CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

EDUARDO J. DUBUC
JORGE G. ZILBER

Weil prolongations of Banach manifolds in an analytic model of *SDG*

Cahiers de topologie et géométrie différentielle catégoriques, tome 46, nº 2 (2005), p. 83-98
<http://www.numdam.org/item?id=CTGDC_2005__46_2_83_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLVI-2 (2005)

WEIL PROLONGATIONS OF BANACH MANIFOLDS IN AN ANALYTIC MODEL OF SDG

by Eduardo J. DUBUC and Jorge G. ZILBER

Résumé. La théorie des points proches des variétés différentielles réelles d'André Weil généralise la notion fondamentale de jet d'Ehresmann, et comme celui-ci, comprend tout le calcul différentiel des dérivées d'ordre supérieur. Dans cet article nous généralisons et développons cette théorie pour le cas des variétés banachiques complexes. Etant donnés une algèbre de Weil W et un ouvert B d'un espace de Banach, l'analyticité et la dimension infinie nous imposent des modifications dans la définition de $B[W]$, le prolongement d'espèce W de B, pour que ce dernier ait les propriétés souhaitées (Définition 2.8). Pour une fonction holomorphe f, nous démontrons une formule explicite en termes des dérivées d'ordre supérieur pour la fonction $f[W]$ induite entre les prolongements d'espèce W. Dans une seconde partie, nous considérons un modèle analytique de la GDS muni d'un plongement j de la catégorie des ouverts d'espaces de Banach, et nous montrons que le calcul différentiel usuel dans cette catégorie correspond au calcul différentiel intrinsèque du topos. Explicitement, nous démontrons les formules $jB[W] \cong (jB)^{D_W}$ et $j(f[W]) = j(f)^{D_W}$, où D_W est l'objet infinitésimal du topos déterminé par l'algèbre de Weil W.

Introduction.

Weil prolongations were introduced for paracompact real C^∞ manifolds as a generalization of Ehresmann's Jet-bundles, and they play a central role in SDG (Synthetic Differential Geometry).

In section 1 we recall some notions and constructions we need in the paper, and in this way we fix notation and terminology.

In section 2 we define and develop Weil prolongations for open sets of complex Banach spaces. We do so in a way that automatically yields the version of Weil prolongations for any Banach manifold. Given a real C^∞ manifold M, and a Weil algebra W, classically the Weil prolongation $M[W]$ is defined as the set of morphisms $M[W] = \{\psi : C^\infty(M) \to W\}$. This definition as such is not adequate for complex Banach manifolds. We introduce a definition that has the desired properties and that coincides with the classical one in the real finite dimensional case (definition 2.8). Then, we give an explicit construction of the Weil bundle $B[W]$ for an open subset of a Banach space B (proposition 2.10), and given an holomorphic function $f : B_1 \to B_2$ between open subsets of complex Banach spaces, we give an explicit formula in terms of higher derivatives for the induced map $f[W] : B_1[W] \to B_2[W]$ between the respective Weil bundles (formula 2.11 and proposition 2.12).

In section 3 we show that the embedding $j : \mathcal{B} \to \mathcal{T}$ of the category of open subsets of complex Banach spaces into the analytic model of SDG developed in [6], [7], is compatible with the differential calculus. That is, we show that under this embedding the usual differential calculus in the category $\mathcal{B}$ corresponds with the intrinsic differential calculus of the topos $\mathcal{T}$. Explicitly, this is subsumed in the formulas $jB[W] \cong (jB)^{D_W}$, and $j(f[W]) = j(f)^{D_W}$, where D_W is the infinitesimal object of the topos that corresponds to the Weil algebra W (theorems 3.19 and 3.20).

1. Recall of some definitions and notation.

Analytic rings were introduced in [5] for the purpose of constructing models of SDG well adapted to the study of analytic spaces. An analytic ring A has an underlying c-algebra that by abuse we also denote A, and the reader can think an analytic ring just as this c-algebra, however, for details see [5].

We consider analytic rings in the Topos $\mathcal{S}h(X)$ of sheaves on a topological space X, see [5][12]. The sheaf C_X of germs of continuous complex valued functions is a local analytic ring in $\mathcal{S}h(X)$. An A-ringed space is (by definition) a pair $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is an analytic ring in $\mathcal{S}h(X)$ furnished with a local morphism $\mathcal{O}_X \to C_X$ (it follows that $\mathcal{O}_X$ is a local analytic ring). Given any point $p \in X$, the fiber is a local analytic ring $\pi : \mathcal{O}_{X,p} \to C_{X,p} \to \mathbb{C}$. If σ is a section defined in (a neighborhood of) p, we shall denote by $[\sigma]_p$ the corresponding element in the ring $\mathcal{O}_{X,p}$, and by $\sigma(p)$ its value, that is, the complex number $\sigma(p) = \pi([\sigma]_p)$.

Consider in $\mathcal{O}_{n,p}$ (ring of germs of holomorphic functions on n variables) the inductive limit topology for the topology of uniform convergence on compact subsets on the rings $\mathcal{O}_n(U)$, $p \in U \subset \mathbb{C}^n$. It can be proved that in this topology a sequence $[f_k]_p$ converges to a limit $[f]_p$, if there is a neighborhood where (for sufficiently large k) all f_k and f are defined and the convergence is uniform. We shall refer to this topology as "the topology of uniform convergence". We shall need the following result of Cartan ([2] 194, or [3] 28. Lemma 6):

1.1. Lemma. *All ideals of the ring $\mathcal{O}_{n,p}$ are closed for the topology of uniform convergence.*

In the finite dimensional case the coordinate projections play an important (and seldom explicitly indicated) role. Here all the continuous linear forms have to be taken into account. The following result from [7] reflects this fact and it is an important tool that we shall need in this paper.

1.2. Lemma. *Let B be an open subset of a complex Banach space C. Let U be an open subset of $\mathbb{C}^n$, let $q \in U$, and let $J_q \subset \mathcal{O}_{n,q}$ be an ideal. Let f and g be holomorphic functions, $U \to B$, such that $f(q) = g(q) = p \in B$. Suppose that for all linear continuous forms $\alpha \in C'$, it holds that $[\alpha \circ f - \alpha \circ g]_q \in J_q$. Then, for all germs $[r]_p \in \mathcal{O}_{B,p}$, it also holds $[r \circ f - r \circ g]_q \in J_q$.* □

We recall now the construction of the topos $\mathcal{T}$ introduced in [6]. We consider the category $\mathcal{H}$ of affine analytic schemes [6]. An object E in $\mathcal{H}$ is an A-ringed space $E = (E, \mathcal{O}_E)$ (by abuse we denote also by the letter E the underlying topological space of the A-ringed space) which is given by two coherent sheaves of ideals R, S in $\mathcal{O}_D$, where D is an open subset of $\mathbb{C}^n$, $R \subset S$. The ideal S determines the set E of points, and the ideal R the structure sheaf. Thus, $E = \{p \in D \mid h(p) = 0 \; \forall \, [h]_p \in S_p\}$, and $\mathcal{O}_E = (\mathcal{O}_D/R)|_E$ (restriction of $\mathcal{O}_D/R$ to E). The arrows in $\mathcal{H}$ are the morphism of A-ringed spaces. We will denote by $\mathcal{T}$ the Topos of sheaves on $\mathcal{H}$ for the (sub canonical) Grothendieck topology given by the open coverings. There is a full (Yoneda) embedding $\mathcal{H} \to \mathcal{T}$.

Consider any open set B of a Banach space C, then the ring $\mathcal{O}(B)$ of complex valued holomorphic functions is an analytic ring, and given any point $p \in B$, the ring $\mathcal{O}_{B,p} = \mathcal{O}_{C,p}$ of germs at p of holomorphic functions is a local analytic ring. The pair $(B, \mathcal{O}_B)$, where $\mathcal{O}_B$ is the sheaf of germs of complex valued holomorphic functions is a A-ringed space. From [7] we have:

1.3. Proposition. *The correspondence $B \mapsto (B, \mathcal{O}_B)$ defines a full embedding $\mathcal{B} \longrightarrow \mathcal{A}$ from the category $\mathcal{B}$ of open sets of Banach spaces and holomorphic functions into the category $\mathcal{A}$ of A-ringed spaces.* □

We warn the reader that this embedding, unlike that in the finite dimensional case, does not preserve finite products (see [7]).

Next we recall, also from [7], the definition of the embedding of the category of open subsets of Banach spaces into the topos $\mathcal{T}$.

1.4. Definition. *Given an arrow $t = (t, \tau) : (E, \mathcal{O}_E) \longrightarrow (B, \mathcal{O}_B)$, we say that (t, τ) has local extensions if for each $x \in E$, there is an open $U \ni x$ in $\mathbb{C}^n$ and an extension $(f, f^*) : (U, \mathcal{O}_U) \longrightarrow (B, \mathcal{O}_B)$, $t = f|_{U \cap E}$, and $\tau = \rho \circ f^*$, where ρ is the quotient map. We denote:*

$$jB(E) = \{(t, \tau) : (E, \mathcal{O}_E) \longrightarrow (B, \mathcal{O}_B) \mid (t, \tau) \textit{ has local extensions}\}.$$

Given an arrow $g : F \longrightarrow E$ in $\mathcal{H}$, if t has local extensions, so does $t \circ g$, and given an arrow $f : B_1 \longrightarrow B_2$ in $\mathcal{B}$, if t has local extensions, so does $(f, f^*) \circ t$. From [7] we have:

1.5. Theorem. *The correspondence $B \mapsto jB$ defines a finite product preserving embedding $\mathcal{B} \longrightarrow \mathcal{T}$ from the category $\mathcal{B}$ of open sets of Banach spaces and holomorphic functions into the topos $\mathcal{T}$.* □

This embedding does preserve products (not an easy fact unlike in the finite dimensional case), it is faithful but not full. However, the global sections functor, when restricted to objects of the form jB, $B \in \mathcal{B}$, is faithful. Thus, the arrows in the topos $\lambda : jB_1 \longrightarrow jB_2$ correspond to certain functions $f = \Gamma(\lambda) : B_1 \longrightarrow B_2$, which are not necessarily holomorphic, but they are G-holomorphic. These functions have been studied in [8], where a complete characterization is given.

2. Weil prolongations of Banach manifolds.

Weil prolongations have been introduced in [11] for paracompact real C^∞ manifolds as a generalization of Ehresmann's Jet-bundles [9], and they play a central role in synthetic differential geometry. Here we develop this concept for open sets of complex Banach spaces (this automatically will yield the version of Weil prolongations for any Banach manifold). Recall the following definition:

2.6. Definition. *A complex Weil algebra is a c-algebra W equipped with a morphism $W \xrightarrow{\pi} \mathbb{C}$ such that:*

1) it is local with maximal ideal $I = \pi^{-1}(0)$.

2) it is finite dimensional as a $\mathbb{C}$-vector space. $W = \mathbb{C} \oplus I$, $I = \mathbb{C}^m$. The integer $m+1$ is the linear dimension of W.

3) I is a nilpotent ideal. The least integer r such that $I^{r+1} = 0$ is the order (or height) of W.

For details about Weil algebras see [1], [5]. Given any Weil algebra W with maximal ideal I, the dimension d of the vector space I/I^2 is the geometric dimension of W, and $W = \mathbb{C}[\xi_1, \xi_2, \dots \xi_d]$, where the ξ_i satisfy a finite set H of polynomial equations, $h(x_1, \dots x_d) = 0$, $h \in H$. Since $\xi_i^{r+1} = 0$, it follows that there is a quotient morphism $\mathcal{O}_{d,0} \to W \cong \mathcal{O}_{d,0}/R$, $[x_i]_0 \mapsto \xi_i$, which determines a (unique) structure of local analytic ring in W [5]. The kernel $R = ((h(x_1, \dots x_d))_{h \in H})$ of this morphism has associated a set $M \subset \mathbb{N}^d$ (where $\mathbb{N}$ indicates the set of non negative integers) of d-multiindexes as described in the following remark [1]:

2.7. Remark. *Let W be any complex Weil algebra as in definition 2.6. Then there is a set M (of cardinality m) of d-multiindexes such that the list of derivatives D^α, for $\alpha \in M$ determines W in the sense that $W \cong \mathcal{O}_{d,0}/R$, $R = \{ [f]_0 \in \mathcal{O}_{d,0} \mid f(0) = 0,\ D^\alpha f(0) = 0\ \forall \alpha \in M \}$.* □

We introduce now a definition of Weil prolongation for Banach manifolds. However, here we consider explicitly only open subsets of Banach spaces (notice that it is a local definition).

2.8. Definition. *Given a complex banach space C, an open subset $B \subset C$, and a Weil algebra W, we define the* prolongation *of B by*

W, denoted $B[W]$, as follows:

$$B[W] = \{(p, \psi) \mid p \in B \text{ and } \psi : \mathcal{O}_{B,p} \to W\}$$

where ψ is a morphism of analytic rings such that there is an open $0 \in V \subset \mathbb{C}^d$ and an holomorphic function $g : V \to B$, such that $g(0) = p$ and $\psi = \rho \circ g^*$, where ρ is the quotient $\mathcal{O}_{d,0} \to W$ (we say that g is a local extension).

Weil prolongations $M[W]$ were first defined for M a finite dimensional paracompact C^∞-manifolds as the set of morphisms $M[W] = \{\psi : C^\infty(M) \to W\}$. In this case this definition coincides with the one given above (see [4], Proposition 1.11). Here, the analytic condition requires a local definition with germs at a point p, and the infinite dimensional condition requires to take as an assumption the existence of local extensions.

The Weil prolongation $B[W]$ is clearly functorial (by composing) in the variable W. It is also functorial in the variable B. More explicitly:

2.9. Proposition. *Let B_1 and B_2 be open subsets of complex Banach spaces, and let f be an holomorphic function $f : B_1 \to B_2$. Consider $(p, \psi) \in B_1[W]$ and $f^* : \mathcal{O}_{B_2, f(p)} \to \mathcal{O}_{B_1, p}$. Then, $(f(p), \psi \circ f^*) \in B_2[W]$ and this defines a map $f[W] : B_1[W] \to B_2[W]$.* □

The projection $(p, \psi) \mapsto p$ is a map $B[W] \to B$ under which $B[W]$ is the jet-bundle whose points contain the information for the value at 0 of an holomorphic function and a prescribed set of its derivatives. In fact, we shall see that the points of $B[W]$ are in bijection with the product of B and m copies of C indexed by the set M of multiindexes in remark 2.7. In particular, $B[W]$ can be considered to be an open subset of a Banach space.

2.10. Proposition. *Let B be an open subset of a complex Banach space C. Then $B[W] \cong B \times \prod C$, where the product is taken over $\alpha \in M$. More explicitly, the map $\omega : B[W] \to B \times \prod C$ defined by $\omega(p, \psi) = (p, (D^\alpha g(0))_{\alpha \in M})$, (where $g : V \to B$ is any local extension as in definition 2.8) is a bijection.*

Proof. Consider the quotient $\rho : \mathcal{O}_{d,0} \to (\mathcal{O}_{d,0}/R) \cong W$. First we show that ω is well defined, then that it is a bijection.

Let h be any other local extension. By definition we have that $g(0) = h(0) = p$ and $\rho \circ g^* = \rho \circ h^* = \psi$. Thus, $\rho([r \circ g]_0) = \rho([r \circ h]_0)$ $\forall [r]_p \in \mathcal{O}_{B,p}$. Then, $[r \circ g - r \circ h]_0 \in R$, that is, $D^\alpha(r \circ g - r \circ h)(0) = 0$ $\forall \alpha \in M$. When $r \in C'$, this means that $(r \circ D^\alpha g - r \circ D^\alpha h)(0) = 0$, that is, $r(D^\alpha g(0)) = r(D^\alpha h(0))$, and, by the Hann-Banach theorem it follows that $D^\alpha g(0) = D^\alpha h(0)$ for all $\alpha \in M$.

Injectivity:

Suppose that $\omega(p, \psi_1) = \omega(q, \psi_2)$. Consider local extensions g of (p, ψ_1) and h of (q, ψ_2). Then $g(0) = p = q = h(0)$, and for each $\alpha \in M$, $(D^\alpha g)(0) = (D^\alpha h)(0)$. We have $\psi_1([r]_p) = \rho([r \circ g]_0)$ and $\psi_2([r]_p) = \rho([r \circ h]_0)$ $\forall [r]_p \in \mathcal{O}_{B,p}$. Let $r \in C'$. Then, $r((D^\alpha g)(0)) = r((D^\alpha h)(0))$, that is $D^\alpha(r \circ g)(0) = D^\alpha(r \circ h)(0)$ $\forall \alpha \in M$. Since also $r \circ g(0) = r \circ h(0)$, we have $[r \circ g - r \circ h]_0 \in R$. Given any $[r]_p \in \mathcal{O}_{B,p}$, by lemma 1.2 we also have $[r \circ g - r \circ h]_0 \in R$, thus $\rho([r \circ g]_0) = \rho([r \circ h]_0)$, so $\psi_1([r]_p) = \psi_2([r]_p)$, which shows $\psi_1 = \psi_2$.

Surjectivity:

Given any $(p, (c_\alpha)_{\alpha \in M})$, let $g : \mathbb{C}^d \to C$ be the function defined by $g(z) = p + \sum_{\alpha \in M} \frac{c_\alpha}{\alpha!} z^\alpha$. Clearly, g is holomorphic, $g(0) = p \in B$ and $D^\alpha(g)(0) = c_\alpha$ for $\alpha \in M$. Take an open subset V of $\mathbb{C}^d$ such that $0 \in V$ and $g : V \to B$. Clearly $(p, \rho \circ g^*) \in B[W]$ and $\omega(p, \rho \circ g^*) = (p, (c_\alpha)_{\alpha \in M})$. □

Next we shall determine an explicit description of the map $f[W]$ under the bijection ω, showing at the same time that it is an holomorphic map of open subsets of Banach spaces.

Let B_1 and B_2 be open subsets of complex Banach spaces C_1 and C_2 respectively, and let f be an holomorphic function, $f : B_1 \to B_2$. Consider for each $\beta \in M$ the set:

$$A_\beta = \{\mu = (\mu_\alpha)_{\alpha \in M},\ \mu_\alpha \in \mathbb{N}, \quad \textit{such that} \quad \sum_{\alpha \in M} \mu_\alpha \alpha \ = \ \beta\}$$

Let

$$|\mu| \ = \ \sum_{\alpha \in M} \mu_\alpha, \quad \mu! \ = \ \prod_{\alpha \in M} \mu_\alpha!, \quad |\alpha| \ = \ \sum_{i=1}^{d} \alpha_i, \quad \alpha! \ = \ \prod_{i=1}^{d} \alpha_i!$$

We say that the set A_β is finite. In fact, for $\alpha \in M$, $|\alpha| \geq 1$, so $|\mu| = \sum_{\alpha \in M} \mu_\alpha \leq \sum_{\alpha \in M} \mu_\alpha |\alpha| = |\sum_{\alpha \in M} \mu_\alpha \alpha| = |\beta|$.

For each $p \in B_1$ there exists a ball $B(p, S)$ and a sequence of continuous homogeneous polynomials P_r of degree r such that $f(c) = \sum_{r \geq 0} P_r(c-p)$ uniformly on $B(p, S)$. Then, there exists a unique multilinear symmetric continuous mapping ϕ_r such that $P_r(c) = \phi_r(c, \dots, c)$, see [10]. We define:

$$\omega(f) : B_1 \times \prod_{\alpha \in M} C_1 \to B_2 \times \prod_{\alpha \in M} .C_2, \qquad \omega(f) = (\omega(f)_0, (\omega(f)_\alpha)_{\alpha \in M})$$

$$(2.11) \qquad \omega(f)_0(p, (v_\alpha)_{\alpha \in M}) = f(p),$$

$$\omega(f)_\beta(p, (v_\alpha)_{\alpha \in M}) = \beta ! \sum_{\mu \in A_\beta} \frac{|\mu|!}{\mu!} \phi_{|\mu|} ((\frac{v_\alpha}{\alpha!}, \dots, \frac{v_\alpha}{\alpha!})_{\alpha \in M}), \quad \beta \in M.$$

(where for each $\alpha \in M$, the dots indicate a vector of μ_α coordinates all equal to $v_\alpha / \alpha!$).

We have that for all $\beta \in M$, $\omega(f)_\beta$ is separately holomorphic, thus it is holomorphic, see [10], and since $\omega(f)_0$ also is holomorphic, it follows that $\omega(f)$ is holomorphic.

2.12. Proposition. *Under the bijection ω, the arrow $f[W]$ is given by the holomorphic function $\omega(f)$. That is, $\omega \circ f[W] = \omega(f) \circ \omega$, the following diagram commutes:*

$$\begin{array}{ccc} B_1[W] & \xrightarrow[\cong]{w} & B_1 \times \prod_{\alpha \in M} C_1 \\ \downarrow{\scriptstyle f[W]} & & \downarrow{\scriptstyle w(f)} \\ B_2[W] & \xrightarrow[\cong]{w} & B_2 \times \prod_{\alpha \in M} C_2 \end{array}$$

Proof. Let $(p, (c_\alpha)_{\alpha \in M}) \in B_1 \times \prod_{\alpha \in M} (C_1)$, and let (p, ψ) be the unique point in $B[W]$ such that $\omega(p, \psi) = (p, (c_\alpha)_{\alpha \in M})$. Let g be the local extension of (p, ψ) defined by $g(z) = p + \sum_{\alpha \in M} \frac{c_\alpha}{\alpha!} z^\alpha$. Then, $f[W](p, \psi) = (f(p), \psi \circ f^*)$, and $f \circ g$ is a local extension of $(f(p), \psi \circ f^*)$. Since $g(0) = p \in B_1$, there is an open subset Y *of* $\mathbb{C}^d$ such that $0 \in Y$, $g(Y) \subset B_1$, and $f \circ g : Y \to B_2$.

We have $\omega(f(p), \psi \circ f^*) = (f(p), (D^\alpha(f \circ g)(0))_{\alpha \in M})$. Thus, $f(c) = \sum_{r \geq 0} P_r(c-p)$ uniformly on a ball $B(p, S) \subset B_1$. Let $Y_1 \ni 0$ be an open subset of Y such that $g(Y_1) \subset B(p, S)$. For $z \in Y_1$,

$$f(g(z)) = \sum_{r\geq 0} P_r(\sum_{\alpha \in M} \frac{c_\alpha}{\alpha\,!} z^\alpha) = \sum_{r \geq 0} \phi_r(\sum_{\alpha \in M} \frac{c_\alpha}{\alpha\,!} z^\alpha, \dots, \sum_{\alpha \in M} \frac{c_\alpha}{\alpha\,!} z^\alpha).$$

Then, by Leibinitz's formula,[10], this is equal to

$$\sum_{r\geq 0} \sum_{|\mu|=r} \frac{r\,!}{\mu!} \phi_r((\frac{c_\alpha}{\alpha!} z^\alpha, \dots, \frac{c_\alpha}{\alpha!} z^\alpha)_{\alpha \in M}).$$

Since ϕ_r is multilinear, this is equal to

$$\sum_{r\geq 0} \sum_{|\mu|=r} \frac{|\mu|\,!}{\mu!} \prod_{\alpha \in M} (z^\alpha)^{\mu_\alpha} \phi_r((\frac{c_\alpha}{\alpha!} \dots, \frac{c_\alpha}{\alpha!})_{\alpha \in M}) =$$

$$\sum_{r\geq 0} \sum_{|\mu|=r} \frac{|\mu|\,!}{\mu!} z^{(\sum_{\alpha \in M} \mu_\alpha \alpha)} \phi_{|\mu|}((\frac{c_\alpha}{\alpha!}, \dots, \frac{c_\alpha}{\alpha!})_{\alpha \in M}).$$

It follows that in the development of $f(g(z))$ around 0, given $\beta \in M$ the coefficient of z^β is obtained by considering all μ such that $\sum_{\alpha \in M} \mu_\alpha \alpha = \beta$, that is, all $\mu \in A_\beta$. So, this coefficient is $\sum_{\mu \in A_\beta} \frac{|\mu|\,!}{\mu\,!} \phi_{|\mu|}((\frac{c_\alpha}{\alpha!}, \dots, \frac{c_\alpha}{\alpha!})_{\alpha \in M})$, and it is is equal to $\frac{D^\beta(f \circ g)(0)}{\beta!}$. Then:

$$D^\beta(f \circ g)(0) = \beta! \sum_{\mu \in A_\beta} \frac{|\mu|!}{\mu!} \phi_{|\mu|}((\frac{c_\alpha}{\alpha!}, \dots, \frac{c_\alpha}{\alpha!})_{\alpha \in M}) = \omega(f)_\beta(p, (c_\alpha)_{\alpha \in M}).$$

It follows that $(f(p), (D^\alpha(f \circ g)(0))_{\alpha \in M}) = \omega(f)(p, (c_\alpha)_{\alpha \in M})$, and thus $\omega(f(p), \psi \circ f^*) = \omega(f)(p, (c_\alpha)_{\alpha \in M}))$. Since this holds for all $(p, (c_\alpha)_{\alpha \in M}) \in B_1 \times \prod_{\alpha \in M} (C_1)$, it follows that $\omega \circ f[W] \circ \omega^{-1} = \omega(f)$, that is, $\omega \circ f[W] = \omega(f) \circ \omega$. □

We end this section describing how Weil algebras determine infinitesimal objects in the topos. A Weil algebra W can be interpreted as an affine (infinitesimal) analytic scheme $D_W \subset (\mathbb{C}^d, \mathcal{O}_d)$, $D_W = (\{0\}, W)$. In $\mathcal{H}$ (or in the topos $\mathcal{T}$) D_W is defined by $D_W = [[(x_1, \dots x_d) \,|\, (h(x_1, \dots x_d) = 0,\ h \in H]] \subset (\mathbb{C}^d, \mathcal{O}_d)$, where H is the set of polynomial equations that define W. We have:

2.13. Proposition. *The assignment* $W \mapsto D_W$ *determines a full embedding* $\mathcal{W}^{op} \longrightarrow \mathcal{H}$ *from the dual of the category* $\mathcal{W}$ *of Weil algebras into the category* $\mathcal{H}$ *of affine analytic schemes.* □

The condition in the definition of $B[W]$ says exactly that the pair (p, ψ) viewed as an arrow $(0, W) \longrightarrow (B, \mathcal{O}_B)$ has local extensions. Thus:

2.14. Remark. *By definition,* $B[W]$ *is the set of arrows* $B[W] = [D_W, jB]$ *in* $\mathcal{T}$, *and under this identification, for any holomorphic function* f, $f[W] = (jf)^*$ *(composing with* jf*).* □

Thus, a map $D_W \to jB$ in $\mathcal{T}$ is a jet of an holomorphic germ (with a shape determined by W), and composition in $\mathcal{T}$ corresponds with composition of jets and functions.

3. Compatibility of Weil prolongations with exponentials in the topos.

In this section we show that the embedding j is compatible with the calculus of all higher derivatives. That is, it is compatible with the construction of the jet bundle $B[W] \to B$ determined by any Weil algebra W. Abusing notation, we can write the equations $jB[W] \cong (jB)^{D_W}$, and $j(f[W]) = j(f)^{D_W}$.

Through all this section we shall consider a Weil algebra W with associated ideal $R \subset \mathcal{O}_{d,0}$, set M of d-multiindexes and set H of polynomial equations, as in definition 2.6 and remark 2.7.

Given a local analytic ring $A = \mathcal{O}_{n,x}/J_x$, where $x \in \mathbb{C}^n$, and $J_x \subset \mathcal{O}_{n,x}$ is any ideal, consider the coproduct (as analytic rings) $A \otimes W = \mathcal{O}_{n+d,(x,0)}/(J_x, R)$, where $(J_x, R) \subset \mathcal{O}_{n+d,(x,0)}$ is the ideal generated by the germs at $(x, 0)$ of the functions of J_x and the functions of R considered as functions of $n+d$ variables. We have:

3.15. Proposition. *Given any* $[f]_{(x,0)} \in \mathcal{O}_{n+d,(x,0)}$:

$$[f]_{(x,0)} \in (J_x, R) \iff [f(-, 0)]_x \in J_x, \;\; [(D^\alpha f)(-, 0)]_x \in J_x \; \forall \alpha \in M.$$

Proof. We consider $f = f(u, z)$ where $u \in \mathbb{C}^n$ and $z \in \mathbb{C}^d$.

$\Rightarrow$) We have $f(u, z) = \sum \gamma_i(u, z)\, h_i(u) + \sum \delta_j(u, z)\, g_j(z)$ where $[h_i]_x \in J_x$, $[g_j]_0 \in R$ and $[\gamma_i]_{(x,0)}, [\delta_j]_{(x,0)} \in \mathcal{O}_{n+d,(x,0)}$. This holds in an open neighborhood of $(x, 0)$. Since D^α indicates derivation with respect to z, it follows that

$$D^\alpha f(u, 0) = \sum (D^\alpha \gamma_i)(u, 0)\, h_i(u) + \sum D^\alpha(\delta_j(u, z)\, g_j(z))(u, 0).$$

Here, for each u, we have that $[\delta_j(u, -)\, g_j]_0 \in R$. It follows that $D^\alpha(\delta_j(u, z)\, g_j(z))(u, 0) = 0$. Thus, $[(D^\alpha f)(-, 0)]_x \in J_x$. Similarly, $f(u, 0) = \sum \gamma_i(u, 0)\, h_i(u)$ (recall that since $[g_j]_0 \in R$, then $g_j(0) = 0$). Thus $[f(-, 0)]_x \in J_x$.

$\Leftarrow$) Consider the development of f around $(x, 0)$, $f(u, z) = f(u, 0) + \sum b_\beta(u)\, z^\beta$, where $b_\beta(u) = \frac{1}{\beta!} D^\beta f(u, 0)$. Given any $\beta \neq 0$, if $\beta \in M$, then $[(D^\beta f)(-, 0)]_x \in J_x$, thus, $[b_\beta]_x \in J_x$. If $\beta \notin M$, then, given any $\alpha \in M$, since $\beta \neq \alpha$, it follows that $D^\alpha(z^\beta)(0) = 0$, thus, $[z^\beta]_0 \in R$. It follows that in all cases $[b_\beta(u)\, z^\beta]_{(x,0)} \in (J_x, R)$. Thus, in the development of f, $[f(-, 0)]_{(x,0)}$ and all $[b_\beta(u)\, z^\beta]_{(x,0)} \in (J_x, R)$. Since this series converges uniformly on a neighborhood of $(x, 0)$, it follows by lemma 1.1 that $[f]_{(x,0)} \in (J_x, R)$. □

Given any analytic ring A in any topos, a Weil c-algebra W (as in definition 2.6) determines an analytic ring structure in A^{m+1} that we shall denote $A[W]$.

In particular, consider an object $E \in \mathcal{H}$ given by two coherent sheaves of ideals I, J in an open subset of $\mathbb{C}^n$, $J \subset I$, $E = Z(I)$ and $\mathcal{O}_{E,x} = \mathcal{O}_{n,x}/J_x$ for $x \in E$. We define the object $(E, \mathcal{O}_E[W])$ to be the A- ringed space with fibers

$$\mathcal{O}_E[W]_x = \mathcal{O}_{E,x}[W] = \{[\sigma_0]_x + \sum [\sigma_i]_x\, \xi_i\,, \quad [\sigma_0]_x, [\sigma_i]_x \in \mathcal{O}_{E,x}\}$$

where the symbols ξ_i satisfy the same set H of polynomial equations that define W. We have:

3.16. Remark. *Let $\pi_x : \mathcal{O}_{n,x} \to \mathcal{O}_{E,x}$ be the quotient map. There is a morphism of analytic rings $\delta_x : \mathcal{O}_{n+d,(x,0)} \to \mathcal{O}_{E,x}[W]$ defined by*

$$\delta_x([f]_{x,0}) = \pi_x[f(-, 0)]_x + \sum_{\alpha \in M} \pi_x([(D^\alpha f)(-, 0)]_x)\, \xi_\alpha$$

which identifies $\mathcal{O}_{E,x}[W]$ with the quotient $\mathcal{O}_{n+d,(x,0)}/(J_x, R)$ (this follows by 3.15 and shows that $(E \times \{0\}, \mathcal{O}_E[W]) \in \mathcal{H}$). □

3.17. Remark. *By construction of coproducts of analytic rings and products in $\mathcal{H}$, we have that $E \times D_W = (E \times \{0\}, \mathcal{O}_{E\times\{0\}})$ where, for each $x \in E$, $\mathcal{O}_{E\times\{0\},(x,0)} = \mathcal{O}_{n+d,(x,0)}/(J_x, R)$. It follows that $E \times D_W = (E \times \{0\}, \mathcal{O}_E[W])$.* □

3.18. Proposition. *Let E be an object in $\mathcal{H}$, let U be an open subset of $\mathbb{C}^n$ such that $E \subset U$, let V be an open subset of $\mathbb{C}^d$ such that $0 \in V$, let B be an open subset of a complex Banach space C, and let g, h be holomorphic functions, $g, h : U \times V \to B$. Then:*

$$(g, g^*)|_{E\times D_W} = (h, h^*)|_{E\times D_W}$$

$$\Longleftrightarrow$$

$$((g(-,0), (g(-,0)^*)|_E = ((h(-,0), (h(-,0)^*)|_E \text{ and } \forall\alpha \in M,$$
$$((D^\alpha g)(-,0), (D^\alpha g)(-,0)^*)|_E = ((D^\alpha h)(-,0), (D^\alpha h)(-,0)^*)|_E.$$

Proof. To simplify the proof it is convenient to adopt the convention that D^0 is the identity operator. Thus, if $\alpha = 0 = (0, 0, \dots, 0)$, $D^\alpha f = f$.

Let $\delta_x : \mathcal{O}_{n+d,(x,0)} \to \mathcal{O}_{n+d,(x,0)}/(J_x, R)$ and $\pi_x : \mathcal{O}_{n,x} \to \mathcal{O}_{n,x}/J_x$ be the quotient maps, and let $x \in E$.

$\Rightarrow$) Clearly $g(x,0) = h(x,0) = p$, and for all $[r]_p \in \mathcal{O}_{B,p}$, $\delta_x([r \circ g]_{(x,0)}) = \delta_x([r \circ h]_{(x,0)})$. Then, $[r \circ g - r \circ h]_{(x,0)} \in (J_x, R)$. Thus, by 3.15, it follows that for $\alpha = 0$ and all $\alpha \in M$, $[D^\alpha((r \circ g) - (r \circ h))(-,0)]_x \in J_x$. When $r \in C'$, this means that $[(r \circ D^\alpha g - r \circ D^\alpha h)(-,0)]_x \in J_x$. Since $J_x \subset I_x$, the value at x of any germ in J_x is 0. Thus, $r((D^\alpha g)(x,0)) = r((D^\alpha h)(x,0))$ (for all $r \in C'$). It follows by the Hahn- Banach theorem that $(D^\alpha g)(x,0) = (D^\alpha h)(x,0) = q_\alpha \in C$. By lemma 1.2, it follows that $[r \circ D^\alpha g(-,0) - r \circ D^\alpha h(-,0)]_x \in J_x$ for all $[r]_{q_\alpha} \in \mathcal{O}_{C,q_\alpha}$. Thus, $\pi_x([r \circ D^\alpha g(-,0)]_x) = \pi_x([r \circ D^\alpha h(-,0)]_x)$. This means that $\pi_x \circ D^\alpha g(-,0)^* = \pi_x \circ D^\alpha h(-,0)^*$ (for all $x \in E$). Thus $(D^\alpha g)(-,0)^*)|_E = (D^\alpha h)(-,0)^*)|_E$.

$\Leftarrow$) For $\alpha = 0$ and each $\alpha \in M$, $(D^\alpha g)(x,0) = (D^\alpha h)(x,0) = q_\alpha \in C$ and $[r \circ D^\alpha g(-,0) - r \circ D^\alpha h(-,0)]_x \in J_x$ for all $[r]_{q_\alpha} \in \mathcal{O}_{C,q_\alpha}$.

When $r \in C'$, this means that $[D^\alpha((r \circ g) - (r \circ h))(-, 0)]_x \in J_x$. Thus, by 3.15, $[r \circ g - r \circ h]_{(x,0)} \in (J_x, R)$ for all $r \in C'$. Using lemma 1.2 for (J_x, R), it follows that $[r \circ g - r \circ h]_{(x,0)} \in (J_x, R)$ for all $[r_p] \in \mathcal{O}_{B,p}$, where p is the point $p = g(x, 0) = h(x, 0) = q_0$. This means that $\delta_x([r \circ g]_{(x,0)}) = \delta_x([r \circ h]_{(x,0)})$. It follows that $\delta_x \circ g^* = \delta_x \circ h^*$ for all $x \in E$. This finishes the proof. □

3.19. Theorem. *Let B be an open subset of a Complex Banach space C. Then, there is an isomorphism $\omega : (jB)^{D_W} \cong j(B \times \prod C)$ in $\mathcal{T}$, where the product is taken over $\alpha \in M$. Moreover, under the identification $B[W] = [D_W, jB]$, this isomorphism on global sections is the bijection ω defined in proposition 2.10.*

Proof. Since the functor j preserves products, it is equivalent to show that for each $E \in \mathcal{H}$, there is a natural (in E) bijection: $[E, (jB)^{D_W}] \cong [E, jB \times \prod (jC)]$. The second statement will be evident by the definition of this bijection.

a) Let ξ be an arrow, $\xi : E \to (jB)^{D_W}$.

That is, ξ is an arrow $E \times D_W \to jB$ in $\mathcal{T}$, which is given by a morphism of A-ringed spaces, $\xi : (E \times \{0\}, \mathcal{O}_{E\times\{0\}}) \to (B, \mathcal{O}_B)$ which has local extensions.

For each $x \in E$ there is an open subset U of $\mathbb{C}^n$ such that $x \in U$, an open subset V of $\mathbb{C}^d$ such that $0 \in V$ and an holomorphic function $g : U \times V \to B$ such that $(g, g^*)|_{E'\times D_W} = \xi|_{E'\times D_W}$, where $E' = U \cap E$. In this way, we have an open covering of E, and, for each E' in the covering, morphisms $(g(-, 0), g(-, 0)^*)|_{E'}$, $((D^\alpha g)(-, 0), (D^\alpha g)(-, 0)^*)|_{E'}, \forall \alpha \in M$.

Given another open E'' in the covering, with holomorphic function h, $(h, h^*)|_{E''\times D_W} = \xi|_{E''\times D_W}$, we have $(g, g^*)|_{(E'\cap E'')\times D_W} = (h, h^*)|_{(E'\cap E'')\times D_W}$. By 3.18 (on the object $(E' \cap E'')$), it follows $(g(-, 0), (g(-, 0)^*)|_{E'\cap E''} = (h(-, 0), (h(-, 0)^*)|_{E'\cap E''}$, and $\forall \alpha \in M$,

$$((D^\alpha g)(-, 0), (D^\alpha g)(-, 0)^*)|_{E'\cap E''} = ((D^\alpha h)(-, 0), (D^\alpha h)(-, 0)^*)|_{E'\cap E''}$$

So, these data is compatible in the intersections. Therefore it determines unique morphisms of A-ringed spaces $\psi : (E, \mathcal{O}_E) \to (B, \mathcal{O}_B)$ and $\beta_\alpha : (E, \mathcal{O}_E) \to (C, \mathcal{O}_C)$ such that, for each E' in the covering, it holds that $\psi|_{E'} = (g(-, 0), (g(-, 0)^*)|_{E'}$, and $\beta_\alpha|_{E'} = ((D^\alpha g)(-, 0), (D^\alpha g)(-, 0)^*)|_{E'}$.

By a similar argument it follows that ψ and β_α do not depend on the covering. Clearly these morphisms have local extensions. Thus, they actually define arrows in the topos, $\psi : E \to jB$, and $\beta_\alpha : E \to jC$, which determine an arrow $(\psi, (\beta_\alpha)_{\alpha \in M}) : E \to jB \times \prod (jC)$ in $\mathcal{T}$. This defines a function $[E, (jB)^{D_W}] \to [E, jB \times \prod (jC)]$. We have to prove now that it is a bijection.

b) (Injectivity). Suppose that we have two arrows ξ_1 and ξ_2 $E \to (jB)^{D_W}$ in $\mathcal{T}$ which determine the same $(\psi, (\beta_\alpha)_{\alpha \in M})$. They correspond to arrows $E \times D_W \to jB$, that is, morphism of A-ringed spaces $E \times D_W \to (B, \mathcal{O}_B)$ with local extensions. For each $x \in E$, let g and h be local extensions of ξ_1 and ξ_2 around $(x, 0)$ respectively. We can assume that they are defined in a same open subset $U \times V \subset \mathbb{C}^n \times \mathbb{C}^d$, $(x, y) \in U \times V$, $g, h : U \times V \to B$, $(g, g^*)|_{E' \times D_W} = \xi_1|_{E' \times D_W}$, $(h, h^*)|_{E' \times D_W} = \xi_2|_{E' \times D_W}$, where $E' = U \cap E$. Since ξ_1 and ξ_2 determine the same $(\psi, (\beta_\alpha)_{\alpha \in M})$, it follows that $(g(-, 0), (g(-, 0)^*)|_{E'} = (h(-, 0), (h(-, 0)^*)|_{E'}$ and $((D^\alpha g)(-, 0), (D^\alpha g)(-, 0)^*)|_{E'} = ((D^\alpha h)(-, 0), (D^\alpha h)(-, 0)^*)|_{E'}$ for all $\alpha \in M$. It follows by 3.18 that $(g, g^*)|_{E' \times D_W} = (h, h^*)|_{E' \times D_W}$, thus, $\xi_1|_{E' \times D_W} = \xi_2|_{E' \times D_W}$. Since the open sets $E' \times D_W$ cover $E \times D_W$, it follows that $\xi_1 = \xi_2$.

c) (Surjectivity). Let $(\psi, (\beta_\alpha)_{\alpha \in M}) : E \to jB \times \prod (jC)$ in $\mathcal{T}$. That is, $\psi : (E, \mathcal{O}_E) \to (B, \mathcal{O}_B)$, $\beta_\alpha : (E, \mathcal{O}_E) \to (C, \mathcal{O}_C)$ are morphisms of A-ringed spaces with local extensions. For each $x \in E$, let g_0, g_α, be a local extension of ψ, β_α respectively, $g_0 : U \to B$, $g_\alpha : U \to C$, U an open subset of $\mathbb{C}^n$, $x \in U$. Let $g : U \times \mathbb{C}^d \to C$ be the function defined by $g(u, z) = g_0(u) + \sum_{\alpha \in M} \frac{g_\alpha(u)}{\alpha!} z^\alpha$. Clearly g is holomorphic and $g(x, 0) \in B$. It follows that there exists an open subset T of $\mathbb{C}^n$, $x \in T$, an open subset V of $\mathbb{C}^d$, $0 \in V$, and $g(T \times V) \subset B$. Consider $g : T \times V \to B$, and the morphism of A-ringed spaces $(g, g*)|_{E' \times D_W} : E' \times D_W \to (B, \mathcal{O}_B)$ where $E' = T \cap E$. Notice that $g(u, 0) = g_0(u)$, and for each $\alpha \in M$, $(D^\alpha g)(u, 0) = g_\alpha(u)$. That is, $g(-, 0) = g_0$ and $(D^\alpha g)(-, 0) = g_\alpha$.

We have an open covering of $E \times D_W$ and, for each $E' \times D_W$ in this covering, a morphism $(g, g*)|_{E' \times D_W} : E' \times D_W \to (B, \mathcal{O}_B)$. Exactly in the same way as before in this proof, it is straightforward to check that these morphisms are compatible in the intersections (use 3.18). Thus, they determine a morphism of A-ringed spaces $\xi : E \times D_W \to (B, \mathcal{O}_B)$

unique such that for each E', the restriction $\xi\,|_{E'\times D_W} = (g,\, g^*)|_{E'\times D_W}$. It is clear that ξ has local extensions and thus it defines an arrow $\xi : E \times D_W \to jB$ in $\mathcal{T}$, that is, $\xi : E \to (jB)^{D_W}$. It is immediate also to check that the construction defined in a) above when applied to ξ yields $(\psi,\, (\beta_\alpha)_{\alpha \in M})$.

Finally, it is straightforward to check the naturality in E of this correspondence. □

Let B_1 and B_2 be open subsets of complex Banach spaces C_1 and C_2 respectively, and let f be an holomorphic function, $f : B_1 \to B_2$. Consider the holomorphic function $\omega(f)$ defined by equation 2.11, then:

3.20. Theorem. *Under the bijection ω of theorem 3.19, the arrow $(jf)^{D_W} : jB_1^{D_W} \to jB_2^{D_W}$ is given by the function $\omega(f)$. Explicitly, $\omega \circ (jf)^{D_W} = j(w(f)) \circ \omega$, that is, the following diagram commutes:*

$$\begin{array}{ccc} (jB_1)^{D_W} & \xrightarrow[\cong]{w} & j(B_1 \times \prod_{\alpha\in M} C_1) \\ \Big\downarrow{\scriptstyle (jf)^{D_W}} & & \Big\downarrow{\scriptstyle j(w(f))} \\ (jB_2)^{D_W} & \xrightarrow[\cong]{w} & j(B_2 \times \prod_{\alpha\in M} C_2) \end{array}$$

Proof. Equivalently, we shall prove the equation $\omega \circ (jf)^{D_W} \circ \omega^{-1} = j(\omega(f))$. Applying the global sections functor Γ this equation becomes the equation $\omega \circ f[W] \circ \omega^{-1} = \omega(f)$ of proposition 2.12 (recall remark 2.14). Thus $\Gamma(\omega \circ (jf)^{D_W} \circ \omega^{-1}) = \Gamma(j(\omega(f)))$. Then, by proposition 1.1 of [8] (see the comments after theorem 1.5) we have $\omega \circ (jf)^{D_W} \circ \omega^{-1} = j(\omega(f))$. □

References

[1] Bunge, M., Dubuc, E.J. *Local Concepts in Synthetic Differential Geometry and Germ Representability* , Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York,(1989).

[2] Cartan H. *Ideaux de fonctions analytiques de n variables complexes* , Annales de l'Ecole Normale, 3e serie, **61** (1944).

[3] Cartan H. *Ideaux et Modules de Fonctions Analytiques de variables complexes*, Bulletin de la Soc. Math. de France, t. **78** (1950).

[4] Dubuc E. J. *Sur les modeles de la geometrie Differentielle Synthetique*, Cahiers de Top. et Geom. Diff. **XX-3** (1979).

[5] Dubuc E. J. , Taubin G. *Analytic Rings*, Cahiers de Top. et Geom. Diff. **XXIV-3** (1983).

[6] Dubuc E. J. , Zilber J. *On Analytic Models of Synthetic Differential Geometry*, Cahiers de Top. et Geom. Diff. Vol **XXXV-1** (1994).

[7] Dubuc, E.J., Zilber, J. *Banach Spaces in an Analytic Model of Synthetic Differential Geometry*, Cahiers de Top. et Geom. Diff. **XXXIX-2** (1998).

[8] Dubuc, E.J., Zilber, J. *Inverse function theorems for Banach spaces in a topos*, Cahiers de Top. et Geom. Diff. **XLI-3** (2000).

[9] Ehresmann, C. *Les prolongements d'une variete differentiable I*, C.R.A.S. Paris 233 (1951), also in *Charles Ehresmann, oeuvres completes et commentes*, Cahiers de Top. et Geom.Diff., supplement au volume **XXIV** (1983)

[10] Mujica, J. *Holomorphic Functions and Domains of Holomorphy in Finite and Infinite Dimensions* , North Holland Mathematics Studies **120** (1986).

[11] Weil, A. *Theorie des points proches sur les varietes differentiables*, Colloque Top. et Geom. Diff., Strasbourg, (1953).

[12] Zilber J. *Local Analytic Rings*, Cahiers de Top. et Geom. Diff. **XXXI-1** (1990).

Eduardo J. Dubuc, **Jorge C. Zilber**

Departamento de Matemática.
Facultad de Ciencias Exactas y Naturales
Ciudad Universitaria
1428 Buenos Aires, Argentina