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« Charles Ehresmann : 100 ans »

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"CHARLES EHRESMANN : 100 ANS"
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Ce fascicule des "*Cahiers*" est consacré à la publication des résumés des conférences données au Colloque International "Charles Ehresmann : 100 ans", organisé à Amiens à l'occasion du 100ème anniversaire de la naissance de Charles (1905-1979). Les articles développés s(er)ont publiés sur le site internet consacré à Charles:

http://perso.wanadoo.fr/vbm-ehr/ChEh

Sur Charles

Charles a été un mathématicien internationalement connu dont les travaux sont à l'origine de plusieurs notions fondamentales en Topologie (espaces fibrés localement triviaux, variétés feuilletées), en Géométrie Différentielle (théorie des jets, connexions, prolongement des variétés différentiables), en Théorie des catégories (structures locales, catégories internes, théorie des esquisses). Ces travaux lui ont valu plusieurs prix de l'Académie des Sciences, et il a été nommé Docteur Honoris Causa de l'Université de Bologne en 1967. Ils sont réimprimés avec des commentaires les actualisant dans les 7 volumes de

"*Charles Ehresmann : Œuvres complètes et commentées*" (Amiens, 1980-83).

Son œuvre a eu et a encore de nombreuses applications non seulement dans diverses branches des Mathématiques, mais aussi en Physique Théorique, en Informatique, en Théorie des Systèmes et en Biologie.

Fondateur des "Cahiers" en 1958, Charles en est resté le Directeur jusqu'à sa mort.
Il a été successivement Professeur à l'Université de Strasbourg, à l'Université de Paris puis Université Paris VII (de 1957 à 1975), enfin à l'Université de Picardie après sa retraite en 1975. Ce colloque a lieu à Amiens car il s'est beaucoup investi dans cette dernière université à partir de 1970, co-dirigeant avec moi l'équipe de recherche "Théorie et Applications des Catégories, Paris-Amiens" qui a organisé à Amiens plusieurs rencontres internationales. En reconnaissance de cette implication qui a impulsé un grand dynamisme à la recherche mathématique à Amiens, son nom a été donné à un amphithéâtre de l'Université de Picardie Jules Verne.

Le Colloque

Il se déroule sur 3 jours (7-9 Octobre 2005). Le premier jour est consacré à des rappels d'intérêt général sur la vie et l'œuvre de Charles, ainsi que ses prolongements. Pendant les deux jours suivants, 3 sessions sont organisées en parallèle :

Session 1. Topologie et Géométrie Différentielle,

Session 2. Théorie des Catégories, sous la forme d'une session du SIC (Séminaire Itinérant des Catégories, organisateur Elisabeth Vaugelade), conjointe à la 82ème session du PSSL (Peripatetic Seminar on Sheaves and Logic).

Session 3. Applications multidisciplinaires sous la forme du Symposium ECHO V dont les co-organisateurs sont : George Farré (Washington), Andrée Ehresmann et Jean-Paul Vanbremeersch (qui avaient déjà organisé ECHO I à Amiens en 1996).

Les résumés des conférences publiés dans ce fascicule sont regroupés en 3 parties : Les résumés suivants directement ce texte sont relatifs à la première journée, en deuxième partie se trouvent ceux des sessions 1 et 2 (il manque les résumés de 3 auteurs qui ne nous sont pas parvenus à temps pour la publication), en troisième partie, ceux de la session 3, ECHO V. Dans chaque partie, les résumés sont ordonnés dans l'ordre alphabétique de leurs auteurs.

Remerciements

Ce Colloque ne pourrait avoir lieu sans l'aide généreuse offerte par : l'Université de Picardie Jules Verne, sa Faculté de Mathématiques et d'Informatique, et son Laboratoire de Mathématiques (LAMFA) qui a pris en charge le SIC, le Conseil Régional de Picardie et Amiens Métropole.

Je voudrais remercier plus particulièrement Michèle Weidenfeld, Doyen de la Faculté de Mathématique et d'Informatique, qui m'a aidée dans toutes les démarches matérielles, et les co-organisateurs de ce Colloque :

- Francis Borceux, Professeur à l'Université Catholique de Louvain, qui a suggéré et permis la tenue d'une session du PSSL à Amiens et qui, par le biais d'invitations conjointes à Louvain-la-Neuve, couvre généreusement les frais de plusieurs participants,
- Elisabeth Vaugelade, qui s'est entièrement occupée du SIC,
- George Farré, Professeur à Georgetown University (Washington) qui a organisé la Session multidisciplinaire ECHO V,
- Jean-Paul Vanbremeersch, qui a créé et continue à gérer le site internet consacré à Charles (ainsi d'ailleurs que celui des "Cahiers").

A.C.E.

Autour d'Ehresmann : Bourbaki, Cavailles, Lautman

par Pierre AGERON

Ce travail se propose de documenter les origines de la singulière pensée mathématique de Charles Ehresmann. Deux pistes ont été suivies: sa participation au groupe Bourbaki, de 1935 à 1950 ; ses relations avec Albert Lautman et Jean Cavaillès, deux philosophes des sciences exécutés par les Allemands en 1944.

1. La participation d'Ehresmann au groupe Bourbaki

Le fonds d'archives de Bourbaki n'étant pas encore mis à disposition, Liliane Beaulieu m'a laissé la liberté de le consulter via la base de données qu'elle élabore actuellement aux Archives Henri Poincaré de Nancy. Je l'en remercie très vivement.

C'est le 18 juillet 1935, dernier jour du congrès fondateur de Besse-en-Chandesse, qu'il fut décidé de proposer à Ehresmann la place abandonnée par Leray "en état de dégonflage". On lui assigna le rapport sur la géométrie, qu'il présenta le 6 juillet 1936 et suscita "quelques échanges de vues sur la notion d'objet géométrique, la notion d'invariant en géométrie". En septembre, il était au congrès "de l'Escorial" où l'on discuta la notion de *structure.* Bizarrement, alors qu'il fut affirmé qu'il s'agissait d'une notion première non susceptible de définition, quelques passages du compte-rendu en donnaient une définition générale précise ainsi que la définition d'une "structure locale" comme classe de structures localement équivalentes en un point d'un espace topologique (par exemple les variétés différentiables). Leur place dans le rapport de géométrie suggère qu'Ehresmann a contribué de manière significative à l'élaboration de ces définitions; il en a lui-même témoigné en privé et Dieudonné l'a ainsi confirmé : "dès avant 1940, il rêvait d'une théorie abstraite de toutes les espèces de structures possibles, ce qui rencontrait quelques réticences au sein du groupe Bourbaki". Tandis qu'Ehresmann s'acquittait de rédactions d'algèbre, Delsarte et Dieudonné rédigèrent le *Fascicule de résultats* de théorie des ensembles de 1939 où apparaît cette fameuse définition de structure. Elle suscitera très tôt, et au sein même du groupe, de très sévères commentaires.

La définition de \"structure locale" n'avait guère sa place dans le livre sur les ensembles. Au congrès de Clermont (août 1942), où Ehresmann présentait un rapport sur les revêtements en vue du livre de topologie générale, il fut noté : "il y a intérêt à étendre la théorie au cas où les espaces sont pourvus d'une structure locale". Ceci conduisait "à insérer dans le même chapitre la définition de ces structures". Ehresmann promit une rédaction en ce sens et mit aussi à l'étude "la question des revêtements du point de vue algébrique (rapports avec la théorie des groupes libres, ou définis par générateurs et relations)". Les circonstances interrompirent ces projets. A la Libération, la participation d'Ehresmann à Bourbaki se fit moins régulière : on blâma son absence au congrès "intercontinental" de Paris (juin 1945) "sous le vain prétexte de copies à corriger". Il promit des rédactions qui ne virent pas le jour. En mars 1947, Samuel exposa à Paris son formalisme des applications universelles affrontant "une assez vive opposition" : Ehresmann fut seul favorable (avec Cartan) à son insertion dans le livre d'algèbre, "parce qu'on y parle du groupe libre (!)". Las des controverses, Ehresmann s'éloigna progressivement de Bourbaki; le dernier congrès auquel il participa fut celui de Nancy (février 1950) et il donna un dernier exposé au séminaire Bourbaki en décembre 1950.

Désormais indépendant, il reprit en 1951 l'étude des structures locales, puis édifia une vaste théorie fonctorielle des structures, réélaboration profonde, mais fidèle, des idées bourbakiennes des années 1930. On y retrouve le point de vue "algébrique" du transport de structure (*Gattungen von lokalen Strukturen*, 1957) et celui de la construction de structures (*Catégorie des foncteurs types*, 1960).

2. Ehresmann et les philosophes : Cavaillès, Lautman

L'intérêt considérable pour l'épistémologie montré par Ehresmann jusque dans les années 1950, qui semble né avec sa lecture de Bergson, renforcé peut-être lors de son séjour auprès de Weyl, a été particulièrement nourri par ses relations amicales et intellectuelles avec Cavaillès et Lautman. Quelques sources publiées permettent de s'en faire une idée : une notice de Dieudonné évoquant leur rencontre à l'Ecole normale supérieure; une lettre de Cavaillès à sa sœur, avec un commentaire de celle-ci, mentionnant l'intérêt d'Ehresmann pour son travail (fin 1938); une intervention d'Ehresmann et la réponse donnée par Lautman après la conférence commune Cavaillès - Lautman de février 1939 ; un compte-rendu par Ehresmann des thèses de Cavaillès publié début 1941; un essai de Lautman écrit fin 1943 dégageant la portée d'un théorème topologique d'Ehresmann ; l'avertissement au livre posthume de Cavaillès cosigné en mai 1946 par Ehresmann et Canguilhem.

Ces sources montrent qu'Ehresmann était alors le mathématicien qui suivait et comprenait le mieux les recherches des deux jeunes philosophes. Les pensées de Cavaillès et Lautman s'opposant franchement, on peut se demander laquelle s'accordait le mieux aux conceptions d'Ehresmann.

Pour Cavaillès, les mathématiques ne sont rien d'autre qu'un développement historique autonome effectué en actes. Dans son compte-rendu précis et élogieux, Ehresmann a écrit son plein accord avec cette thèse. Elle a certainement influencé son approche de la question des fondements logiques, audacieuse et pragmatique: "la contradiction n'est que l'expérience d'un échec", relevait-t-il chez Cavaillès.

Dans l'ontologie de Lautman, les êtres mathématiques s'intègrent à des théories qui sont autant d'ébauches de solutions d'idées-problèmes dialectiques. Ehresmann appréciait cet accent mis sur les problèmes généraux qui font l'unité des mathématiques, mais contesta la domination de la métaphysique, estimant que les idées dialectiques ont vocation à être exprimées \"en termes mathématiques". Là où Lautman, en précurseur, repérait deux manifestations différentes d'une même structure dialectique (représentations d'un groupe / interprétations d'un système d'axiomes), Ehresmann assignait au mathématicien le travail d'unification conceptuelle des deux théories. Et tel fut bien le véritable sens de ses travaux.

Some of the things I learnt from Charles Ehresmann

by Jean BENABOU

The contributions of Charles Ehresmann to mathematics are so numerous and cover such a wide variety of domains, some of which he totally created or renovated, that it would be vain to try to describe even the major ones. Vain also to list all of his ideas which were directly influential to my own work.

Thus I shall mainly concentrate on what he taught me about mathematics and being a "working mathematician".

I first met Ehresmann in 1956. I was at that time a student, undecided about what I wanted to do, and knowing very little mathematics. And, even that little, I had learnt by the painful reading of a few volumes of Bourbaki. At the time the "ambiant ideology" was that Bourbaki was the ultimate standard by which mathematics should be judged, and I was not ready to spend many years and a lot of work in such a formal environment.

Thus to me Ehresmann was quite a revelation. Here was a mathematician, of great talent, who could explain deep concepts by going to the core, with almost no formalism at all, with elegance and efficiency.

Let me mention a few examples: The infinitesimal structure of differential geometry described by his notion of *jets*. The intrinsic definition of connexion without the so called tensor calculus and its tedious multi-indices. Most striking for me, both for mathematical and philosophical reasons, was the theory of *localic structures*. I had never believed, and I still do not believe, that the space I live in, or the time I feel flowing were made of points. And here was a perfectly precise, and workable, notion of space not built up from points. I had started doing a little work in differential geometry, but the idea of localic structures appealed so much to me that I abandoned this work and explored the beautiful world of locales. And even to-day, when I have learnt the more general notion of Grothendieck topologies, I tend to think of them as "generalized locales".

I could say, forgetting many important things, that what I learnt from him, and which has most influenced my work, is: "find what is essential, and, if is really essential, it should be simple".

I shall finish by saying that Charles Ehresmann belonged to a species which unfortunately tends to disappear, apart from being an eminent mathematician, in his relations with his students he was, not merely a professor, he was a Master.

Et si l'intuition devenait rigueur ?
La réponse de la géométrie différentielle synthétique

par Francis BORCEUX

Une fonction $f\colon \mathbb{R} \to \mathbb{R}$ est continue en un point $a \in \mathbb{R}$ quand pour tout ε il existe un δ tel que ... bla, bla, bla!

Mais intuitivement, la fonction f est continue en a lorsqu'une variation infinitésimale au voisinage de a provoque une variation infinitésimale au voisinage de $f(a)$. Et même si nous écrivons toujours nos démonstrations en termes de ε et δ, le cheminement intuitif de notre raisonnement – que nous nous gardons bien d'avouer – procède pourtant souvent en termes d'infinitésimaux.

Mais diable! N'y a-t-il pas moyen de donner un sens précis à la notion d'infinitésimal, de manière à pouvoir l'utiliser en toute rigueur dans les preuves?

Une première réponse, apportée par l'analyse non-standard, est de dire qu'un infinitésimal est une quantité tellement petite qu'elle est inférieure à tout nombre réel strictement positif. Et grâce à la magie de la logique mathématique, on construit effectivement un ensemble $\mathcal{R}$ – appelé ensemble des nombres réels non standards – qui contient tous les nombres réels usuels, mais aussi des éléments infiniment petits, plus petits que tous les réels strictement positifs. Et sur cette droite réelle non-standard $\mathcal{R}$ on peut développer toute l'analyse en termes d'infinitésimaux ... et retrouver comme corollaires tous les bons vieux théorèmes classiques d'analyse sur la droite réelle usuelle.

Une autre idée, due à *F.W. Lawvere* et *A. Kock*, est la suivante. Si x est petit, x^2 est encore plus petit. Si x est très, très petit, x^2 devient vraiment minuscule. Appelons donc *infiniment petit* un nombre x tel que $x^2 = 0$. On cherche alors à remplacer la droite réelle $\mathbb{R}$ par un anneau $\mathbf{R}$ dans lequel

$$D = \{x | x^2 = 0\}$$

est considéré comme l'ensemble des éléments infiniment petits.

Mais si l'on veut développer l'analyse dans un tel contexte, alors on doit en particulier disposer d'un développement de Taylor pour de bonnes fonctions $f\colon D \to \mathbf{R}$, à savoir

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \cdots$$

Et puisque x est de carré nul, cela se réduit à

$$f(x) = f(0) + xf'(0)$$

et une telle fonction f définie sur les infiniment petits est donc nécessairement du premier degré.

Cela nous conduit à l'axiome suivant, qui est l'axiome de base de ce que l'on appelle aujourd'hui la *géométrie différentielle synthétique.* Il existe des versions encore plus puissantes de cet axiome.

Axiome de **Lawvere–Kock.**
Il existe un anneau commutatif $\boldsymbol{R}$ *contenant le corps* $\mathbb{Q}$ *des nombres rationnels et tel que si*

$$D = \{x \in \boldsymbol{R} | x^2 = 0\}$$

alors toute fonction

$$f\colon D \to \boldsymbol{R}$$

s'écrit de manière unique sous la forme

$$f(x) = a + bx, \quad a, b \in \boldsymbol{R}.$$

Observons d'emblée que si

$$f\colon \mathbf{R} \to \mathbf{R}$$

est une fonction quelconque et $a \in \mathbf{R}$, on peut considérer la fonction

$$f(a + -)\colon D \to \mathbf{R}, \quad x \mapsto f(a + x)$$

qui donc s'écrit de manière unique

$$f(a + x) = \alpha_a + \beta_a x, \quad \alpha_a, \beta_a \in \mathbf{R}.$$

Faisant $x = 0$ on trouve $\alpha_a = f(a)$ et bien sûr on définit

$$f'\colon \mathbf{R} \to \mathbf{R}, \quad a \mapsto \beta_a.$$

Cela définit la dérivée de f et – itérant le processus – on conclut que toute fonction $f\colon \mathbf{R} \to \mathbf{R}$ est infiniment dérivable!

Absurde, pas vrai? Et il y a pire! Car des anneaux $\mathbf{R}$ satisfaisant l'axiome de *Lawvere-Kock* ... il est facile de prouver que cela n'existe pas! Catastrophe? Pas vraiment! Car pour établir cette "catastrophe", il faut utiliser le principe du *tiers exclu*, c'est-à-dire accepter la validité du principe de démonstration par l'absurde. Mais si l'on s'en tient à une logique dans laquelle on s'impose de ne faire que des preuves constructives, de tels anneaux $\mathbf{R}$ existent bel et bien et l'on peut en construire explicitement, dans des univers logiques constructifs que l'on appelle *topos*.

Bon, d'accord, cela existe. Mais quel intérêt y a-t-il à développer l'analyse ou la géométrie différentielle dans un contexte où toutes les fonctions $f\colon \mathbf{R} \to \mathbf{R}$ sont infiniment dérivables? Cela n'a plus rien à voir avec l'analyse et la géométrie classique. Et bien si justement! Par exemple, tous les théorèmes de géométrie différentielle que l'on peut prouver constructivement à partir de l'axiome de *Lawvere-Kock*, sont automatiquement de bons vieux théorèmes classiques de géométrie différentielle, valables pour les variétés différentiables paracompactes ... de classe C^∞.

Et oui! Si en géométrie différentielle classique on ne s'intéresse qu'aux variétés de classe C^∞, qui finalement sont les plus fréquentes, on peut tout aussi bien travailler avec les réels de *Lawvere-Kock*. Et qu'est-ce que cela simplifie drôlement la vie!

A Mathematician's Itinerary
by Andrée C. EHRESMANN

The aim of this abstract is not to give an overview of Charles' works, but to try to make clear the thread of ideas which led him naturally from Geometry to Category Theory; I'll concentrate more on his itinerary from the late fifties on, which I know from first hand for having been a constant witness of his research. All the references are to *"Charles Ehresmann : Oeuvres completes et commentées"* (abbreviated in O). 3 periods will be distinguished in his work: 1. Topology and Geometry from his thesis (1934, under the direction of Elie Cartan) to the mid fifties; 2. Intermediate period with 3 seminal papers in 1957-59; 3. Category theory in the sixties and seventies.

1. Topology and Geometry

After his thesis (O I, 3-86) which is a study of the topology of homogeneous spaces, Charles worked on locally homogeneous spaces (O I, 87-104), raising the problem: how to 'localize' a structure? His answer is the theory of structures associated to a pseudogroup of transformations G, obtained by gluing the maps of an atlas compatible with G.
Among these structures, in the forties he developed the theory of fibre bundles (O I 105-154), introducing in particular the notion of a principal bundle and of the associated bundles, and giving an elegant definition of connections in their frame (O I, 179-203). These have important applications in Physics. He also introduced foliated manifolds (1951, O I, 155-178), developed in the thesis of his student Reeb.
Fibre bundles are at the root of the theory of prolongations of manifolds he developed in the early fifties as a foundation for Differential Geometry, based on a coordinate-free notion of derivative, namely the notion of infinitesimal jets which has now become folklore. (O I, 207-235; 343-369). These different works led him to identify a unifying notion of "local structure" (O I, 411-420 ; O II, 4-20).

2. Three seminal papers

In the late fifties, he wrote 3 papers which can be seen as the basis of all his subsequent works. They represent the turning point when he explicitly used the notion of a category (he had implicitly used categories in the early fifties under the form of groupoids, and in describing the properties of the categories of jets).

2.1. *"Gattungen von Lokalen Strukturen"* (1957, O I, 126-154) generalizes the notion of a pseudogroup of transformations into that of a local category (in modern terms, a category internal to a category of locales), and he describes how such a local category can act on a local space, determining what he calls a "species of local structures". The main result is the "Theorem of complete enlargement" which describes the construction of the complete species of local structures associated to a given species of local structures, and generalizes the gluing process by which structures are associated to a pseudogroup.
In 1960-63, the concepts and results of this paper were extended in a series of papers (O II, 75-92, 155-180, 257-316; 435-562; summarized in O II, 117-125) to different kinds of

ordered categories and species of structures. These generalizations were needed for applications in Analysis and in a general theory of foliations (1961, O II, 563-626).

2.2. *"Catégories topologiques et categories différentiables"* (1959, O I, 237-250) defines the notions of a topological or differentiable category and of its actions, which he proposes to consider as defining generalized fibre bundles. To recover a principal fibre bundle, a topological groupoid must be *locally trivial* (it means that each unit x has an open neighbourhood on which there exists a local section s of the codomain map such that dom$\circ s$ is the constant map on x). Charles proves that the theory of locally trivial groupoids is equivalent to that of principal bundles, and that the spaces on which such a groupoid acts correspond to the associated fibre bundles. Later he reduced the theory of prolongations of manifolds to the differentiable actions of some categories of jets (O I. 251-274).

2.3. *"Catégorie des foncteurs type"* (1960, O IV, 101-116) is more abstract. It gives a categorical basis to the notion of a structure as defined by Bourbaki. For this, Charles introduces the double category of natural transformations and constructs "typical functors" by completing appropriate sub-categories of the double category of its squares.

3. Category Theory

3.1. *Structured categories.* At the beginning of 1963 (O III, 1-4; 431-434), Charles realized that the notions of ordered categories, topological or differentiable categories and double categories introduced in the 3 preceding papers could be unified by introducing the notion of what he called "p-structured categories" and "p-structured species of structures", where p is a concrete functor; in modern term, they correspond to internal categories and their internal actions in a concrete category. Their theory was made precise and developed in several papers (O III, 21-120; 125-279). where general theorems are given relying on specific properties of p. In particular different kinds of completions are constructed, generalizing the processes of complete enlargement and prolongations of manifolds to various situations (O III, 443-447, 467-485; O IV, 34-100, 117-279).

3.2. *Sketches.* In 1966, searching for a definition of a category as an "algebraic" structure, Charles introduced a general notion of sketch, of which one example is the sketch of a category (O III 591-677), the models of which (he called "generalized structured categories") are the categories internal to a category. (Remark that Benabou had given a different definition of an internal category in 1964, but valid only in categories with pullbacks, that is not the case for the category of differentiable manifolds and for some categories of ordered spaces Charles considered.) In later papers (O IV 19-31 and our joint 1972 paper in O IV, 407-518), to obtain general theorems the notion of a sketch was restricted to that of a mixed sketch, or even of a projective sketch (or projective topology as defined by Benabou, 1967). Among these main theorems: the construction of the prototype and of the type of a sketch, and the construction of monoidal structures on categories of models of a projective sketch. Finally we applied and refined this last result in the frame of multiple categories, in a series of 4 long papers (O IV, 545-764) the fourth one completed when we were in the hospital during Charles' last illness.

Though his categorical works might seem very abstract, up to the end Charles thought of them as parts of a general theory of Geometry, conceived as “the theory of more or less rich structures, in which are generally intertwined algebraic and topological structures”.

Charles Ehresmann, au carrefour des structures locales et algébriques

par René GUITART

En la présente occasion je voudrais me faire porte-parole de la génération de 1968 des élèves d'Ehresmann, génération formée en réalité conjointement par Charles et Andrée Ehresmann — qui travaillaient en commun depuis 1962 — et exprimer notre amitié et nos plus profonds remerciements à eux deux. Nous nous souvenons de nombreuses soirées auxquelles Andrée et Charles nous conviaient — souvent avec des mathématiciens de passage au *Séminaire* — où amicalement les discussions roulaient sur de nombreux sujets. Parce qu'il aimait partager chaleureusement ses enthousiasmes, nous savons qu'Ehresmann admirait le Tāj Mahal, Vermeer de Delf, Goethe, Bergson. Mais nous savons d'abord que par-dessus tout il aimait la compréhension mathématique des choses — sa recherche et son partage - qu'il pensait que le plus difficile en mathématique consiste à trouver la bonne définition, que les théories abouties devraient *in fine* tenir en quelques pages, quoique le processus d'invention de structures soit sans fin. Nous savons incidemment que — en profond admirateur d'Archimède — il n'aimait pas que l'on dénigre l'idée d'espace affine — alias le calcul des équilibres de points matériels — au profit de l'espace vectoriel, celui-ci n'existant géométriquement qu'attaché au choix d'un point d'origine dans ledit espace affine, et ne prenant sa valeur naturelle que croisé avec le groupoïde des translations entre les points. Son enseignement était ainsi souvent de telles *simples indications*. Ehresmann était très attentif à ses élèves, méditait longuement ce qu'il allait leur dire, leur indiquer, personnellement en privé ou bien en cours, qui pourrait les inciter à écrire effectivement, à poursuivre par eux-mêmes : "écrivez, écrivez ..." encourageait-il. Nous avons tous le souvenir de ses cours, très préparés en fait, mais où, au bout de quelque minutes il commençait à modifier encore une fois ce qui avait été soigneusement prévu, bref à faire des mathématiques dans l'instant, à penser en direct devant nous, et *en géomètre* bien sûr. Un jour, lors d'une discussion sur l'analyse non-standard et le rapport possible avec le calcul des jets, il m'avertit: "Guitart, vous raisonnez comme un logicien". C'est que pour lui la rigueur devait œuvrer à valoriser les intuitions géométriques, à les rendre vraies, plutôt qu'à les barrer en les rejetant au nom de schèmes de pensées présupposés, voire de vérités établies ; faire des mathématiques c'était chercher toujours l'exact déploiement de *nouvelles* vérités.

Á cette époque là, Ehresmann nous instruisait de deux idées : *structures locales* et *structures algébriques* — des problèmes associés : construction de complétions d'atlas, construction de structures libres — et des outils de calculs afférents : recollements et quasi-quotients. Sa pratique inventive était sous condition d'une pulsation entre le local et l'algébrique, et dans la volonté de penser mathématiquement cette pulsation : il œuvrait à convaincre que le cadre des groupoïdes et des catégories en offrait la possibilité. Le but était plus profond que d'empiler les données algébriques sur un cadre local — comme jadis avec ses fibrés — mais bien de penser l'un comme de même nature que l'autre. Ce faisant se dégageait déjà la problématique générale suivante, celle de la construction de *prototype* et de la théorie générale des *esquisses*.

Les *structures locales* — où les variétés, les espaces fibrés et les feuilletages, les revêtements, les espaces étalés, trouvaient à s'unifier — étaient définies depuis 1951. Une *espèce de structure locale* W est une espèce de structure telle que pour toute structure s sur un ensemble E, $s \in W(E)$, il existe une *structure induite* sur certaines de ses parties, ces parties formant les ouverts d'une topologie, les inductions étant transitives, et avec l'*axiome du recollement* : si $E = \cup_{i \in I} E_i$ et si pour tout i de I on a une structure s_i sur E_i telles que $E_i \cap E_j$ soit ouvert dans E_i et E_j et que les structures induites de s_i et s_j sur $E_i \cap E_j$ soient identiques, alors il existe sur E une unique structure s, notée ${}^{W}\!\int_E ((s_i)_{i \in I}) = s$, ou brièvement $\int s_i = s$, dont les E_i soient des ouverts et induisant les s_i. En fait, à une structure locale s sur E est associé le *pseudogroupe* $\Gamma_W(E, s)$ de ses automorphismes locaux, qui est un groupoïde ordonné, qui constitue l'essentiel de la structure locale, comme anciennement le groupe d'un espace d'une géométrie de Klein en était le cœur. Par là — après avoir remplacé dans la définition les topologies par des *treillis locaux*, ce qui dispensait de l'usage des points — Ehresmann va, dans les années 60, développer la suite de la théorie du local comme une théorie des groupoïdes ordonnés, puis des catégories ordonnées. On appréciera aujourd'hui le lien et l'écart avec le point de vue de la théorie des topos de Grothendieck, ultérieure aux structures locales et treillis locaux.

Au début des années 60, Ehresmann projetait un livre sur les catégories ordonnées. Il modifia ce projet car, notant le rôle de l'oubli p_{Ord} : Ord $\to$ Ens, il visa quelque chose de plus vaste, via l'axiomatisation des propriétés d'un tel foncteur, et, examina, pour des foncteurs quelconques $p : \mathcal{C} \to$ Ens l'idée d'une donnée c qui soit une *catégorie p-structurée*. Ce qui incluait comme exemples les catégories doubles, et, via les groupoïdes ordonnés, les structures locales, et aussi les groupes et groupoïdes topologiques ou différentiables indispensables à ses théories sur les prolongements de structures et calculs de jets. Cette idée est alors au centre de sa mise en scène. En fait, rétrospectivement, il y a d'un côté, en omettant c, un lien entre l'étude de p et la théorie des *foncteurs fibrants* — ce qui chez Ehresmann avait deux aspects : *catégories d'opérateurs* et *foncteurs à structures quasi-quotients* — et de l'autre côté, en omettant le foncteur p et en ne gardant que la catégorie $\mathcal{C}$ et la donnée c, il s'agit de ce que l'on appelle aujourd'hui une *catégorie c interne* à $\mathcal{C}$ (dont la première définition est due à Bénabou en 1964). Ce qui se pense naturellement aussi dans l'idée d'*esquisse (mixte)* d'une espèce de structure, introduite en 1966.

Une *esquisse projective* est un couple $\sigma = (S, P)$ où S est un graphe multiplicatif, et P un ensemble de cônes projectifs $p = (p_k : \Pi \to B_k)_{k \in K}$ dans S. Une catégorie $\mathcal{S}$ de structures est dite esquissée par σ si elle est isomorphe à la catégorie Ens^σ des réalisations R de σ dans Ens, qui sont les 'foncteurs' $R : S \to \text{Ens}$ transformant chaque $p = (p_k : \Pi \to B_k)_{k \in K} \in P$ en un cône limite projective dans Ens. Ce qui fait que toute flèche $\omega : \Pi \to B$ de S devient, dans la réalisation R, une 'composition' ω_R dite 'en R'. Ce que l'on écrit ainsi :

$$R : S \to \text{Ens}, \quad \omega : \Pi \to B,$$

$$R(\Pi) \simeq \underset{\leftarrow\, k \in K}{\lim} R(B_k), \quad \omega_R = R(\omega).(\simeq^{-1}) : \underset{\leftarrow\, k \in K}{\lim} R(B_k) \to R(B).$$

Ainsi par $\omega_R = R(\omega).(\simeq^{-1})$ des éléments x_k des $R(B_k)$, pour les k de K, supposés compatibles entre eux c'est-à-dire bien agencés — soit tels que pour toute flèche $t : k \to k'$ dans K on ait $x_{k'} = R(B_t)(x_k)$ –, peuvent être "composés" pour former $\omega_R((x_k)_{k \in K})$ qui est un élément x de $R(B)$, ce qui est noté, en rappelant en pré-exposant que ω est une loi de σ :

$${}^\sigma\omega_R((x_k)_{k \in K}) = x,$$

ce que l'on rapprochera de

$${}^W\!\!\int_E ((s_i)_{i \in I}) = s$$

vu plus haut, avec ${}^W\!\!\int$, recollement de W, à la place de ${}^\sigma\omega$, opération de σ, et E à la place de R, pour comprendre que les esquisses projectives sont analogues aux espèces de structures locales.

Dans ce cadre on trouve aussi les lois de compositions binaires : si $K = \{1, 2\}$, alors il n'y a pas de condition de compatibilité, et $\omega_R((x_k)_{k \in \{1,2\}}) = x$ est noté simplement $x_1 ._\omega x_2 = x$ ou $x_1.x_2 = x$. Et plus généralement on trouve les structures algébriques 'partout définies' unisortes de Lawvere, et les multisortes, de Bénabou. L'*esquisse projective*, suite aux *topologies projectives* de Bénabou et aux *catégories marquées* de Chevalley, est le milieu naturel ou coexistent l'opération algébrique $x_1.x_2 = x$ et le recollement $\int s_i = s$. Ainsi sont de même nature le fait algébrique que 2 fois 3 fasse 6 et le fait géométrique qu'en collant deux segments $[0, 1]$ et $[0', 1']$ par leurs extrémités — en posant $0 \equiv 0'$ et $1 \equiv 1'$ — on obtienne un cercle S^1.

On obtient des esquisses projectives pour des lois partielles, et, par exemple il existe une esquisse σ_{Cat} de la structure de catégorie, de sorte qu'une catégorie C soit assimilable à une réalisation $C : \sigma_{\text{Cat}} \to \text{Ens}$, et qu'ainsi $\text{Cat} \simeq \text{Ens}^{\sigma_{\text{Cat}}}$. Alors les catégories c internes à $\mathcal{C}$ sont les réalisations de σ_{Cat} dans $\mathcal{C}$, et forment la catégorie $\text{Cat}(\mathcal{C}) = \mathcal{C}^{\sigma_{\text{Cat}}}$. Il y a aussi bien sûr une esquisse σ_{Grd} pour les groupoïdes, une esquisse σ_{Ord} pour les ordres. Et via ces esquisses, les structures locales peuvent se spécifier comme des groupoïdes internes aux ordres, voire comme réalisations dans Ens de l'esquisse $\sigma_{\text{Grd}} \otimes \sigma_{\text{Ord}}$. Mais le rapport du local à l'algébrique est aussi autre : les esquisses projectives sont comme des espèces de structures locales généralisées. Ce qu'accentue le mouvement vers la théorie des prototypes.

Du côté des structures algébriques il y a l'existence de structures libres, par exemple la construction des groupes libres et des monoïdes libres, comme $\mathbb{N} = \{1\}^*$, avec l'idée de mot ; et du côté des structures locales, il y a la complétion d'atlas, par exemple la complétion d'ensembles ordonnés, comme $\mathbb{R} = \hat{\mathbb{Q}}$, avec l'idée de coupure. Les deux s'intègrent aux *théorèmes d'existence de p-structures libres.* Puis au niveau des esquisses, cela se fond dans un même énoncé : si $\mu : \sigma \rightarrow \tau$ est un morphisme d'esquisses projectives, alors le foncteur de composition avec μ, soit $\mathrm{Ens}^{\mu} : \mathrm{Ens}^{\tau} \rightarrow \mathrm{Ens}^{\sigma}$, admet un adjoint à gauche L_{μ}.

En 1968, dans l'amphithéâtre Hermite à l'IHP, Ehresmann nous enseignait comment construire universellement, comme complétion, le prototype $\Theta(\sigma)$ d'une esquisse σ, l'esquisse universellement associée à σ et qui soit un *prototype* c'est-à-dire telle que le graphe multiplicatif sous-jacent soit une catégorie et que les cônes distingués y soient des limites projectives. En fait, il y a, sous des conditions limitant les tailles, une esquisse projective $\sigma_{\mathrm{Esquisse}}$ dont les réalisations soient les esquisses projectives, et avec Esquisse $\simeq \mathrm{Ens}^{\sigma_{\mathrm{Esquisse}}}$, et l'on peut appliquer le théorème d'existence des L_{μ} ci-avant à la théorie des esquisses. Notamment, si l'on désire des esquisses ayant telle propriété, on peut le faire universellement par ce moyen, dès que ladite propriété est elle-même projectivement esquissable. La construction des prototypes peut se faire par ce moyen, puisque le fait d'être un prototype s'esquisse, par une esquisse projective $\sigma_{\mathrm{Prototype}}$ avec un morphisme $\mu : \sigma_{\mathrm{Esquisse}} \rightarrow \sigma_{\mathrm{Prototype}}$.

Pour ne pas laisser de place au malentendu je souligne bien que parmi les outils que j'utilise pour m'expliquer ici, les constructions $\sigma \otimes \tau$, L_{μ}, $\sigma_{\mathrm{Esquisse}}$ et $\sigma_{\mathrm{Prototype}}$, sont des résultats de Conduché, Foltz, Burroni, Lair, au tournant des années 1970, et qu'en 1968 Ehresmann ne parlait donc pas en ces termes. Mais ensuite ces manières de dire lui sont devenues usuelles, notamment dans les travaux avec Andrée sur les catégories multiples.

En tous cas, en 1968, l'idée de complétion, qui venait du côté des structures locales, trouvait naturellement et nécessairement à se répéter pour les esquisses projectives ou mixtes, au titre de la question des prototypes, ce qui confirme que les esquisses soient à penser comme des espèces de structures locales généralisées.

Et puis, si la théorie des esquisses projectives est bien comme je le suggère un élargissement minimal de la théorie des espèces de structures locales qui permette de penser comme *de même nature* le local et l'algébrique, alors en retour se dégage le problème d'importer en théorie des esquisses ce qui semblait spécifique de la théorie des structures locales. Ainsi — pour terminer par une *simple indication* à la manière d'Ehresmann — on demandera tout spécialement de construire pour une réalisation pointée $(R; B, x)$ d'une esquisse projective σ, soit une réalisation R de σ, un objet B de S, et un élément x de $R(B)$, une structure $\Gamma_{\sigma}(R; B, x)$ qui jouerait en théorie des esquisses un rôle analogue à celui du pseudogroupe $\Gamma_W(E, s)$ des isomorphismes locaux pour une structure locale (E, s). De manière à étendre aux structures algébriques ce qu'Ehresmann a fait pour les structures locales et Klein pour les géométries élémentaires. Comme aurait dit Ehresmann, il faudra y revenir une autre fois.

RESUMES DES SESSIONS : CATEGORIES, TOPOLOGIE, GEOMETRIE

SIC et PSSL

Two sessions of "Charles Ehresmann : 100 ans" are devoted to

"Topology and Differential Geometry",

and to

"Category theory",

domains in which Charles' works have had a deep influence.
They consist in the 82nd session of the PSSL and in a session of the SIC.

Here after we publish together the abstracts of the lectures given in them, in the alphabetic order of their (first) author.

Locally small Cartesian functors

by Jean BENABOU

0. Introduction The notion of locally internal fibration is fairly well known and its relevance is acknowledged. The purpose of this work is to generalize this notion in two directions. The "intended meaning" of this generalization is described naively as follows.

Let $F : \mathbf{X} \to \mathbf{X}'$ be a functor. We assume nothing about local smallness of $\mathbf{X}$ and $\mathbf{X}'$. For each triple (X, Y, f') with X, Y objects of $\mathbf{X}$ and $f' : FX \longrightarrow FY$ in $\mathbf{X}'$ we denote by $F^{-1}(f')$ the "class" $\{f : X \longrightarrow Y \in \mathbf{X} \mid Ff = f'\}$. If all these classes "are sets" we say that F is locally internal. More generally if we are interested in a class of "small" sets, e.g. finite sets we say that F is locally small if all the $F^{-1}(f')$ are small sets.

Clearly a category $\mathbf{X}$ is locally small iff the functor $\mathbf{X} \to \mathbf{1}$ is. On the other hand F can be locally small for "locally big" $\mathbf{X}$ and $\mathbf{X}'$, e.g. if F is full and faithfull, provided the one element set 1 is small.

1. Calibrations A *calibration* of a category $\mathbf{B}$ is a class $\mathcal{P}$ of maps of $\mathbf{B}$, called *proper maps*, satisfying :

($\mathcal{P}_0$) $\mathcal{P}$ contains all isos and is closed under composition.

($\mathcal{P}_1$) $\mathcal{P}$ is stable by pull-backs in the following sense :

for every $\begin{array}{c} \downarrow p \\ \xrightarrow[f]{} \end{array}$ with $p \in \mathcal{P}$ the pull-back $p' \begin{array}{c} \longrightarrow \\ \downarrow \quad \downarrow \\ \longrightarrow \end{array} p$ exists and p' is in $\mathcal{P}$.

These axioms are minimal but many important calibrations will be shown to satisfy other properties. We list the principal ones :

(Sep) $\mathcal{P}$ is *separated* if for every $p : T \to S \in \mathcal{P}$ the diagonal $T \to T \times_S T$ is in $\mathcal{P}$.

(Eq) $\mathcal{P}$ is *equational* if every split monomorphism is proper.

(Def) $\mathcal{P}$ is *definable* if for every $f \in \mathbf{B}$ there exists a map m such that the pull-back

$f' \begin{array}{c} \xrightarrow{m'} \\ \downarrow \quad \downarrow \\ \xrightarrow[m]{} \end{array} f$ exists, f' is proper, and m is universal for this property.

We give three important examples

(i) If $\mathbf{B}$ has pull-backs, the total calibration $\mathcal{T}_{\mathbf{B}}$ of $\mathbf{B}$, where all maps are proper.

(ii) If $\mathbf{B}$ is an elementary topos equipped with a topology j the calibration $\mathcal{S}_h(j)$ of sheaves :

A map $p : T \to S$ is proper if p is a sheaf of $\mathbf{B}/S$ for the induced topology, i.e. "internally" p is a family (T_s) of sheaves of $\mathbf{B}$, indexed by the object S.

(iii) Every presheaf category $\hat{\mathbf{B}}$ has a canonical calibration $Rep(\mathbf{B})$, where p is proper iff it is a representable morphism of presheaves in the sense of [**G.D**]

These three calibrations are definable, the first two are separated, and the third is iff $\mathbf{B}$ has pull-backs. Most of the axioms of [**J.M.**] fail for the three of them.

⚠ *We assume from now on that* $\mathbf{B}$ *has pull-backs and* $\mathcal{P}$ *is a fixed separated calibration of* $\mathbf{B}$

2. Proper spans If $P : \mathbf{X} \longrightarrow \mathbf{B}$ is a fibration, $\mathcal{P}$ can be lifted to a separated calibration $P^\star(\mathcal{P})$ of $\mathbf{X}$ by defining a map f of $\mathbf{X}$ to be proper iff f is cartesian and $P(f) \in \mathcal{P}$. (One has to verify the stability by pull-backs). In particular $P^\star(\mathcal{T}_{\mathbf{B}})$ is the class $\mathcal{K} = \mathcal{K}(\mathcal{P})$ of all cartesian maps of $\mathbf{X}$.

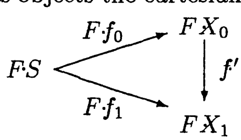
We say that a span $X_0 \xleftarrow{f_0} S \xrightarrow{f_1} X_1$ is *cartesian* (resp. *proper*) if f_0 is.

These spans can be composed in the usual way, because of $(\mathcal{P}_0)$ and $(\mathcal{P}_1)$, thus determine a bicategory $\mathcal{K}.Span(\mathbf{X})$ (resp. $\mathcal{P}.Span(\mathbf{X})$)

Every cartesian functor $F : \mathbf{X} \to \mathbf{X}'$ preserves proper maps and the composition of these spans hence induces homomorphisms of bicategories, still denoted by F,

$$\mathcal{K}.Span(\mathbf{X}) \to \mathcal{K}.Span(\mathbf{X}') \quad and \quad \mathcal{P}.Span(\mathbf{X}) \to \mathcal{P}.Span(\mathbf{X}')$$

3. Local smallness If $F : \mathbf{X} \to \mathbf{X}'$ is a cartesian functor, for all $X_0,\ X_1 \in \mathbf{X}$ and $f' : FX_0 \to FX_1 \in \mathbf{X}'$ we denote by $\mathbf{Hom}_{f'}(X_0, X_1)$ the full subcategory of $\mathcal{K}.Span(X_0, X_1)$ having as objects the cartesian spans which make commutative the triangle :



We say that F is *locally internal* (*l.i.*) if all these categories have a terminal object. If moreover these terminal objects are proper spans we say that F is *locally small* (*l.s.*). The usual notion of *l.i.* fibration is obtained for $\mathbf{X}' = \mathbf{B}, P' = id_{\mathbf{B}}$ and $F = P$.

4. First properties If F is *l.s.* and G is a cartesian functor such that FG is defined then G is *l.s.* iff FG is. (The implication G *l.s.* $\Rightarrow$ FG *l.s.* does not require the separation of $\mathcal{P}$). In particular every cartesian functor between *l.s.* fibrations is *l.s.*

5. Cleavages If we use cleavages, and neglect many canonical isomorphisms a more "classical" definition of an *l.i.* functor can be given as follows :

For each $I \in \mathbf{B}$; X_0, X_1 in the fiber $\mathbf{X}_I$ and $f' : FX_0 \to FX_1$ in the fiber $\mathbf{X}'_I$ there exists a map $h : H \to I \in \mathbf{B}$, and an $f : h^\star X_0 \to h^\star X_1$ in the fiber $\mathbf{X}_H$ such that Ff is the composite :

$$Fh^\star X_0 \xrightarrow{\sim} h^\star FX_0 \xrightarrow{h^\star(f')} h^\star FX_1 \xrightarrow{\sim} Fh^\star X_1$$

and the pair (h, f) is universal for this property. We can denote H by $\underline{Hom}_{f'}(X_0, X_1)$. Moreover F is $l.s.$ iff the map h is in $\mathcal{P}$.

6. Final remarks "en vrac".

(i) To make 5 precise we need to use many "coherence diagrams". The intrinsic notion of cartesian span, preserved by cartesian functors, permits to avoid tedious verifications. It can be used for other purposes, e.g. descent.

(ii) As one can expect, many further properties assumed, or proved, for $\mathcal{P}$ or **B**, will be shown to induce properties of local smallness defined by $\mathcal{P}$.

(iii) Locally small cartesian functors generalize representable morphisms of presheaves, and we shall see that "with a grain of salt" they can be described in these terms.

(iv) For locally small functors, the internal langage of **B** "understands" many properties of these functors, thus they have a "foundational flavour" in the sense of [Be. 3]

BIBLIOGRAPHY

[Be.1] J. Bénabou. Introduction to bicategories, Springer LNM 40, 1967, 1-77.

[Be.2] J. Bénabou. Catégories fibrées. Draft of lectures in Louvain-la-Neuve, 1983, unpublished, but widely circulated, written by J.R. Roisin. (in French).

[Be.3] J. Bénabou. Fibered categories and the foundation of naive category theory, J.Symbolic logic 50(1) 10-37 1985.

[G.D] A. Grothendieck - J. A. Dieudonné. Éléments de Géométrie algébrique. I. Springer Grundlehren Band 166, 1971.

[J.M] A. Joyal - I. Moerdijck. Algebraic set Theory. LMS Lecture Notes 220, 1995.

Action groupoids in protomodular categories

by Dominique BOURN

It is well known that, a group X being given, the group $AutX$ of automorphisms of X has the following property: given any other group G, the set $Gp(G, AutX)$ of group homomorphisms between G and $AutX$ is in bijection with the set of actions of the group X on the group G, which is itself, via the semidirect product, in bijection with the set of isomorphic classes $SExt(G, X)$ of split exact sequences:

$$1 \longrightarrow X \overset{k}{\rightarrowtail} H \underset{s}{\overset{g}{\rightleftarrows}} G \longrightarrow 1$$

This is a universal property which can be described in a very general way: consider any category $\mathbb{C}$ which is pointed (i.e. finitely complete with a zero object) and protomodular (see the precise definition below). *An object X of $\mathbb{C}$ is said to have a* split extension classifier *when there is a split extension:*

$$X \overset{\gamma}{\rightarrowtail} D_1(X) \underset{s_0}{\overset{d_0}{\rightleftarrows}} D(X)$$

which is universal in the sense that any other split extension as above determines a unique morphism χ such that, in the following diagram, the right hand side square is a pulback and the left hand side square commutes:

$$\begin{array}{ccccc} X & \overset{k}{\rightarrowtail} & H & \underset{s}{\overset{g}{\rightleftarrows}} & G \\ {\scriptstyle 1_X}\downarrow & & {\scriptstyle \chi_1}\downarrow & & \downarrow{\scriptstyle \chi} \\ X & \underset{\gamma}{\rightarrowtail} & D_1(X) & \underset{s_0}{\overset{d_0}{\rightleftarrows}} & D(X) \end{array}$$

This implies that the map χ_1 is uniquely determined by the map χ, and it is the reason of the assumption of protomodularity. Indeed a category $\mathbb{C}$ is protomodular [3] when given any split epimorphism (g, s) and any pullback diagram:

$$\begin{array}{ccc} U & \overset{\bar{h}}{\longrightarrow} & X \\ {\scriptstyle \bar{g}}\downarrow & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle s} \\ V & \underset{h}{\longrightarrow} & Y \end{array}$$

the pair $(s, \bar{h})$ is jointly strongly epic. This insures that, in the diagram defining the split extension classifier, the pair (k, s) is jointly epimorphic and the map χ_1 uniquely determined by the map χ. Of course the category Gp is the leading and guiding example of a pointed protomodular category. A pointed protomodular category $\mathbb{C}$ will be said to be *action representative* when such a split extension classifier does exist for any object X. We just observed that this is the case for the category Gp. This is also classically true for the category R-Lie of Lie algebras on a ring R, with $D(X) = Der(X)$ the Lie algebra of derivations of X. Many other examples and counterexamples are given in [2] in the more restricted context of semi-abelian categories.

Let X be an object with split extension classifier. Then by $j_X : X \to D(X)$ we shall denote the classifying map of the following upper split extension which makes the following diagram commute:

$$\begin{array}{ccccc} X & \overset{r_X}{\rightarrowtail} & X \times X & \underset{s_0}{\overset{p_0}{\rightleftarrows}} & X \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle \tilde{j}_X} & & \downarrow{\scriptstyle j_X} \\ X & \underset{\gamma}{\rightarrowtail} & D_1(X) & \underset{s_0}{\overset{d_0}{\rightleftarrows}} & D(X) \end{array}$$

We are now going to introduce the *action groupoid* associated with X: *Let X be any object with a split extension classifier. Then this classifier is underlying a groupoid structure $D_\bullet(X)$ endowed with a canonical discrete fibration $j_\bullet$: $\nabla X \to D(X)$, where ∇X is the indiscrete equivalence relation associated with X. We shall call this $D_\bullet(X)$ the* action groupoid *of the object X.*

Proof. Let us consider the following split extension, with $R[d_0]$ the kernel equivalence relation of $d_0 : D_1(X) \to D(X)$, and s_1 (according to the simplicial notations) the unique map such that $p_0.s_1 = s_0.d_0$ and $p_1.s_1 = 1_{D_1(X)}$:

$$X \overset{s_1.\gamma}{\rightarrowtail} R[d_0] \underset{s_0}{\overset{d_0}{\rightleftarrows}} D_1(X)$$

It determines a unique pair (d_1, δ_2) of arrows making the following commutative square a pullback:

$$\begin{array}{ccc} R[d_0] & \overset{\delta_2}{\longrightarrow} & D_1(X) \\ {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \\ D_1(X) & \underset{d_1}{\longrightarrow} & D(X) \end{array}$$

Since any protomodular category $\mathbb{C}$ is Mal'cev, this is sufficient (see [4]) to produce the following internal groupoid $D_\bullet(X)$:

$$R[d_0] \begin{array}{c} \xrightarrow{\delta_2} \\ \xrightarrow{d_1} \\ \xrightarrow[d_0]{} \end{array} D_1(X) \begin{array}{c} \xrightarrow{d_1} \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{array} D(X)$$

Moreover, we have $d_1.\tilde{j}_X = j_X.p_1$ since the two maps clearly classify the same split extension. This produces an internal functor $j_\bullet : \nabla X \to D(X)$ which is actually a discrete fibration. □

When $\mathbb{C} = Gp$, this action groupoid is just the internal groupoid associated with the canonical crossed module $X \to AutX$. In $\mathbb{C}$, this discrete fibration $j_\bullet$ has still a universal property (see also [1]): given any other internal groupoid:

$$R[z_0] \; \begin{array}{c} \xrightarrow{\zeta_2} \\ \xrightarrow{z_1} \\ \xrightarrow[z_0]{} \end{array} \; Z_1 \; \begin{array}{c} \xrightarrow{z_1} \\ \xleftarrow{s_0} \\ \xrightarrow[z_0]{} \end{array} \; Z_0$$

endowed with a discrete fibration $k_\bullet : \nabla X \to Z_\bullet$, there is a unique internal functor $\chi_\bullet = (\chi_0, \chi_1) : Z_\bullet \to D_\bullet$ such that $\chi_\bullet.k_\bullet = j_\bullet$. This implies that $\chi_\bullet$ is itself a discrete fibration ($\mathbb{C}$ being protomodular).

Actually this last universal property still makes sense even when the category $\mathbb{C}$ is no longer pointed, and consequently the notion of action groupoid is still valid in any protomodular category $\mathbb{C}$. The point of this communication is to claim that: 1) when the pointed protomodular category $\mathbb{C}$ is action representative, the non pointed protomodular slice category $\mathbb{C}/X$ is still action representative in this new sense, 2) the fibration $()_0 : Grd \to Set$ which associates with any groupoid its object of objects (and whose fibre above 1 is precisely Gp) has any of its (non pointed and protomodular, see [3]) fibre action representative, 3) when $\mathbb{C}$ is essesentially affine [3], it is not only Naturally Mal'cev [5], but also action representative.

References

[1] F. Borceux and D. Bourn *Split extension classifier and centrality*, Preprint 2005.

[2] F. Borceux, G. Janelidze and G.M. Kelly *On the representability of actions in a semi-abelian category*, to appear in Theory and Applications of Categories, 2005.

[3] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, Springer Lecture Notes Math. 1488, 1991, 43-62.

[4] A. Carboni, M.C. Pedicchio and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, CMS Conference Proceedings, **13**, 1992, 97-109.

[5] P.T. Johnstone, *Affine categories and naturally Mal'cev categories*, J. Pure Appl. Algebra, **61**, 1989, 251-256.

Groupoid Atlases: an Introduction

by Ronnie BROWN, A. BAK, G. MINIAN and T. PORTER

Groupoid atlases generalise *global actions*. The latter were developed by Bak for giving an approach to higher algebraic **K**-theory more combinatorial than the usual topological one, and which is also able to handle a wider class of examples, such as quadratic theories. Global actions were also found to deal helpfully with identities among relations for a group given with a family of subgroups.

Global actions give rise to groupoid atlases via the action groupoid construction. The generalisation to groupoid atlases also allows for applications to Kripke frames, and so to multiagent systems.

The intuition is that global actions and groupoid atlases allow for the patching of these algebraic structures, analogously to the idea of a manifold.
Thus this theory fits within C. Ehresmann's general aim, of developing mathematical structures for describing and dealing with local-to-global situations.

The main aim is a homotopy theory for these structures.

The paper will give some basic definitions, examples, and references, and indicate the full theory.

References

H. Abels and S. Holz, Higher generation by subgroups, *J. Alg.* 160 (1993), 311-341.
A. Bak, Global Actions: The algebraic counterpart of a topological space, Uspeki Mat. Nauk., English Translation: *Russian Math. Surveys* 525 (1997), 955-996.
A. Bak, R. Brown, G. Minian, T. Porter, *Global actions, groupoid atlases, and related topics*, UWB Preprint 99.27, 72 pages.
T. Porter, Geometric aspects of multiagent systems, *Electronic notes in computer science* 81 (2003), 26 pages.

On two non-discrete localic generalizations of π_0

by Marta BUNGE

Charles Ehresmann [1] introduced local structures in the 1950's, in the truly pioneer fashion that characterized much of his work in several areas. His reasons for so doing are well explained in [2].

In this lecture I wish to emphasize that locales have a special status in topos theory, not only on account of the fact that every (Grothedieck) topos is covered by a localic one, but also because several basic constructions in algebraic topology, even when applied to non-localic toposes, miraculously produce localic toposes. To this end, I will discuss two unpublished results, each based on a certain localic generalization of the discrete locale π_0 which is available only in the locally connected case.

The first result (given by myself at the Workshop in the Ramifications of Category Theory, Firenze, 2003) falls within the homotopy theory of toposes, and is a construction of the locale of paths components of an arbitrary topos. The locale of paths components leads to a new construction of the fundamental groupoid of a topos by paths.

The second result (ongoing joint work with Jonathon Funk) concerns an extension of the theory of singular coverings of toposes to the non-locally connected case. In the absence of (a discrete locale of) components, we are led to work with a (zero-dimensional) locale of quasicomponents, which we subtly construct and utilize in the theory of complete spreads.

References

C. Ehresmann, Structures locales, *Ann. di Mat.*, 133-142, 1952. Reprinted in "Charles Ehresmann : Oeuvres completes et commentées", Partie I (ed. A.C. Ehresmann), Amiens 1984, 411-420.

C. Ehresmann, Notice sur les travaux scientifiques de Charles Ehresmann, Paris 1955, printed in *Charles Ehresmann: Oeuvres Complètes et Commentéees,* Partie I, Amiens, 1984, 471-488.

Une autre approche des ω-catégories
par Albert BURRONI

Dans ([Str 87]), R.Street construit un objet cosimplicial de la catégorie des ω-catégories (strictes) $\mathcal{O} : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}_\omega$ (pour $n \in \mathbb{N}$, la ω-catégorie $\mathcal{O}_n = \mathcal{O}([n])$ est l'« oriental » de dimension n). De cet objet cosimplicial $\mathcal{O}$ on tire un foncteur « nerf » $N : \mathbf{Cat}_\omega \longrightarrow \mathbf{Simpl}$ qui associe fidèlement à toute ω-catégorie C un ensemble simplicial $N(C) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Ens}$ défini, pour $n \in \mathbb{N}$, par $N(C)_n = \mathrm{Hom}_{\mathbf{Cat}_\omega}(\mathcal{O}_n, C)$. Ainsi une étude géométrique de C à travers son nerf $N(C)$ est proposée. Mais la définition de Street est de nature combinatoire et difficile à maîtriser. Nous proposons, ici, une approche entièrement différente de cette construction (une version manuscrite différente, mais plus étendue est accessible en [Bu 00]). L'idée de base est simple et naturelle et donne lieu à des calculs catégoriques et algorithmiques faciles (sur ordinateur). Ce type d'idée appliquée à des catégories comme **Esp** (espaces topologiques), **Cat**, **Simpl**, ... redonne les définitions classiques, et devrait pouvoir s'appliquer également à de nouvelles catégories comme, par exemple, celle des ω-catégories laxes. Un des ingrédients utiles à ce type de construction, appliquée aux objets d'une catégorie $\mathcal{C}$, est celui d'une monade $\Lambda_{\mathcal{C}} = (\Lambda, \eta, \mu)$ sur $\mathcal{C}$ où, pour tout objet C de $\mathcal{C}$, l'objet ΛC peut être interprété comme un « cône » de base C. le morphisme $\eta C : C \longrightarrow \Lambda C$ étant le plongement de C sur la base du cône (le rôle de μ est secondaire : il est lié aux opérateurs de dégénérescences). Si la catégorie $\mathcal{C}$ est munie d'un objet initial $\emptyset$, on associe à cette monade l'objet cosimplicial $\mathcal{O}^{(\Lambda)} : \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$ d'où l'on tire un foncteur nerf $N^{(\Lambda)} : \mathcal{C} \longrightarrow \mathbf{Simpl}$ en posant, pour $n \in \mathbb{N}$, $\mathcal{O}_n^{(\Lambda)} = \Lambda^{(n+1)}\emptyset$ et où les opérateurs faces sont donnés par les transformations naturelles $\eta^i \Lambda \eta^{n-i}$. Pour $\mathcal{C} = \mathbf{Cat}_\omega$, la signification de ce cône ΛC est de compléter librement C par un objet ω-initial (ou objet initial laxe). La suite des orientaux $\mathcal{O}_n$ s'obtient ainsi par itérations successives du foncteur Λ à partir de $\emptyset$ (dimension -1).

La notion de cône d'une ω-catégorie C dérive de celle de cylindre $I \otimes C$ obtenu comme produit monoïdal de C et de $I = (0 \longrightarrow 1)^*$, catégorie à deux objets et un seul morphisme non trivial. Un ω-foncteur (strict) $I \otimes C \longrightarrow D$ est interprétable comme une tranformation naturelle laxe $F_0 \Longrightarrow F_1 : C \longrightarrow D$ entre deux ω-foncteurs (stricts). Le cône ΛC est obtenu en réduisant la première face du cylindre $I \otimes C$ à un point. Pour les transformations naturelles laxes cela revient à imposer à F_0 d'être constant. Ces concepts nous servirons de guide pour une définition du cône Λ

Voici quelques rappels terminologiques sur les ω-catégories (appelées ∞-catégories dans [Bu 93]) ; pour simplifier, toute n-catégorie ($n \in \mathbb{N}$) est identifiée à une ω-catégorie dont les cellules de dimension stictement supérieure à n sont des identités).

Un ω-graphe est défini par une suite d'ensembles $\mathrm{c}_n(C)$ $(n \in \mathbb{N})$ et une suite double de couples d'applications de la forme $\mathrm{s}_i^n, \mathrm{t}_i^n : \mathrm{c}_n(C) \rightrightarrows \mathrm{c}_i(C)$ telles que $\mathrm{s}_j^i \mathrm{s}_i^n = \mathrm{s}_j^i \mathrm{t}_i^n$ et $\mathrm{t}_j^i \mathrm{s}_i^n = \mathrm{t}_j^i \mathrm{t}_i^n$ $(0 \leq j < i < n)$. Une ω-catégorie (stricte) est la donnée d'un ω-graphe muni, pour tout $n \in \mathbb{N}$ de n lois de composition et n lois d'unité, notées respectivement $*_i^n$ et id_i^n $(0 \leq i < n)$. Les axiomes bien connus d'associativité, d'échange et de neutralité (stricts) doivent être satisfaits.

On construit la ω-catégories cône ΛC par générateurs et relations, dimension par dimension. Les outils sont les suivants : pour tout $n \in \mathbb{N}$, soit $\mathbf{e}_n$, la n-catégorie qui représente le foncteur d'oubli $\mathrm{c}_n : \mathbf{Cat}_n \longrightarrow \mathbf{Ens}$, i.e. $\mathrm{c}_n(C) \simeq \mathrm{Hom}_n(\mathbf{e}_n, C)$. Soit $\partial_n \mathbf{e}_{n+1}$ le n-squelette de $\mathbf{e}_{n+1}$. Soient $j_n : \partial \mathbf{e}_{n+1} \longrightarrow \mathbf{e}_{n+1}$ l'inclusion canonique et $q_n : \partial \mathbf{e}_{n+1} \longrightarrow \mathbf{e}_n$ le quotient dont la restriction à $\partial_{n-1}\partial_n \mathbf{e}_{n+1}$ est une inclusion, et considérons un ω-foncteur de la forme $\varphi : X \cdot \partial_n \mathbf{e}_{n+1} \longrightarrow C$ où $X \cdot \partial_n \mathbf{e}_{n+1}$ est un coproduit de X copies de $\partial_n \mathbf{e}_{n+1}$. On notera $C[\varphi]$ et C/φ les sommes amalgamées dans $\mathbf{Cat}_\omega$ explicitées par les diagrammes suivant :

$$\begin{array}{ccccc}
X \cdot \mathbf{e}_{n+1} & \xleftarrow{X \cdot j_n} & X \cdot \partial_n \mathbf{e}_{n+1} & \xrightarrow{X \cdot q_n} & X \cdot \mathbf{e}_n \\
\downarrow & & \downarrow \varphi & & \downarrow \\
C[\varphi] & \xleftarrow{\hat{j}_n} & C & \xrightarrow{\hat{q}_n} & C/\varphi
\end{array}$$

Le cône ΛC est alors la limite inductive d'une double suite C_n et C'_n de ω-catégories. $C_{-1} = C$, $\hat{j}_0 : C_{-1} \longrightarrow C'_0 = C[\lambda_0]$ (adjonction d'un unique objet λ_0). Supposons déjà définie la suite $C_{-1} \xrightarrow{\hat{j}_0} C'_0 \xrightarrow{\hat{q}_0} C_0 \xrightarrow{\hat{j}_1} \cdots C_{n-1} \xrightarrow{\hat{j}_n} C'_{n-1} \xrightarrow{\hat{q}_n} C_n$, on pose $C'_{n+1} = C_n[\lambda_{n+1}]$ où : $\lambda_{n+1} : \mathrm{c}_n(C) \cdot \partial_n \mathbf{e}_{n+1} \longrightarrow C_n$ est déterminée, pour tout $x \in \mathrm{c}_n(C)$, par les formules :

$$\begin{aligned}
\mathrm{s}_n \lambda_{n+1} x &= \lambda_n \mathrm{t}_{n-1} x \quad (\text{si} \quad n > 0), \qquad \mathrm{s}_0 x = \lambda_0 \quad (\text{sinon}), \\
\mathrm{t}_n \lambda_{n+1} x &= x *_0 (\lambda_1 \mathrm{s}_0 x) *_1 (\lambda_2 \mathrm{s}_1 x) *_2 \cdots *_{n-2} (\lambda_{n-1} \mathrm{s}_{n-2} x) *_{n-1} (\lambda_n \mathrm{s}_{n-1} x).
\end{aligned}$$

et de façon analogue $C_{n+1} = C'_{n+1}/\rho_{n+2}$ doit conduire, dans ΛC_{n+}, aux égalités :

$$\begin{aligned}
\lambda_{n+1}(y *_j x) &= (\mathrm{t}_{j+1} y) *_0 (\lambda_1 \mathrm{s}_0 x) *_1 (\lambda_2 \mathrm{s}_1 x) \ldots \\
&\qquad *_{j-1} (\lambda_j \mathrm{s}_{j-1} x) *_j (\lambda_{n+1} x) *_{j+1} (\lambda_{n+1} y), \\
\lambda_{n+1}(\mathrm{id}^n x) &= \mathrm{id}^{n+1} \lambda x.
\end{aligned}$$

On tire de ces formules les algorithmes de calculs des $\mathcal{O}_n$ (voir version longue).

Références

[Bu 93] A.BURRONI. *Higher-dimensional word problems with application to equationnal logic.* T.C.S. **115**(1) (1993) 41-42.

[Bu 00] A.BURRONI. *A new calculation of the orientals of Street.* Oxford (mai 2000) (manuscript). http ://www.math.jussieu.fr/ burroni/

[Str 87] R.STREET. *The algebra of oriented simplexes.* J.P.A.A. **49** (1987) 283-335.

Cartesian Bicategories II

by A. CARBONI, G.M. KELLY and R.J. WOOD

We recall that in [1] a locally ordered bicategory $\mathbf{B}$ was said to be *cartesian* if the subbicategory of left adjoints, Map$\mathbf{B}$, had finite products; each hom-category $\mathbf{B}(B,C)$ had finite products; and a certain derived tensor product on $\mathbf{B}$, extending the product structure of Map$\mathbf{B}$, was pseudofunctorial. It was shown that cartesian structure provides an elegant base for sets of axioms characterizing bicategories of

i) relations in a regular category

ii) ordered objects and order ideals in an exact category

iii) additive relations in an abelian category

iv) relations in a Grothendieck topos.

Notable was an axiom, here called *groupoidalness*, that captures the *discrete* objects in a cartesian locally ordered bicategory and gives rise to a very satisfactory approach to duals.

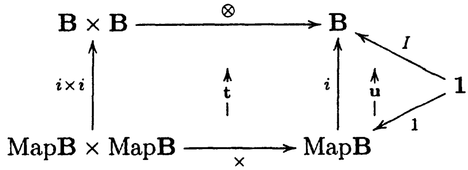
It was predicted in [1] that the notion of cartesian bicategory would be developable without the restriction of local orderedness, so as to capture

v) spans in a category with finite limits

vi) profunctors in an elementary topos.

It is this development of the unrestricted notion of cartesian bicategory that is our present concern. It turns out that the description of cartesianness given in the first sentence above, which was only an alternative characterization in [1], carries over more easily to the general case than the original definition in [1].

Merely assuming of a bicategory $\mathbf{B}$ that Map$\mathbf{B}$ has finite products (in the sense appropriate for bicategories) and that each hom-category $\mathbf{B}(B,C)$ has finite products — in which case we say that $\mathbf{B}$ is *precartesian* — it is possible to define canonical lax functors $\otimes$ and I and lax natural transformations $\mathbf{t}$ and $\mathbf{u}$ as below, where $\times$ and $\mathbf{1}$ are the pseudofunctors providing the finite products for Map$\mathbf{B}$ and i is the inclusion.



On objects $B \otimes C$ is the product in Map$\mathbf{B}$ while, for general arrows $R: B \to D$ and $S: C \to E$, $\otimes$ is described by the functorial formula $R \otimes S = p^*Rp \wedge r^*Sr$, where we write p and r for first and second projections in Map$\mathbf{B}$, p^* and r^* for their right adjoints, and $- \wedge -$ for the binary product in a hom-category. The lax functor $I: \mathbf{1} \to \mathbf{B}$ is given by the monad on the terminal object I of MapB whose underlying arrow $\top : I \to I$ is the terminal object in $\mathbf{B}(I, I)$. Now a bicategory $\mathbf{B}$ is *cartesian* if it is precartesian and moreover the constraints $\otimes^\circ : 1_{B\otimes C} \to 1_B \otimes 1_C$, $\tilde{\otimes} : (T \otimes U)(R \otimes S) \to (TR) \otimes (US)$ of $\otimes$ and the constraint $I^\circ : 1_I \to \top$ of I are invertible (the last implying the invertibility of the constraint $\tilde{I} : \top\top \to \top$). Invertibility of these 2-cells makes $\otimes$ and I pseudofunctors and it also makes $\mathbf{t}$ and $\mathbf{u}$ into invertible pseudonatural transformations. Note that when MapB has finite products, the existence of finite products in the $\mathbf{B}(B, C)$ is equivalent to the existence of finite products in the Grothendieck bicategory arising from the pseudofunctor $\mathbf{B}(-,-) : (\mathrm{Map}\mathbf{B})^{\mathrm{op}} \times \mathrm{Map}\mathbf{B} \to \mathbf{CAT}$ and their preservation by the associated pseudofibration; this explains the close connection between cartesian biategories and the cartesian objects studied in [2] and [3].

The structure $(\mathbf{B}, \otimes, I)$ admits canonical constraints, of associativity and so on, that make it a monoidal bicategory. Thus $\otimes$ may be regarded as an 'outer level' composition and the *dual* B° of an object B can be defined in terms of arrows $N : I \to B^\circ \otimes B$ and $E : B \otimes B^\circ \to I$ satisfying the *triangle equations* to within coherent isomorphisms. In particular the groupoidal condition of [1] can be expressed in the general context, and the groupoidal objects have duals for which $B^\circ = B$. An object B is *posetal* if the unit $\eta_B : 1_B \to d_B^* d_B$ of the adjunction for the diagonal map $d_B : B \to B \otimes B$ is invertible. The *discrete* objects in a general cartesian bicategory are those which are both groupoidal and posetal, these playing a major role in the characterizations of the bicategories of v) and vi) above. Our characterizations require further axioms involving existence of Eilenberg-Moore objects for comonads and Kleisli objects for monads.

References

[1] A. Carboni and R.F.C Walters. *Cartesian bicategories I.* J. Pure Appl. Algebra 49 (1987), 11–32.

[2] A. Carboni, G.M. Kelly, and R.J. Wood. *A 2-categorical approach to change of base and geometric morphisms I.*, Cahiers top. et géom diff. XXXII-1 (1991), 47–95.

[3] A. Carboni, G.M. Kelly, D. Verity and R.J. Wood. *A 2-categorical approach to change of base and geometric morphisms II.*, TAC v4 n5 (1998), 73–136.

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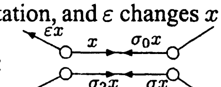
Cellular maps as an introduction to the combinatorial topology of surfaces

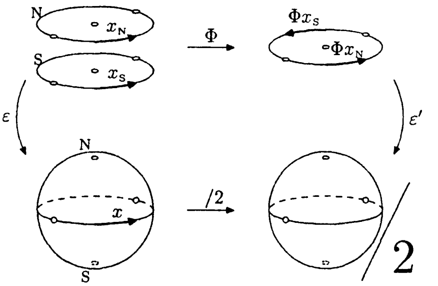
by Pierre DAMPHOUSSE

The classical point of view on *cellular maps* on a surface X is the datum of a graph c realized on X, the complement of which is a union of open disks. Two maps (X, c) and (X', c') are *isomorphic* if there is a homeomorphism $X \longrightarrow X'$ inducing a homeomorphism $c \longrightarrow c'$. This naive point of view is however not sufficient to yield a complete description of cellular maps.

Let the *polygon of order* n, written P_n, be the closed unit disk minus the n n-th roots of 1, and the *polygon of order zero* be $\mathbb{R} \times [0, 1) - \mathbb{Z} \times \{0\}$. We define a *cellular map* as the datum of a continuous surjection ϵ from a sum of polygons $\coprod_i \mathrm{P}^{(i)}_{k_i}$ (the *cut-out*) on a surface X (the *underlying surface*) such that (1) ϵ maps homeomorphically each connected component of $\coprod_i \mathrm{P}^{(i)}_{k_i}$ on its image, (2) ϵ maps homeomorphically each connected component of $\partial \coprod_i \mathrm{P}^{(i)}_{k_i}$ on its image, and (3) for each $t \in \partial \coprod_i \mathrm{P}^{(i)}_{k_i}$, $\epsilon^{-1}\epsilon(t)$ is made up of exactly two points. The covering maps $\mathrm{P}_{kn} \longrightarrow \mathrm{P}_n$ and $\mathrm{P}_0 \longrightarrow \mathrm{P}_n$ $(n, k \neq 0)$ and $x + iy \longmapsto (1 - y)\mathrm{e}^{\frac{i(x+k_0)\pi}{n}}$ $(k_0 \in \mathbb{N})$, are called *windings*; a morphism between two maps ε and ε' is a sum of windings between their cut-outs which is compatible with ϵ and ϵ'. The figure shows a morphism from a map on the sphere to a map on the projective plane (spherical coordinates are used for the points on the sphere). The category of maps and their morphisms is written $\mathcal{C}$.

The classical point of view considers oriented edges of a map, which we call *arcs*, as for example arc x on the figure. But the important things seem to be the oriented edges of the cut-out, which we call *double-arcs*. This amounts to consider that we orient arcs longitudinally and transversally; for example, choosing x_N to represent x is morally seeing x as going (transversally) from north to south. The set of all double-arcs of a map C is written $C^{\ddagger}$. This set has a natural structure as we now describe. Let $\mathbb{K} = \langle \sigma_0, \sigma_2 : \sigma_0^2 = \sigma_2^2 = (\sigma_0\sigma_2)^2 = 1 \rangle$ be KLEIN's four-group —we will use σ for $\sigma_0\sigma_2$—, and $\mathbb{E} = \langle \varepsilon : \varepsilon^2 = 1 \rangle$. The free product $\mathbb{G} = \mathbb{E} * \mathbb{K}$ canonically acts on the double-arcs of any map. σ_0 reverses the longitudinal orientation, σ_2 reverses the transversal orientation, and ε changes x to the unique double arc with same "source" but reversed orientation : 

. Let us observe that the four elements ε, σ_0, σ_2, σ have act without fixed points. A *maquette* is a $\mathbb{G}$-set upon which ε, σ_0, σ_2, σ act without fixed points. Therefore, to each map C, corresponds its maquette $C^{\ddagger}$. Maquettes and $\mathbb{G}$-invariant maps form a category $\mathcal{M}$, and it is near trivial that $C \longmapsto C^{\ddagger}$ extends to a functor $\mathcal{C} \longrightarrow \mathcal{M}$; we write it $(\,)^{\ddagger}$.



$$\Phi(\rho,\theta) = \begin{cases} (\rho,\theta) & \text{si } (\rho,\theta) \in N \\ (\rho,\theta+\pi) & \text{si } (\rho,\theta) \in S \end{cases}$$

$$\varepsilon(\rho,\theta) = \begin{cases} (\theta, \arccos\rho) & \text{si } (\rho,\theta) \in N \\ (\theta, -\arccos\rho) & \text{si } (\rho,\theta) \in S \end{cases}$$

$$\varepsilon'(\rho,\theta) = (\theta, \arccos\rho) \bmod (\pi,\pi)$$

/2 : antipodality.

FIGURE :A morphism between two maps, arcs and double arcs.

With these definitions and observations at hand, we now state :

THEOREM 1. (Classification theorem) — $()^{\dagger}: \mathcal{C} \to \mathcal{M}$ *is an equivalence of categories ; therefore, maquettes completely classify cellular maps.*

There are many consequences to this theorem. It is clear that a map is connected if and only if its maquette is transitive. The full subcategory $\mathcal{C}_c$ of connected maps is naturally equivalent to the full subcategory $\mathcal{M}_t$ of transitive maquettes, which has obviously an initial object, that is $\mathbb{G}$ itself. Hence $\mathcal{C}_c$ has also one, which means that there exists a connected map which is a universal covering of all connected maps. This is a non trivial topological fact. There is no room here to describe this universal map.

THEOREM 2. (Existence of a universal map) — *There exists a connected map which is a covering of all connected maps.*

$\mathcal{C}$ contains the full subcategory of oriented maps ; therefore, there is an analogue for $\mathcal{M}$. Since the presentation $\mathbb{G} = \langle \varepsilon, \sigma_0, \sigma_1 : \varepsilon^2 = \sigma_0^2 = \sigma_1^2 = (\sigma_0\sigma_2)^2 = 1 \rangle$ has only words of even length, elements of $\mathbb{G}$ which can be described by words of even (resp. odd) length can only be described by words of even lengths (resp. odd lengths) in this presentation. Hence, $\mathbb{G}$ has an "alternate subgroup", which we write $\mathbb{G}_+$.

THEOREM 3. (On orientability) — *A map C is orientable if and only if in its maquette any of its orbits is the sum of two $\mathbb{G}_+$-orbits. In this case, choosing a $\mathbb{G}_+$-orbit in each $\mathbb{G}$-orbit is equivalent to orienting C.*

Theorem 2 is a non trivial topological result from an obvious algebraic observation. Theorem 3 is an algebraic result hinted at by a topological fact. There are many other results in both directions which are a consequence of theorem 1. Among these results, we state that any map has a minimal regular covering (all "countries" are of the same order, all "vertices" are of the same order) which is finite, that a maquette is naturally the quotient of a "two-sheeted maquette" (the analogue of the two-sheeted covering), that $\mathcal{C}$ has products and limits easy to describe, that the group of outer automorphisms of $\mathbb{G}$ is $\mathfrak{S}_3$ and that, by its actions on the actions of $\mathbb{G}$, it shows natural constructions other than duality, that there is a natural arithmetics of the embeddings of $\mathcal{C}$ in itself ($\mathcal{C}$ is "fractal" in a technical sense), etc. These results and others are developped and used in a book on combinatorial topology being prepared. Some of the theorems are to be found in "Pierre Damphousse, Topological cartography, *PhD thesis*, Université de Paris-Sud, centre d'Orsay, 1981".

On centres and lax centres for promonoidal categories

by Brian DAY, Elango PANCHADCHARAM and Ross STREET

In celebration of the hundredth anniversary of Charles Ehresmann's birth

Our purpose is to highlight the notions of lax braiding and lax centre for a monoidal category and more generally for a promonoidal category. Lax centres are lax braided. Generally the centre is a full subcategory of the lax centre, however it is sometimes the case that the two coincide. There is always an adjunction involving the (lax) centre of a presheaf category and the presheaf category on the (lax) centre. In important cases the adjunction is an equivalence of (lax) braided monoidal categories.
One reason for being interested in the lax centre of a monoidal category is that, if an object of the monoidal category is equipped with the structure of monoid in the lax centre, then tensoring with the object defines a monoidal endofunctor on the monoidal category. This has applications in cases where the lax centre can be explicitly identified (as in the presheaf cases mentioned above).

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Bifiltered colimits of categories

by Ed. J. DUBUC, A. JOYAL and R. STREET

We introduce the notions of *2-filtered 2-category* and *pseudo 2-filtered 2-category.* Filtered or pseudofiltered categories, considered as trivial 2-categories, are 2-filtered or pseudo 2-filtered respectively. We give an explicit construction of the bicolimit of a category valued 2-functor defined on a 2-pseudofiltered 2-category and prove some of its properties. The following definitions are better understood using diagrams and 2-cell diagrams, but we lack the space here to do so.

0.1 Definition. A 2-category is defined to be *pseudo 2-filtered* when:

*F*1. *Given* $A \xleftarrow{f} D \xrightarrow{g} B$,
there exists

$$A \xrightarrow{u} C \xleftarrow{v} B\ ,\ \gamma : uf \xrightarrow{\cong} vg.$$

*F*2. *Given* $A \xleftarrow{f} D \xrightarrow{g} B$, *and two*

$$A \xrightarrow{u} C \xleftarrow{v} B\ ,\ \gamma : uf \xrightarrow{\cong} vg,$$
$$A \xrightarrow{u'} C' \xleftarrow{v'} B\ ,\ \gamma' : u'f \xrightarrow{\cong} v'g.$$

there exists

$$C \xrightarrow{w} W \xleftarrow{w'} C'\ ,\ \alpha : wu \xrightarrow{\cong} w'u',\ \beta : wv \xrightarrow{\cong} w'v',$$

such that $(wuf \xrightarrow{w\gamma} wvg \xrightarrow{\beta g} w'v'g) = (wuf \xrightarrow{\alpha f} w'u'f \xrightarrow{w'\gamma'} w'v'g)$

0.2 Definition. A 2-category is defined to be *2-filtered* when it is pseudo 2-filtered, non empty, and connected (as a category). It follows that in addition to F1 and F2 it satisfies:

*F*0. *Given* A, B, *there exist* $A \xrightarrow{u} C \xleftarrow{v} B$.

Let $\mathcal{A}$ be a pseudo 2-filtered 2-category and $F : \mathcal{A} \to \mathcal{C}at$ a category valued 2-functor.

0.3 Construction LL. We shall now define a category $\mathcal{L}$ which is to be the bicolimit (that is, the universal pseudocone) of F. The objects are pairs (x, A) with $x \in FA$. A *premorphism* is a triple $(u, \xi, v) : (x, A) \to (y, B)$, where $A \xrightarrow{u} C \xleftarrow{v} B$ and $\xi : (Fu)x \to (Fv)y$ in FC.

We abuse notation and write $F \xrightarrow{x} A$ in $(\mathcal{C}at^{\mathcal{A}})^{op}$ for the natural transformation $\mathcal{A}[A, -] \to F$ defined by $x_C(A \xrightarrow{u} C) = (Fu)x \in FC$. In this way, a premorphism consists of:

$$A \xleftarrow{x} F \xrightarrow{y} B\ ,\quad A \xrightarrow{u} S \xleftarrow{v} B\ ,\ \xi : ux \longrightarrow vy.$$

Two premorphisms $(u,\ \xi,\ v)$, $(u',\ \xi',\ v')$ are *equivalent* if there exists $S \xrightarrow{w} W \xleftarrow{w'} S'$, $\alpha : wu \xrightarrow{\cong} w'u'$, $\beta : wv \xrightarrow{\cong} w'v'$, such that: $(wux \xrightarrow{w\xi} wvy \xrightarrow{\beta y} w'v'g) = (wux \xrightarrow{\alpha x} w'u'x \xrightarrow{w'\xi'} w'v'y)$.

Equivalence is indeed an equivalence relation, and morphisms are defined to be equivalent classes of premorphisms.

We now look at composition: $(x,\ A) \xrightarrow{(u,\xi,v)} (y,\ B) \xrightarrow{(h,\zeta,k)} (z,\ C)$

$$A \xleftarrow{x} F \xrightarrow{y} B\,,\quad A \xrightarrow{u} S \xleftarrow{v} B\,,\quad \xi : ux \longrightarrow vy.$$
$$B \xleftarrow{y} F \xrightarrow{z} C\,,\quad B \xrightarrow{h} T \xleftarrow{k} C\,,\quad \zeta : hy \longrightarrow kz.$$

Let $S \xrightarrow{r} W \xleftarrow{s} T$, $\gamma\ :\ rv \xrightarrow{\cong} sh$ be given by axiom F1 on $S \xleftarrow{v} B \xrightarrow{h} T$. Then $(x,\ A) \xrightarrow{(h,\zeta,k)\circ(u,\xi,v)} (z,\ C)$ is given by:

$$A \xrightarrow{ru} W \xleftarrow{sk} C\ ,\quad rux \xrightarrow{r\xi} rvy \xrightarrow{\gamma y} shy \xrightarrow{s\zeta} skz.$$

This composition is, up to equivalence, independent of the choice of representatives $(u,\ \xi,\ v)$, $(h,\ \zeta,\ k)$, and independent of the choice of r, s, γ. Associativity and identities hold, so the construction above actually defines a category.

0.4 Theorem. *The following data defines a pseudocone* $F \xrightarrow{\lambda} \mathcal{L}$ *:*

$$x \in FA,\ \xi \in FA,\ A \xrightarrow{u} B :$$
$$\lambda_A(x) = (x,\ A)$$
$$\lambda_A(\xi) = (id_A,\ \xi,\ id_A)$$
$$\lambda_u x = (u,\ id_{(Fu)x},\ id_B)$$

$$\begin{array}{ccc} FA & \xrightarrow{\lambda_A} & \\ {\scriptstyle Fu}\downarrow & \cong_{\lambda_u} & \mathcal{L} \\ FB & \xrightarrow{\lambda_B} & \end{array}$$

which induces an equivalence of categories $Cat[\mathcal{L},\ \mathcal{X}] \xrightarrow{\cong} \mathcal{PC}[F,\ \mathcal{X}]$.

0.5 Theorem. *Given any finite diagram* $\mathcal{I} \longrightarrow \mathcal{L}$*, there exist* $A \in \mathcal{A}$ *and a factorization*

$$\begin{array}{ccccc} & & FA & & \\ & \nearrow & \cong & \searrow^{\lambda_A} & \\ \mathcal{I} & & \longrightarrow & & \mathcal{L} \end{array}$$

0.6 Corollary. *If the 2-functor takes its values in the category* Cat_{fl} *of categories with finite limits and finite limit preserving functors, then the category* $\mathcal{L}$ *has finite limits, the functors* λ_A *preserve finite limits, and induce an equivalence of categories* $Cat_{fl}[\mathcal{L},\ \mathcal{X}] \xrightarrow{\cong} \mathcal{PC}_{fl}[F,\ \mathcal{X}]$.

A few points on Directed Algebraic Topology

by Marco GRANDIS (*)

Dedicated to Charles Ehresmann, on the centennial of his birth

Directed Algebraic Topology studies 'directed spaces' in some sense, where paths and homotopies cannot generally be reversed; for instance: simplicial and cubical sets, ordered topological spaces, 'spaces with distinguished paths', 'inequilogical spaces', etc. Its present applications deal mostly with the analysis of concurrent processes (see [2, 3] and references there), but its natural range should cover non reversible phenomena, in any domain.

Here, after a review of a series of paper devoted to this subject ([4] to [8]), we shall give some hints at future developments. A wider literature can be found in the papers mentioned above.

Directed spaces can be studied with directed versions of the classical tools of Algebraic Topology. Thus, the directed homology groups $\uparrow H_n(X)$ introduced in [5, 6] are preordered abelian groups. Similarly, the fundamental category $\uparrow \Pi^1(X)$ [4, 7] replaces the classical fundamental groupoid and allows one to study situations where all directed loops are trivial. The study of higher fundamental categories has begun [8].

Directed Algebraic Topology has thus a deep interaction with ordinary and higher dimensional category theory. Unexpected links with non-commutative Geometry have appeared, brought about by orbit spaces or spaces of leaves which would be trivial in ordinary topology but can be realised in both domains. For instance, the well-known irrational-rotation C*-algebras A_ϑ represent 'non-commutative spaces' which can also be modelled by cubical sets and inequilogical spaces, so that the classification of the former by ordered K-theory [10, 11] corresponds to the classification of the latter by directed homology [5]. Other links have appeared between the notion of root of a category developed by A.C. Ehresmann [1], for modelling biological systems, and our study of the fundamental category [7].

Finally, we shall examine some links with Differential Geometry and describe the beginning of a formal setting for Directed Algebraic Topology, based on Kan ideas (formalising the (co)cylinder functor, cf. [9]) rather than on Quillen model structures, which do not seem to be adapted to abstract privileged directions and directed homotopies.

I am grateful for the opportunity of presenting these results in this Conference and on this Journal. The position of Directed Algebraic Topology, at the confluence of Topology, Geometry and Category Theory, can presumably be viewed as coherent with the research lines pursued by Charles and Andrée C. Ehresmann, in their work and with this Journal.

Main references

[1] A.C. Ehresmann, Localization of universal problems. Local colimits, *Appl. Categ. Structures* 10 (2002), 157-172.

[2] L. Fajstrup, M. Raussen, E. Goubault, E. Haucourt, Components of the fundamental category, *Appl. Categ. Structures* 12 (2004), 81-108.

[3] E. Goubault, Geometry and concurrency: a user's guide, in: Geometry and concurrency, *Math. Structures Comput. Sci.* 10 (2000), n° 4, 411-425.

[4] M. Grandis, Directed homotopy theory, I. The fundamental category, *Cahiers Top.. Géom. Différ. Catég.* 44 (2003), 281-316.

[5] M. Grandis, Directed combinatorial homology and non-commutative tori (The breaking of symmetries in algebraic topology), *Math. Proc. Cambridge Philos. Soc.* 138 (2005), 233-262.

[6] M. Grandis, Inequilogical spaces, directed homology and non-commutative geometry, *Homology Homotopy Appl.* 6 (2004), 413-437.

[7] M. Grandis, *The shape of a category up to directed homotopy*, Preprint, Dip. Mat. Univ. Genova 509 (2004).

[8] M. Grandis, *Modelling fundamental 2-categories for directed homotopy,* Preprint, Dip. Mat. Univ. Genova 527 (2005).

[9] D.M. Kan, Abstract homotopy I, *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 1092-1096.

[10] M. Pimsner - D. Voiculescu, Imbedding the irrational rotation C*-algebra into an AF-algebra, *J. Operator Th.* 4 (1980), 93-118.

[11] M.A. Rieffel, C*-algebras associated with irrational rotations, *Pacific J. Math.* 93 (1981), 415-429.

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Moving Logic, from Boole to Galois

by René GUITART

If a group G acts on a set X and if $(\wedge_g, \neg_g)_{g \in G}$ is a family of boolean structures on X such that $gx \wedge_{gh} gy = g(x \wedge_h y)$ and $\neg_{gh} gx = g(\neg_h x)$, then we speak of a *G-moving boolean logic* or shortly of a G-moving logic. In fact every such datum is equivalent to the datum of the action of G on X and of one boolean structure $(\wedge, \neg)$ on X: then we recover $(\wedge_g, \neg_g)$ by $x \wedge_g y = g(g^{-1}x \wedge g^{-1}y)$ and $\neg_g x = g(\neg g^{-1}x)$.

If $V = (\wedge_i, \neg_i)_{i \in I}$ is a family of boolean structures on a set X then we say that a function $f : X^k \to X$ is a *V-moving boolean* function if f can be defined using constants in X, $\wedge_i$, $\neg_i$, with possibly various $i \in I$ occuring in it.

Theorem 1. *For every finite set X of cardinal 2^n with n odd (resp. even), there is a family V of 4 (resp. 3) boolean structures on X — different but isomorphic, and with the same 'false'$= 0$ and the same addition '$+$' — such that, for every integer k, every function $f : X^k \to X$ is a V*-moving boolean function.

As for *Specular Logic* [3], moving logics and moving functions can be used for *discourse analysis*. But here we just want to show how it links boolean and galoisian calculi, and how in this way Theorem 1 is proved.

For every integer n the set $2^n = \{0,1\}^n$ is equipped with a boolean structure and a field structure, $\mathbb{B}^n$ and $\mathbb{F}_{2^n}$, both unique up to isomorphisms. If the addition $+$ is fixed as being the same in $\mathbb{B}^n$ and in $\mathbb{F}_{2^n}$, then a natural question arises: what is the link between multiplication "$\times$" in $\mathbb{F}_{2^n}$ with zero 0 and unit 1 and conjonctions "$\wedge$" in $\mathbb{B}^n$ with 'false' $= 0$ and 'true' $= t$? In one direction, it is clear: if $\varphi = (e_1, \ldots, e_n)$ is a basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$, then we get a conjonction $\wedge_\varphi$ by $(\sum_1^n x_i \times e_i) \wedge_\varphi (\sum_i^n y_i \times e_i) = \sum_i^n x_i \times y_i \times e_i$, when for all $i \leq n$, x_i and y_i are 0 or 1; and negation, disjunction and implication by $\neg_\varphi(x) = x + t_\varphi$, with $t_\varphi = \sum_1^n e_i$, $x \vee_\varphi y = \neg_\varphi(\neg_\varphi x \wedge_\varphi \neg_\varphi y)$, $x \Rightarrow_\varphi y = (\neg_\varphi x) \vee_\varphi y$. So we get a boolean structure Boole$_\varphi$ associated with a basis φ, with false 0 and true t_φ, and of course the operations of Boole$_\varphi$, as every functions $f : \mathbb{F}_{2^n}^k \to \mathbb{F}_{2^n}$, can be expressed with $\times$ and $+$. In fact, the crucial point between boolean and galoisian calculi is that $x \wedge_\varphi x = x$, wheras $x \times x \neq x$, but $x \times x$ and x are *undiscernible*: $x^2 \sim x$; this is the meaning of the fact that the Frobenius map $x \mapsto x^2$ generates the Galois group of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$. So, in order to go in the other direction — i.e. to come back from boolean structures to polynomial functions in a Galois field of characteristic 2 —

our proposed method is first to get the product $\times$ with one boolean structure and the Frobenius, and then to get the Frobenius as a moving boolean function. We first do that for $n = 2, 3$ and show in these cases that *every function is moving boolean.*

In the case $n = 2$, $\mathbb{F}_4 = \mathbb{F}_2[X]/(X^2+X+1)$. With u and v the two imaginary roots of X^2+X+1 over $\mathbb{F}_2$, $u \times v = 1$, $u+v = 1$, and $\mathbb{F}_4 = \{0, 1, u, v\}$. Ordered bases of $\mathbb{F}_4$ over $\mathbb{F}_2$ determine the group $\mathrm{GL}_2(\mathbb{F}_2) \simeq \mathcal{S}_3$, for which we consider spanning by $p = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which are, with respect to the basis $\kappa = (u, v)$ with $t_\kappa = 1$, the matrices of the basis $\alpha = (1, v)$ with $t_\alpha = u$, and $\beta = (u, 1)$ with $t_\beta = v$. So $p(x) = ux^2$ and $q(x) = vx^2$.

Theorem 2 [1]. *In $\mathbb{F}_4$ we have $x \wedge_\varphi y = x^2y^2 + t_\varphi(x^2y + xy^2)$, and in particular $x \wedge_\kappa y = x^2y^2 + x^2y + xy^2$. With $\wedge = \wedge_\kappa$, we have $x \times y = x^2 \wedge y + x \wedge y^2 + x^2 \wedge y^2$. Every fonction on $\mathbb{F}_4^k$ with values in $\mathbb{F}_4$ is a composition of constants, $\wedge$, $\neg$, and $(-)^2$. Furthermore, we have $p(x) + q(x) = x^2$, and every function is a composition of constants, $\wedge$, $\neg$, and p, q. As in fact $x^2 = x \wedge_\kappa 1 + x \wedge_\alpha 1 + x \wedge_\beta 1$, we get also that every function is a $\{\kappa, \alpha, \beta\}$-moving* boolean fonction.

We now consider the case $n = 3$, $\mathbb{F}_8 = \mathbb{F}_2[X]/(X^3+X^2+1)$. With a, b, c the three imaginary roots of $X^3 + X^2 + 1$ over $\mathbb{F}_2$, a^{-1}, b^{-1}, c^{-1} are the roots of $X^3 + X + 1$, $abc = 1$, $ab + bc + ca = 0$, $a + b + c = 1$, $a^{-1} = c + 1 = bc$, $b^{-1} = a + 1 = ca$, $c^{-1} = b + 1 = ab$, $a^2 = b$, $b^2 = c$, $c^2 = a$, $a + a^{-1} = b$, $b + b^{-1} = c$, $c + c^{-1} = a$, $\mathbb{F}_8 = \{0, 1, a, b, c, a^{-1}, b^{-1}, c^{-1}\}$. Ordered bases of $\mathbb{F}_8$ over $\mathbb{F}_2$ are organized as the simple group $\mathrm{GL}_3(\mathbb{F}_2) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$, of order 168, which is the group of automorphisms of the Klein's quartic $\mathrm{X}(7) = \{[x : y : z] \in \mathrm{P}_2(\mathbb{C}); x^3y + y^3z + z^3x = 0\}$ (the most symmetric riemannian surface of genus 3). Now, in $\mathrm{GL}_3(\mathbb{F}_2)$ we consider the order-seven matrices $r = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $s = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $i = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, which are, with respect to the unique normal basis $\kappa = (a, b, c)$, the matrices of the three other strictly auto-dual bases $\rho = (a^{-1}, c^{-1}, 1)$, $\sigma = (1, b^{-1}, a^{-1})$, $\iota = (b^{-1}, 1, c^{-1})$. So $r(x) = a^{-1}x^4 + x^2 + c^{-1}x$, $s(x) = b^{-1}x^4 + x^2 + a^{-1}x$, and $i(x) = c^{-1}x^4 + x^2 + b^{-1}x$.

The actions of r, s, i on $1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\ldots$, $7 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, are the 7−cycles $r = [1746325]$, $s = [1647235]$, $i = [1564327]$, with a visible ternary symetry (Fig. 1) realizable in $\mathcal{S}_7$ with $j = (142)(356)$: $jrj^{-1} = s$, $jsj^{-1} = i$, $jij^{-1} = r$.

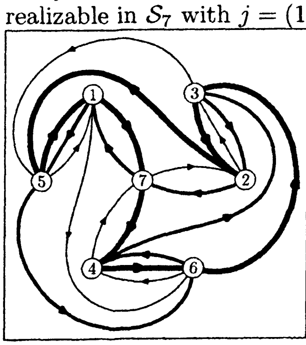

Figure 1: symetry of r, s, i

Theorem 3 [2]. *$\mathrm{GL}_3(\mathbb{F}_2)$ is generated by r, s and i with the relations $(srir^{-1})^2 = 1$, $(is^3i^{-1})^7 = 1$, $((is^3i^{-1})(srir^{-1}))^3 = 1$, $((is^3i^{-1})^4(srir^{-1}))^4 = 1$. And if $w(r, s, i) = 1$ is satisfied, with $w(r, s, i)$ any word in r, s, i, then also $w(s, i, r) = 1$, $w(i, r, s) = 1$; we speak here of a* borromean spanning *of* $\mathrm{GL}_3(\mathbb{F}_2)$.

Theorem 4. *For $\varphi = (f_1, f_2, f_3)$ a basis of $\mathbb{F}_8$ with matrix m we have $x \wedge_\varphi y = m(m^{-1}x \wedge_\kappa m^{-1}y)$, and if with $q = (f_1 + f_2 + f_3)(f_1f_2 + f_2f_3 + f_3f_1)(f_1f_2f_3)^{-1}$ we take $t = (q + 1)^4$, $\lambda = (f_1 + f_2 + f_3)t^{-1}$, $\mu = 1 + t^2 + t^3 + t^4 + t^5 + t^6 + t^7$, then we get $\neg_\varphi(x) = x + t$,*

$x \wedge_\varphi y = x^4y^4 + \lambda^{-5}\mu[x^4y^2 + x^2y^4] + +\lambda^{-4}(t+1)[x^4y + xy^4] + \lambda^{-2}t[x^2y + xy^2]$.
In particular we have $x \wedge_\kappa y = x^4y^4 + 1[x^4y^2 + x^2y^4] + 0[x^4y + xy^4] + 1[x^2y + xy^2]$,

$x \wedge_\rho y = x^4y^4 + (a+1)[x^4y^2 + x^2y^4] + (a+1)[x^4y + xy^4] + a[x^2y + xy^2]$, $t_\rho = a$,

$x \wedge_\sigma y = x^4y^4 + (b+1)[x^4y^2 + x^2y^4] + (b+1)[x^4y + xy^4] + b[x^2y + xy^2]$, $t_\sigma = b$,

$x \wedge_\iota y = x^4y^4 + (c+1)[x^4y^2 + x^2y^4] + (c+1)[x^4y + xy^4] + c[x^2y + xy^2]$, $t_\iota = c$.

Theorem 5. *As a kind of counterpart of the borromean spanning of* $\mathrm{GL}_3(\mathbb{F}_2)$ *we get on* $\mathbb{F}_8$ *a symetric system of six projectors 'by intersection', with associated logical expressions for* $x \mapsto x^4$ *and its inverse* $x \mapsto x^2$*:*

$x \wedge_\rho b = cx^4 + b^{-1}x^2 + c^{-1}x$, $x \wedge_\sigma c = ax^4 + c^{-1}x^2 + a^{-1}x$, $x \wedge_\iota a = bx^4 + a^{-1}x^2 + b^{-1}x$,

$x \wedge_\rho c = a^{-1}x^4 + ax^2 + c^{-1}x$, $x \wedge_\sigma a = b^{-1}x^4 + bx^2 + a^{-1}x$, $x \wedge_\iota b = c^{-1}x^4 + cx^2 + b^{-1}x$,

$x^4 = x \wedge_\rho b + x \wedge_\sigma c + x \wedge_\iota a$, $x^2 = x \wedge_\rho c + x \wedge_\sigma a + x \wedge_\iota b$.

Theorem 6. *In* $\mathbb{F}_8$, *with* $\wedge = \wedge_\kappa$, *the product is* $x \times y = x^2 \wedge y^2 + x \wedge y^4 + x^4 \wedge y + x^4 \wedge y^2 + x^2 \wedge y^4$, *and so every function on* $\mathbb{F}_8^k$ *with values in* $\mathbb{F}_8$ *is a composition of constants,* $\wedge$, $\neg$, *and* $(-)^2$. *Furthermore, we have* $r(x) + s(x) + i(x) = x^2$, *and every function is a composition of constants,* $\wedge$, $\neg$, r, s, i. *As* $x^2 = x \wedge_\rho c + x \wedge_\sigma a + x \wedge_\iota b$, *every function is also a* $\{\kappa, \rho, \sigma, \iota\}$-moving boolean function.

Now, we are ready for the general case (and then Theorem 1):

Theorem 7. *In* $\mathbb{F}_{2^n}$, *every function is a composition of constants,* $\wedge, \neg$ *and* $(-)^2$, *for* $(\wedge, \neg)$ *a boolean structure on* $\mathbb{F}_{2^n}$ *associated to a normal basis, and* $(-)^2$ *the Frobenius map. There is a subset* V *of* $\mathrm{GL}_n(\mathbb{F}_2)$ *— of cardinal* 4 *if* n *is odd, and* 3 *if* n *is even — such that for every integer* k, *every function* $\mathbb{F}_{2^n}^k \to \mathbb{F}_{2^n}$ *is a* V*-moving boolean function.*

The idea of the proof is inspired by the cases $n = 2, 3$. $\mathbb{F}_{2^n}$ is now equipped with a *normal* basis $\beta = (b_1, \ldots, b_n)$, with $b_i = (b_n)^{2^i}$. With some $\gamma_{i,j,k} \in \mathbb{F}_2$ (to be precised only for computing an explicit result, as we did for $n = 2, 3$), $(\sum_i x_i b_i) \times (\sum_j x_i b_j) = \sum_{i,j,k} \gamma_{i,j,k} x_i y_j b_k$. We get for example $(x \wedge_\beta y^{2^q})^{2^l} \wedge_\beta b_{i+l} = x_i y_{i-k} b_{i+l}$. The general operator $x \mapsto y \wedge_\varphi z$, for an arbitrary basis φ with matrix A with respect to β and an arbitrary z can be written as $x \mapsto A(A^{-1}z)^d A^{-1}x$ (with $(A^{-1}z)^d$ the diagonal matrix given by $((A^{-1}z)^d)_{i,i} = (A^{-1}z)_i$) and is the general projector $x \mapsto Px$, with P linear and $P^2 = P$. And here, in the context of $\mathbb{F}_2$, every linear map is a sum of projectors; in particular with π_i the matrix with 1 at (i,i) and $(i+1,i)$, for $i < n$, and π_n the matrix with 1 at (n,n) and $(1,n)$, we get $x^2 = x + (\sum_{(i<n)\,\&\,(i\,\mathrm{odd})} \pi_i) + (\sum_{(i\leq n)\,\&\,(i\,\mathrm{even})} \pi_i) + (\pi_n)\,\&\,(n\,\mathrm{odd})$.

[1] *Théorie cohomologique du sens*, SIC, 8 nov. 2003, compte-rendu, 2004-10/Mars 2004, LAMFA-CNRS UMR 6140, 39-47 [version allongée le 9 février 2004, 22 p].

[2] *Borroméanité du groupe* G_{168} *de la quartique de Klein*, en préparation, 2005.

[3] L'idée de Logique Spéculaire, *Journées Mathématiques Catégories, Algèbres, Esquisses et Néo-esquisses (C.A.E.N.)*, Univ. de Caen, 27-30 septembre 1994, 127-132.

Diagrammes d'espaces fibrés principaux et sphères d'homologie.

par Horst IBISCH

Le but de ce travail est de généraliser la théorie de la classification homotopique des espaces fibrés principaux aux espaces fibrés principaux doubles dont la structure est beaucoup plus riche. Comme première application, nous obtenons une relation nouvelle entre les sphères d'homologie entière et les groupes d'homotopie encore peu connus de certains groupes de transformations d'espaces euclidiens.

Les définitions fondamentales et les premiers résultats dans le domaine des espaces fibrés principaux et leurs espaces classifiants sont dus à C. EHRESMANN à partir de 1934 ([2],[3] et al.).

I. Espaces fibrés principaux doubles et leur classification homotopique

Définition. Soit K un groupe topologique, G et H des sous-groupes de K ; soit X un espace topologique. On désigne par $C_K(X; G,H)$ la classe des triplets (η,Φ,ζ) sur X, où η est un G-fibré principal sur X, ζ est un H-fibré principal sur X et $\Phi : \eta\times_G K \to \zeta\times_H K$ est un K-isomorphisme sur X. Les triplets (η,Φ,ζ) seront aussi appelés *espaces fibrés principaux doubles*.

Deux triplets (η,Φ,ζ) et (η',Φ',ζ') sont *isomorphes sur* X s'il existe un G-isomorphisme $\sigma : \eta \to \eta'$ et un H-isomorphisme $\tau : \zeta \to \zeta'$ sur X tels que le carré de K-isomorphismes

$$\begin{array}{ccc} \eta\times_G K & \xrightarrow{\Phi} & \zeta\times_H K \\ \downarrow{\scriptstyle \sigma\times_G K} & & \downarrow{\scriptstyle \tau\times_H K} \\ \eta'\times_G K & \xrightarrow{\Phi'} & \zeta'\times_H K \end{array}$$

est commutatif. On note $k_K(X; G,H)$ l'ensemble des classes d'isomorphie sur X des espaces fibrés principaux doubles sur X.

Exemple. Si K = G = H, $k_K(X; K,K) = k_K(X)$ est l'ensemble des classes d'isomorphie des K-fibrés principaux sur X.

Théorème 1. *Le foncteur* $k_K(- ; G,H)$: Top $\to$ Ens *est représentable. Il existe un espace classifiant* B(K; G,H) *et une bijection naturelle*

$$I : [X; B(K;G,H)] \xrightarrow{\cong} k_K(X; G,H)$$

entre l'ensemble des classes d'homotopie des applications continues X $\to$ B(K; G,H) *et* $k_K(X; G,H)$.

Plus précisément, il existe un fibré principal double universel $(\gamma_0,\Phi_\gamma,\gamma_1)$ de base B(K ;G,H) et la bijection ***I*** associe à toute application continue f: X → B(K; G,H) le fibré principal double induit $(f^*(\gamma_0), f^*(\Phi_\gamma), f^{*\varsigma}(\gamma_1))$.

Dans le cas K = G = H, B(K; G,H) est homotopiquement équivalent à l'espace classifiant de J.W. MILNOR [4] des K-fibrés principaux.

Théorème 2. (1) *Il existe une suite exacte de groupes d'homotopie*

$$\ldots \to \pi_i(B(K;G,H)) \to \pi_{i-1}(G) \,\square\, \pi_{i-1}(H) \xrightarrow{\varphi} \pi_{i-1}(K) \to \pi_{i-1}(B(K;G,H)) \to \ldots$$
$$\ldots \to \pi_0(B(K;G,H)) .$$

L'homorphisme φ *a la forme* $\varphi(a\square b) = i_{Gi}(a) - i_{Hi}(b)$, où i_G: G → K *et* i_H: H → K *sont les inclusions des sous-groupes* G *et* H *dans* K.

(2) *Si* i_H: H → K *est une équivalence d'homotopie pointée*, i_{Hi} $\pi_i(H) \to \pi_i(K)$ *est un isomorphisme et la suite exacte dégénère en un isomorphisme* $\pi_i(B(K; G,H)) \to \pi_{i-1}(G)$ *pour* $i \geq 1$.

II. Groupes de transformation d'espaces euclidiens

Soit $m > n$ et soit K= H(R^m)= H(m) le groupe des homéomorphismes de R^m muni de la topologie de la convergence compacte. Soit G = H(R^m,R^n) = H(m,n) le sous-groupe topologique des homéomorphismes des paires (R^m,R^n) et soit H = H.(R^m) = H.(m) le sous-groupe topologique des homéomorphismes de R^m fixant l'origine.
Les théorèmes 1 et 2 donnent alors l'isomorphisme

$$\pi_{i+1}(B(H(m); H(m,n),H.(m)) \xrightarrow{\cong} \pi_i(H(m,n)) \quad \text{pour } i \geq 0.$$

Pour le cas $m = n+2$, R.C. KIRBY et L.C.SIEBENMANN [5] ont montré que les groupes d'homotopie $\pi_i(H(n+2,n))$ sont liés à des questions importantes de la topologie géométrique : pour $n \neq 2$, toute n-sous-variété topologique M^n localement plate d'une variété topologique Q^{n+2} admet un (micro-)fibré normal.
La raison en est que le groupe de transformations H(n+2,n) est homotopiquement proche du groupe orthogonal $O(n+2,n) \approx O(2)$:

$$\pi_i(H(n+2,n)/O(n+2,n)) = 0 \quad \text{pour } i \leq n \neq 2.$$

Autrement dit, pour $n \neq 2$, on connaît les groupes d'homotopie

$$\pi_i(H(n+2,n)) \approx \begin{cases} Z/2 & \text{pour } i=0 \\ Z & \text{pour } i=1 \\ 0 & \text{pour } 2 \leq i \leq n \end{cases} \qquad (n \neq 2)$$

Pour $n = 5$ le premier groupe non connu est $\pi_4(H(5,3))$; nous allons l'examiner à l'aide de l'isomorphisme de la partie I.

III. Faisceaux de germes de 3-structures dans S^5 et sphères d'homologie de dimension 3.

On considère le pseudo-groupe P(5,3) des transformations de la forme

$$f\colon (U, U\cap R^3) \xrightarrow{\cong} (V, V\cap R^3)$$

où U,V sont des ouverts de S^5.
P(5,3) opère sur l'atlas $A(S^5)$ des cartes locales topologiques de S^5 et le quotient $A(S^5)/P(5,3)$ admet une structure canonique de faisceau sur S^5. On l'appelle le faisceau $F_3(S^5)$ des germes de 3-structures sur S^5. Soit $\Gamma(S^5, F_3(S^5))$ l'espace des sections continues de $F_3(S^5)$. Le groupe $H(S^5)$ des homéomorphismes de S^5 agit dans $\Gamma(S^5,F_3(S^5))$. Le quotient $\Gamma(S^5, F_3(S^5))/ H(S^5) = \mathbf{S}_3 (S^5)$ sera appelé *l'espace* des classes d'équivalence *des 3-structures sur* S^5.

Théorème 3. *L'ensemble* $\mathbf{H}_3$ *des classes d'homéomorphie des* 3-*sphères homologiques est canoniquement plongé dans* $\mathbf{S}_3(S^5)$.

Ce théorème est une conséquence du célèbre Théorème de la Double Suspension des Sphères d'Homologie ([1] Corollary 3D (Double Suspension Theorem) p. 184), issu d'un résultat fondamental de R.D. EDWARDS en théorie de décomposition des variétes.

Théorème 4. *Il existe une injection* J: $\mathbf{S}_3(S^5) \rightarrow \pi_5(H(5); H(5,3),H.(5))$.

Corollaire: $\mathbf{H}_3 \approx \pi_4(H(5,3))$.

La structure du groupe $\pi_4(H(5,3))$ apparaît donc riche et importante pour la topologie géométrique.

[1] R. J. DAVERMAN, *Decomposition of manifolds.* Acad. Press Inc. New York 1986.
[2] C. EHRESMANN, Sur la topologie de certains espaces homogènes, *Ann. of Math.* (2) 35 (1934),396-443. Réimprimé dans *Charles Ehresmann : Œuvres complètes et(commentées,* Partie I (Amiens 1983).
[3] C. EHRESMANN et J. FELDBAU, Sur les propriétés d'homotopie des espaces fibrés, *C.R. Acad.Sc. Paris*, 213,1941, 762-764. Re-publié dans *Charles Ehresmann : Œuvres complètes et(commentées,* Partie I (Amiens 1983).
[4] J.W. MILNOR, Construction of universal bundles II, *Ann. of Math.* (2) 63(1956), 430-436.
[5] R.C.KIRBY and L.C.SIEBENMANN, Normal bundles for codimension 2 locally flat embeddings, *Lecture Notes in Math.* 438: Geometric Topology (Proceedings of the Geometric Topology Conference held at Park City, Utah, February 19-22, 1974) Springer Verlag 1975, 310-324.

The Gödel incompleteness theorem, a categorical approach

by André JOYAL

In 1973, I found a categorical proof that the consistency of primitive recursive arithmetic cannot be proved within primitive recursive arithmetic. The result is a special case of the general Gödel incompleteness theorem. The general theorem states that if a recursively axiomatisable theory is consistent and powerful enough to contain primitive recursive arithmetic, then the theory is incomplete. In 1973 in Paris, I promised to Charles Ehresmann to send him a categorical proof of the general theorem. Unfortunately, I never found the time to work on the general problem and my proof of the special case remained unpublished. I would like to take the opportunity of Ehresmann 100^{th} anniversary to fill my promise, at least partially, by describing my partial results.

I will discuss the work that has been done on arithmetic universe: S. Rolland (1976), J.R.B. Cockett (1990), A. Morisson (1996) and M.E. Maietti (2002-2004). I will formulate the general incompleteness theorem in categorical terms.

A synthetic theory of envelopes of families of curves or surfaces

by Anders KOCK

We examine critically some of the existing descriptions of the envelope of a 1-parameter family of surfaces. An old, natural, description is that the envelope is the union of the characteristic curves (defined as the "limit intersection curves" of surfaces from the family). This description fell in disrepute during the 20th century. We claim that this relegation of the limit intersection curves made some other aspect of the theory less justified, namely the definite article "*the*" in the phrase "the envelope".
We intend to re-establish the limit intersection curves (characteristic curves) on a rigourous basis, -- thereby also re-establishing the old obvious argument for the definiteness of envelopes.

The method will be that of Synthetic Differential Geometry; this is a theory, where the geometric notion of when two points in space are *neighbours* (first order neighbours, second order neighbours, ...) is made rigorous. In a family of surfaces, where the family is parametrized by some "space", it makes sense to *intersect* two neighbour surfaces, and even to intersect a surface of the family with *all* its neighbour surfaces. This will, under suitable non-singularity assumptions, be a curve in the given surface, a *characteristic* curve for the family. These characteristic curves together make up the envelope, but are defined *prior* to the envelope itself.

C. Ehresmann and Differential Geometry

by Paulette LIBERMANN

Under the influence of his Master Elie Cartan, C. Ehresmann was interested by many aspects of Geometry. Trying to understand Cartan's works on 'infinite groups', the equivalence problem, connections, the theory of the moving frame, C. Ehresmann introduced fundamental concepts in Differential Geometry such as principal fiber bundles (with their connections), fibrations, groupoids, jets (holonomic and non-holonomic jets), Lie pseudogroups, Lie groupoids... He was a pioneer in many fields, for instance he introduced the definition of a manifold by means of an atlas of local charts compatible with a pseudogroup of transformations.

Besides his written works, C. Ehresmann gave many ideas in his lectures or in private conversations with his pupils and other mathematicians, for instance his influence on R. Thom was very important.

His Seminars were famous. During the war, C. Ehresmann was teaching in Clermont-Ferrand; after the war the Seminar took place in Strasbourg and then in Paris. Mathematicians from many countries attended the Seminars (for instance de Rham, Whitney, Remmert, Dold, Kuiper, Kobayashi, Nomizu, Wu Wen Tsun, Haefliger); young mathematicians learnt a lot from these Seminars.

C. Ehresmann gave lectures in many countries and organized International Conferences.

I shall explain with more details two notions which are less known:

1. non-holonomic and semi-holonomic jets;
2. second order foliations.

Variance Vs. Volition

by F.E.J. LINTON

Because of the symmetry inherent in the Cartesian monoidal structure of the category of sets, there is no difficulty in treating on an equal footing a category and its opposite. Consequently, once one understands what to mean by a (covariant) functor from one category to another, contravariant functors can be explained with the greatest of ease as either covariant functors from the first category to the opposite of the second, or, equally effectively, as covariant functors from the opposite of the first category to the second itself. The only small snag here is that it is the orientation of the maps in the target category that determines the natural direction of the maps from one functor to another in the functor category. But this problem already rears its head in the identification of covariant functors from **A** to **B** with covariant functors from $\mathbf{A}^{op}$ to $\mathbf{B}^{op}$, and is dealt with simply by making a well-motivated choice.

Symmetry comes into play again in defining the product of two (or more) categories, and, hence, in explaining what to mean by functors of two (or more) variables, of fixed, or mixed, variance: one takes them, for example, to be (covariant) functors from a product $\mathbf{A}^{op}\times\mathbf{B}$ to **C** .

A much larger and less easily surmountable snag arises once one wants to generalize the observations above to the setting of categories enriched in a closed, or monoidal, or, most generally, multilinear category **V** whose structure (like that of almost any endofunctor category $\mathbf{A}^{\mathbf{A}}$ under the monoidal operation of composition) manifests no inherent symmetry whatsoever. Here, given a **V**-category **A**, there may well be *no* appropriate understanding at all for what the **V**-category opposite of **A** should be. Hand in hand with this difficulty comes the further annoyance that, despite the easy availability of a notion of covariant **V**-functor from one **V**-category to another, absence of symmetry totally stymies any attempt to elucidate what a *contravariant* **V**-functor between two **V**-categories should be. By the same token, absence of symmetry makes it impossible to explain what to mean by a **V**-functor of several variables, of fixed, or mixed, variance.

On the other hand, even when the multilinear base category **V** is not itself a **V**-category (*i.e.*, is not closed) – even if, say, **V** is non-symmetrically monoidal, like our $\mathbf{A}^{\mathbf{A}}$ example earlier – it miraculously remains possible to explain what to understand by **V**-valued **V**-functors – of *either* variance – defined on an arbitrary **V**-category **A** . What is more, even a notion of **V**-valued **V**-functor of mixed variance, defined on a pair of **V**-categories **A** and **B**, contravariant in **A** and covariant in **B**, is readily available: this comes down to a system F consisting of a rule assigning to each object A of **A** and each object B of **B** a value-object $F(A, B)$ of **V** , along with further information providing **V**-multilinear maps

$$F: < \mathbf{B}(B, B'), F(A, B), \mathbf{A}(A', A) > \rightarrow F(A', B')$$

all satisfying conditions analogous to the usual associativity and unit laws. Such an F would be termed a **V**-valued **V**-functor of the two variables A and B , contravariant in **A** and covariant in **B**. A prime example of such a functor is the actual **V**-valued hom-functor of the **V**-category **A** itself.

What is more, despite the likely absence of any **V**-category on which such a **V**-valued **V**-functor of two variables might be defined, there is available a "law of exponents" of sorts, identifying **V**-valued **V**-functors, of mixed variance, contravariant in **A** and covariant in **B**, with actual (covariant) **V**-functors on **B** – taking values (when **A** is small enough, and **V** complete enough) in a sort of presheaf **V**-category $\mathbf{V}^{(\mathbf{A}\mathrm{op})}$ of contravariant **V**-valued **V**-functors on **A**. Applied to **A**'s own **V**-valued hom-functor, this identification yields a Yoneda representation $\mathbf{A} \to \mathbf{V}^{(\mathbf{A}\mathrm{op})}$, which (sure enough) turns out to be **V**-full-and-faithful. Even more surprisingly, there is a twist on this "law of exponents" permitting identification of such **V**-valued **V**-functors of mixed variance with, instead, single covariant **V**-functors from **A** to a **V**-category best described, intuitively, as the **V**-opposite $(\mathbf{V}^{\mathbf{B}})^{\mathrm{op}}$ of the category of covariant **V**-valued **V**-functors on **B** . **Note**: NOT defined on $\mathbf{A}^{\mathrm{op}}$ – there may well not BE any **V**-category serving as opposite for **A** ; and NOT taking values in $\mathbf{V}^{\mathbf{B}}$ – there may well not be even a reasonable candidate (no matter how small **B** or how complete **V** may be) for a "covariant **V**-valued **V**-functors on **B**" **V**-category $\mathbf{V}^{\mathbf{B}}$; BUT defined on **A** and taking values in $(\mathbf{V}^{\mathbf{B}})^{\mathrm{op}}$. This identification, applied to the **V**-valued hom functor of **A**, yields the "other" Yoneda representation, $\mathbf{A} \to (\mathbf{V}^{\mathbf{A}})^{\mathrm{op}}$, which, like the former one, is again **V**-full-and-faithful.

An amusing corollary of these considerations and others like them is the following moral regarding the direction of morphisms in Kleisli categories, and the variance of Lawvere-style algebras as (roughly speaking) "representable" functors on such Kleisli categories. One way to concoct the Kleisli category of a **V**-adjoint pair $F \dashv U: \mathbf{X} \to \mathbf{A}$ (with $F: \mathbf{A} \to \mathbf{X}$ left adjoint to $U: \mathbf{X} \to \mathbf{A}$) is to form the *full image* of F, that is, the **V**-category with same objects as A but **V**-objects-of-morphisms $[A, B] = \mathbf{X}(F(A), F(B))$. Another way, even in the conceivable absence of F, is to use as morphisms from A to B a suitable **V**-object of **V**-natural transformations between U^A and U^B (here $U^A: \mathbf{X} \to \mathbf{V}$ is the **V**-functor given by $U^A(X) = \mathbf{A}(A, U(X))$). Now while it has traditionally been thought that it should be the transformations from U^A to U^B – mimicking the direction of A-ary operations – that ought to serve as the Kleisli-maps from A to B, we see here that, it being not $\mathbf{V}^{\mathbf{X}}$ but $(\mathbf{V}^{\mathbf{X}})^{\mathrm{op}}$ that stands any chance of being a **V**-category, we really need to use $(\mathbf{V}^{\mathbf{X}})^{\mathrm{op}}(U^A, U^B)$, whose "elements" are the **V**-natural transformations from U^B to U^A (!), as the Kleisli maps from A to B. Fortunately, this observation is consistent with (rather than opposite to) the one provided by the full image of F, for, by Yoneda and adjointness,

$$[A, B] = \mathbf{X}(F(A), F(B)) \approx (\mathbf{V}^{\mathbf{X}})^{\mathrm{op}}(Y(F\,A), Y(F\,B)) \approx (\mathbf{V}^{\mathbf{X}})^{\mathrm{op}}(U^A, U^B)\,.$$

And the **V**-category of Lawvere-style algebras "over" the Kleisli **V**-category **K** can then be taken as the full **V**-subcategory of the **V**-category $\mathbf{V}^{(\mathbf{K}\mathrm{op})}$ of contravariant **V**-valued **V**-

functors on **K** whose compositions with **A** → **K** are explicitly representable (which is to say: as the pullback of the diagram

$$\begin{array}{ccc} & & \mathbf{V}^{(\mathbf{Kop})} \\ & & \downarrow \\ \mathbf{A} & \rightarrow & \mathbf{V}^{(\mathbf{Aop})} \end{array}$$

of **V**-categories and **V**-functors).

Historical remarks. Much of what is to be told here has appeared in the author's ancient articles in Springer LNM # 99 and 195, or was promulgated orally on the occasion of the esteemed Professor Charles Ehresmann's 70th birthday celebration at Chantilly/Fontainebleau in the summer of 1975. But the oral seeds seem never to have taken root, having instead been simply scattered by the winds of time, so it has seemed worthwhile to broadcast them once again, hoping that this time they will fall on more fertile soil.

Technical remarks. It is of course a gross abuse of language to speak of either of these functor categories $\mathbf{V}^{(\mathbf{Aop})}$, $(\mathbf{V}^{\mathbf{A}})^{op}$ as **V**-categories. The actual structure available is, at best, for each pair of **V**-functors F,G , a "job-description" for the desired **V**-ish hom-object ***n.t.***(F, G), in the form of a suitable **Sets**-valued functor on $\mathbf{V}^{op}$ – or better, on $(\mathbf{M}_0(\mathbf{V}))^{op}$, the (opposite of the) strictly associative monoidal category $\mathbf{M}_0(\mathbf{V})$ (of finite strings of objects of **V**) that serves (in SLNM 195) as the *multilinear structure* of **V** – with **Sets** any available category of "sufficiently large" sets. Any object of **V** representing this "job-description" functor will serve as the desired **V**-natural transformations object.

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Three Synthetic Approaches to Total Differentials

by NISHIMURA Hirokazu

In teaching differential calculus of several variables, mathematicians are expected to exhort freshmen or sophomores majoring in science, engineering etc. to understand that it is not partial derivatives but total differentials that are of intrinsic meaning, while partial derivatives are used for computational purposes. If we want to discuss not only first-order total differentials but higher-order ones, we have to resort to the theory of jets initiated by Ehresmann more than half a century ago, though it is not easy to generalize it beyond the scope of finite-dimensional smooth manifolds so as to encompass differentiable spaces and suitable infinite-dimensional manifolds.

The then moribund notion of nilpotent infinitesimals in differential geometry was retrieved by Lawvere in the middle of the preceding century, while Robinson revived invertible infinitesimals in analysis, and Grothendieck authenticated nilpotent infinitesimals in algebraic geometry. The new trend in differential geometry enunciated by Lawvere as synthetic differential geometry has finally matured enough to yield a solid and elegant foundation to the theory of jets, which has never occupied a central position in orhtodox differential geometry.

In this talk we present three approaches to jets or total differentials. The first one is repetitive, while the other two approaches are non-repetitive. In the classical repetitive approach to the theory of jet bundles there are three methods of repetition, which yield three kinds of jet bundles, namely, non-holonomic, semi-holonomic and holonomic ones. The former two have been endowed with truly repetitive definitions, while the last one, which is undoubtedly most important, has failed to get genuinely repetitive guises, for it must resort to such a non-repetitive coordinate-dependent construction as Taylor expansion. Our first synthetic approach successfully gives a repetitive definition to holonomicity. Our two non-repetitive approaches to total differentials make use of the space of first-order nilpotent infinitesimals and that of n^{th}-order nilpotent infinitesimals so as to capture the notion of n^{th}-order total differentials respectively. It is interesting to note that, assuming only microlinearity, there is a canonical projection from the former to the latter, while there is a canonical projection from our repetitive holonomic total differentials to our non-repetitive ones of the former kind. It is more interesting to note that if coordinates are available, these two canonical projections are bijective, so that our three synthetic approaches to total differentials coincide admitting classical coordinate descriptions.

∞-catégorification de structures équationnelles

par Jacques PENON

• La procédure de catégorification que nous avons introduite dans [2] était suffisamment élémentaire pour pouvoir être itérée (elle permet, entre autres, rappelons-le, de retrouver les n-catégories non strictes). Cependant, elle ne peut s'appliquer qu'à une classe assez réduite de structures : celles définissables par des monades cartésiennes. La structure de groupe ou celle de groupe abélien, par exemple, ne rentrent pas dans cette classe. On aimerait pourtant arriver à catégorifier n'importe quelle structure équationnelle (= théorème de Lawvere), ne serait-ce que pour comparer ces structures ainsi catégorifiées de façon générale avec celles obtenues *ad hoc* dans de nombreux exemples. La méthode que nous proposons ici a ce degré de généralité. Cependant, elle ne peut s'itérer comme celle signalée plus haut. A la place, nous lui avons préféré un procédé de ∞–catégorification (ce dernier intégrant, en effet, d'un seul coup, la catégorification et ses multiples itérations) afin de prévenir de futures catégorifications répétées. Cette façon de faire a aussi le mérite de construire un affaiblissement structurel sans avoir recours, dans aucune étape, à des passages aux quotients, ce qui semble inévitable lors d'une simple catégorification. Notre procédure reprend, dans les grandes lignes, celle que nous avions introduite dans [3] pour la mise au point des prolixes (une version du concept multiforme de ∞–catégorie non-stricte). Elle ne fait d'ailleurs que préciser, dans un cadre approprié, ce que M. Batanin avait suggéré dans [1].

• On peut résumer la procédure de ∞–catégorification d'une théorie équationnelle **T** comme suit. On commence par ∞–catégorifier strictement cette théorie équationnelle. C'est-à-dire qu'on construit un adjoint K au foncteur d'oubli de ∞–Cat(Mod(**T**)) dans ∞–**Gr** (où Mod(**T**) désigne la catégorie des modèles de **T**, ∞-Cat(Mod(**T**)) est la catégorie des ∞–catégories internes dans Mod(**T**) et ∞–**Gr** celle des ∞–graphes ou ensembles globulaires). On affaiblit ensuite le résultat de cette ∞–catégorification stricte par étirement. De façon plus précise, on commence par plonger ∞–Cat(Mod(**T**)) dans la catégorie L-∞-**Mag** des L-∞-magmas (les ∞–magmas ont été introduits dans [3], ils sont ici en plus munis, en chaque dimension, des lois de composition provenant du langage L de la théorie **T**, celles-ci étant seulement astreintes à des conditions de positionnement) puis, un objet **G** de ∞–**Gr** étant donné, on construit dans L-∞-**Mag** l'étirement libre (voir la définition d'un étirement dans [3]) associé à la flèche universelle $\mathbf{G} \to \mathrm{K}(\mathbf{G})$. Il suffit ensuite de ne conserver de cet étirement que sa structure de ∞–graphe sous-jacente pour obtenir une monade sur ∞-**Gr**, dont les algèbres sont pour nous les **T**-modèles ∞-catégorifiés.

• La procédure de ∞–catégorification étant obtenue, on en déduit aussi une n-catégorification (on revient au cas fini) plus utile dans la pratique (le cas $n = 1$ ou $n = 2$ étant déjà, pour le moment, largement suffisant).

[1] M.A. Batanin, On the Penon method of weakening of algebraic structures, *J. Pure Appl. Algebra,* 172 (2002).
[2] D. Bourn & J. Penon, Catégorification de structures définies par monade cartésienne, *Cahiers Top. et Géom. Diff. Categ.* XLVI-1 (2005), 2-52.
[3] J. Penon, Approche polygraphique des ∞–catégories non strictes, *Cahiers Top. et Géom. Diff. Catég.* XL-1 (1999), 31-80.

Posets, granular topologies, and Štefan Foliations

by Jean PRADINES

Looking for local models of manifolds with edges and corners having better categorical properties than the classical ones (a question which will not be tackled here, and will be developed elsewhere), one is led to endow $\mathbf{R}^{(N)}$ with the canonical Štefan foliation admitting for its leaves all the open quadrants (with varying dimensions) (the fine topology of this foliation is indeed a locally finite-dimensional manifold topology).
Though such a foliation may look rather trivial among Štefan foliations, it is in a certain sense typical, and it is as far as possible from being regular (in the usual sense), since *all* its leaves are singular, and each leaf is a stratum.
Its space of leaves draws attention on those topological spaces (which will be called here briefly *granular*) for which any intersection of open sets in still open. Unless they are discrete, such topologies are never T_1 (*i.e.* accessible in Bourbaki's terminology) (and *a fortiori* T_2, *i.e.* Hausdorff), hence of course they are very far from being manifold topologies. The term "granular" used here is suggestive of the property that each point has a smallest neighbourhood, which will be called its *vicinity*.
These spaces define a full subcategory of the category of (continuous maps between) general topological spaces, which turns out to be canonically equivalent (indeed in a two-fold way) to the category of (order-preserving maps between) preordered sets, hence also with the category of (functors between) those special small categories which may be called *graph-like* for convenience.
This correspondence (which uses the right or left topologies) is described in a sequence of exercises in Bourbaki TOP, only in the special case of ordered sets and T_0 (or Kolmogoroff) spaces. Considering preordered sets (hence including also, in a unified point of view, spaces endowed with equivalence relations, or equivalently graph-like groupoids) brings a significant increase in flexibility.
The interplay between the three equivalent structures yields a trilingual dictionary which brings together various notions in a sometimes rather unexpected way. Certain constructions happen to be much easier and natural under one or other viewpoint.
For instance Bourbaki stresses in ENS IV the fact that initial or final structures may not exist in the category of ordered sets, and gives (in an exercise of ENS III) counter-examples for the quotient. However enlarging the category to preordered sets and using the topological/algebraic interpretation gives easy solutions for the search of initial/final structures (in the sense of Bourbaki), notably for the quotient.
It should be noted that in the present setting the suitable topology for an infinite product happens to be the so-called box topology (which differs from the classical product topology).
We intend to explore somewhat this correspondence for its own intrinsic interest, and in view of studying those Štefan topologies on $\mathbf{R}^{(N)}$ which are less fine than the above-mentioned canonical one. These will be used elsewhere for the announced local models of generalized angular manifolds.

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Volume XLVI-3 (2005)*

APPLICATIONS MULTIDISCIPLINAIRES

ECHO V

The following abstracts correspond to the third session of "Charles Ehresmann : 100 ans" which consists in the fifth Symposium ECHO, for

"Emergence, Complexity, Hierarchy, Organization",

organized by George Farre, Andrée C. Ehresmann and Jean-Paul Vanbremeersch. The first ECHO had already been organized in Amiens in 1996 by the same persons.

This session is devoted to multidisciplinary applications, for Charles' works have had a deep influence in several domains outside Mathematics, in particular Physics (e.g., fiber bundles, connections, jets, Lie groupoids) and Computer Science (sketch theory). Though he has not worked directly on such applications, he was interested by problems raised in Physics, in particular the role of time. Several of the notions he has introduced and/or developed have proved appropriate for representing natural phenomena, that justifies his idea that Mathematics "is the key for the understanding of the whole Universe", as he said in an address at Lawrence, Kansas, 1966 (printed in the paper "Unity of Mathematics", in his "Oeuvres", Part III).

Weak Quantum Theory: Axiomatic Framework and Applications

by Harald ATMANSPACHER

The concepts of complementarity and entanglement are considered with respect to their significance in and beyond physics. An axiomatically formalized, weak version of quantum theory, more general than the ordinary quantum theory of physical systems, is described [1]. Its mathematical structure generalizes the algebraic approach to ordinary quantum theory. It is found that the key formal feature leading to complementarity and entanglement is the non-commutativity of observables.

The usual Hilbert space quantum mechanics can be recovered by re-introducing the necessary features. This refers to the addition of observables (von Neumann algebra), the Hilbert space representation (norm topology), a probability interpretation (Born rule), Planck's action (universal commutator) and appropriate dynamical laws (Schrödinger equation).

Two concrete applications refer to the complementarity of Liouville dynamics and information dynamics [2] and entanglement phenomena in the perception of bistable stimuli [3]. Both will be described in some detail. In current work we try to extend the approach to cognitive operations related to learning and pattern recognition.

References:

[1] H. Atmanspacher, H. Römer, and H. Walach (2002): Weak quantum theory: complementarity and entanglement in physics and beyond, *Foundations of Physics* 32, 379–406.

[2] H. Atmanspacher and H. Scheingraber (1987): A fundamental link between system theory and statistical mechanics. *Foundations of Physics* 17, 939–963.

[3] H. Atmanspacher, T. Filk, and H. Römer (2004): Quantum Zeno features of bistable perception. *Biological Cybernetics* 90, 33–40.

Towards abstract matter

by Nils A. BAAS

In this talk we will introduce a new kind of structure which we will call Abstract Matter (AM). It is an abstract mathematical structure which in many ways will reflect and model the way in which matter - both organic and inorganic - is being built. It is the starting point of a general theory of structure and organization which will be useful in designing models and experiments of how to synthesize various kinds of matter.

Abstract Matter is an extension and explication of our previous notion of a hyperstructure for which we refer to [1 - 7]. In 1995 we introduced in [4, 5, 6,] the notion of Categorical Matter. This previous notion of Categorical matter is included in the new notion of Abstract Matter. The talk is a continuation of the ideas presented at the ECHO symposium in 1996 [4] and extended in [10].

The basic idea is to get a good notion and theory of higher order structures. Mathematically this is closely related to the theory of higher categories (n-categories) which was initiated by Charles Ehresmann [12]. Much of his work was ahead of its time and may serve as an inspiration for the growing interest today in higher categories and higher order structures.

Hyperstructures were introduced as a framework to combine hierarchies, higher order structures and emergence in a general way in order to include examples from physics, biology and other fields as well. Any kind of complex system or structure is built up from some elementary pieces and constructed layer by layer. The problem is to find a good framework for describing this in a non-trivial way. In our opinion a good definition is the clue to further progress in our understanding of such systems. In the present approach we will emphasize the notion of bonds, interactions, relations and relationships at one level and investigate how new levels and higher order structures are being created.

In mathematics higher order structures have mostly been studied in special forms in logic and set theory. At the time we were developing the notion of hyperstructures, the mathematical notion of higher categories (n-categories) was revived and a new and very extensive development started [11,13,14]. Our study and use of higher categories in purely mathematical research [8] has been another motivation for introducing hyperstructures. Extensions of dynamical systems to higher order dynamical systems have been initiated in [9] in the form of higher order cellular automata.

We will define hyperstructures in a precise mathematical way. Examples, motivation and an outline of future developments will be given. Sets with such a hyperstructure we call pieces of Abstract Matter - extending our previous notion of Categorical Matter. We suggest that this may be very useful as a model in simulations and constructions of new hierarchical materials which one wants to synthesize by modern nanotechnological methods.

References

1. N.A. Baas, Emergence, Hierarchies, and Hyperstructures, Artificial Life III, *Santa Fe Institute Studies in the Sciences of Complexity*, ed. C.G. Langton, Addison-Wesley, 1994 (515-537).
2. N.A. Baas, Hyperstructures as Tools in Nanotechnology, *Nanobiology* 3(1) (1994), 49-60.
3. N.A. Baas, A Framework for Higher Order Cognition and Consciousness, In *Toward a Science of Consciousness*, edited by S. Hameroff, A. Kaszniak and Scott, MIT-Press, 1996 (633-648).
4. N.A. Baas, Thoughts on Higher Order Structures, *ECHO*-Amiens 1996, Amiens 1996 (47-52).
5. N.A. Baas, *Categorical matter*, Preprint, NTNU, 1996.
6. N.A. Baas, Self-Organization and Higher Order Structures, In *From individual to collective dynamics*, Gordon and Breach, 1997 (71 - 82).
7. N.A. Baas, A.C. Ehresmann and J-P. Vanbremeersch, Hyperstructures and Memory Evolutive Systems, *International Journal of General Systems* 33(5), pp 553-568.
8. N.A. Baas, B.I. Dundas and J. Rognes, Two-Vector Bundles and Forms of Elliptic Cohomology. (X-archives, math. AT/0306027, June 2003.) In *Topology, geometry on Quantum Field Theory*, ed. U. Tillmann, London Math Soc. LN 308 CUP, (18-45), 2004.
9. N.A. Baas and T. Helvik, Higher Order Cellular Automata, To appear in *Adv. of Complex Systems.*
10. N.A. Baas, *Abstract Matter*, Preprint, NTNU, Trondheim, Norway, 2003, under revision and extension.
11. J. Baez and J. Dolan, Higher-dimensional algebra and topological quantum field theory, *J. Math. Phys.* 36(11) 1995, 6073 - 6105.
12. *Charles Ehresmann, Oeuvres complètes et commentées*, ed. A.C. Ehresmann, Amiens, 1981-83.
13. A. Grothendieck, *Pursuing stacks*, manuscript, 1983.
14. T. Leinster, Higher Operads, Higher Categories, *London Math. Soc. LN* 298, CUP, 2004.

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Intelligent Inverse-anthropomorphic Emergence

by Ron COTTAM, Willy RANSON and Roger VOUNCKX

History and pre-historical conjecture suggest that our species has progressively modified its view of itself from one of subjugation to a higher intelligence towards one of seeing mankind as simply participating in a somewhat rigidly defined natural environment. An historically early indication of this transformation can be seen in the modification of 'world view' which followed from Aristotle's rejection of the underlying perfection of Plato's teaching and the consequent birth of the scientific attitude to nature. However, the Middle Ages saw an understandable return to a European society governed by the Church, in the absence of an extensive understanding of natural causes and a resultant insecurity.

Following on from the 'Enlightenment' and the blooming of technology, a pendulum swing made God-fearing man a god in the 19th century and anthropomorphic descriptions of nature held sway, until the Great Wars of the 20th century brought massive investment in technological success and the instatement of Science as a deity. The second half of the 20th century saw the entry of Quantum Mechanics into technology, the discovery of Order in chaos and the determinism of genetics, bringing with them enthronement of the randomness of neo-Darwinism and the ubiquitous relativism of Post-Modern thought. Contemporary anti-anthropomorphic wisdom rejects human association with nature other than via the pre-supposed arbitrariness of evolution.

Our purpose in this paper is to develop a next stage beyond this progression from subjugate entity to irrelevant organism, by rejecting anti-anthropomorphism and continuing the pendulum swing towards acceptance that we are products of our environment and that those most human characteristics of consciousness and intelligence which we hold so dear derive from the addition of network complexity to inanimate physics.

The evolutions of survivability, consciousness, intelligence, wisdom and evolution itself are broadly equivalent. Key to establishing this equivalence is the adoption of a viewpoint which rejects both 19th century anthropomorphism and late 20th century anti-anthropomorphism in favour of an inverse-anthropomorphic stance which presupposes the continuity of evolvability and 'intelligence' between blind inanimate dependence on Newton's Laws and human technological control. Newton was lucky enough (as he himself pointed out) to hit on *energy* as the ground for a successful mechanical world view. As an initial approximation this has turned out to be an amazingly accurate foundation, for reasons we discuss in this paper, and it has proved possible in science to modify the concept of energy to take account of various inter-model difficulties which have popped up over the years. Quantum Electrodynamics is the most accurate energy-based physical model ever, and its predictions hold up under extremely close examination, but unfortunately the extension of quantum mechanical systems to larger scale reveals a breakdown in logical completeness. As Shakespeare wrote in *Hamlet*: "Something is rotten in the state of Denmark"!

Following on from the scientific bombshell of Deterministic Chaos in the 1960s, it is evident that only a very small part of our environment can be easily described by the kinds of logical formality which are currently available to science and mathematics, and that the vast majority of our interesting surroundings are subject to the large scale unpredictability of small causes within extreme nonlinearity. We consider that *energy* should be included in this category, and that the logical incompleteness of large quantum mechanical systems is evidence that although small scale Quantum Electrodynamics may be very accurate, there is a very small discrepancy between 'real' energy and our 'abstract' model of it which accounts for the appearance in large information processing networks of the phenomenon we call *consciousness*.

We suggest that the random nature of purely Darwinian selection has been progressively modified towards a more directed form through a number of mechanisms which first simulate '*intelligence*' – e.g. in amoebas and the Venus flytrap – then later implement it – e.g. in insects and animals. The study of inhomogeneously birational entity-ecosystem hierarchies indicates that the capability we call intelligence is associated with inter-scalar coupling in large networks. Although individual scales may be formally modelled, the coupling between them depends on all the scales present, and is not locally modellable. This injects a sense of Aristotle's 'final cause' into natural hierarchy.

The development of highly multiscalar systems brings with it not only the stability requirement for extensive cross-scalar correlation, it also provides for the stabilization of and development of hyperscalar correlation. We conclude that 'intelligence' in the absence of self-observation is unlikely; that self-observation in the absence of scalar development is impossible; that emergence of scale corresponds to the emergence of a "theory of self" in infants; and that the attainment of "wisdom" in humans is associated with the development of cervical hyperscalarity.

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The Blind Men and the Quantum

by John G. CRAMER

For the eight decades since the initial formulations of quantum mechanics by Heisenberg and Schroedinger, the field has been blighted by problems of interpretation involving questions about the meaning of the mathematics, the role of observers, and the nature of physical reality. A host of mutually contradictory quantum interpretations have arisen, including the orthodox Copenhagen Interpretation, the observer-independent Many Worlds Interpretation, and the author's own time-symmetric Transactional Interpretation. Because it is the mathematical formalism of quantum mechanics that makes testable predictions, it has been considered impossible to perform experimental tests that could support or falsify these rival interpretations, all of which describe the same mathematics. However, a new quantum optics experiment casts doubt on this assumption by illustrating that if an interpretation can be found to be inconsistent with the mathematics of quantum mechanics itself, it can be falsified or at least placed in doubt. We will discuss selected interpretations of quantum mechanics in the context of this new experimental insight.

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Learning Algorithms for Single-Layer Nonlinear Neural Networks based on the Nonlinear Threshold Logic Resolving the Parity Problem

by Daniel M. DUBOIS

In 1943, W.S. McCulloch and W. Pitts [5] proposed the first formal neuron based on a linear threshold logic. D.O. Hebb [4] proposed a learning rule based on the modification of the synaptic weights of neurons. R. Rosenblatt [7] realized the first neural network called the Perceptron, with an error correction based learning. B. Widrow and M. E. Hoff [9] proposed the delta rule based learning algorithm. M. Minsky and S. Papert [6] wrote a book where they criticized the Perceptron because it is not able to learn the exclusive OR, XOR (Parity 2 Problem). Since this time, the learning of the Parity Problem will be the main benchmark for any neural network. D.E. Rumelhart et al [8] proposed the back-propagation learning algorithm for the Multi-Layer Perceptron, with hidden neurons, able to learn the Parity Problem.
In 1990, Daniel Dubois [1] resolved the XOR problem with one single non-linear neuron

$$y = 2X(1 - X/2), \text{ where } X = x_1 + x_2, \text{ with } x_1 \in \{0,1\}, x_2 \in \{0,1\},$$

Parity 2 Problem : XOR Boolean Table

x_1	x_2	Y
0	0	0
0	1	1
1	0	1
1	1	0

and, in 1991, he showed that all the Boolean rules with two inputs can be modeled with the following single non-linear neuron [2]

$$y = \alpha + 4\mu X(1 - \beta X), \text{ where } X = s_1 x_1 + s_2 x_2,$$
$$\text{with } x_1 \in \{0,1\}, x_2 \in \{0,1\}$$

and where α, β, μ are parameters and s_1, s_2 the synaptic weights.
In 1993, Daniel Dubois and Germano Resconi [3] introduced the theory of what they called the non-linear threshold logic, based on the Dubois non-linear neuron. They demonstrated that the Parity problem is resolved without hidden neurons.
The general non-linear neuron is given by

$$y = w_0 + w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots + w_{12} x_1 x_2 + w_{13} x_1 x_3 + w_{23} x_2 x_3 + \ldots + w_{123} x_1 x_2 x_3 + \ldots$$

where w_i, w_{ij}, w_{ijk}, ... are the synaptic weights, or by its dual representation

$$y = W_0 + W_1 X + W_2 X^2 + W_3 X^3 + \ldots, \text{ with } X = s_1 x_1 + s_2 x_2 + s_3 x_3 + \ldots$$

where W_i are global weights, and s_j are the synaptic weights.
The XOR problem (Parity 2) is resolved by the neuron

$$y = x_1 + x_2 - 2x_1x_2$$

The Parity 3 problem is resolved by

$$y = x_1 + x_2 + x_3 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + 4x_1x_2x_3$$

or by its dual representation

$$y = 10/3\ X - 9/3\ X^2 + 2/3\ X^3, \text{ with } X = x_1 + x_2 + x_3$$

This paper will deal with learning algorithms for Single-Layer Nonlinear Artificial Neural Networks (NANN) based on these non-linear neurons resolving the Parity Problem. Numerical simulations of such learning algorithms will be presented and discussed.

References

1. **Daniel Dubois, 1990.** *Self-Organisation of fractal objects in XOR rule-based multi-layer Networks*, in Neural Networks & their Applications, Neuro-Nîmes 90, Proceedings of the third International Workshop, EC2, 1990, pp. 555-557.
2. **Daniel Dubois, 1991.** *Mathematical fundamentals of the fractal theory of artificial intelligence*, **Invited paper**, in New Mathematical tools in artificial intelligence, Communication & Cognition - Artificial Intelligence, 1991, vol 8, N°1 pp. 5-48.
3. **Daniel M. Dubois and Germano Resconi, 1993.** *Mathematical Foundation of a Non-linear Threshold Logic: a new Paradigm for the Technology of Neural Machines*, ACADEMIE ROYALE DE BELGIQUE, Bulletin de la Classe des Sciences, 6ème série, Tome IV, 1-6, 23 pages, 1993.
4. **D.O. Hebb, 1949.** *The Organization of Behaviour*, Wiley, New York 1949.
5. **W.S. McCulloch and W. Pitts, 1943.** *A logical Calculus of the Ideas Immanent in Nervous Activity*, Bulletin of mathematical Biophysics, vol. 5 (1943) 115-133.
6. **M. Minsky and S. Papert, 1969.** *Perceptrons* , the MIT Press, Cambridge 1969.
7. **R. Rosenblatt, 1958.** *Principles of Neurodynamics*, Spartan Books, New York 1962.
8. **D.E. Rumelhart et al, 1986.** *Learning representations by back-propagating errors*, Nature, vol. 323 (1986) 323.
9. **B. Widrow and M. E. Hoff, 1960.** *Adaptive switching circuits.* In *1960 IRE WESCON Convention Record*, pages 96-104, New York, 1960. IRE.

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Emergence, cognition, consciousness

by Andrée C. EHRESMANN and J.-P. VANBREMEERSCH

Several of the concepts introduced by Charles Ehresmann in Geometry have had applications outside of mathematics and he thought the same would apply to his works in category theory. The Memory Evolutive Systems (MES) we have developed in a series of papers since 1986 (summarized on our Internet site http://perso.wanadoo.fr/vbm-ehr, developed in a volume in print, Elsevier) justify this expectation by giving a model for natural complex systems, such as biological, neural or social systems. Here we'll insist on the case of cognitive systems, to model the emergence of higher cognitive processes and analyze the "neuronal correlates of consciousness".

1. Memory Evolutive Systems

The modeling of a complex system relies on the following operations:
1. *Description* of its objects, of their interrelations, how they are composed to transfer information and how they evolve over time: The system is modeled by an *Evolutive System* (ES), that is a family of categories (representing the state of the system at a given time) indexed by Time, with (partial) '*transition functors*' between them modeling the change of states (fibration over Time).
2. *Organization* into a hierarchy of different complexity levels: An object is complex if it binds an internal organization in patterns of simpler objects coordinated between them to perform its specific function; the complex object is then represented by the *colimit* of these patterns. The ES is *hierarchical* if its units are separated into levels, with an object of level $n+1$ being the colimit of a pattern of lower levels objects.
3. *Change and learning*: The changes correspond to the addition or suppression of some objects, and the binding together of patterns of interacting objects to form new complex objects (becoming the colimit of the pattern). In particular the system learns to recognize features of its environment and to develop adapted answers, thus memorizing a hierarchy of representations. This is modeled by the *complexification process* of a category with respect to a procedure (or 'strategy') having such objectives (adapted from the passage from a sketch to a prototype, C. Ehresmann). A sequence of complexifications leads to the long term storage of information, procedures and their result in a sub-ES, called the *Memory*, from which they can be later recalled and possibly modified for adaptation.
4. *Self-regulation*: An autonomous system is internally regulated by a modular organization at the lower levels which gives rise to a coherent global dynamics without a central conductor. A MES has a net of cooperating but possibly conflicting local modules, represented by sub-ES called *coregulators* (CR). Each CR operates a stepwise process at its own time-scale, to gather partial information in its landscape, to select an admissible procedure and send its commands to effectors, finally to evaluate its results. At each time, the the global operative procedure realized on the system is obtained after an equilibration process, the *interplay among the procedures* of the various CRs, possibly causing *fractures* to some CRs whose procedures have to be discarded for coherence. Whence a *dialectics between CRs* with heterogeneous time-scales and complexity levels.

2. Cognitive systems

The brain of an animal is modeled by the ES of neurons Neur representing his neurons linked by (classes of) synaptic paths. This ES is extended over time by adding more conceptual but functional components, called *cat-neurons*, modeling (classes of) synchronous assemblies of neurons. The *MES of cat-neurons* is obtained from Neur by successive complexifications, leading to the formation of more and more complex cat-neurons representing mental objects and cognitive processes. Here are some of its characteristics.
1. *Binding problem*: The construction of a complexification determines not only what are the added new objects, but also the complex links formed between them. In the MES of cat-neurons, it explicitly describes the 'natural' interactions between synchronous (super-) assemblies of neurons, thus solving the *binding problem* raised by neuroscientists. It also displays the emerging properties of a complex object (say, a synchronous assembly of neurons), not reducible to those of its lower levels components (its neurons).
2. *Emergence*: We find the condition for iterated complexifications to lead to the emergence of objects and processes of strictly *increasing complexity orders* (such as higher cognitive processes up to consciousness), which have both robustness and plasticity. It is the existence of 'multifold' objects able to switch between various internal organizations. This degeneracy property (in the terminology of Edelman, 1989) is modeled by the *Multiplicity Principle* (MP): 2 patterns may have the same colimit though their components are not directly linked. If a category satisfies the MP, so do its successive complexifications. As the laws of quantum physics imply that MP is satisfied in the category of atoms, it extends to all natural systems obtained by successive complexification of (sub-categories) of it; in particular higher cognitive or 'mental' processes emerge from physical states of the brain, thus supporting an *emergentist monism* (in the sense of Mario Bunge, 1979).
3. *Semantics, AC, consciousness*: Higher animals are able not only to store representations of particular perceptions, behaviors or events, but also to classify them through the detection of specific classes of invariance. This classification (described through the limit operation) gives rise to a *semantic memory*. It allows for the development, from birth on, of a part of the memory, the *Archetypal Core* (AC), in which the most important experiences of different modalities (perceptual, motor, affective,...) are integrated, strongly interconnected and can be self-activated along specific *fans* (which equip the AC with a Grothendieck topology). The AC forms the basis of the self and allows for the development of *consciousness* characterized as the formation of extended landscapes in which time can be internalized and retro- and prospection processes realized.

This model for cognition uses category theory rather than differential equations or dynamic systems as most usual models do. This is justified by the particular status of category theory which offers a framework for a "mathematical structuralism" reflecting the mathematician's processes. In fact, our model shows that the main categorical operations mirror the capacities of our mind determined by evolution: (i) description of a category = analysis of a class of objects and their interrelations; (ii) colimit = recognition of patterns of coordinated objects; (iii) complexification = formation and learning of more and more complex objects; (iv) universal problems = search for optimal solutions as does evolution.

Evolution of matter as a science
by George L. FARRE

If the theory of evolution is to be treated as a genuine science, then a number of constraints, both methodological and theoretical, have to be stipulated and respected. Chief among these are those associated with the representation of nature, which is the primary objective.

The representation of nature has two constituents, the *givens of nature* and the *elements of its representation.* The first are discrete events localized in Space and Time where they form sets of events **S(e_i)**, with (e_i) standing for the individual events. The second is a relational structure, i.e. a structural matrix, and its relata, neither one of which is local in Space and Time. They transform the singular events, or raw data, into the structural relata of the perspicuous representation of the local phenomena.

The history of the evolution of matter, anchored on solid phenomena, reveals a continuing diversification of emergent systems. It began with the Hot Bang **(HB)**, an explosive enrgy surge some **13.5 10^{12}** years ago, followed almost immediately by a cascade of virtual material systems engaged in rapid series of energy transformations, the outcome of which was the emergence of stable neutral matter, the atoms. Around **3.5 10^5** years after the **HB**, the atoms put an end to the dominance of the radiant energy that characterized the initial fire ball, thereby opening the expanding cosmos to eventual observations, spurred on by the rapid decrease of the cosmological energy density.

The science of matter is the manifestation of the evolution of the energy issued from the Quantum Vacuum **(QV)**, with consequences observable to human observers. The evolution of neutral matter began in earnest with the emergence of the simplest atoms, H, D, He, Li, C, N, followed by more complex ones prepared in galaxies and supernovae, and by molecules, both inorganic and organic, and then by nucleated cells that marked the emergence of life forms. In the course of time, more complex material systems capable of rational language emerged, making possible complex societies, global organizations, and the alteration of their environments.

The complex structure of natural matter may be used to define the *quantum unit of evolution* **(QUE)**. Its basic characteristics are: first, its *locality*, i.e. its unitary behavior in Space and Time; second, the closed *dynamical architecture* of its internally entangled energy fields; and third, the *Heisenberg energy surface* **(HES)**, that separates the internal energy domain $\mathbf{D_{int}}$ from the external domain of action $\mathbf{D_{ext}}$ of the newly emergent system. The **HES** is opaque to transactional energy processes of the type discovered by Fermi and Bethe (the so called "exchange forces") from either side of it, the energy densities $\mathbf{E_1}$ and $\mathbf{E_2}$ of the two domains being unequal ($\mathbf{E_1 > E_2}$). This feature of **HES** rules out the possibility of deriving the external wave function Ψ from the internal one $\mathbf{\Phi}$, and consequently the hope of reducing the complex energy dynamics of material systems to those of their subatomic elements, thereby making possible both the relative stability of evolved matter and its remarkable diversity in nature.

The **QUE** becomes considerably more complex in the life era, as shown by the behavior of living matter which reveals the effects of a new internal ability to image, store and evaluate various features of the environment in terms of actions. In the case of humans, this new feature is often called the *Cartesian Energy Function* (**CEF**) because the stored holographic images are mapped and represented by means of a conceptual language.

The constraints that made the objectivity of the science of matter possible and kept its representations focused on what there is, point to the limits intrinsic to this science in both space and time. Some of these will be sighted in the short coda, and other speakers may also address these issues. Thus goes the evolution of science.

Short References

Atmanspacher, Harald: *Complexity and Meaning as a Bridge Across the Cartesian Cut,* .**Journal of Consciousness Studies, I/2** (1994).

Cramer, John. C.: *The Transactional Interpretation of Quantum Mechanics*, **RMP 58/3,** 647 (1987).

Ehresmann, Andrée, Vanbremeersch Jean-Paul: *Memory Evolutive Systems* **Elsevier Press** (2005) (In Press)

Eigen, Manfred: *Steps Towards Life: A Perspective on Evolution,* **Oxford U.P.** (1992).

Heelan, Patrick: *Space-Perception and the Philosophy of Scienc*e **U. California Press** (1988).

Josephson, Brian: *Limits to the Universality of Quantum Mechanics* in **Foundations of Physics, 18/12** (1988).

Pribram, Karl: *Brain and Perception: Holonomy and Structure in Figural Processing* **Lawrence Erlbaum,** Hillsdale N.J. USA (1991).

Williams, W.S.C.: *Nuclear and Particle Physics* **Oxford, Clarendon Press,** (1991).

Some Implications of the Curie and Rosen Symmetry Principles in Quantum Physics

by John E. GRAY and Allen D. PARKS

Symmetry in the guise of group theory has played a fundamental role in physics in the last hundred years. For a general survey of physical applications of group theory, Wigner [5] is one of the best sources. The role of symmetry in a more general sense in physics has more obscure origins. One hundred years ago Pierre Curie [1] wrote a paper on symmetry in physics that can be efficiently encoded as the koan "the effect is at least as symmetric as the cause". Rosen [4] has expanded upon this notion and has summarized the relationship of symmetry to physics in terms of six core symmetry principles. Of particular interest here are Rosen's **Symmetry Principle for Processes**, which states that the "initial" symmetry group (that of the cause) is a subgroup of the "final" symmetry group (that of the effect), and his **Special Symmetry Evolution Principle**, which states that *as a quasi-isolated system evolves, the populations of the equivalence subspaces (equivalence classes) of the sequence of the states through which it passes cannot decrease, but either remain constant or increase.*

These principles have been used implicitly in physics over the years, but largely without recognition of their importance. Parks [3] wrote a paper on finite cyclic quantum state machines (FCQSM) that explored a number of properties of such machines. Here, we extend this work and show that these machines exhibit symmetries consistent with Curie's Principle and Rosen's Symmetry Evolution Principle, but inconsistent with Rosen's Symmetry Principle for Processes.

More specifically, a FCQSM is the closed cyclic evolution of a state $|\Psi_0\rangle$ produced by the sequential m-fold application of a unitary transformation $\hat{U}(\Delta t)$, i.e.

$$|\Psi_0\rangle \overset{\hat{U}}{\Rightarrow} |\Psi_1\rangle \overset{\hat{U}}{\Rightarrow} |\Psi_2\rangle \overset{\hat{U}}{\Rightarrow} \dots \overset{\hat{U}}{\Rightarrow} |\Psi_{m-1}\rangle \overset{\hat{U}}{\Rightarrow} |\Psi_m\rangle = |\Psi_0\rangle$$

Thus, for a given integer k, we have the state evolution rule:

$$\hat{U}^k(\Delta t)\,|\Psi_0\rangle = \hat{U}^{k \bmod m}(\Delta t)\,|\Psi_0\rangle = |\Psi_{k \bmod m}\rangle$$

This implies that the group generated by $\hat{U}(\Delta t)$ is isomorphic to the finite cyclic group $\mathbf{Z}_m$ of order m. Consequently, the processing of states in an $(n+1)$-dimensional Hilbert space by such a FCQSM is equivalent to the continuous free left action of the group $\mathbf{Z}_m$ upon a $(2n+1)$-dimensional sphere. This approach lends itself to a covering space description of FCQSMs from which it is easily verified that Rosen's Special Symmetry Evolution Principle applies for all FCQSMs (fixed fiber cardinality). In addition, two other types of symmetry can be identified: dynamical symmetries (automorphisms of $\mathbf{Z}_m$) and process symmetries (covering space automorphisms). It is shown that every FCQSM simulates its group of process symmetries and that *the associated process is more symmetric than the dynamics which produces it* (Curie's Principle). It is also determined that not every FCQSM simulates its group of dynamical symmetries (in the sense that the dynamic symmetry group is isomorphic to a subgroup of the group of process symmetries). Within this context, such FCQSMs violate Rosen's Symmetry Principle for Processes. Thus, one may classify FCQSMs according to whether or not they are consistent with this principle and we pose the question "*does this symmetry based dichotomy indicate the existence of a fundamental difference between these machine classes?*"

This question and the analysis of its answer falls within the broad domain category theory and conceptual mathematics [2] which has become increasingly important in physics (see Baez website). In the language of categories, we note that the homotopy functor is employed to define the induced topological complexity index associated with FCQSM processing and its relationship to dynamic and process symmetry, as well as entropy, is established by this paper. Additionally, we observe that the concept of the internal simulation of internal dynamics versus external dynamics simulation has some implications with respect to machine intelligence (internal simulation of external behavior of the world being a form of intelligence) as well as other forms of intelligence. In the full paper, we explore all of these issues in some detail.

References

[1] P. Curie, Sur la symétrie dans les phénomènes physiques d'un champ électrique et d'un champ magnétique, *J. Phys.* (3rd ser.) 3, 393-415 (1894); English translation, J. Rosen and P. Copie, On symmetry in physical phenomena, symmetry of an electric field and a magnetic field, reprinted in J. Rosen (editor), *Symmetries in Physics: Selected Reprints* (American Association of Physics Teachers, Stony Book, NY, 1982), pp 17-25.

[2] F. W. Lawvere and S. H. Schanuel, *Conceptual Mathematics: A first introduction to categories*, Cambridge University Press, 1997.

[3] A. D. Parks, Finite cyclic quantum state machines: a topological perspective, *J. Phys. A: Math Gen.* 34 (2001), 5731-5743.

[4] J. Rosen, *Symmetry in Science: An Introduction to the General Theory,* Springer-Verlag New York, 1995.

[5] E. P. Wigner, *Symmetries and Reflections: Scientific Essays*, Ox Bo Press, Connecticut, 1979.

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Embodied Consciousness and Quantum Science
by Patrick A. HEELAN

I take 'ernbodied consciousness' to be a code word for a study of consciousness in the phenomenological tradition of E. Husserl, M. ,Heidegger, and M. Merleau-Ponty, the neuro-psychology tradition of K. Pribram, F. Varela, T. Metzinger, M. Donald, and A. Noe, and the quantum macro-theory of G. Farre. H. Atmanspacher, and other scholars some of whom are present at this meeting.

The study of 'ernbodied consciousness' involves coordination between the *externality* of the physical body in its environment and the *internality* of consciousness. 1 take the former to include the use of scientific methods (theory and instrumental observation), and the latter to use hermeneutical and phenomenological methods in reflecting on human awareness. Since all objective knowing is revealed to us only within consciousness, *externality* is captive to *internality*. The general ontological principle involved is; *The Being of Knowing is the Knowing of Being*, where 'the Being of Knowing' is Heidegger's *Dasein*, that is the human subject actively and caringly engaged with the world we navigate (or simply, *World*) and 'the Knowing of Being' is Heidegger's *alethia* or the process of revealing *Being* as *Truth.* All of the italicized terms are relative to a human point of view. The captivity of *externality* by *internality* leads to the recognition that the revelation of truth about the world is an ongoing cumulative historical and cultural process. It accumulates in physical long-term memory (part of *externality* from which place it is accessible to consciousness (*internality*) as needed.

Long-term memory comprises symbols filled with common cultural meanings; these reside in bodily neurological functions (*engrams*) and in the World (*exograms*). Among the latter are conventional gestures, taken-for-granted community activities such as sports, dancing, music, etc., but in the case of humans the most powerful among the exograms are linguistic symbols. Some of these define the categories fundamental to customary ways of living in the World and others belong to the structure of the World to which they ultimately refer. While such external memory symbols are entities (*Vorhanden*) in the world and can be studied by scientific methods, they are also carriers of meaning (*Zuhanden*) and as such relate to consciousness since meanings are given only to consciousness. The cumulative historical and cultural process of human knowing then has the structure of a *hermeneutical spiral* (or *circle*) that revolves through four ontological phases: 1. questioning by focused engagement with the world, 2. making and testing of heuristic intelligibilities, 3. affirming what is revealed, and finally, 4. deciding what is to be done. In its many and diverse historical revolutions, the hermeneutical spiral opens to human consciousness new windows on the World, re-focuses and possibly closes some old windows. Through these windows new worldly horizons are made accessible and new forms of human culture are invented. Horizons are the place of nested objects ordered in space and time according to scientific and other lawful cultural relationships.
The study concludes that perception and measurement, whether classical or quantum, are structured by the same core set of neuro- psycho- teleo-logical transcendental functions

operating within embodied consciousness that constitute the *hermeneutical circle*. These are the structures that constitute what we call *'intentionality'*. Intentionality, as we said above, is structured dynamically according to the *hermeneutical spiral*. Among the *functions of intentionality* are *virtual group-theoretic transformation functions*; they are *mimetic* functions in search of invariants or symmetries in the World relative to *mimetic behaviors*. These functions are *hermeneutical*: they are search-engines for categorial meanings within appropriate horizons of the World. They are *intentional*: they are oriented towards the 'objectivation' of given objects relative to the dynamics of the inquiry. These objects ontologically are ideal forms or symmetries as given internally, or local individuals 'factually' as given externally as present in the World. Symmetry-making involves *entanglement*'; this is the ontological inseparability of (categorial) unity and (sensory) diversity in the sensory-motor flux producible by the group-theoretic functions. Symmetry-breaking occurs when the embodied subject is confronted with the sensory 'facticity' of engagement with the World and accepts the contingent sensory clue as categorial. Categories and local individuals can each be symbolized, internally or externally, as operators on the embodied subject's sensory-motor flux. As externally labeled in language or theory they become part of the theoretical structure of the World represented usually by the grammar of descriptive language. Symmetry-breaking involves *'disentanglement'* -- the reduction of possible unity in diversity to a local 'factual' presence experienced within the contingent exchange of information between subject and World.
The general formal structure of the intentional processes described above exemplifies in a striking way the same Hilbert space structure that characterizes the quantum theory; in which the Hilbert space of states of the embodied subject (being conscious of the World) is partitioned by operators for horizons, categories, and facts. This raises further questions relative to the proposed quantum structure of human embodied consciousness. This question was first posed by F. London and E. Bauer, and later taken up by E. Wigner and J. von Neumann. It is being carried forward today by H. Atmanspacher, G. Farre, and other scholars some of whom are present at this meeting.

References

Atmanspacher, H. & G. Dalenoort (eds.), 1994. *Inside versus outside: Endo- and Exo-Concepts of Observation and Knowledge in Physics, Philosophy and Cognitive Science.* Berlin: Springer.

Cassirer, E., 1944. The Concept of Group and the theory of Perception, *Philosophy and Phenomenological Research* 5, 1-15.

Donald, M., 2001. :*A mind so rare: The Evolution of Human Consciousness.* New-York: Norton.

Elvee, R. (ed.). 1982. *Mind in Nature / Nobel Conference XVII.* New York: Harper & Row.

Farre, G. 2002a. The quantum theory of evolution: A sketch, *Cybernetics & Systems I* , 29.

Farre G. 2002b, The Evolution of Matter: A quantal theory of evolution, *American Institute of Physics Proceeding* **627**, 17.

Gadamer, H-G.,.1975. *Truyth and Method.* New York: Crossroads Press.

Heelan, P., 2002. Phenomenology and the Philosophy of the Natural Sciences, In *Phenomenology World-wide*, T. Tvmieniecka, ed. Dordrecht & Boston, M.A: Kluwer, 631-641..

Heelan, P., 1998, Scope of Hermeneutics in the Philosophy of Natural Science, *Studies in the History and Philosophy of Science* 29, 273-298.

Heelan, P., 1983/88. *Space Perception and the Philosophy of Science.* Berkeley. CA: Univ. of California Press.

Heelan, P., 1965. *Quantum Mechanics and Objectivity: The physical Philosophy of Heisenberg.* The Hague: Nijhoff.

Heelan, P.. and Jay Schulkin, 1998. Hermeneutical Philosophy and Pragmatism: A Philosophy of Science, *Synthese* 115 (1998), 269-302.

Heidegger, M., 1962. *Being and Time.* Trans. from the German by John Macquarrie. Oxford: Blackwell.

Heidegger, M., 1977. *Questions concerning Technology and other Essays.* New York: Harper and Row.

Husserl, E., 1960. *Cartesian Meditations.* Trans. D. Cairns. The Hague: Nijhoff.

Husserl, E., 1976. *Crisis of European Sciences and Transcendental Philosophy.* Evanston, IL: Northwestern Univ. Press.

Husserl, E., 1983. *Ideas: General Introduction to Pure Phenomenology [Ideas I].* Dordrecht &. Boston: Kluwer.

London, F. and E. Bauer. 1939. *La théorie de l'observation en mécanique quantique.* Actualités scientiques et industrielles 775. Paris: Herman.

Lonergan, B., 1971. *Method in Theology.* Toronto: Toronto University Press.

Merleau-Ponty, M., 1962. *Phenomenology of Perception.* New York: Humanities Press.

Noe, A., 2004. *Action in Perception.* Cambridge, MA: MIT Press.

Pribram. K., 1991. *Brain and Perception.* Hillsdale, NJ: Erlbaum.

Pribram, K., 1971. *Languages of the brain.* Englewood Cliffs, NJ: Prentice-Hall.

Snell, B., 1982. *The Discovery of Mind in Greek Philosophy and Literature.* New York: Dover.

Varela, F. and E. Thompson, 1991. *The Embodied Mind.* Cambridge: MIT Press.

Wheeler, J.A. and W. Zurek (eds.), 1983. *Quantum Theory and Measurement.* Princeton: Princeton University Press.

Wigner, È., 1967. *Symmetries and Reflections.* Bloomington, IN: Indiana University Press.

Concepts and Reality
by Brian D. JOSEPHSON

Concepts, sometimes taking a more precise form in the shape of theories, are mental constructs that drive the thinking process, and are valid to the degree that the thinking that derives from them accords with reality; typically, concepts and theories are valid only within a particular domain. Quantum theory, though at first sight a single theory, is in reality a succession of theories, whose content and concepts change form as the science develops, new content and concepts being added from time to time as we try to expand the domain of validity.

The question remains whether the upper bound of all such theories can include everything. In fact paralleling (though not closely analogous to) measurability in the mathematical theory of integration, the quantifiability typically demanded of a theory by physicists implies a limitation to its domain of applicability, a point once briefly appreciated by Nils Bohr, and more recently by Robert Rosen. Outside the conventional quantum mechanics lie areas of investigation such as biology and cognition, and in addition non local cognition and semiosis. each of which have their own characteristic modes of description reflecting some consistent aspect of reality, which may be perceived as anomalous when only partially understood.

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Category theory and living systems
by Paul KAINEN

We apply some recent results on commutativity of diagrams to the notion of a weak adjoint functor. We then consider these ideas in the context of a category-theoretic model for living systems.

A diagram scheme is a directed graph. A diagram in some category is a diagram scheme in the underlying undirected graph, and a face of the diagram is a pair of directed paths with common source and sink. The face *commutes* if composition of the morphisms in each path gives the same resultant morphism. A diagram commutes if each of its faces commutes.

It is shown in [6] that in a groupoid category, for diagrams on the scheme of the d-dimensional hypercube, when commutativity fails, the failure occurs on at least d - 1 faces. For example, if only one face is not known to be commutative in a 3-dimensional cube while the others are commutative, then by the well-known "cube lemma" the unknown face is also commutative - so failure of commutativity can only occur on at least two faces.

Adjointness of two functors F,G can be expressed as follows: For any morphism in either of the two domain categories there is a commutative square in which the objects are hom-sets and two of the parallel arrows are induced by composition with the morphism and by composition with the functorial image of the morphism. The other two sides are natural isomorphisms. The notion of *weak* adjointness substitutes a natural *epimorphism*. Weak adjointness was introduced by the author applied to a problem in algebraic topology [3], and it has been explored in connection with typed graphs [1].

A categorical model for biology has been proposed by various authors. The approach suggested in [4] involves the use of adjointness. Suppose $F : \boldsymbol{C} \rightarrow \boldsymbol{D}$ with G in the reverse direction, and assume that F is the left adjoint and G the right adjoint, where $\boldsymbol{C}$ may be thought of as a physical category and $\boldsymbol{D}$ as some sort of abstract biological category. Then F will be playing the role of perception and G of action. According to adjointness, F will preserve colimits. We note that the categorical model proposed in [2] also has the feature that perception preserves colimits. An example of such preservation could be contained in the psychophysical concept of Gestalt. However, the adjoint model also includes the preservation of limits by action, which we applied to the idea of muscular coordination.

But even weak adjointness is sufficient to guarantee the preservation of commutative squares. If the natural transformation ν given on object pairs C,D by

$$\nu_{\mathrm{C;D}} : hom_{\boldsymbol{C}}(\mathrm{C}, G\mathrm{D}) \rightarrow hom_{\boldsymbol{D}}(\mathrm{FC},\mathrm{D})$$

is epi, then by the cube lemma (and some similar constructions due to C. Ehresmann) the image under F of a commutative diagram in $\boldsymbol{C}$ is also commutative. Hence, the veridicality of perception can be described in this categorical language.

We will consider the corresponding problem for action.

REFERENCES

[1] Banach, R. & Corradini, A., *Abstract diagrams and an opfibration account of typed graph transformation*, 1999, http://www.cs.man.ac.uk/banach/some.pubs/Abs.Diag.Op.Ty.Gr.Tr.pdf

[2] Ehresmann, A.C. & Vanbremeersch J.-P., Hierarchical evolutive systems, *Bull. Math. Biol.* 49 (1987) 13-50.

[3] Kainen, P. C., Weak adjoint functors, *Math. Zeit.*, 122 (1971) 1-9.

[4] Kainen, P.C., Functorial cybernetics of attention, Chap. 57 in *Neurocomputers and Attention*, vol. II, A. V. Holden and V. I. Kryukov, Eds., Manchester University Press, 1990.

[5] Kainen, P. C., Categorical coordination of muscular action. In *Proc. 8th International Conference on Cybernetics and Systems*, C. N. Manikopoulos, Ed., Vol. I, NJIT Press, Newark, NJ, 1990, 314-319.

[6] Kainen, P. C., Isolated squares in hypercubes and robustness of commutativity, *Cahiers de Topol. et Géom.Diff. Cat.*, XLIII (2002) 213-220.

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Rencontre de la psychiatrie avec la Théorie des catégories

par P. MARCHAIS

II y a loin de la psychiatrie à la théorie des catégories. Néanmoins, leur rencontre s'est avérée possible lors d'une progression naturelle des démarches cliniques.
Cet exposé a pour objectif de montrer comment une approche clinique peut conduire l'observateur à passer d'une étude classique à une conception ensembliste, puis catégorique du trouble mental.

Le cheminement suivi

Lc trouble mental est polymorphe et mouvant, d'où la difficulté de son étude. En outre, le phénomène psychique n'est pas réductible à une logique formelle et le langage psychiatrique est souvent métaphorique. Le renouvellement de sa connaissance impose donc un cheminement adapté. Il s'est fait en trois temps à l'aide d'une logique du flou et de graphes nécessaires pour aborder la complexité psychique.

Une première phase descriptive classique de repérage a consisté à réunir des symptômes et une étiopathogénie dans une spatio-temporalité bidimensionnelle, afin de constituer des formes circonscrites: manie, mélancolie, schizophrénie, délire, etc. Or, en 1950, le passage d'un service fermé d'hôpital psychiatrique en service ouvert d'hôpital général nous a montré l'existence de modifications des troubles mentaux.

Une deuxième phase a donc consisté à traduire la mouvance de ces troubles. Une perspective dynamique a ainsi cherché à extraire des invariants fonctionnels et à recourir à une combinatoire dans un monde virtuel pluridimensionnel, ce qui permettait de mobiliser les formes préalables. Elle a transformé les formes circonscrites en processus organisés en ensembles et sous-ensembles (excitation, dépression, déstructuration...). Le recours à un module d'observation en niveaux subdivisés en sous-ensembles (somato-instinctif, émotivo-affectif et intellectuels), qui est intégré à une échelle de divers milieux (éducatif, social, culturel...), permet de reformuler la psychiatrie sur un mode ensembliste. C'est l'approche systémale. Considérant le psychisme comme un système, elle en extrait les composants : niveaux d'organisation, intégrations, communications, rétroactions, auto- et hétérorégulations. Des rigidités conceptuelles, liées au modèle de référence, ne permettant pas de traduire toutes les mouvances cliniques ont incité à recourir aux hyperensembles pour envisager des hypersystèmes qui offrent notamment une meilleure compréhension de l'histoire complexe de la psychiatrie et de ses dynamiques sous-jacentes.

Une troisième phase précisant la formation structurale de ces courants en a résulté. Elle a conduit à envisager des formes virtuelles de dysfonctionnements psychiques à partir de fonclions représentées graphiquement par un fléchage systémal.

La jonetion avec la théorie des catégories

Celle-ci a spontanément surgi lors d'un exposé de A. C. Ehresmann et J.-P. Vanbremeersch sur la théorie des systèmes évolutifs à mémoire qui a permis d'enrichir cette approche systémale.
Elle s'est confirmée lors d'une étude sur les liens entre l'angoisse et l'anxiété, symptômes banals qui s'intriquent el s'engendrent mutuellement, mais dont le mode structural évolutif n'avait jamais pu être bien précisé jusqu'à présent.

Cette rencontre s'est ensuite développée lors d'un rapprochement analogique de nos recherches sur le principe controversé du tiers-inclus (qui avait abouti à la notion d'un noyau fonctionnel central dans le système psychique) avec le noyau archétypal isolé par la théorie des systèmes évolutifs à mémoire.
Le fléchage systémal permet aussi de mieux comprendre l'existence de phénomènes complexes comme les "états mixtes" qui mêlent l'excitation et la dépression, les états dits "atypiques" qui prêtent à confusion, ainsi que la structuration de troubles classiques.

Cette méthode incite encore à préciser des opérateurs psychiques essentiels mal définis à ce jour, comme ceux d'intégration el de décision.
Enfin, l'élude particulièrement complexe des phénomènes de conscience, incite à en analyser des topiques abordées à partir de graphes, les unes relevant de démarches logiques naturelles, les autres de démarches analytiques métaphoriques.

Conclusions

La rencontre entre la psychiatrie et la théorie des catégories se fait dans un monde virtuel sur un mode éminemment analogique. Cette théorie incite d'un point de vue conceptuel à pénétrer davantage dans la structuration et le déterminisme des troubles mentaux et des outils d'observation. Elle permet ainsi d'ouvrir une ligne de recherche nouvelle dans le domaine particulièrement complexe des troubles mentaux.
Dès lors, la psychiatrie est conviée à un objectif voisin de celui du mathématicien qui est de "comprendre la structure de toute chose", et par là est induite à envisager une meilleure analyse des soubassements d'une interdisciplinarité qui la concerne.

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Minding quanta
by Karl H. PRIBRAM

The revolution in science inaugurated by quantum physics made us aware, as never before, of taking into consideration the role of observation and measurement in the construction of data. A personal experience illuminates the extent of this revolution. Once, Eugene Wigner remarked that in quantum physics we no longer have observables (invariants) but only observations. Tongue in cheek I asked whether that meant that quantum physics is really psychology, expecting a gruff reply to my sassiness. Instead, Wigner beamed a happy smile of understanding and replied "yes, yes, that's exactly correct". In a sense, therefore, if one takes the reductive path in science one ends up with psychology, not particles of matter.

Another clue to this "turning reductive science on its head" is the fact that theoretical physics is, in some non-trivial sense, a set of esthetically beautiful mathematical formulations that are looking for confirmation (see George Chapline: Is theoretical physics the same as mathematics?)

David Bohm pointed out that, were we to look at the cosmos without the lenses of our telescopes, we would see a hologram. Holograms were the mathematical invention of Dennis Gabor who developed them in order to increase the resolving power of electron microscopy. Emmet Leith developed the hologram for laser light photography, a development that has, in popularity, overshadowed the mathematical origin of the invention.

I have extended Bohm's insight into the importance of lenses to observing a space-time image to the lens in the optics of the eye (Pribram 1991). In fact, the receptor mechanisms of the ear, the skin and tongue work in a similar fashion. Without these lenses and lens-like operations all of our perceptions would be entangled as in a hologram. Holography is based on taking a space-time image and spreading it (the transformation rule is called a spread function; the Fourier transform is the one used by Gabor) over the extent of the recording medium. Thus, whole has become totally entangled in the part.

In optics, the small aperture of the pupil produces the same transformation as does the lens. When the pupil has been chemically dilated as during an eye examination, focus is lost and the experienced vision becomes blurred. However, if a pinhole or slit in a piece of cardboard is placed in front of the dilated eye, ordinary vision is restored. One can accomplish this in a crude fashion if one needs to read some directions by tightly curling one's index finger producing a slit. Also, in experiments during which we map receptive fields of cells in the brain we drift dots or slit-like lines and edges in front of a stationary eye. In my laboratory we used dots, single lines, double lines and gratings and found differences in the recorded receptive fields when more than one dot or line was used. The differences were due to interactions produced in the visual system of the brain when the stimulating dots or lines moved together against a random background.

What I'm proposing is that the difference in the observation of interference effects (an entangled holographic record) in the two slit experiment vs. the observation of an objective (particle) in the single slit experiment is due to the difference in the measurement apparatus. This, of course, is not a new proposal: it is the essence of the argument made initially by Bohr and accepted by quantum physicists for almost a century. What I am adding is that the measuring apparatus, the slits, are mimicking the biology of how we ordinarily observe the world we live in.

The presentation will fill out and develop the consequences of these claims.

Dirac's Hole Theory

by Andrew VOGT

In an attempt to unify quantum mechanics and relativity, Dirac in the 1930's proposed a revolutionary series of ideas. Electromagnetism became intrinsically a multi-particle theory, with such startling features as creation and annihilation of particles and a second species of matter - anti-matter. His theory also gave an account of spin, previously an ad hoc observable.

Dirac presented his ideas in terms of an infinite sea of negative energy particles hypothesized as the ground for ordinary phenomena. Occasionally under the influence of the electromagnetic field a negative energy particle might jump to a positive energy, leaving a hole in the infinite sea. The phenomenon that resulted would be a pair of particles - a positive energy particle with a certain charge and another positive energy particle of the opposite charge. As time evolved this pair might be annihilated and/or further such pairs might come into existence, always in such a way that total charge and other quantum numbers are preserved.

In this presentation we give an account of the essence of his ideas using a category-theoretic framework. We consider the category of Hilbert spaces and bounded linear transformations and certain functorial constructions in this category. This allows us to focus on Dirac's main ideas without becoming distracted by concrete details.

If time permits, we shall also discuss computational approaches developed by Feynman and others that led to the great experimental success of quantum electrodynamics.

1. S. S. Schweber, *An Introduction to Quantum Field Theory*, Row, Peterson and Company, Evanston, Illinois, 1961.
2. J. S. Schwinger (ed.), *Selected papers on Quantum Electrodynamics,* Dover, New York, 1958.

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