$.*

*Proof.* Let $m \leq c_K(p)$ and admit that there exists $n \in \mathcal{N}(m)$ with $n \wedge p = o_K$. Then $p \leq \overline{n}$, hence $c_K(p) \leq c_K(\overline{n})$. Thus, since $m \wedge c_K(\overline{n}) = o_K$, we have $m \wedge c_K(p) = o_K$. But this is a contradiction with $o_K < m \leq c_K(p)$.
Vice versa, let $n \wedge p > o_K$ for each $n \in \mathcal{N}(m)$ and let $m$ be an atom of sub$K$. Admit that $m \not\leq c_K(p) = \overline{\overline{c_K(p)}}$. Then $m \wedge \overline{c_K(p)} > o_K$, which yields $m \wedge \overline{c_K(p)} = m$. Hence $m \leq \overline{c_K(p)} = \overline{c_K(\overline{\overline{p}})}$, thus $\overline{p} \in \mathcal{N}(m)$. But $\overline{p} \wedge p = o_K$, which is a contradiction. Therefore $m \leq c_K(p)$. □

**Corollary 2.6** *Let $K$ be a $\mathcal{K}$-object such that $subK$ is a Boolean algebra and let $p \in subK$. If $c_K(p)$ equals a join of atoms of $subK$, then $c_K(p) = \bigvee\{m \in subK;\ m$ is an atom and $n \wedge p > o_K$ for each $n \in \mathcal{N}(m)\}$.*

**Corollary 2.7** *Let $K, L$ be $\mathcal{K}$-objects with $|K| = |L|$ and let $subK (= subL)$ be an atomic Boolean algebra. If $\mathcal{N}_{c_L}(m) \subseteq \mathcal{N}_{c_K}(m)$ for each $m \in subK$, then $c_K \leq c_L$.*

**Remark 2.8** a) In Proposition 2.5 and Corollary 2.6, the condition that sub$K$ is a Boolean algebra can be replaced by the weaker condition that $\overline{\overline{p}} = p$ and $\overline{\overline{c_K(p)}} = c_K(p)$.
b) Having defined $c_K$-neighborhoods ($K \in \mathcal{K}$ an object), we can define $c$-open subobjects of $K$ as those $m \in$ sub$K$ that fulfill $m \in \mathcal{N}_{c_K}(m)$ or equivalently, $m \in \mathcal{N}_{c_K}(p)$ whenever $p \in$ sub$K$ and $p \leq m$. Then, obviously, an arbitrary element $m \in$ sub$K$ is $c$-open if and only if $\overline{m}$ is $c$-closed. It is also evident that the inverse image of a $c$-open subobject under a $\mathcal{K}$-morphism is $c$-open (in consequence of the conditions 1° and 2° in the previous section). But $c$-open subobjects have already been introduced in [17] as those $m \in$ sub$K$ that satisfy $m \wedge c_K(p) \leq c_K(m \wedge p)$ whenever $p \in$ sub$K$. This concept of $c$-openness is stronger than the above defined one (as $c$ is grounded) and both concepts coincide provided that $c$ is additive and sub$K$ is a Boolean algebra. In this note, we do not need any concept of $c$-openness and so we will avoid using it.

**Definition 2.9** Let $\mathcal{G}$ be a (possibly large) complete lattice with the least element 0. A subclass $\mathcal{R} \subseteq \mathcal{G}$ is called a *raster* on $\mathcal{G}$ provided that

(1) $\mathcal{R}$ is an *upper class* of $\mathcal{G}$ (i.e., $x \in \mathcal{R}$ implies $y \in \mathcal{R}$ for every $y \in \mathcal{G}$, $y \geq x$), and

(2) $\mathcal{R}$ is *centered* (i.e., $\mathcal{R} \neq \emptyset$ and every finite meet of elements of $\mathcal{R}$ is different from 0).

Given a (possibly large) complete lattice $\mathcal{G}$, we will use the usual concepts of *filter, ultrafilter, filter base, ultrafilter base* and *filter subbase* naturally extended to subclasses of $\mathcal{G}$. Thus, every filter on $\mathcal{G}$ is a raster on $\mathcal{G}$. Conversely, every raster on $\mathcal{G}$ which is a filter base on $\mathcal{G}$ is a filter on $\mathcal{G}$. Of course, filter subbases on $\mathcal{G}$ coincide with centered subclasses of $\mathcal{G}$. If $\mathcal{R}$, $\mathcal{S}$ are rasters on $\mathcal{G}$, then $\mathcal{S}$ is said to be *finer* than $\mathcal{R}$, and $\mathcal{R}$ is said to be *coarser* than $\mathcal{S}$, provided that $\mathcal{R} \subseteq \mathcal{S}$.

For any subclass $\mathcal{B} \subseteq \mathcal{G}$ we put $[\mathcal{B}] = \{y \in \mathcal{G};\ \exists x \in \mathcal{B} :\ x \leq y\}$ and $\widehat{\mathcal{B}} = \{x;\ x$ is a finite meet of elements of $\mathcal{B}\}$. Thus, if $\mathcal{B}$ is centered, then $[\mathcal{B}]$ is a raster on $\mathcal{G}$. If $\mathcal{B}$ is a raster on $\mathcal{G}$, then $\widehat{\mathcal{B}}$ is a filter base on $\mathcal{G}$, so that $[\widehat{\mathcal{B}}]$ is a filter on $\mathcal{G}$. Therefore, for each raster $\mathcal{R}$ on $\mathcal{G}$ there is a filter on $\mathcal{G}$ which is finer than $\mathcal{R}$. It follows that maximal (with respect to "coarser") rasters on $\mathcal{G}$ coincide with ultrafilters on $\mathcal{G}$. As the Axiom of Choice for conglomerates is supposed, each raster on $\mathcal{G}$ is coarser than an ultrafilter on $\mathcal{G}$.

For each $\mathcal{X}$-object $X$ we denote by $\mathbf{R}_X$ the conglomerate of all rasters on sub$X$. Thus, $\mathbf{R}_X = \emptyset$ if and only if $X$ is a trivial object (because otherwise $\{\mathrm{id}_X\} \in \mathbf{R}_X$). Given a $\mathcal{K}$-object $K$, we write briefly $\mathbf{R}_K$ instead of $\mathbf{R}_{|K|}$.

Let $X,\ Y$ be $\mathcal{X}$-objects and $\mathcal{B} \subseteq$ sub$X$ a subclass. As usual, if $f : X \to Y$ is an $\mathcal{X}$-morphism, we put $f(\mathcal{B}) = \{f(r);\ r \in \mathcal{B}\}$. We then clearly have $[f[\mathcal{B}]] = [f(\mathcal{B})]$. If $\mathcal{B}$ is a centered subclass or a filter base or an ultrafilter base respectively, then so is $f(\mathcal{B})$ (for ultrafilter bases this follows from the fact that a centered class $\mathcal{R} \subseteq$ sub$X$ is an ultrafilter if and only if $m \in \mathcal{R}$ or $\overline{m} \in \mathcal{R}$ for each $m \in$ sub$X$). If $\mathcal{E}$ is stable under pullbacks and $f \in \mathcal{E}$, then $f(\mathcal{B}) \in \mathbf{R}_Y$ whenever $\mathcal{B} \in \mathbf{R}_X$.

Let $X = \prod_{i \in I} X_i$ be a product in $\mathcal{X}$ and let $\mathcal{B}_i \subseteq$ sub$X_i$ for each $i \in I$. Then we put $\prod_{i \in I} \mathcal{B}_i = \{\prod_{i \in I} m_i;\ m_i \in \mathcal{B}_i$ for each $i \in I\}$. If in $\mathcal{X}$ the non-trivial objects are stable under products and if $\mathcal{B}_i$ is centered for each $i \in I$, then $\prod_{i \in I} \mathcal{B}_i \in$ sub$X$ is centered too (because $\bigwedge_{j \in J} \prod_{i \in I} m_i^j \geq \prod_{i \in I} \bigwedge_{j \in J} m_i^j$ whenever $m_i^j \in \mathcal{B}_i$ for each $j \in J$ and each $i \in I$, and the domain of $\bigwedge_{j \in J} m_i^j$ is non-trivial for each $i \in I$), hence $[\prod_{i \in I} \mathcal{B}_i] \in \mathbf{R}_X$.

The following, quite obvious fact will be used later on.

**Remark 2.10** If each non-trivial $\mathcal{X}$-object has a point and $\mathrm{card}(\mathrm{sub}1_{\mathcal{X}}) \geq 2$, then in $\mathcal{X}$ the non-trivial objects are stable under products.

Let $K$ be a $\mathcal{K}$-object and $m \in \mathrm{sub}K$, $m > o_K$. By Proposition 2.2, $\mathcal{N}(m)$ is a raster on sub$K$ (and $\mathcal{N}(m)$ is even a filter provided that $c$ is additive and sub$K$ is a Boolean algebra). But $\mathcal{N}(m)$ need not be a filter on sub$K$ in general. For this reason the notion of a raster is introduced and convergence is defined further on in terms of superclasses of $\mathcal{N}(m)$.

# 3 Raster convergence

**Definition 3.1** Let $K$ be a $\mathcal{K}$-object, $m \in \mathrm{sub}K$ and $\mathcal{R} \in \mathbf{R}_K$.

(a) We say that $\mathcal{R}$ *converges* to $m$, and write $\mathcal{R} \to m$, if $\mathcal{N}(p) \subseteq \mathcal{R}$ for each $p \in \mathrm{sub}K$ with $o_K < p \leq m$.

(b) $m$ is called a *clustering* of $\mathcal{R}$ provided that $m \leq c_K(r)$ for each $r \in \mathcal{R}$.

Clearly, there holds:

**Proposition 3.2** *Let $K$ be a $\mathcal{K}$-object and $m \in \mathrm{sub}K$. Then*

*(1) $\mathcal{R} \to o_K$ for each $\mathcal{R} \in \mathbf{R}_K$.*

*(2) $\mathcal{N}(m) \to m$ whenever $m$ is an atom of sub$K$.*

*(3) For any $\mathcal{R} \in \mathbf{R}_K$ and any $m \in \mathrm{sub}K$, from $\mathcal{R} \to m$ it follows that $\mathcal{R} \to p$ for each $p \in \mathrm{sub}K$, $p \leq m$.*

*(4) Let the lattice sub$K$ be atomic, let $\mathcal{R} \in \mathbf{R}_K$ and let $m \in \mathrm{sub}K$. If $\mathcal{R} \to a$ for each atom $a \in \mathrm{sub}K$ with $a \leq m$, then $\mathcal{R} \to m$.*

*(5) For any $\mathcal{R} \in \mathbf{R}_K$ and any $m \in \mathrm{sub}K$, from $\mathcal{R} \to m$ it follows that $\mathcal{S} \to m$ whenever $\mathcal{S} \in \mathbf{R}_K$ is finer than $\mathcal{R}$.*

*(6) $o_K$ is a clustering of every raster $\mathcal{R} \in \mathbf{R}_K$.*

*(7) Let $\mathcal{R} \in \mathbf{R}_K$ and $m, n \in \mathrm{sub}K$. If $m$ is a clustering of $\mathcal{R}$ and $n \leq m$, then $n$ is a clustering of $\mathcal{R}$ too.*

**Example 3.3** Let $\mathcal{K}$ = **Top**, let $K$ be a $\mathcal{K}$-object, $\mathcal{R} \in \mathbf{R}_K$ be a filter, and $m : M \to |K|$ be an inclusion (in **Set**). Then $\mathcal{R} \to m$ (respectively, $m$ is a clustering of $\mathcal{R}$) if and only if $\mathcal{R}$ converges to $x$ (respectively, $x$ is a cluster point of $\mathcal{R}$) - in the usual topological sense - for each $x \in M$.

As an immediate consequence of Proposition 2.5, Definition 3.1 and Proposition 3.2 we get

**Proposition 3.4** *Let $K$ be a $\mathcal{K}$-object, $m, p \in subK$, and let $m$ be an atom of $subK$. If $m \leq c_K(p)$, then there exists $\mathcal{R} \in \mathbf{R}_K$ such that $\mathcal{R} \to m$ and $n \wedge p > o_K$ for each $n \in \mathcal{R}$, and vice versa provided that $subK$ is a Boolean algebra.*

**Corollary 3.5** *Let $K$ be a $\mathcal{K}$-object such that $subK$ is a Boolean algebra and let $p \in subK$. If $c_K(p)$ equals a join of atoms of $subK$, then $c_K(p) = \bigvee\{m \in subK;\ m$ is an atom such that there exists $\mathcal{R} \in \mathbf{R}_K$ with $\mathcal{R} \to m$ and $n \wedge p > o_K$ for each $n \in \mathcal{R}\}$.*

**Proposition 3.6** *Let $K$ be a $\mathcal{K}$-object such that $subK$ is a Boolean algebra, let $\mathcal{R} \in \mathbf{R}_K$ and let $m \in subK$ be a join of atoms. If there exists $\mathcal{S} \in \mathbf{R}_K$ with $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{S} \to m$, then $m$ is a clustering of $\mathcal{R}$, and vice versa provided that $subK$ is atomic, $c$ is additive and $\mathcal{R}$ is a filter.*

*Proof.* For $m = o_K$ the statement is trivial. Let $m > o_K$ and let there exist $\mathcal{S} \in \mathbf{R}_K$ with $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{S} \to m$. Then, for an arbitrary atom $p \in$ sub$K$ with $p \leq m$, we have $\mathcal{S} \to p$. From $\mathcal{N}(p) \subseteq \mathcal{S}$ it follows that $r \wedge n > o_K$ whenever $r \in \mathcal{R}$ and $n \in \mathcal{N}(p)$. By Proposition 2.5, $p \leq c_K(r)$ for each $r \in \mathcal{R}$. Hence $p$ is a clustering of $\mathcal{R}$, i.e., $p \leq \bigwedge\{c_K(r);\ r \in \mathcal{R}\}$. Consequently, $m$ is a clustering of $\mathcal{R}$.
Conversely, let sub$K$ be atomic, $c$ be additive and $\mathcal{R}$ be a filter. Suppose that $m$ is a clustering of $\mathcal{R}$ and let $p \leq m$ be an arbitrary atom of sub$K$. Put $\mathcal{B} = \{r \wedge n;\ r \in \mathcal{R},\ n \in \mathcal{N}(p)\}$. By Proposition 2.5, $r \wedge n > o_K$ whenever $r \in \mathcal{R}$ and $n \in \mathcal{N}(p)$. As $\mathcal{R}$ is a filter and by Proposition 2.2(9), $\mathcal{N}(p)$ is a filter too, $\mathcal{B}$ is a filter base. We have $\mathcal{R} \subseteq [\mathcal{B}]$ and $\mathcal{N}(p) \subseteq [\mathcal{B}]$, hence $[\mathcal{B}] \to p$. By Proposition 3.2(4), $[\mathcal{B}] \to m$. The proof is complete. □

**Corollary 3.7** *Let $K \in \mathcal{K}$ be an object such that $subK$ is a Boolean algebra, let $\mathcal{R} \in \mathbf{R}_K$ and let $m \in subK$ be a join of atoms. If $\mathcal{R} \to m$, then $m$ is a clustering of $\mathcal{R}$.*

**Corollary 3.8** *Let $c$ be additive, $K$ be a $\mathcal{K}$-object such that sub$K$ is an atomic Boolean algebra, and let $\mathcal{R} \in \mathbf{R}_K$ be an ultrafilter. Then $\mathcal{R} \to m$ if and only if $m$ is a clustering of $\mathcal{R}$.*

In the last section, we will need the following result.

**Lemma 3.9** *Let $c$ be additive, $K$ be a $\mathcal{K}$-object such that sub$K$ is a Boolean algebra, and let $m, p \in$ sub$K$ where $m$ is an atom. Then the following conditions are equivalent:*

*(1)* $m \leq c_K(p)$.

*(2) There exists a filter base $\mathcal{B} \subseteq$ subK with $[\mathcal{B}] \to m$ such that $q \leq p$ for each $q \in \mathcal{B}$.*

*(3) There exists a filter $\mathcal{F} \in \mathbf{R}_K$ such that $\mathcal{F} \to m$ and $p \in \mathcal{F}$.*

*Proof.* Let (1) be true. Then, by Proposition 2.5, $n \wedge p > o_K$ for each $n \in \mathcal{N}(m)$. By Proposition 2.2(9), $\mathcal{N}(m) \in \mathbf{R}_K$ is a filter. Thus, $\mathcal{B} = \{n \wedge p;\ n \in \mathcal{N}(m)\}$ is a filter base and $q \leq p$ for each $q \in \mathcal{B}$. Since $\mathcal{N}(m) \to m$ (by Proposition 3.2(2)) and $\mathcal{N}(m) \subseteq [\mathcal{B}]$, we have $[\mathcal{B}] \to m$. Thus, (1)$\Rightarrow$(2).
The implication (2)$\Rightarrow$(3) is obvious.
Assume that (3) is true. Then $n \wedge p > o_K$ for each $n \in \mathcal{F}$ and by Proposition 3.4, we get $m \leq c_K(p)$. Hence (3)$\Rightarrow$(1). □

**Theorem 3.10** *Let $f : K \to L$ be a $\mathcal{K}$-morphism, $m \in$ subK and $\mathcal{R} \in \mathbf{R}_K$. If $\mathcal{R} \to m$, then $[f(\mathcal{R})] \to f(m)$.*

*Proof.* Let $\mathcal{R} \to m$, $p \in$ sub$L$, $o_L < p \leq f(m)$, and let $n \in \mathcal{N}(p)$. Since $f(f^{-1}(p)) \leq p$, we have $n \in \mathcal{N}(f(f^{-1}(p)))$. Thus, by Proposition 2.4, $f^{-1}(n) \in \mathcal{N}(f^{-1}(p))$. Since $f^{-1}(p) \wedge m \leq f^{-1}(p)$, we have $f^{-1}(n) \in \mathcal{N}(f^{-1}(p) \wedge m)$. By Lemma 1.1, $o_K < f^{-1}(p) \wedge m \leq m$. Thus, $\mathcal{N}(f^{-1}(p) \wedge m) \subseteq \mathcal{R}$ because $\mathcal{R} \to m$. It follows that $f^{-1}(n) \in \mathcal{R}$, hence $f(f^{-1}(n)) \in f(\mathcal{R}) \subseteq [f(\mathcal{R})]$. As $n \geq f(f^{-1}(n))$, there holds $n \in [f(\mathcal{R})]$. Therefore, $\mathcal{N}(p) \subseteq [f(\mathcal{R})]$, which yields $[f(\mathcal{R})] \to f(m)$. □

Let $K = \prod_{i \in I} K_i$ be a product in $\mathcal{K}$ and let $\mathcal{R} \in \mathbf{R}_K$. By Theorem 3.10, given $m \in$ sub$K$, $\mathcal{R} \to m$ implies $[\mathrm{pr}_i(\mathcal{R})] \to \mathrm{pr}_i(m)$ for each $i \in I$. If the converse implication is also valid, we say that the raster $\mathcal{R}$ is *convergence-compatible* with the product $K$. The following statement provides a useful criterion for the convergence-compatibility of filters:

**Proposition 3.11** *Let $K = \prod_{i \in I} K_i$ be a product in $\mathcal{K}$. For any $p \in subK$, $o_K < p$, and any $n \in \mathcal{N}(p)$, let there exist a finite subset $I_0 \subseteq I$ and a subobject $n_i \in subK_i$ with $n_i \in \mathcal{N}(n_i)$ for each $i \in I_0$ such that $p \leq \bigwedge_{i \in I_0} pr_i^{-1}(n_i) \leq n$. Then every filter on subK is convergence-compatible with K.*

*Proof.* Let $\mathcal{R} \in \mathbf{R}_K$ be a filter, $m \in \text{sub}K$ and $[\text{pr}_i(\mathcal{R})] \to \text{pr}_i(m)$ for each $i \in I$. Suppose also that $p \in \text{sub}K$, $o_K < p \leq m$, and let $n \in \mathcal{N}(p)$. Then there is a finite subset $I_0 \subseteq I$ and a subobject $n_i \in \text{sub}K_i$ with $n_i \in \mathcal{N}(n_i)$ for each $i \in I_0$ such that $p \leq \bigwedge_{i \in I_0} \text{pr}_i^{-1}(n_i) \leq n$. Consequently, $\text{pr}_i(p) \leq \bigwedge_{j \in I_0} \text{pr}_i \text{pr}_j^{-1}(n_j) \leq \text{pr}_i \text{pr}_i^{-1}(n_i) \leq n_i$ for each $i \in I_0$. It follows that, for each $i \in I_0$, $n_i \in \mathcal{N}(\text{pr}_i(p))$ because $n_i \in \mathcal{N}(n_i)$. Since $o_{K_i} < \text{pr}_i(p) \leq \text{pr}_i(m)$, we have $n_i \in [\text{pr}_i(\mathcal{R})]$ for each $i \in I_0$. Thus, whenever $i \in I_0$, $n_i \geq \text{pr}_i(r)$ for some $r \in \mathcal{R}$. Hence $\text{pr}_i^{-1}(n_i) \geq \text{pr}_i^{-1}\text{pr}_i(r) \geq r$, so that $\text{pr}_i^{-1}(n_i) \in \mathcal{R}$ for each $i \in I_0$. Therefore, $\bigwedge_{i \in I_0} \text{pr}_i^{-1}(n_i) \in \mathcal{R}$, which yields $n \in \mathcal{R}$. It follows that $\mathcal{N}(p) \subseteq \mathcal{R}$. We have shown that $\mathcal{R} \to m$. This proves the statement. □

**Example 3.12** For $\mathcal{K} = \mathbf{Top}$, the assumptions of Proposition 3.11 are clearly fulfilled. Thus, we get the well-known fact that filters are convergence-compatible with products of topological spaces.

**Proposition 3.13** *Let in $\mathcal{X}$ the non-trivial objects be stable under products and let $K = \prod_{i \in I} K_i$ be a product in $\mathcal{K}$. For each $i \in I$, let $\mathcal{R}_i \in \mathbf{R}_{K_i}$, $m_i \in subK_i$ and $\mathcal{R}_i \to m_i$. If $[\prod_{i \in I} \mathcal{R}_i] \in \mathbf{R}_K$ is convergence-compatible with K, then $[\prod_{i \in I} \mathcal{R}_i] \to \prod_{i \in I} m_i$.*

*Proof.* By the assumptions, $[\prod_{i \in I} \mathcal{R}_i] \in \mathbf{R}_K$. Let $r_i \in \mathcal{R}_i$, $r_i : R_i \to K_i$ for each $i \in I$, and put $R = \prod_{i \in I} R_i$. Then, for each $i \in I$, $\text{pr}_i \circ \prod_{i \in I} r_i = r_i \circ p_i$ where $p_i : R \to R_i$ is the projection. Thus, $\text{pr}_i(\prod_{i \in I} r_i)$ is the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $r_i \circ p_i$. Now, the diagonalization property results in $\text{pr}_i(\prod_{i \in I} r_i) \leq r_i$ for each $i \in I$. Analogously, we have $\text{pr}_i(\prod_{i \in I} m_i) \leq m_i$ for each $i \in I$. Hence $[\text{pr}_i(\prod_{i \in I} \mathcal{R}_i)] \supseteq \mathcal{R}_i$, which yields $[\text{pr}_i(\prod_{i \in I} \mathcal{R}_i)] \to m_i$ for each $i \in I$. Consequently, $[\text{pr}_i(\prod_{i \in I} \mathcal{R}_i)] \to \text{pr}_i(\prod_{i \in I}(m_i))$ for each $i \in I$. Since $[\prod_{i \in I} \mathcal{R}_i]$ is convergence-compatible with $K$, we have $[\prod_{i \in I} \mathcal{R}_i] \to \prod_{i \in I} m_i$. □

# 4 Raster separation and raster compactness

**Definition 4.1** A $\mathcal{K}$-object $K$ is said to be

(a) *raster separated* (with respect to $c$) provided that, whenever $m, p \in$ sub$K$ are atoms and $\mathcal{R} \in \mathbf{R}_K$, from $\mathcal{R} \to m$ and $\mathcal{R} \to p$ it follows that $m = p$,

(b) *raster compact* (with respect to $c$) provided that each $\mathcal{R} \in \mathbf{R}_K$ has a clustering different from $o_K$ (or equivalently, has the property $\bigwedge\{c_K(r);\ r \in \mathcal{R}\} > o_K$).

**Example 4.2** In the case $\mathcal{K} =$ **Top**, a topological space is raster-separated (respectively, raster compact) if and only if it is a Hausdorff space (respectively, compact in the usual sense).

**Theorem 4.3** *Let $K$ be a $\mathcal{K}$-object. If for any pair $p, q \in subK$ of different atoms there exist $m \in \mathcal{N}(p)$ and $n \in \mathcal{N}(q)$ such that $m \wedge n = o_K$, then $K$ is raster separated, and vice versa provided that $c$ is additive and sub$K$ is a Boolean algebra.*

*Proof.* Assume that, for any pair $p, q \in$ sub$K$ of different atoms, there exist $m \in \mathcal{N}(p)$ and $n \in \mathcal{N}(q)$ such that $m \wedge n = o_K$. Let $\mathcal{R} \in \mathbf{R}_K$ be a raster with $\mathcal{R} \to r$ and $\mathcal{R} \to s$ where $r, s \in$ sub$K$ are atoms. Then $\mathcal{N}(r) \subseteq \mathcal{R}$ and $\mathcal{N}(s) \subseteq \mathcal{R}$, hence $m \wedge n > o_K$ whenever $m \in \mathcal{N}(r)$ and $n \in \mathcal{N}(s)$. Therefore, $r = s$. We have shown that $K$ is raster separated.
Conversely, let $c$ be additive and let sub$K$ be a Boolean algebra. Suppose there is a pair $p, q \in$ sub$K$ of different atoms such that $m \wedge n > o_K$ whenever $m \in \mathcal{N}(p)$ and $n \in \mathcal{N}(q)$. Put $\mathcal{B} = \mathcal{N}(p) \cup \mathcal{N}(q)$. Then $\mathcal{B} \subseteq$ sub$K$ is centered because $\mathcal{N}(p)$ and $\mathcal{N}(q)$ are filters (by Proposition 2.2(9)). Since we have both $[\mathcal{B}] \to p$ and $[\mathcal{B}] \to q$, $K$ is not raster separated. □

**Theorem 4.4** *Let $K$ be a $\mathcal{K}$-object such that sub$K$ is a Boolean algebra. If $r = \bigwedge\{c_K(n);\ n \in \mathcal{N}(r)\}$ for each atom $r \in$ sub$K$, then $K$ is raster separated, and vice versa provided that $c$ is additive and sub$K$ is atomic.*

*Proof.* Let $r = \bigwedge\{c_K(n);\ n \in \mathcal{N}(r)\}$ for each atom $r \in$ sub$K$. Let $p, q \in$ sub$K$ be different atoms. Then $q \not\leq p = \bigwedge\{c_K(n);\ n \in \mathcal{N}(p)\}$. Hence, there is $n \in \mathcal{N}(p)$ with $q \not\leq c_K(n)$. Thus, $q \leq \overline{c_K(n)}$ and we have

$\overline{n} \in \mathcal{N}(q)$. As $n \wedge \overline{n} = o_K$, $K$ is raster separated by Theorem 4.3.
Conversely, let $c$ be additive, let sub$K$ be atomic, and let $K$ be raster separated. Given an atom $r \in$ sub$K$, we have $r \leq \bigwedge\{c_K(n);\ n \in \mathcal{N}(r)\}$. Admit that $r < \bigwedge\{c_K(n);\ n \in \mathcal{N}(r)\}$. Then there is an atom $p \in$ sub$K$, $p$ different from $r$, such that $p < \bigwedge\{c_K(n);\ n \in \mathcal{N}(r)\}$. Thus, by Theorem 4.3, there are $m \in \mathcal{N}(p)$ and $n \in \mathcal{N}(r)$ with $m \wedge n = o_K$. Consequently, $\overline{n} \geq m$. This yields $\overline{n} \in N(p)$, i.e., $p \leq \overline{c_K(n)}$. Therefore, $p \not\leq c_K(n)$, which is a contradiction. □

Now, we will proceed to the study of raster compactness. Definition 4.1(b) and Proposition 3.6 immediately result in

**Proposition 4.5** *Let $K$ be a $\mathcal{K}$-object such that sub$K$ is a Boolean algebra. If for each $\mathcal{R} \in \mathbf{R}_K$ there exist $\mathcal{S} \in \mathbf{R}_K$ with $\mathcal{S} \supseteq \mathcal{R}$ and an atom $m \in$ sub$K$ such that $\mathcal{S} \to m$, then $K$ is raster compact, and vice versa provided that sub$K$ is atomic, $c$ is additive and $\mathbf{R}_K$ is replaced by the conglomerate of all filters on sub$K$.*

**Corollary 4.6** *Let $c$ be additive and $K$ be a $\mathcal{K}$-object such that sub$K$ is an atomic Boolean algebra. Then $K$ is raster compact if and only if each ultrafilter $\mathcal{R} \in \mathbf{R}_K$ converges to an atom of sub$K$.*

**Theorem 4.7** *Let $K$ be a $\mathcal{K}$-object. If $K$ is raster compact, then $\bigwedge \mathcal{T} > o_K$ for each centered class $\mathcal{T} \subseteq$ sub$K$ of c-closed subobjects of $K$, and vice versa provided that c is idempotent.*

*Proof.* Let $K$ be raster compact. Admit that there exists a centered class $\mathcal{T} \subseteq$ sub$K$ of $c$-closed subobjects of $K$ such that $\bigwedge \mathcal{T} = o_K$. Then $[\mathcal{T}] \in \mathbf{R}_K$ and $\bigwedge\{c_K(p);\ p \in [\mathcal{T}]\} \leq \bigwedge\{c_K(p);\ p \in \mathcal{T}\} = \bigwedge \mathcal{T} = o_K$. Hence the only clustering of $[\mathcal{T}]$ is $o_K$, which is a contradiction.
Conversely, let $c$ be idempotent and let $\bigwedge \mathcal{T} > o_K$ for each centered class $\mathcal{T} \subseteq$ sub$K$ of $c$-closed subobjects of $K$. Let $\mathcal{R} \in \mathbf{R}_K$ and let $\mathcal{S} \subseteq \mathcal{R}$ be the class of all $c$-closed elements of $\mathcal{R}$. Then $\mathcal{S}$ is centered, hence $\bigwedge \mathcal{S} > o_K$. Since $\bigwedge \mathcal{S} = \bigwedge\{c_K(r);\ r \in \mathcal{R}\}$, we have $\bigwedge\{c_K(r);\ r \in \mathcal{R}\} > o_K$. As $\bigwedge\{c_K(r);\ r \in \mathcal{R}\}$ is clearly a clustering of $\mathcal{R}$, $\mathcal{R}$ is raster compact. □

**Theorem 4.8** *Let c be idempotent and hereditary, and let $m : M \to K$ be a c-closed embedding in $\mathcal{K}$. If $K$ is raster compact, then $M$ is raster compact too.*

*Proof.* Let $K$ be raster compact and $\mathcal{T} \subseteq \text{sub}M$ be a centered class of $c$-closed subobjects of $M$. Let $t \in \mathcal{T}$ be an arbitrary element. Then $c_M(t) = m^{-1}(c_K(m \circ t))$ because $c$ is hereditary. It follows that $m^{-1}(c_K(m \circ t)) = t$ as $c_M(t) = t$. Thus, since $c_K(m) = m$, we have $(c_K(m))^{-1}(c_K(m \circ t)) = t$. But we also have $c_K(m \circ t) \le c_K(m)$ because $m \circ t \le m$. Hence $c_K(m \circ t) = c_K(m) \circ t = m \circ t$. Therefore, $m \circ t$ is $c$-closed. Further, for any $t_1, t_2, ..., t_k \in \mathcal{T}$ $(k \in \mathbb{N})$ we have $m \circ t_1 \wedge m \circ t_2 \wedge ... \wedge m \circ t_k \ge m(t_1 \wedge t_2 \wedge ... \wedge t_k) > o_K$. Consequently, $m \circ \mathcal{T} \in \text{sub}K$ is centered. It follows that $\bigwedge(m \circ \mathcal{T}) > o_K$. Thus, as $\bigwedge(m \circ \mathcal{T}) = m \circ \bigwedge \mathcal{T}$, there holds $\bigwedge \mathcal{T} > o_M$. By Theorem 4.7, $M$ is raster compact. □

In the proof of Theorem 4.8, the hereditariness of $c$ has been used only for showing that a composition of a pair of $c$-closed subobjects is $c$-closed. As $c$ is supposed to be idempotent, Theorem 4.8 remains valid when replacing the hereditariness of $c$ with the so-called *weak hereditariness* of $c$ (see [14], 2.4).

**Theorem 4.9** *Let $\mathcal{E}$ be stable under pullbacks, $c$ be idempotent and $f : K \to L$ be a $\mathcal{K}$-morphism. If $K$ is raster compact and $f \in \mathcal{E}$, then $L$ is raster compact too.*

*Proof.* Let $\mathcal{T} \subseteq \text{sub}L$ be a centered class of $c$-closed subobjects of $L$. Then $f^{-1}(\mathcal{T})$ is a centered class of $c$-closed subobjects of $K$ and therefore, $\bigwedge f^{-1}(\mathcal{T}) > o_K$. Since $f^{-1}(\bigwedge \mathcal{T}) = \bigwedge(f^{-1}(\mathcal{T})) > o_K$, we have $\bigwedge \mathcal{T} = f(f^{-1}(\bigwedge \mathcal{T})) > o_L$. Now, the statement follows from Theorem 4.7. □

**Theorem 4.10** *Let $c$ be additive, idempotent and hereditary. Let $m : M \to K$ be an embedding in $\mathcal{K}$ where $M$ is raster compact, $K$ is raster separated and sub$K$ is an atomic Boolean algebra. Then $m$ is $c$-closed.*

*Proof.* If $m = o_K$, then the statement is trivial. Let $m > o_K$ and let $p \le c_K(m)$ be an atom. By Proposition 2.5, $n \wedge m > o_K$ for each $n \in \mathcal{N}(p)$. Put $\mathcal{T} = \{m^{-1}(c_K(m \circ m^{-1}(n))); \; n \in \mathcal{N}(p)\}$. As $c$ is hereditary, for each $n \in \mathcal{N}(p)$ there holds $m^{-1}(c_K(m \circ m^{-1}(n))) = c_M(m^{-1}(n))$. Thus, each element of $\mathcal{T}$ is a $c$-closed subobject of $M$ and for any $n \in \mathcal{N}(p)$ we have $m^{-1}(c_K(m \circ m^{-1}(n))) \ge m^{-1}(n)$. Consequently, given $n_1, n_2, ..., n_k \in \mathcal{N}(p)$ $(k \in \mathbb{N})$, there holds $m^{-1}(c_K(m \circ m^{-1}(n_1))) \wedge m^{-1}(c_K(m \circ m^{-1}(n_2))) \wedge ... \wedge m^{-1}(c_K(m \circ m^{-1}(n_k))) \ge m^{-1}(n_1) \wedge$

$m^{-1}(n_2) \wedge ... \wedge m^{-1}(n_k) = m^{-1}(n_1 \wedge n_2 \wedge ... \wedge n_k) > o_M$ because $m \wedge n_1 \wedge n_2 \wedge ... \wedge n_k > o_K$ (as $n_1 \wedge n_2 \wedge ... \wedge n_k \in \mathcal{N}(p)$ by Proposition 2.2(9)). It follows that $\mathcal{T}$ is centered. Since $m \circ m^{-1}(n) = m(m^{-1}(n)) \leq n$, we have $\bigwedge \mathcal{T} \leq \bigwedge\{m^{-1}(c_K(n));\ n \in \mathcal{N}(p)\} = m^{-1}(\bigwedge\{c_K(n);\ n \in \mathcal{N}(p)\})$. Thus, by Theorems 4.4 and 4.7, $o_M < \bigwedge \mathcal{T} \leq m^{-1}(p)$. As $p \wedge m = m \circ m^{-1}(p)$, we get $p \wedge m > o_K$. Hence $p \leq m$ because $p$ is an atom. Therefore, $c_K(m) \leq m$. □

**Corollary 4.11** *Let $\mathcal{E}$ be stable under pullbacks and $\mathcal{K}$ have embeddings and $(\mathcal{E}, Emb_{\mathcal{M}})$-factorization structure. Let c be additive, idempotent and hereditary and $f : K \to L$ be a $\mathcal{K}$-morphism where K is raster compact and L is raster separated with the property that subL is an atomic Boolean algebra. Then f is c-preserving.*

*Proof.* Let $m \in$ sub$K$. Then there is a $\mathcal{K}$-object $M$ such that $c_K(m) : M \to K$ is an embedding in $\mathcal{K}$. As $c_K(m)$ is $c$-closed, $M$ is raster compact by Theorem 4.8. Further, there is a $\mathcal{K}$-object $N$ such that $f(c_K(m)) : N \to L$ is an embedding in $\mathcal{K}$. Let $e : |M| \to |N|$ be the $\mathcal{E}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ c_K(m)$. By the assumptions, $e$ is a $\mathcal{K}$-morphism. Thus, $N$ is raster compact by Theorem 4.9. According to Theorem 4.10, $f(c_K(m))$ is $c$-closed. From $f(m) \leq f(c_K(m))$ it follows that $c_L(f(m)) \leq c_L(f(c_K(m))) = f(c_K(m))$. As the converse inequality is clearly fulfilled, $f$ is $c$-preserving. □

**Remark 4.12** Let the assumptions of Corollary 4.11 be satisfied and let $\mathcal{K}$ have the property that each $\mathcal{K}$-morphism which is a $c$-preserving $\mathcal{X}$-isomorphism is a $\mathcal{K}$-isomorphism. Then $f$ is a $\mathcal{K}$-isomorphism whenever it is an $\mathcal{X}$-isomorphism. Moreover, let $|K| = |L|$ and suppose that $c_K \leq c_L$ implies that $\mathrm{id}_{|K|}$ is a $\mathcal{K}$-morphism $\mathrm{id}_{|K|} : K \to L$. Then $c_K \leq c_L$ implies $c_K = c_L$ by Corollary 4.11 (putting $f = \mathrm{id}_{|K|}$). Thus, given an $\mathcal{X}$-object $X$, in the class of all $c_K$ with $K$ a raster separated $\mathcal{K}$-object such that $|K| = X$, $c_K$ with $K$ raster compact are maximal (provided that the class is nonempty).

As an immediate consequence of Theorem 4.9 we get

**Corollary 4.13** *Let $\mathcal{E}$ be stable under pullbacks and c be idempotent. Let $K = \prod_{i \in I} K_i$ be a product in K such that all projections $pr_i : K \to K_i$, $i \in I$, belong to $\mathcal{E}$. If K is raster compact, then $K_i$ is raster compact for each $i \in I$.*

The following statement is a converse of Corollary 4.13.

**Theorem 4.14** *Let in $\mathcal{X}$ the non-trivial objects be stable under products, let c be additive and $K = \prod_{i \in I} K_i$ be a product in $\mathcal{K}$ such that sub$K$ is a Boolean algebra and sub$K_i$ is an atomic Boolean algebra for each $i \in I$. Let each ultrafilter on sub$K$ be convergence-compatible with $K$. If $K_i$ is raster compact for each $i \in I$, then $K$ is raster compact too.*

*Proof.* Let $K_i$ be raster compact for each $i \in I$ and let $\mathcal{R} \in \mathbf{R}_K$ be an ultrafilter. Then $[\mathrm{pr}_i(\mathcal{R})]$ is an ultrafilter for each $i \in I$. As sub$K$ is atomic, sub$K_i$ is atomic too for each $i \in I$ by Lemma 1.2. By Corollary 4.6, for each $i \in I$ there is an atom $m_i \in$ sub$K_i$ such that $[\mathrm{pr}_i(\mathcal{R})] \rightarrow m_i$. Since in $\mathcal{X}$ the non-trivial objects are stable under products, we have $\prod_{i \in I} m_i > o_K$. Thus, there is an atom $m \leq \prod_{i \in I} m_i$. Then $\mathrm{pr}_i(m) \leq m_i$ and thus $[\mathrm{pr}_i(\mathcal{R})] \rightarrow \mathrm{pr}_i(m)$ for each $i \in I$. As $\mathcal{R}$ is convergence-compatible with $K$, we have $\mathcal{R} \rightarrow m$. Hence, $K$ is raster compact by Corollary 4.6. □

**Remark 4.15** If $\mathcal{K} = \mathbf{Top}$, the assumptions of all statements of this section are fulfilled and these statements give well-known results. The property presented in Theorem 4.10 is sometimes referred to as the *absolute closedness* of $M$, and that of Remark 4.12 as *minimality* of compact spaces (note that topologies on a given set are usually ordered conversely to the corresponding Kuratowski closure operators). Notice also that Theorem 4.14 is *Tichonoff's theorem* transposed to our settings.

# 5 $c$-separation and $c$-compactness

Our investigation of raster separation and raster compactness with respect to $c$ would be rather incomplete without discussing their relationships to $c$-separation and $c$-compactness, respectively. The aim of this section is to study these relations.

**Definition 5.1** [10] A $\mathcal{K}$-object $K$ is called

(a) *c-separated* if the diagonal morphism $\delta_K : |K| \rightarrow |K| \times |K|$ is $c$-closed,

(b) *c-compact* if the projection $\mathrm{pr}_L : K \times L \rightarrow L$ is $c$-preserving for every $\mathcal{K}$-object $L$.

**Remark 5.2** It is well known [10] that a $\mathcal{K}$-object $K$ ic $c$-separated if and only if, for each pair $f, g : K \to L$ of $\mathcal{K}$-morphisms and each $m \in \text{sub}K$, $f \circ m = g \circ m$ implies $f \circ c_L(m) = g \circ c_L(m)$.

**Theorem 5.3** *Let c be additive and K be a $\mathcal{K}$-object such that sub$(K \times K)$ is an atomic Boolean algebra and both the projections $pr_i : |K| \times |K| \to |K|$, $i = 1, 2$, fulfill $pr_i \circ m \in \mathcal{M}$ whenever $m \in$ sub$(K \times K)$ is an atom. If K is raster separated, then it is c-separated.*

*Proof.* Let $K$ be raster separated. If $c_{K\times K}(\delta_K) = o_K$, then the statement is trivial. Let $c_{K\times K}(\delta_K) > o_K$ and let $m \in \text{sub}(K \times K)$ be an atom with $m \leq c_{K\times K}(\delta_K)$. Then, by Lemma 3.9, there is a filter base $\mathcal{B} \subseteq \text{sub}(K \times K)$ with $[\mathcal{B}] \to m$ such that $q \leq \delta_K$ for each $q \in \mathcal{B}$. Consequently, for each $q \in \mathcal{B}$ there exists $t_q \in \mathcal{M}$ such that $q = \delta_K \circ t_q$. We have $\mathcal{B} = \{\delta_K \circ t_q;\ q \in \mathcal{B}\}$, hence $\text{pr}_i(\mathcal{B}) = \{\text{pr}_i(\delta_K \circ t_q);\ q \in \mathcal{B}\}$ for $i = 1, 2$. From $\text{pr}_1 \circ \delta_K \circ t_q = \text{pr}_2 \circ \delta_K \circ t_q$ it follows that $\text{pr}_1(\delta_K \circ t_q) = \text{pr}_2(\delta_K \circ t_q)$ for each $q \in \mathcal{B}$. This yields $[\text{pr}_1(\mathcal{B})] = [\text{pr}_2(\mathcal{B})]$. By Theorem 3.10, $[\text{pr}_i(\mathcal{B})] \to \text{pr}_i(m)$ for $i = 1, 2$. Thus, since $\text{pr}_i(m) = \text{pr}_i \circ m$, $i = 1, 2$, are atoms of $\text{sub}(|K| \times |K|)$ by Lemma 1.2, we have $\text{pr}_1 \circ m = \text{pr}_2 \circ m$. This results in $m \leq \delta_K$ because $\delta_K$ is an equalizer of $\text{pr}_1$ and $\text{pr}_2$. Therefore, $c_{K\times K}(\delta_K) \leq \delta_K$, i.e., $\delta_K$ is $c$-closed. Hence $K$ is $c$-separated. □

**Theorem 5.4** *Let $K$ be a $\mathcal{K}$-object such that all atoms of sub$K$ have the same domain (up to isomorphisms), sub$(K \times K)$ is a Boolean algebra, and each raster on $K \times K$ is convergence compatible with $K \times K$. If $K$ is c-separated, then it is raster separated.*

*Proof.* Let $K$ be $c$-separated, $\mathcal{R} \in \mathbf{R}_K$, $\mathcal{R} \to m$ and $\mathcal{R} \to p$ where $m, p \in \text{sub}(K \times K)$ are atoms. Put $\mathcal{S} = [\delta_K(\mathcal{R})]$ and let $\text{pr}_1, \text{pr}_2 : |K| \times |K| \to |K|$ be the two projections. Then $\mathcal{S} \in \mathbf{R}_{K\times K}$ and $[\text{pr}_i(\mathcal{S})] \supseteq \mathcal{R}$ for $i = 1, 2$. So, we have $[\text{pr}_1(\mathcal{S})] \to m$ and $[\text{pr}_2(\mathcal{S})] \to p$. Putting $q = \langle m, p\rangle$ we get an atom $q \in \text{sub}(K \times K)$ by Lemma 1.3. As $m = \text{pr}_1 \circ q = \text{pr}_1(q)$ and $p = \text{pr}_2 \circ q = \text{pr}_2(q)$, the convergence compatibility of $\mathcal{S}$ results in $\mathcal{S} \to q$. Let $s \in \mathcal{S}$. Then there exists $r \in \mathcal{R}$ with $s \geq \delta_K(r)$. Thus, we have $s \wedge \delta_K \geq \delta_K(r) \wedge \delta_K = \delta_K(r) > o_K$ because $r > o_K$. By Proposition 3.4, $q \leq c_{K\times K}(\delta_K) = \delta_K$. Consequently, there exists $t \in \mathcal{M}$ with $q = \delta_K \circ t$. We have $m = \text{pr}_1 \circ q = \text{pr}_1 \circ \delta_K \circ t = \text{pr}_2 \circ \delta_K \circ t = \text{pr}_2 \circ q = p$. Therefore, $K$ is raster separated. □

**Theorem 5.5** *Let $c$ be additive and sub$L$ be an atomic Boolean algebra for each $\mathcal{K}$-object $L$. Let $K$ be a $\mathcal{K}$-object satisfying the following condition:*

> *Given a $\mathcal{K}$ object $L$, an atom $y \in$ sub$L$ and a subobject $m \in$ sub$(K \times L)$ with $pr_L(c_{K\times L}(m)) \wedge y = o_L$, for each atom $x \in$ sub$K$ there are subobjects $u_x \in$ sub$K$ and $v_x \in$ sub$L$, $u_x$ c-closed, such that $u_x \wedge x = o_K$, $c_L(v_x) \wedge y = o_L$, and $c_{K\times L}(m) \leq pr_K^{-1}(u_x) \vee pr_L^{-1}(v_x)$.*

*If $K$ is raster compact, then it is c-compact.*

*Proof.* Let $K$ be raster compact, $L$ be a $\mathcal{K}$-object and $m \in \mathrm{sub}(K \times L)$. If $\mathrm{pr}_L(c_{K\times L}(m)) = \mathrm{id}_L$, then we clearly have $c_L(\mathrm{pr}_L(m)) \leq \mathrm{pr}_L(c_{K\times L}(m))$. Let $\mathrm{pr}_L(c_{K\times L}(m)) < \mathrm{id}_L$. Then $\overline{\mathrm{pr}_L(c_{K\times L}(m))} > o_L$. Let $y \in \mathrm{sub}L$ be an atom with $y \leq \overline{\mathrm{pr}_L(c_{K\times L}(m))}$, i.e., with $\mathrm{pr}_L(c_{K\times L}(m)) \wedge y = o_L$. For each atom $x \in \mathrm{sub}K$, let $u_x \in \mathrm{sub}K$ and $v_x \in \mathrm{sub}L$ be the subobjects from the condition of the statement. Then $\bigwedge\{u_x;\ x \in \mathrm{sub}K$ is an atom$\} = o_K$ (because otherwise there is an atom $x_0 \in \mathrm{sub}K$ with $x_0 \leq u_x$ for each atom $x \in \mathrm{sub}K$, which is a contradiction with $u_{x_0} \wedge x_0 = o_K$). Thus, by Theorem 4.7, there is a finite set $\{x_1, ..., x_k\}$ of atoms of sub$K$ such that $\bigwedge_{i=1}^k u_{x_i} = o_K$. Put $v = \bigvee_{i=1}^k v_{x_i}$. Then $c_L(v) \wedge y = c_L(\bigvee_{i=1}^k v_{x_i}) \wedge y = \bigvee_{i=1}^k (c_L(v_{x_i}) \wedge y) = o_L$. Consequently, $\overline{v} \in \mathcal{N}(y)$. Further, we have $c_{K\times L}(m) \leq \bigwedge_{i=1}^k (\mathrm{pr}_K^{-1}(u_{x_i}) \vee \mathrm{pr}_L^{-1}(v_{x_i})) \leq \bigwedge_{i=1}^k \mathrm{pr}_K^{-1}(u_{x_i}) \vee \bigvee_{i=1}^k \mathrm{pr}_L^{-1}(v_{x_i}) \leq \mathrm{pr}_L^{-1}(\bigvee_{i=1}^k v_{x_i}) = \mathrm{pr}_L^{-1}(v)$, hence $\mathrm{pr}_L(c_{K\times L}(m)) \leq v$. This yields $\overline{v} \leq \overline{\mathrm{pr}_L(c_{K\times L}(m))}$, i.e., $\overline{v} \wedge \mathrm{pr}_L(c_{K\times L}(m)) = o_L$. It follows that $\overline{v} \wedge \mathrm{pr}_L(m) = o_L$. By Proposition 2.5, $y \wedge c_L(\mathrm{pr}_L(m)) = o_L$. Consequently, $y \in \overline{c_L(\mathrm{pr}_L(m))}$. We have shown that $\overline{c_L(\mathrm{pr}_L(m))} \geq \overline{\mathrm{pr}_L(c_{K\times L}(m))}$. Therefore, $c_L(\mathrm{pr}_L(m)) \leq \mathrm{pr}_L(c_{K\times L}(m))$ and the proof is complete. □

**Theorem 5.6** *Let $c$ be idempotent and $K$ be a $\mathcal{K}$-object with the properties that sub$K$ is a Boolean algebra and for any centered subclass $\mathcal{F} \subseteq$ sub$K$ of c-closed subobjects of $K$ there exist a $\mathcal{K}$-object $L$ and a c-dense subobject $m : |K| \to |L|$ of $L$ such that the following conditions are satisfied:*

*(1) sub$(K \times L)$ is atomic.*

*(2) For any atom $z \in$ sub$(K \times L)$, from $p \in \mathcal{N}(pr_K(z))$ and $q \in \mathcal{N}(pr_L(z))$ it follows that $p \times q \in \mathcal{N}(z)$.*

*(3) There exists a subobject $y \in$ sub$L$ with $y > o_L$, $y \wedge m = o_L$, and $y \vee m(s) \in \mathcal{N}(y)$ for each $s \in \mathcal{F}$.*

*If $K$ is c-compact, then it is raster compact.*

*Proof.* Let $K$ be $c$-compact and $\mathcal{F} \subseteq \text{sub}K$ be a centered class of $c$-closed subobjects of $K$. Put $d = \langle \text{id}_K, m \rangle$. Then $m = \text{pr}_L(d) \leq \text{pr}_L(c_{K\times L}(d))$. As $m$ is dense, we have $y \leq c_L(m)$. Consequently, $y \leq c_L(\text{pr}_L(c_{K\times L}(d))) = \text{pr}_L(c_{K\times L}(d))$ because $\text{pr}_L : K \times L \to L$ is $c$-preserving and $c$ is idempotent. Thus, by Lemma 1.1, $c_{K\times L}(d) \wedge \text{pr}_L^{-1}(y) > o_{K\times L}$. Let $z \in \text{sub}(K \times L)$ be an atom with $z \leq c_{K\times L}(d) \wedge \text{pr}_L^{-1}(y)$. Then $z \leq c_{K\times L}(d)$ and $\text{pr}_L(z) \leq y$. Put $a = \text{pr}_K(z)$ and $q_s = y \vee m(s)$ for each $s \in \mathcal{F}$. By Lemma 1.2, $a \in \text{sub}K$ is an atom. Let $p \in \mathcal{N}(a)$. Since $q_s \in \mathcal{N}(y)$, we have $p \times q_s \in \mathcal{N}(z)$ for each $s \in \mathcal{F}$. By Proposition 2.5, $w \wedge d > o_{K\times L}$ for each $w \in \mathcal{N}(z)$. Thus, $(p \times (y \vee m(s))) \wedge d > o_{K\times L}$ for each $s \in \mathcal{F}$. Hence, there is an atom $v_s \in \text{sub}(K \times L)$ with $v_s \leq (p \times (y \vee m(s))) \wedge d$ for each $s \in \mathcal{F}$. As $v_s \leq d$, there is an element $u_s \in \text{sub}K$, $u_s > o_K$, with $v_s = d \circ u_s$. We have $\text{pr}_K \circ v_s = \text{pr}_K \circ \langle \text{id}_K, m \rangle \circ u_s = u_s$ and $\text{pr}_L \circ v_s = \text{pr}_L \circ \langle \text{id}_K, m \rangle \circ u_s = m \circ u_s$. From $v_s \leq p \times (y \vee m(s))$ it follows that $u_s \leq \text{pr}_K(p \times (y \vee m(s)))$ and $m(u_s) \leq \text{pr}_L(p \times (y \vee m(s)))$ (for each $s \in \mathcal{F}$). Now, using the $(\mathcal{E}, \mathcal{M})$-diagonalization property, we get $u_s \leq p$ and $m(u_s) \leq y \vee m(s)$, i.e., $u_s \leq m^{-1}(y) \vee m^{-1}(s) = m^{-1}((y \vee m(s)) \wedge m) = m^{-1}((y \wedge m) \vee (m(s) \wedge m)) = m^{-1}(m(s) \wedge m) = m^{-1}(m(s)) = s$. Consequently, $p \wedge s \geq u_s > o_K$ for each $s \in \mathcal{F}$. Therefore, by Proposition 2.5, $a \leq c_K(s) = s$ for each $s \in \mathcal{F}$. Hence $\bigwedge \mathcal{F} > o_K$ and by Theorem 4.7, $K$ is raster compact. □

**Example 5.7** If $\mathcal{K} = \textbf{Top}$, the assumptions of each of the Theorems 5.3–5.6 are satisfied and the Theorems then give well-known results. Theorems 5.3 and 5.5 are also valid for example for the larger category $\mathcal{K} = \textbf{PrTop}$ of pretopological spaces in the sense of Čech (i.e., closure spaces from [6]). The assumptions of Theorem 5.5 are satisfied whenever $\mathcal{K}$ is a full topological subcategory of **Top** (the subobjects $u_x$ and $v_x$ are then obtained as complements of certain open neighborhoods of $x$ and $y$, respectively - see [15]). As for Theorem 5.6, its assumptions are satisfied, for example, whenever $\mathcal{K}$ is the category of $T_1$-spaces or the category of normal spaces (the topological space $L$ is then defined to be the space with $|L| = |K| \cup \{y\}$ where $y \notin |K|$ is a point and the open sets in $L$ are just the open sets in $K$ and the sets of the form $\{y\} \cup T \cup X$ where $T$ is a finite intersection of elements of $\mathcal{F}$ and $X \subseteq |K|$ is a subset - see [15] again). On the other hand, there hardly exist topological categories which are not subcategories of **Top** and fulfill the conditions of Theorem 5.5 or 5.6.

**Remark 5.8** a) The assumptions of Theorems 5.3 and 5.4 are quite natural (especially if $\mathcal{K}$ is a construct), thus there is a strong relationship between raster separation and $c$-separation. But this is not true for raster compactness and $c$-compactness in general (if $c$ is not a Kuratowski closure operator). Theorems 5.5 and 5.6 are presented here mainly for the sake of completeness, and are by no means our major concern in this paper.
b) The results of section 4 show that raster compactness behaves more decently than $c$-compactness because the former preserves all basic properties of the usual topological compactness transposed to our setting. This is a consequence of the fact that raster compactness is closer to the classical Lebesgue definition of compactness than $c$-compactness which is based on the Kuratowski-Mrówka characterization. But, in contrast to $c$-compactness, raster compactness can be considered only for such a category $\mathcal{K}$ with a closure operator which (together with its underlying category $\mathcal{X}$) fulfills all the conditions assumed in the first section

# References


[1] J. Adámek, H. Herrlich and G.E. Strecker, *Abstract and Concrete Categories*, Wiley & Sons, New York, 1990.

[2] G. Castellini and E. Giuli, Closure operators with respect to a functor, *Appl. Cat. Struct.* **9** (2001), 525-537.

[3] G. Castellini, Connectedness with respect to a closure operator, *Appl. Cat. Struct.* **9** (2001), 285-302.

[4] G. Castellini and D. Hajek, Closure operators and connectedness, *Topology Appl.* **55** (1994), 29-45.

[5] E. Čech, Topological Spaces, in: *Topological papers of Eduard Čech*, Academia, Prague, 1968, pp. 432-472.

[6] E. Čech, *Topological Spaces* (Revised by Z.Frolík and M.Katětov), Academia, Prague, 1966.

[7] M.M. Clementino, On connectedness via closure operator, *Appl. Categorical Structures* **9** (2001),539-556.

[8] M.M. Clementino and W. Tholen, Tychonoff's Theorem in a category, *Proc. Amer. Math. Soc.* **124** (1996), 3311-3314.

[9] M.M. Clementino and W. Tholen, Separation versus connectedness, *Topology Appl.* **75** (1997),143-181.

[10] M.M. Clementino, E. Giuli and W. Tholen, Topology in a category: compactness, *Portugal. Math.* **53** (1996), 397-433.

[11] M.M. Clementino, E. Giuli and W. Tholen, What is a quotient map with respect to a closure operator?, *Appl. Categorical Structures* **9** (2001), 139-151.

[12] D. Dikranjan and E. Giuli, Closure operators I, *Topology Appl.* **27** (1987), 129-143.

[13] D. Dikranjan and E. Giuli, Compactness, minimality and closedness with respect to a closure operator, in: *Proc. Int. Conf. on Categorical Topology, Prague 1988*, World Scientific, Singapore, 1989, 279-335.

[14] D. Dikranjan and W. Tholen, *Categorical Structure of Closure Operators*, Kluwer Academic Publishers, Dordrecht, 1995.

[15] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[16] E. Giuli, On $m$-separated projection spaces, *Appl. Categorical Structures* **2** (1994), 91-100.

[17] E. Giuli and W. Tholen, Openness with respect to a closure operator, *Appl. Categorical Structures* **8** (2000), 487-502.

[18] G. Grätzer, *General Lattice Theory*, Birkhäuser Verlag, Basel, 1978.

[19] J. Šlapal, Net spaces in categorical topology, *Annals New York Acad. Sci.* **806** (1996), 393-412.

[20] J. Šlapal, Compactness in categories of $S$-net spaces, *Appl. Categorical Structures* **6** (1998), 515-525.

[21] J. Šlapal, Compactification of limit $S$-net spaces, *Monatsh. Math.* **131** (2000), 335-341.

[22] J. Šlapal, Convergence structures for categories, *Appl. Categorical Structures* **9** (2001), 557-570.

[23] J. Šlapal, Compactness with respect to a convergence structure, *Quaestiones Math.* **25** (2002), 19-27.

ERALDO GIULI, DIPARTIMENTO DI MATEMATICA, UNIVERSITÁ DI L'AQUILA, 67100 L'AQUILA, ITALY
*e-mail address*: giuli@univaq.it

JOSEF ŠLAPAL, DEPARTMENT OF MATHEMATICS, BRNO UNIVERSITY OF TECHNOLOGY, 616 69 BRNO, CZECH REPUBLIC
*e-mail address*: slapal@fme.vutbr.cz