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MARCO GRANDIS

ROBERT PARE

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ADJOINT FOR DOUBLE CATEGORIES ^(*)

by Marco GRANDIS and Robert PARE

RESUME. Cet article poursuit notre étude de la théorie générale des catégories doubles faibles, en traitant des adjonctions et des monades. Une adjonction double générale, telle qu'elle apparaît dans des situations concrètes, présente un foncteur double colax adjoint à gauche d'un foncteur double lax. Ce couple ne peut pas être vu comme une adjonction dans une bicatégorie, car les morphismes lax et colax n'en forment pas une. Mais ces adjonctions peuvent être formalisées dans une catégorie double intéressante, formée des catégories doubles faibles, avec les foncteurs doubles lax et colax comme flèches horizontales et verticales, et avec des cellules doubles convenables.

Introduction

This is a sequel to a paper on '*Limits in double categories*' [GP], referred to as Part I. It was proved there that, in a double category \mathbb{A} , all (double) limits can be constructed from (double) products, equalisers and *tabulators*, the latter being the double limit of a vertical arrow. The reference I.1 (or I.1.2) applies to Section 1 of Part I (or its Subsection 1.2).

Here we study adjoints for (weak, or pseudo) double categories. The general situation, defined in 3.1, is a *colax/lax adjunction* $F \rightarrow R$, where F is a colax double functor while R is lax. (See Kelly [Ke], dealing with adjunctions between *op-D-functors* and *D-functors*, in the context of algebras for a doctrine D .)

Interesting examples of this type are provided by extending an ordinary adjunction $F \rightarrow R$ between abelian categories to their double categories of morphisms, relations and inequality cells: the right exact functor F has a colax extension, the left exact functor R a lax extension, and we get a colax/lax adjunction, which is

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pseudo/lax (resp. *colax/pseudo*) when the original left (resp. right) adjoint is exact (5.4). This and other examples are examined in Section 5.

Of course, the composition of a lax and a colax double functor has poor properties and *no comparison cells for vertical composition of arrows*. Thus, a colax/lax adjunction $F \dashv R$ will be defined through the *interplay of the comparison cells of F and R*, instead of the lacking comparison cells of RF and FR (cf. 3.2); in other words, it *cannot be seen as an adjunction in a bicategory*. Nevertheless, it has a formal description, as an 'orthogonal adjunction' within $\mathbb{D}bl$, a *strict* double category studied in Section 2 and consisting of pseudo double categories, with *lax* and *colax* double functors as horizontal and vertical arrows, respectively, and suitable double cells.

More particular adjunctions, of type *pseudo/lax* (resp. *colax/pseudo*), are studied in Section 4 and reduced to adjunctions in the 2-category $LxDbl$ (resp. $CxDbl$) of double categories, lax (resp. colax) double functors and horizontal transformations; both of them sit inside $\mathbb{D}bl$ (end of 2.2).

Limits and colimits are well behaved with *unitary* adjunctions (6.2, 6.3), and a lax functorial choice of \mathbb{I} -limits in \mathbb{A} (Part I) amounts to a unitary lax double functor $\mathbb{A}^{\mathbb{I}} \rightarrow \mathbb{A}$ right adjoint to the diagonal (6.5). Finally, we study 'double monads'. The classical 1-dimensional theory of monads, when extended to weak double categories, splits into two 'standard' cases, treated in Section 7: *colax* monads have standard *Eilenberg-Moore* algebras and are linked with *colax/pseudo* adjunctions, while *lax* monads have standard *Kleisli* algebras and are linked with *pseudo/lax* adjunctions. On the other hand, the construction of Eilenberg-Moore algebras for a *lax* monad should be performed via coequalisers of free algebras, extending a similar construction for 2-categories (cf. Guitart [Gu], Carboni-Rosebrugh [CR]); this will not be dealt with here.

The theory of double categories, established by Ehresmann [E1, E2], has not yet been extensively developed. The interested reader can see [BE, Da, DP1, DP2, Da, BM], and [BMM] for applications in computer science.

Outline. The first section studies the connections between horizontal and vertical morphisms in a double category: horizontal morphisms can have orthogonal *companions* and orthogonal *adjoints*. Then, in Section 2, lax and colax double functors between weak double categories are organised in the *strict* double category $\mathbb{D}bl$, as horizontal and vertical arrows, respectively. The theory of adjunctions between weak double categories, from their definition to various characterisations, examples and relations with double limits, is dealt with in the next four sections. We end with studying lax and colax monads, in Section 7.

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1. Orthogonal companions and adjoints

This section studies the connections between horizontal and vertical morphisms in a double category: horizontal morphisms can have vertical *companions* and vertical *adjoints*. Such phenomena, which are interesting in themselves and typical of double categories, already appeared to a limited extent in Part I (cf. 1.5). \mathbb{A} is always a (unitary) pseudo double category.

1.1. Basics. For double categories, we generally use the same terminology and notation as in Part I. The composite of horizontal arrows $f: A \rightarrow A'$, $g: A' \rightarrow A''$ is written gf , while for vertical arrows $u: A \rightarrow B$, $v: B \rightarrow C$ we write $u \circ v$ or $v \circ u$. The boundary of a double cell α , consisting of two horizontal arrows and two vertical ones

$$(1) \quad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ u \downarrow & \alpha & \downarrow v \\ B & \xrightarrow{g} & B' \end{array} \quad .$$

will be displayed as $\alpha: (u \overset{f}{g} v)$, or sometimes as $\alpha: u \rightarrow v$. If f, g are identities, α is said to be a *special cell* and displayed as $\alpha: (u \overset{A}{B} v)$ or $\alpha: u \rightarrow v$. The horizontal and vertical compositions of cells are written as $(\alpha | \beta)$ and $(\frac{\alpha}{\gamma})$, or more simply as $\alpha | \beta$ and $\alpha \otimes \gamma$. Horizontal identities, of an object or a vertical arrow, are written as $1_A, 1_u: (u \overset{A}{B} u)$; vertical identities as $1_A^*, 1_f^*: (A \overset{f}{f} A')$.

Let us recall that, in a *pseudo double category* \mathbb{A} (I.1.9; I.7.1), or weak double category, the horizontal structure behaves categorically, while the composition \otimes of vertical arrows is associative up to *comparison cells* $\alpha(u, v, w): (u \circ v) \circ w \cong u \circ (v \circ w)$; these are *special cells* (i.e., their horizontal arrows are identities), and actually *special isocells* – horizontally invertible. On the other hand, it will be useful to assume that vertical identities are *strict*, i.e. behave as strict units, a constraint which can generally be met without complication (cf. I.3). Thus, a weak double category \mathbb{A} contains a category A_0 (objects, horizontal arrows, their composition)

and a category \mathbb{A}_1 (vertical arrows, cells, their horizontal composition), plus the remaining structure making it a *pseudo* category object in \mathbf{Cat} (cf. I.7).

It will also be useful to recall that, in \mathbb{A} , the existence of a special isocell $\lambda: u \cong u'$ yields an equivalence relation for parallel vertical arrows, consistent with vertical composition. Similarly, the relation $\alpha \cong \alpha'$ (there exist two special isocells $\lambda: u \cong u'$, $\mu: v \cong v'$ such that $(\alpha|\mu) = (\lambda|\alpha')$) is an equivalence relation for double cells having the same horizontal arrows, consistent with vertical composition, but not with the horizontal one (because the vertical arrows of equivalent cells are not fixed). Therefore, the double graph \mathbb{A}/\cong is just a 1-dimensional category: the *strictification* of \mathbb{A} is more complicated (I.7.5).

The expression 'profunctor-based examples' will refer to the following pseudo double categories, treated in Part I: \mathbf{Cat} (formed of categories, functors and profunctors, I.3.1), \mathbf{Set} (sets, mappings and spans, I.3.2), \mathbf{Pos} (preordered sets, monotone mappings and order ideals, I.3.3), \mathbf{Mtr} (generalised metric spaces, weak contractions and metric profunctors, I.3.3), \mathbf{Rel} (sets, mappings and relations, I.3.4), \mathbf{RelAb} (abelian groups, homomorphisms and relations, I.3.4; see also 5.4), \mathbf{Rng} (unitary rings, homomorphisms and bimodules, I.5.3). In \mathbf{Cat} , a profunctor $u: A \rightarrow B$ is defined as a functor $u: A^{op} \times B \rightarrow \mathbf{Set}$.

We also consider Ehresmann's double category of quintets \mathbf{QA} on a 2-category \mathbf{A} ([E1, E2]; I.1.3), where a double cell $\alpha: (u \overset{f}{g} v)$ is defined as a 2-cell $\alpha: vf \rightarrow gu$ of \mathbf{A} . If \mathbf{A} is just a category (with trivial cells), such double cells reduce to commutative squares, and \mathbf{QA} will be written as $\square \mathbf{A}$.

A weak double category \mathbb{A} contains a *bicategory* \mathbf{VA} of vertical arrows and special cells, as well as (because of unitarity) a 2-*category* \mathbf{HA} of horizontal arrows and 'vertically special' cells (I.1.9).

1.2. Orthogonal companions. In the pseudo double category \mathbb{A} , the horizontal morphism $f: A \rightarrow B$ and the vertical morphism $u: A \rightarrow B$ are made *companions* by assigning a pair (η, ε) of cells as below, called the *unit* and *counit*, satisfying the identities $\eta|\varepsilon = 1_f^*$, $\eta \otimes \varepsilon = 1_u$

$$(1) \quad \begin{array}{ccc} A & \equiv & A \\ 1 \downarrow & \eta & \downarrow u \\ A & \xrightarrow{f} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ u \downarrow & \varepsilon & \downarrow 1 \\ B & \equiv & B \end{array}$$

Given f , this is equivalent (by *unitarity*, again) to saying that the pair (u, ϵ) satisfies the following universal property:

(a) for every cell $\epsilon': (u' \xrightarrow{f} B)$ there is a unique cell $\lambda: (u' \xrightarrow{A} u)$ such that $\epsilon' = \lambda|\epsilon$

$$(2) \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ u' \downarrow & \epsilon' & \downarrow 1 \\ A' & \xrightarrow{g} & B \end{array} = \begin{array}{ccc} A & \xrightarrow{=} & A \xrightarrow{f} B \\ u' \downarrow & \lambda & \downarrow u \quad \epsilon \quad \downarrow 1 \\ A' & \xrightarrow{g} & B \xrightarrow{=} B \end{array}$$

In fact, given (η, ϵ) , we can (and must) take $\lambda = \eta \otimes \epsilon'$; on the other hand, given (a), we define $\eta: (A \xrightarrow{A} u)$ by the equation $\eta|\epsilon = 1_f^\bullet$ and deduce that $\eta \otimes \epsilon = 1_u$ because $(\eta \otimes \epsilon) | \epsilon = (\eta|\epsilon) \otimes \epsilon = \epsilon = (1_u | \epsilon)$. Similarly, also (u, η) is characterised by a universal property

(b) for every cell $\eta': (A \xrightarrow{B} u')$ there is a unique cell $\mu: (u \xrightarrow{B} u')$ such that $\eta' = \eta|\mu$.

Therefore, if f has a vertical companion, this is determined up to a unique special isocell, and will often be written f_* . Companions compose in the obvious (covariant) way: if $g: B \rightarrow C$ also has a companion, then $g_* f_*: A \rightarrow C$ is companion to $gf: A \rightarrow C$, with unit $(\frac{\eta|1}{1^\bullet|\eta'}) : (A \xrightarrow{A} gf_* f_*)$.

Companionship is preserved by *unitary* lax or colax double functors (cf. 2.1).

We say that A has *vertical companions* if every horizontal arrow has a vertical companion. All our profunctor-based pseudo double categories (1.1) have vertical companions, given by the obvious embedding of horizontal arrows into the vertical ones. For instance, in $\mathbb{C}at$, the vertical companion to a functor $f: A \rightarrow B$ is the associated profunctor $f_*: A \rightarrow B$, $f_*(a, b) = B(f(a), b)$. Also $\mathbb{Q}A$ (1.1) has companions: for a map f , take the same arrow (or any 2-isomorphic one). In $\mathbb{R}ng$, f_* is the bimodule B , with A -structure induced by f .

Companionship is simpler for horizontal *isomorphisms*. If f is one and has a companion u , then its unit and counit are also horizontally invertible and determine each other:

$$(3) \quad (\epsilon | 1_g^\bullet | \eta) = \eta \otimes \epsilon = 1_u \quad (g = f^{-1}),$$

as it appears rewriting $(\epsilon | 1_g^\bullet | \eta)$ as follows, and then applying middle-four interchange

$$\begin{array}{ccccccc}
 A & -f \rightarrow & B & -g \rightarrow & A & \equiv & A \\
 1 \downarrow & 1_f^\bullet & \downarrow 1 & 1_g^\bullet & \downarrow 1 & \eta & \downarrow u \\
 A & -f \rightarrow & B & -g \rightarrow & A & -f \rightarrow & B \\
 u \downarrow & \varepsilon & \downarrow 1 & 1_g^\bullet & \downarrow 1 & 1_f^\bullet & \downarrow 1 \\
 B & \equiv & B & -g \rightarrow & A & -f \rightarrow & B
 \end{array}$$

Conversely, the existence of a horizontally invertible cell $\eta: (A \xrightarrow{f} u)$ implies that f is horizontally invertible, with companion u and counit as above.

1.3. Orthogonal adjoints. Transforming companionship by vertical (or horizontal) duality, the arrows $f: A \rightarrow B$ and $v: B \rightarrow A$ are made *orthogonal adjoints* by a pair (α, β) of cells as below

$$(1) \quad \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 1 \downarrow & \alpha & \downarrow v \\
 A & \equiv & A
 \end{array} \quad \begin{array}{ccc}
 B & \equiv & B \\
 v \downarrow & \beta & \downarrow 1 \\
 A & \xrightarrow{f} & B
 \end{array}$$

with $\alpha\beta = 1_f^\bullet$ and $\beta\alpha = 1_v$. Then, f is the *horizontal adjoint* and v the *vertical* one. (In the general case, there is no reason of distinguishing 'left' and 'right', unit and counit; see the examples below). Again, given f , these relations can be described by universal properties for (v, β) or (v, α)

- (a) for every cell $\beta': (v \xrightarrow{g} B)$ there is a unique cell $\lambda: (v \xrightarrow{g} v)$ such that $\beta' = \lambda\beta$,
- (b) for every cell $\alpha': (A \xrightarrow{f} v')$ there is a unique cell $\mu: (v \xrightarrow{g} v')$ such that $\alpha' = \alpha\mu$.

The vertical adjoint of f is determined up to a special isocell and will often be written f^* . Vertical adjoints compose, contravariantly: f^*g^* gives $(gf)^*$.

\mathcal{A} has vertical adjoints if each horizontal arrow has a vertical adjoint. All our profunctor-based examples are so. For instance, in $\mathcal{C}at$, the vertical adjoint to a functor $f: A \rightarrow B$ is the associated profunctor $f^*: B \rightarrow A$, $f^*(b, a) = B(b, f(a))$; in $\mathcal{R}el$, the vertical adjoint of a function $f: A \rightarrow B$ is the reversed relation $f^\#: B \rightarrow A$, with $\alpha: 1 \leq f^\#f$, $\beta: ff^\# \leq 1$. On the other hand, $\mathcal{Q}A$ does not have (all) vertical adjoints, since our data amount to an adjunction in the 2-category \mathcal{A} , $v \dashv f$, with $\alpha: vf \rightarrow 1$, $\beta: 1 \rightarrow fv$. In $\square \mathcal{A}$, this means that f is iso and $v = f^{-1}$.

1.4. Proposition. *Let $f: A \rightarrow B$ have a vertical companion $u: A \rightarrow B$. Then $v: B \rightarrow A$ is vertical adjoint to f if and only if $u \dashv v$ in the bicategory \mathbf{VA} (of vertical arrows and special cells, 1.1).*

Proof. Given four cells $\eta, \varepsilon, \alpha, \beta$ as above (1.2, 1.3), we have two special cells $\eta \otimes \alpha: 1^\bullet \rightarrow u \otimes v$, $\beta \otimes \varepsilon: u \otimes v \rightarrow 1^\bullet$, which are easily seen to satisfy the triangle identities in \mathbf{VA} . The converse is similarly obvious. \square

1.5. Theorem (Horizontal invariance). *In a pseudo double category \mathbb{A} , the following properties are equivalent:*

- (a) every horizontal iso in \mathbb{A} has a vertical companion,
- (b) every horizontal iso in \mathbb{A} has a vertical adjoint,
- (c) every horizontal iso in \mathbb{A} is a sesqui-isomorphism (I.2.3),
- (d) \mathbb{A} is horizontally invariant (I.2.4).

(The last two definitions are recalled in the proof. Property (d) ensures that double limits in \mathbb{A} are 'vertically determined', by the Invariance Theorem, I.4.6).

Proof. First, let us recall that a map $f: A \rightarrow B$ is said to be a sesqui-isomorphism (I.2.3) if there exist two horizontally invertible cells η, α as in the left diagram

$$(1) \quad \begin{array}{ccc} A & \xlongequal{\quad} & A \\ 1 \downarrow & \eta & \downarrow u \\ A & \xrightarrow{-f-} & B \\ 1 \downarrow & \alpha & \downarrow v \\ A & \xlongequal{\quad} & A \end{array} \qquad \begin{array}{ccc} B & \xlongequal{\quad} & B \\ v \downarrow & \beta & \downarrow 1 \\ A & \xrightarrow{-f-} & B \\ u \downarrow & \varepsilon & \downarrow 1 \\ B & \xlongequal{\quad} & B \end{array}$$

Then, we already know (by 1.2 and horizontal duality) that: f is a horizontal iso; it has a vertical companion u , via η and $\varepsilon = (\eta^{-1} | 1_f^\bullet)$; it has a vertical adjoint v , via α and $\beta = (\alpha^{-1} | 1_f^\bullet)$. Note also that, in the diagram above, u and v form a *vertical equivalence*: $u \otimes v \cong 1_A^\bullet$, $v \otimes u \cong 1_B^\bullet$. We also know that it is equivalent to assign η or ε ; as well as α or β (when all of them are horizontally invertible).

Conversely, if two inverse horizontal isos f, g have companions $u = f_*$, $v = g_*$, then we know that all their units and counits are horizontally invertible (1.2); f and g are easily seen to be sesqui-isomorphisms (defining α with the unit of g).

We have thus proved that (a) and (c) are equivalent; by horizontal duality, also (b) is equivalent to (c).

Finally, the last property, *horizontal invariance* (I.2.4), means that, given two horizontal isos f, g and a vertical arrow x as below, there exists a horizontally invertible cell λ

$$(2) \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ y \downarrow & \lambda & \downarrow x \\ X & \xrightarrow{g} & Y \end{array}$$

In fact, assuming (a) and (b), λ can be obtained as a vertical composite of three horizontally invertible cells, $\varepsilon: (u \overset{f}{\underset{B}{\rightrightarrows}} B)$, 1_u and $\beta: (v \overset{Y}{\underset{g}{\rightrightarrows}} Y)$, where $u: A \rightarrow B$ is companion to f and $v: Y \rightarrow X$ is vertical adjoint to g . Conversely, assuming (d), each horizontal iso has two horizontally invertible cells $\eta: (y \overset{f}{\underset{B}{\rightrightarrows}} B)$, $\beta: (v \overset{B}{\underset{f}{\rightrightarrows}} B)$, whence it is a sesqui-isomorphism. \square

1.6. Orthogonal flipping. Assume that the horizontal map $f: A \rightarrow B$ has a vertical companion $f_*: A \rightarrow B$; then there is a bijective correspondence between cells φ and cells ψ , as below, whose boundaries are obtained by 'flipping' f to f_* or vice versa

$$(1) \quad \begin{array}{ccccc} & & h & & A \\ & & \rightarrow & & \downarrow f_* \\ \bullet & \xrightarrow{h} & A & \xrightarrow{f} & B \\ u \downarrow & & \varphi & & \downarrow v \\ \bullet & \xrightarrow{k} & \bullet & & \bullet \end{array} \quad \begin{array}{ccccc} & & h & & A \\ & & \rightarrow & & \downarrow f_* \\ \bullet & \downarrow u & \psi & & B \\ & & \downarrow v & & \downarrow v \\ \bullet & \xrightarrow{k} & \bullet & & \bullet \end{array}$$

The correspondence is obtained through the cells η, ε of the companions f, f_* (and bijectivity follows from the unitarity of \mathbb{A})

$$(2) \quad \varphi = \left(\psi \left| \begin{array}{c} \varepsilon \\ 1_v \end{array} \right. \right), \quad \psi = \left(\begin{array}{c} 1_h \\ \varphi \end{array} \middle| \eta \right).$$

By horizontal and vertical duality, the previous statement has three other forms, which establish a bijective correspondence between cells φ_i and ψ_i as below (in the last two cases, flipping f to its vertical adjoint f^*)

$$\begin{array}{c}
 (3) \quad \begin{array}{ccc}
 \bullet & \longrightarrow & \bullet \\
 \downarrow & \varphi_1 & \downarrow \\
 A & \xrightarrow{f} & B \longrightarrow \bullet
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \longrightarrow \bullet \\
 \downarrow & \varphi_2 & \downarrow \\
 \bullet & \longrightarrow & \bullet
 \end{array}
 \quad
 \begin{array}{ccc}
 \bullet & \longrightarrow & \bullet \\
 \downarrow & \varphi_3 & \downarrow \\
 \bullet & \longrightarrow & A \xrightarrow{f} B
 \end{array}
 \\
 \\
 (4) \quad \begin{array}{ccc}
 \bullet & \longrightarrow & \bullet \\
 \downarrow & \psi_1 & \downarrow \\
 A & & \bullet \\
 f_* \downarrow & & \downarrow \\
 B & \longrightarrow & \bullet
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \longrightarrow & \bullet \\
 f^* \downarrow & \psi_2 & \downarrow \\
 A & & \bullet \\
 \downarrow & & \downarrow \\
 \bullet & \longrightarrow & \bullet
 \end{array}
 \quad
 \begin{array}{ccc}
 \bullet & \longrightarrow & \bullet \\
 \downarrow & \psi_3 & \downarrow \\
 \bullet & & B \\
 \downarrow & & \downarrow f^* \\
 \bullet & \longrightarrow & A
 \end{array}
 \end{array}$$

Starting from a given cell, and applying the flipping process to various arrows, successively, one can *often* show that the final result does not depend on the order of such steps, because of the *normality* of the ternary compositions involved (1.1). For instance, if the maps f and g have vertical companions (resp. vertical adjoints) in \mathbb{A} , to assign a cell $\varphi: (u \overset{f}{\underset{g}{\rightrightarrows}} v)$ is equivalent to assigning a special cell $\varphi_\#$, its *companion* (resp. $\varphi^\#$, its *adjoint*)

$$(5) \quad \varphi_\# = \eta_f \otimes \varphi \otimes \varepsilon_g: u \otimes g_* \rightarrow f_* \otimes v, \quad \varphi^\# = \beta_f \otimes \varphi \otimes \alpha_g: f^* \otimes u \rightarrow v \otimes g^*$$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 u \downarrow & \varphi & \downarrow v \\
 B & \xrightarrow{g} & B'
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \equiv & A \\
 u \downarrow & & \downarrow f_* \\
 B & \varphi_\# & A' \\
 g_* \downarrow & & \downarrow v \\
 B' & \equiv & B'
 \end{array}
 \quad
 \begin{array}{ccc}
 A' & \equiv & A' \\
 f^* \downarrow & & \downarrow v \\
 A & \varphi^\# & B' \\
 u \downarrow & & \downarrow g^* \\
 B & \equiv & B
 \end{array}$$

1.7. Theorem. (a) *The functor of quintets, $\mathbb{Q}: 2\text{-Cat} \rightarrow \mathbf{Dbl}$ (with values in the category of double categories and strict double functors) has a right adjoint \mathcal{C} , constructed with companion pairs.*

(b) *The functor $\mathbb{Q}^v: 2\text{-Cat} \rightarrow \mathbf{Dbl}$, obtained by composing \mathbb{Q} with vertical duality of double categories, has a right adjoint \mathcal{A} constructed with pairs of orthogonal adjoints.*

Proof. (a) The right adjoint \mathcal{C} associates to a double category \mathbb{D} the 2-category $\mathbb{C}\mathbb{D}$ whose arrows are *companion pairs* $(f, u; \eta, \varepsilon): A \rightarrow B$ in \mathbb{D} (with composition as in 1.2), and whose cells

(1) $\varphi: (f, u; \eta, \varepsilon) \rightarrow (g, v; \eta', \varepsilon'): A \rightarrow B,$

consist of a double cell $\varphi: (A \xrightarrow{f} B)$ or – equivalently – of its companion, the special cell $\varphi\#: (v \xrightarrow{A} u)$

$$(2) \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ 1 \downarrow & \varphi & \downarrow 1 \\ A & \xrightarrow{g} & B \end{array} \qquad \begin{array}{ccc} A & \xlongequal{\quad} & B \\ v \downarrow & \varphi\# & \downarrow u \\ B & \xlongequal{\quad} & B \end{array}$$

For this adjunction, the counit double functor $\mathbb{Q}\mathbb{C}\mathbb{D} \rightarrow \mathbb{D}$ is the identity on objects; moreover, it sends a (declared) horizontal arrow $(f, u, \eta, \varepsilon): A \rightarrow B$ to f , a (declared) vertical arrow $(f, u, \eta, \varepsilon): A \rightarrow B$ to u , and a double cell of quintets

$$(3) \quad \begin{array}{ccc} A & \xrightarrow{(h, x, \rho, \sigma)} & B \\ (f, u, \eta, \varepsilon) \downarrow & \swarrow \varphi\# & \downarrow (g, v, \eta', \varepsilon') \\ C & \xrightarrow{(k, y, \rho', \sigma')} & D \end{array} \qquad \varphi\#: (u \otimes y \xrightarrow{A} x \otimes v) \text{ in } \mathbb{D},$$

to its pre-companion $\varphi: (u \xrightarrow{h} v)$ (as considered at the end of 1.6).

(b) Similarly, the functor \mathcal{A} right adjoint to \mathbb{Q}^v yields a 2-category $\mathcal{A}\mathbb{D}$ whose maps are *pairs of orthogonal adjoints* $(f, u; \alpha, \beta): A \rightarrow B$; the counit $\mathbb{Q}^v\mathcal{A}\mathbb{D} \rightarrow \mathbb{D}$ sends a vertical arrow $(f, u; \alpha, \beta): A \rightarrow B$ to $u: B \rightarrow A$, and so on. \square

1.8. Theorem (Orthogonal completions). *Let \mathcal{A} be a 2-category.*

(a) $\mathbb{Q}\mathcal{A}$ is the companion completion of $\mathbb{H}\mathcal{A}$, the double category with trivial vertical arrows and cells as in \mathcal{A} . Precisely, if the pseudo double category \mathbb{X} has vertical companions and $F: \mathbb{H}\mathcal{A} \rightarrow \mathbb{X}$ is a double functor, there is an essentially unique pseudo double functor $G: \mathbb{Q}\mathcal{A} \rightarrow \mathbb{X}$ which extends F and preserves companions.

(b) By vertical duality, $\mathbb{Q}^v\mathcal{A}$ is the completion of $\mathbb{H}\mathcal{A}$ by vertical adjoints.

Proof. It is sufficient to prove (a). Take in \mathbb{X} a unitary choice of vertical companions f_* , with special isocells $\lambda(f, g): (gf)_* \rightarrow f_* \otimes g_*$. To extend F , we define $G: \mathbb{Q}\mathcal{A} \rightarrow \mathbb{X}$ on a (declared) vertical arrow $u: A \rightarrow B$ of $\mathbb{Q}\mathcal{A}$ and a cell $\alpha: (u \xrightarrow{f} v)$ (i.e., $\alpha: vf \rightarrow gu: A \rightarrow D$ in \mathcal{A}), as

$$(1) \quad G(u) = (Fu)_*: FA \rightarrow FB, \qquad G\alpha: (Gu \xrightarrow{Ff} Fg) v,$$

where $G\alpha$ is companion to $F\alpha: (1^{\bullet} \begin{smallmatrix} Fv \cdot Ff \\ Fg \cdot Fu \end{smallmatrix} 1^{\bullet})$, a 'vertically special' cell of \mathbb{X} . G is then a pseudo double functor, with special comparison isocells $\gamma(u, v) = \lambda(Fu, Fv): G(u \otimes v) \rightarrow Gu \otimes Gv$. \square

2. Lax and colax double functors

The strict double category \mathbb{Dbl} is a crucial, interesting structure where our adjunctions will live; it consists of pseudo double categories, lax and colax double functors, with suitable cells. Comma double categories are also considered.

2.1. Lax functors. Let us recall their definition. Note that, while a pseudo double category is always assumed to be unitary, lax double functors are *not* (because there are important examples of lax right adjoints which are not unitary, cf. 5.2, 5.3b).

A *lax* double functor $R: \mathbb{A} \rightarrow \mathbb{X}$ between pseudo double categories amounts to assigning:

(a) two functors $R_i: \mathbb{A}_i \rightarrow \mathbb{X}_i$ for $i = 0, 1$ (cf. 1.1), consistent with domain and codomain,

(b) for any object A in \mathbb{A} , a special cell, the *identity comparison* $\rho_A: 1_{RA}^{\bullet} \rightarrow R(1_A^{\bullet}): RA \rightarrow RA$ (also denoted $R[A]$),

(c) for any vertical composition $u \otimes v: A \rightarrow B \rightarrow C$ in \mathbb{A} , a special cell, the *composition comparison* $\rho(u, v): Ru \otimes Rv \rightarrow R(u \otimes v): RA \rightarrow RC$ (also denoted $R[u, v]$),

satisfying the following axioms:

(i) (*naturality*) for a horizontal map $f: A \rightarrow A'$ in \mathbb{A} , $(1_{Rf}^{\bullet} \mid \rho_{A'}) = (\rho_A \mid R(1_f^{\bullet}))$

$$(1) \quad \begin{array}{ccccc} RA & \longrightarrow & RA' & \equiv & RA' \\ 1 \downarrow & & 1_{Rf}^{\bullet} & \downarrow 1 & \rho & \downarrow & = & 1 \downarrow & \rho & R(1) \downarrow & R(1_f^{\bullet}) & \downarrow R(1) \\ RA & \longrightarrow & RA' & \equiv & RA' & & & RA & \equiv & RA & \longrightarrow & RA' \end{array}$$

(ii) (*naturality*) for a vertical composition of cells $a \otimes b$, we have $(\begin{smallmatrix} Ra \\ Rb \end{smallmatrix} \mid \rho(u', v')) = (\rho(u, v) \mid R(a \otimes b))$

$$(2) \quad \begin{array}{ccccc} RA & \longrightarrow & RA' & \equiv & RA' \\ Ru \downarrow & & Ra & \downarrow & Ru' \\ RB & \longrightarrow & RB' & \xrightarrow{\rho} & \\ Rv \downarrow & & Rb & \downarrow & Rv' \\ RC & \longrightarrow & RC' & \equiv & RC' \end{array} \quad = \quad \begin{array}{ccccc} RA & \equiv & RA & \longrightarrow & RA' \\ Ru \downarrow & & \rho & \downarrow & R(a \otimes b) \\ RB & & R(u \otimes v) & & \downarrow R(u' \otimes v') \\ Rv \downarrow & & \downarrow & & \downarrow \\ RC & \equiv & RC & \longrightarrow & RC' \end{array}$$

(iii) (*coherence laws for identities*) for a vertical map $u: A \rightarrow B$, the following diagrams of special cells are commutative

$$(3) \quad \begin{array}{ccc} Ru & \xrightarrow{1} & Ru \\ \rho_A \otimes 1 \searrow & & \nearrow \rho(1^*, u) \\ & R1_A^* \otimes Ru & \end{array} \quad \begin{array}{ccc} Ru & \xrightarrow{1} & Ru \\ 1 \otimes \rho_B \searrow & & \nearrow \rho(u, 1^*) \\ & Ru \otimes R1_B^* & \end{array}$$

(iv) (*coherence hexagon for associativity*) for consecutive u, v, w in \mathbb{A} , the following diagram of special cells is commutative (α denotes the associativity isocells, in \mathbb{A} and \mathbb{X})

$$(4) \quad \begin{array}{ccc} (Ru \otimes Rv) \otimes Rw & \xrightarrow{\alpha_R} & Ru \otimes (Rv \otimes Rw) \\ \rho \otimes 1 \downarrow & & \downarrow 1 \otimes \rho \\ R(u \otimes v) \otimes Rw & & Ru \otimes R(v \otimes w) \\ \rho \downarrow & & \downarrow \rho \\ R((u \otimes v) \otimes w) & \xrightarrow{R\alpha} & R(u \otimes (v \otimes w)) \end{array}$$

The lax double functor R is said to be *unitary* if its unit comparisons ρ_A are identities; then, by (iii), also the cells $\rho(1^*, u)$ and $\rho(u, 1^*)$ are.

A *colax* double functor $F: \mathbb{A} \rightarrow \mathbb{X}$ has comparison cells in the opposite direction, $\varphi_A: F(1_A^*) \rightarrow 1_{FA}^*$ and $\varphi(u, v): F(u \otimes v) \rightarrow Fu \otimes Fv$. A *pseudo* double functor is a lax one, whose comparison cells are special isocells (horizontally invertible); or, equivalently, a colax one satisfying the same condition. Note that a pseudo double functor can always be made unitary.

2.2. The double category \mathbb{Dbl} . Lax and colax double functors do not compose well. But they can be organised in a *strict* double category \mathbb{Dbl} , crucial for our study, where orthogonal adjunctions will provide our general notion of double

adjunction (Section 3) while companion pairs amount to pseudo double functors (Section 4).

Its objects are the *pseudo* double categories $\mathbb{A}, \mathbb{B}, \dots$; its horizontal arrows are the *lax* double functors R, S, \dots ; its vertical arrows are the *colax* double functors F, G, \dots . A cell α

$$(1) \quad \begin{array}{ccc} \mathbb{A} & \xrightarrow{R} & \mathbb{B} \\ F \downarrow & \alpha \swarrow \searrow & \downarrow G \\ \mathbb{C} & \xrightarrow{S} & \mathbb{D} \end{array}$$

is – roughly speaking – a 'horizontal transformation' $\alpha: GR \rightarrow SF$ (as stressed by the arrow we are placing in the square). But this is an abuse of notation, since the composites GR and SF are neither lax nor colax (just morphisms of double graphs, respecting the horizontal structure): the coherence conditions of α will require the individual knowledge of the four 'functors' and their comparison cells.

Precisely, the cell α consists of the following data: two lax double functors R, S , two colax double functors F, G

- (a) $R: \mathbb{A} \rightarrow \mathbb{B}$, $\rho_A: 1_{\mathbb{A}}^{\bullet} \rightarrow R(1_{\mathbb{A}}^{\bullet})$, $\rho(u, v): Ru \otimes Rv \rightarrow R(u \otimes v)$ (*lax*),
 $S: \mathbb{C} \rightarrow \mathbb{D}$, $\sigma_C: 1_{\mathbb{C}}^{\bullet} \rightarrow S(1_{\mathbb{C}}^{\bullet})$, $\sigma(u, v): Su \otimes Sv \rightarrow S(u \otimes v)$ (*lax*),
- (b) $F: \mathbb{A} \rightarrow \mathbb{C}$, $\varphi_A: F(1_{\mathbb{A}}^{\bullet}) \rightarrow 1_{\mathbb{C}}^{\bullet}$, $\varphi(u, v): F(u \otimes v) \rightarrow Fu \otimes Fv$ (*colax*),
 $G: \mathbb{B} \rightarrow \mathbb{D}$, $\gamma_B: G(1_{\mathbb{B}}^{\bullet}) \rightarrow 1_{\mathbb{D}}^{\bullet}$, $\gamma(u, v): G(u \otimes v) \rightarrow Gu \otimes Gv$ (*colax*),
- (c) maps $\alpha_A: GR(A) \rightarrow SF(A)$ and cells α_u in \mathbb{D} (for $A, u: A \rightarrow A'$ in \mathbb{A})

$$(2) \quad \begin{array}{ccc} \text{GRA} & \xrightarrow{\alpha_A} & \text{SFA} \\ \text{GRu} \downarrow & \alpha_u & \downarrow \text{SFu} \\ \text{GRA}' & \xrightarrow{\alpha_{A'}} & \text{SFA}' \end{array}$$

satisfying the naturality conditions (c0), (c1) (the former is redundant, being implied by the latter) and the coherence conditions (c2), (c3)

- (c0) $\alpha_{A'} \cdot \text{GR}f = \text{SF}f \cdot \alpha_A$ (for $f: A \rightarrow A'$ in \mathbb{A}),
- (c1) $(\text{GR}a \mid \alpha v) = (\alpha u \mid \text{SF}a)$ (for $a: (u \xrightarrow{f} v)$ in \mathbb{A}),

(c2) $(G\rho A \mid \alpha 1_A^\bullet \mid S\varphi A) = (\gamma RA \mid 1_{\alpha A}^\bullet \mid \sigma FA)$ (for A in \mathbb{A}),

(c3) $(G\rho(u, v) \mid \alpha w \mid S\varphi(u, v)) = (\gamma(Ru, Rv) \mid (\alpha u \otimes \alpha v) \mid \sigma(Fu, Fv))$
 (for $w = u \otimes v$ in \mathbb{A}),

$$\begin{array}{ccccccc}
 \text{GRA} & \equiv & \text{GRA} & \longrightarrow & \text{SFA} & \equiv & \text{SFA} \\
 \downarrow G(Ru \otimes Rv) & & \downarrow G\rho & & \downarrow \alpha w & & \downarrow S(Fu \otimes Fv) \\
 \text{GRA}'' & \equiv & \text{GRA}'' & \longrightarrow & \text{SFA}'' & \equiv & \text{SFA}'' \\
 \\
 \text{GRA} & \equiv & \text{GRA} & \longrightarrow & \text{SFA} & \equiv & \text{SFA} \\
 \downarrow G(Ru \otimes Rv) & & \downarrow \gamma R & & \downarrow \sigma F & & \downarrow S(Fv \otimes Fv) \\
 \text{GRA}'' & \equiv & \text{GRA}'' & \longrightarrow & \text{SFA}'' & \equiv & \text{SFA}''
 \end{array}$$

The horizontal and vertical composition are both defined *via the horizontal composition* of \mathbb{D}

(3)

$$\begin{array}{ccccccc}
 \bullet & - & R & \rightarrow & \bullet & - & R' & \rightarrow & \bullet \\
 F \downarrow & & \alpha \swarrow & & \downarrow G & & \beta \swarrow & & \downarrow H \\
 \bullet & - & S & \rightarrow & \bullet & - & S' & \rightarrow & \bullet \\
 F' \downarrow & & \gamma \swarrow & & \downarrow G' & & \delta \swarrow & & \downarrow H' \\
 \bullet & - & T & \rightarrow & \bullet & - & T & \rightarrow & \bullet
 \end{array}$$

(4) $(\alpha \mid \beta)(u) = (\beta Ru \mid S'\alpha u),$ $\left(\frac{\alpha}{\gamma}\right)(u) = (G'\alpha u \mid \gamma Fu),$

$$\begin{array}{ccccccc}
 \text{HR}'\text{RA} & \rightarrow & \text{S}'\text{GRA} & \rightarrow & \text{S}'\text{SFA} & & \text{G}'\text{GRA} & \rightarrow & \text{G}'\text{SFA} & \rightarrow & \text{TF}'\text{FA} \\
 \downarrow & & \downarrow \beta Ru & & \downarrow S'\alpha u & & \downarrow & & \downarrow G'\alpha u & & \downarrow \gamma Fu & & \downarrow \\
 \text{HR}'\text{R}'\text{A}' & \rightarrow & \text{S}'\text{G}'\text{R}'\text{A}' & \rightarrow & \text{S}'\text{S}'\text{F}'\text{A}' & & \text{G}'\text{G}'\text{R}'\text{A}' & \rightarrow & \text{G}'\text{S}'\text{F}'\text{A}' & \rightarrow & \text{TF}'\text{F}'\text{A}'
 \end{array}$$

Thus, these compositions are both strictly associative. They satisfy the middle-four interchange law: in (3), computing $(\alpha \mid \beta) \otimes (\gamma \mid \delta)$ and $(\alpha \otimes \gamma) \mid (\beta \otimes \delta)$ on $u: A \rightarrow A'$, we obtain the cells

$$\begin{array}{c}
 \text{(5)} \quad \begin{array}{ccccccccc}
 \text{H'HR'RA} & \longrightarrow & \text{H'S'GRA} & \longrightarrow & \text{H'S'SFA} & \longrightarrow & \text{T'G'SFA} & \longrightarrow & \text{T'TF'FA} \\
 \downarrow & & \text{H'\beta Ru} & & \downarrow & & \text{H'S'\alpha u} & & \downarrow & & \delta \text{SFu} & & \downarrow & & \text{T'\gamma Fu} & & \downarrow \\
 \text{H'HR'RA}' & \longrightarrow & \text{H'S'GRA}' & \longrightarrow & \text{H'S'SFA}' & \longrightarrow & \text{T'G'SFA}' & \longrightarrow & \text{T'TF'FA}'
 \end{array} \\
 \\
 \text{(6)} \quad \begin{array}{ccccccccc}
 \text{H'HR'RA} & \longrightarrow & \text{H'S'GRA} & \longrightarrow & \text{T'G'GRA} & \longrightarrow & \text{T'G'SFA} & \longrightarrow & \text{T'TF'FA} \\
 \downarrow & & \text{H'\beta Ru} & & \downarrow & & \delta \text{GRu} & & \downarrow & & \text{T'G'\alpha u} & & \downarrow & & \text{T'\gamma Fu} & & \downarrow \\
 \text{H'HR'RA}' & \longrightarrow & \text{H'S'GRA}' & \longrightarrow & \text{T'G'GRA}' & \longrightarrow & \text{T'G'SFA}' & \longrightarrow & \text{T'TF'FA}'
 \end{array}
 \end{array}$$

where the equality $(\text{H'S'\alpha u} \mid \delta \text{SFu}) = (\delta \text{GRu} \mid \text{T'G'\alpha u})$, for the central pastings above, amounts to the naturality of the $\mathbb{D}\text{bl}$ -cell $\delta: (G \overset{S}{\dashv} H)$ on the cell αu .

Finally, to show that the cells defined in (4) are indeed coherent, let us verify the condition (c3) for $(\alpha \mid \beta)$, with respect to a vertical composition $w = u \otimes v$ in \mathbb{A} . Writing cells as arrows between their vertical arrows, our property amounts to the commutativity of the outer diagram below, in \mathbb{D}

$$\begin{array}{c}
 \text{(7)} \quad \begin{array}{ccccc}
 & & \text{HR'Rw} & \xrightarrow{\beta \text{Rw}} & \text{S'GRw} & \xrightarrow{\text{S'\alpha w}} & \text{S'SFw} \\
 & & \nearrow \text{HR'\rho} & & \nearrow \text{S'G\rho} & & \searrow \text{S'S\phi} \\
 \text{HR'(Ru}\otimes\text{Rv)} & \xrightarrow{\beta(\text{Ru}\otimes\text{Rv})} & \text{S'G(Ru}\otimes\text{Rv)} & & \text{S'(Fu}\otimes\text{Fv)} \\
 \nearrow \text{H\rho'R} & & \searrow \text{S'\gamma R} & & \nearrow \text{S'\sigma F} \\
 \text{H(R'Ru}\otimes\text{R'Rv)} & & \text{S'(GRu}\otimes\text{GRv)} & \xrightarrow{\text{S'(\alpha u}\otimes\text{\alpha v)}} & \text{S'(SFu}\otimes\text{SFv)} \\
 \searrow \text{S'S\phi} & & \nearrow \sigma \text{GR} & & \nearrow \sigma \text{SF} \\
 \text{HR'Ru}\otimes\text{HR'Rv} & \xrightarrow{\beta \text{Ru}\otimes\beta \text{Rv}} & \text{S'GRu}\otimes\text{S'GRv} & \xrightarrow{\text{S'\alpha u}\otimes\text{S'\alpha v}} & \text{S'SFu}\otimes\text{S'SFv}
 \end{array}
 \end{array}$$

and, indeed, the two hexagons commute by (c3), for α and β ; the upper parallelogram commutes by naturality of β ; the lower one by consistency of S' with the cells $\alpha u, \alpha v$ (2.1(ii)).

Within $\mathbb{D}\text{bl}$, we have the strict 2-category \mathbf{LxDbl} of *pseudo double categories, lax double functors and horizontal transformations*: namely, $\mathbf{LxDbl} = \mathbf{H}(\mathbb{D}\text{bl})$ is the restriction to trivial vertical arrows (1.1). Similarly, we have the strict 2-categories \mathbf{CxDbl} ($= \mathbf{H}(\mathbb{D}\text{bl}^!)$) and \mathbf{PsDbl} , whose vertical arrows are the colax or pseudo double functors, respectively. Even more interestingly, inside $\mathbb{D}\text{bl}$ we have *horizontal transformations from lax to colax double functors* $\alpha: \mathbb{R} \rightarrow \mathbb{F}: \mathbb{A} \rightarrow \mathbb{B}$ (take $G = S = \text{id}_{\mathbb{B}}$ in (1)) and their compositions; as well as *from colax*

to lax ones, $\alpha: G \rightarrow S: C \rightarrow D$ (take $R = S = \text{id}_C$ in (1)). This will be of use (for instance, in 4.2).

Viewing bicategories in \mathbf{LxDbl} , as vertical pseudo double categories, we have a 2-category of bicategories, lax functors and *special* transformations $\alpha: R \rightarrow S$, whose components are identities and special cells

$$(8) \quad \alpha_A = 1: RA \rightarrow SA, \quad \alpha_u: (Ru \begin{smallmatrix} RA \\ RA' \end{smallmatrix} Su),$$

which is only possible if R and S coincide on objects. Fixing the class of objects, this is precisely the 2-category considered by Carboni and Rosebrugh to define lax monads of bicategories ([CR], Prop. 2.1). Note, on the other hand, that lax functors and lax transformations of bicategories (or 2-categories) do not form a bicategory.

2.3. The double category \mathbf{Mcat} . The strict double category \mathbf{Dbl} has a full double subcategory \mathbf{Mcat} of monoidal categories (viewed as vertical double categories on one formal object $*$, vertical arrows $A: * \rightarrow *$ and cells $a: A \rightarrow A'$; the vertical composition is the tensor product).

The horizontal arrows of \mathbf{Mcat} are the *monoidal functors* (lax with respect to tensor product); the vertical arrows are the *comonoidal functors* (which are *colax*). A cell $\alpha: (F_S^R G)$ associates to every object A in \mathbf{A} an arrow $\alpha_A: GRA \rightarrow SFA$ in \mathbf{D} , satisfying the naturality condition (c1) and the coherence conditions (c2, 3) of 2.2; these amount to the commutativity of the diagrams below (where the lax monoidal functor R has comparison arrows $\rho = \rho(*): I \rightarrow RI$, $\rho(A, A'): RA \otimes RA' \rightarrow R(A \otimes A')$, and so on; the identity of tensor products are always written as $I = 1_*$)

$$(1) \quad \begin{array}{ccc} GRA & \xrightarrow{\alpha_A} & SFA \\ \text{GRa} \downarrow & & \downarrow \text{SFa} \\ GRA' & \xrightarrow{\alpha_{A'}} & SFA' \end{array} \quad \begin{array}{ccc} & & \alpha_I \\ & \nearrow \text{G}\rho & \text{GRI} \xrightarrow{\quad} \text{SFI} \\ & & \searrow \text{S}\phi \\ \text{GI} & \xrightarrow{\quad \gamma} & I \xrightarrow{\quad \sigma} & \text{SI} \end{array}$$

$$\begin{array}{ccc}
 & \xrightarrow{\alpha(A \otimes A')} & \\
 GR(A \otimes A') & & SF(A \otimes A') \\
 \nearrow G\rho & & \searrow S\varphi \\
 G(RA \otimes RA') & & S(FA \otimes FA') \\
 \searrow \gamma R & & \nearrow \sigma F \\
 GRA \otimes GRA' & \xrightarrow{\alpha A \otimes \alpha A'} & SFA \otimes SFA'
 \end{array}$$

The horizontal and vertical composition are defined as above (2.2.4)

$$\begin{aligned}
 (2) \quad (\alpha \mid \beta)(A) &= (S'\alpha A) \circ (\beta RA): HR'RA \rightarrow S'GRA \rightarrow S'SFA, \\
 \left(\frac{\alpha}{\gamma}\right)(A) &= (\gamma FA) \circ (G'\alpha A): G'GRA \rightarrow G'SFA \rightarrow TF'FA.
 \end{aligned}$$

We prove now that internal monoids provide an interesting lax double functor from $\mathbb{M}cat$ to Cat .

2.4. Theorem. *There exists a lax double functor $Mon: \mathbb{M}cat \rightarrow Cat$. In this transformation, a monoidal category \mathbf{V} is sent to the category $Mon(\mathbf{V})$ of monoids in it, a monoidal functor $R: \mathbf{V} \rightarrow \mathbf{W}$ (which preserves monoids) lifts to a functor $Mon(R): Mon(\mathbf{V}) \rightarrow Mon(\mathbf{W})$, while a comonoidal functor $F: \mathbf{V} \rightarrow \mathbf{V}'$ induces a profunctor $Mon(F): Mon(\mathbf{V}) \rightarrow Mon(\mathbf{V}')$.*

Proof. The beginning of the statement being obvious, let us define the associated profunctor $Mon(F): Mon(\mathbf{V}) \rightarrow Mon(\mathbf{V}')$. Given two monoids M, N in \mathbf{V} and \mathbf{V}' , respectively, the set $Mon(F)(M, N)$ consists of all morphisms $f: FM \rightarrow N$ in \mathbf{V}' which make the following diagrams commute

$$(1) \quad \begin{array}{ccc}
 F(I) \xrightarrow{\varphi} I & & F(M \otimes M) \xrightarrow{\varphi} FM \otimes FM \xrightarrow{f \otimes f} I \\
 F e \downarrow & & F m \downarrow & & \downarrow e \\
 F(M) \xrightarrow{f} N & & F(M) \xrightarrow{f} N
 \end{array}$$

Now, given a cell α in $\mathbb{M}cat$, as in the left diagram below, we define the corresponding cell $Mon(\alpha)$ between functors and profunctors

$$(2) \quad \begin{array}{ccc} \mathbf{V} & \xrightarrow{\mathbf{R}} & \mathbf{W} \\ \mathbf{F} \downarrow & \alpha \swarrow & \downarrow \mathbf{G} \\ \mathbf{V}' & \xrightarrow{\mathbf{S}} & \mathbf{W}' \end{array} \quad \begin{array}{ccc} \mathbf{Mon}(\mathbf{V}) & \xrightarrow{\mathbf{Mon}(\mathbf{R})} & \mathbf{Mon}(\mathbf{W}) \\ \mathbf{Mon}(\mathbf{F}) \downarrow & \mathbf{Mon}(\alpha) & \downarrow \mathbf{Mon}(\mathbf{G}) \\ \mathbf{Mon}(\mathbf{V}') & \xrightarrow{\mathbf{Mon}(\mathbf{S})} & \mathbf{Mon}(\mathbf{W}') \end{array}$$

by a natural transformation whose general component on M, N is

$$(3) \quad \mathbf{Mon}(\alpha)(M, N): \mathbf{Mon}(\mathbf{F})(M, N) \rightarrow \mathbf{Mon}(\mathbf{G})(\mathbf{R}M, \mathbf{S}N),$$

$$(f: \mathbf{F}M \rightarrow N) \mapsto (\mathbf{S}f \cdot \alpha M: \mathbf{G}(\mathbf{R}M) \rightarrow \mathbf{S}(\mathbf{F}M) \rightarrow \mathbf{S}N).$$

Finally, given a comonoidal functor $F': \mathbf{V}' \rightarrow \mathbf{V}''$ composable with F , one constructs the special cell $\mu(F, F')$ in \mathbb{C} at (the laxity comparison for vertical composition), as the following natural transformation of profunctors

$$(4) \quad \mu(F, F'): \mathbf{Mon}(\mathbf{F}) \otimes \mathbf{Mon}(\mathbf{F}') \rightarrow \mathbf{Mon}(\mathbf{F}'\mathbf{F}): \mathbf{Mon}(\mathbf{V}) \rightarrow \mathbf{Mon}(\mathbf{V}''),$$

$$\mu(F, F')(M, P)[f, g] = (g \cdot F'(f): F'\mathbf{F}(M) \rightarrow F'N \rightarrow P),$$

where $f: \mathbf{F}M \rightarrow N$ and $g: F'N \rightarrow P$ are in \mathbf{V}' and \mathbf{V}'' , while the class $[f, g]$ belongs to the composition-colimit $(\mathbf{Mon}(\mathbf{F}) \otimes \mathbf{Mon}(\mathbf{F}'))(M, P)$. \square

2.5. Commas. Given a *colax* double functor F and a *lax* double functor R with the same codomain, we can construct the *comma* pseudo double category $F \Downarrow R$, where the projections P and Q are strict double functors, and π is a cell of $\mathbb{D}bl$

$$(1) \quad \begin{array}{ccc} \mathbf{F} \Downarrow \mathbf{R} & \xrightarrow{\mathbf{P}} & \mathbf{A} \\ \mathbf{Q} \downarrow & \pi \swarrow & \downarrow \mathbf{F} \\ \mathbf{X} & \xrightarrow{\mathbf{R}} & \mathbf{C} \end{array}$$

An object is a triple $(A, X; c: \mathbf{F}A \rightarrow \mathbf{R}X)$; a horizontal morphism $(a, x): (A, X; c) \rightarrow (A', X'; c')$ comes from a commutative square of \mathbb{C} , as in the left diagram

$$(2) \quad \begin{array}{ccc} \mathbf{F}A & \xrightarrow{c} & \mathbf{R}X \\ \mathbf{F}a \downarrow & = & \downarrow \mathbf{R}x \\ \mathbf{F}A' & \xrightarrow{c'} & \mathbf{R}X' \end{array} \quad \begin{array}{ccc} \mathbf{F}A & \xrightarrow{c} & \mathbf{R}X \\ \mathbf{F}u \downarrow & \gamma & \downarrow \mathbf{R}v \\ \mathbf{F}B & \xrightarrow{d} & \mathbf{R}Y \end{array}$$

their composition is obvious. A vertical arrow $(u, v; \gamma): (A, X; c) \rightarrow (B, Y; d)$ comes from a cell $\gamma: (Fu \underset{d}{\overset{c}{\rightrightarrows}} Rv)$ in \mathbb{C} , as in the right diagram above; their composition *does* require F colax and R lax

$$(3) \quad \begin{array}{ccccccc} \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{-c} & \bullet & \xlongequal{\quad} & \bullet \\ \downarrow F(u \otimes u') & & \begin{array}{c} Fu \downarrow \\ F[u, u'] \end{array} & \begin{array}{c} \gamma \\ \downarrow \\ Rv \end{array} & & \begin{array}{c} \downarrow \\ R[v, v'] \end{array} & \downarrow R(v \otimes v') \\ \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{-d} & \bullet & \xlongequal{\quad} & \bullet \\ \downarrow Fu'' & & \begin{array}{c} Fu'' \downarrow \\ 1_{Fu''} \end{array} & \begin{array}{c} \delta \\ \downarrow \\ Rv' \end{array} & & \begin{array}{c} \downarrow \\ 1_{Rv''} \end{array} & \downarrow Rv'' \\ \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{-e} & \bullet & \xlongequal{\quad} & \bullet \end{array}$$

A cell (α, ξ) is a pair of cells $\alpha: (u \underset{b}{\overset{a}{\rightrightarrows}} u')$, $\xi: (v \underset{y}{\overset{x}{\rightrightarrows}} v')$ in \mathbb{A} and \mathbb{X} , such that $F\alpha$ and $R\xi$ are coherent with γ, γ' in \mathbb{C}

$$(4) \quad \begin{array}{ccc} (A, X; c) & \xrightarrow{(a, x)} & (A', X'; c') \\ (u, v; \gamma) \downarrow & (\alpha, \xi) & \downarrow (u', v'; \gamma') \\ (B, Y; d) & \xrightarrow{(b, y)} & (B, Y; d') \end{array} \quad (F\alpha \mid \gamma') = (\gamma \mid R\xi).$$

Their horizontal and vertical compositions are obvious.

The associativity isocell for three consecutive vertical arrows $(u, v; \gamma)$, $(u', v'; \delta)$, $(u'', v''; \epsilon)$ is the pair $(\alpha(u), \xi(v))$ of associativity isocells of \mathbb{A}, \mathbb{X} for the triples $\mathbf{u} = (u, u', u'')$, $\mathbf{v} = (v, v', v'')$

$$(5) \quad (\alpha(\mathbf{u}), \xi(\mathbf{v})): ((u, v; \gamma) \otimes (u', v'; \delta)) \otimes (u'', v''; \epsilon) \rightarrow (u, v; \gamma) \otimes ((u', v'; \delta) \otimes (u'', v''; \epsilon)),$$

Denoting by Φ, Φ' the pasted cells (6), (7) of the two ternary composites, the coherence of the preceding cell (5) is expressed by the equality $(F\alpha(\mathbf{u}) \mid \Phi') = (\Phi \mid R\xi(\mathbf{v}))$, which comes from the coherence axioms on F, R and \mathbb{C} (write $u_1 = (u \otimes u') \otimes u''$, $u_2 = u \otimes (u' \otimes u'')$, and similarly v_1, v_2)

$$(6) \quad \begin{array}{ccccccc} \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{-c} & \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet \\ \downarrow F(u_1) & & \begin{array}{c} F(u \otimes u') \downarrow \\ F[u \otimes u', u''] \end{array} & \begin{array}{c} \downarrow \\ 1_{Fu''} \end{array} & \begin{array}{c} \downarrow \\ 1_{Fu''} \end{array} & \begin{array}{c} \gamma \otimes \delta \\ \downarrow \\ Rv \end{array} & \begin{array}{c} \downarrow \\ R[v, v'] \end{array} & \begin{array}{c} \downarrow \\ R(v \otimes v') \end{array} & \downarrow R(v_1) \\ \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{-e} & \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet \\ \downarrow Fu'' & & \begin{array}{c} Fu'' \downarrow \\ 1_{Fu''} \end{array} & \begin{array}{c} \downarrow \\ 1_{Fu''} \end{array} & \begin{array}{c} \downarrow \\ 1_{Fu''} \end{array} & \begin{array}{c} \epsilon \\ \downarrow \\ Rv'' \end{array} & \begin{array}{c} \downarrow \\ 1_{Rv''} \end{array} & \begin{array}{c} \downarrow \\ Rv'' \end{array} & \downarrow Rv'' \\ \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{-f} & \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet \end{array}$$

$$(7) \quad \begin{array}{ccccccc} \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{-c} & \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet \\ \downarrow & & \text{Fu} \downarrow & \downarrow & \downarrow & \gamma & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{F}(u_2) \downarrow & & \text{F}[u, u' \otimes u''] & \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{-d} & \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{\quad} & \text{R}(v_2) \\ & & \text{F}(u' \otimes u'') \downarrow & \downarrow & \downarrow & \delta \otimes \varepsilon & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet & \xrightarrow{-f} & \bullet & \xlongequal{\quad} & \bullet \end{array}$$

Finally, P and Q are projections; the components of π on objects and vertical arrows are:

$$(8) \quad \pi(A, X; c) = c: FA \rightarrow RX, \quad \pi(u, v; \gamma) = \gamma: (Fu \xrightarrow{\pi(A, X; c)} Rv)$$

2.6. Theorem (Universal properties of commas). (a) For a pair of lax double functors S, T and a cell α as below (in $\mathbb{D}bl$) there is a unique lax double functor $L: \mathbb{Z} \rightarrow F \Downarrow R$ such that $S = PL$, $T = QL$ and $\alpha = (\beta | \pi)$ where the cell β is defined by $1: QL \rightarrow T$ (a horizontal transformation of lax double functors)

$$(1) \quad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{S} & \mathbb{A} \\ 1 \downarrow & & \alpha \swarrow \downarrow F \\ \mathbb{Z} & \xrightarrow{T} \mathbb{X} \xrightarrow{R} & \mathbb{C} \end{array} = \begin{array}{ccc} \mathbb{Z} & \xrightarrow{S} & \mathbb{A} \\ -L \downarrow & & -P \downarrow \\ \mathbb{Z} & \xrightarrow{T} \mathbb{X} \xrightarrow{R} & \mathbb{C} \end{array}$$

Moreover, L is pseudo if and only if both S and T are.

(b) A similar property holds for a pair of colax G, H and a cell $\alpha': (G \xrightarrow{1} R \Downarrow H)$.

Proof. (a) L is defined as follows on items of \mathbb{Z} : an object Z, a horizontal arrow f, a vertical arrow u, a cell φ

$$(2) \quad L(Z) = (SZ, TZ; \alpha Z: FSZ \rightarrow RTZ), \quad L(f) = (Sf, Tf), \\ L(u) = (Su, Tu; \alpha u), \quad L(\varphi) = (S\varphi, T\varphi).$$

The comparison special cells $L[-]$, for Z and $w = u \otimes v$ in \mathbb{Z} , are constructed with the laxity cells $S[-]$ and $T[-]$ (and are invertible if and only if the latter are)

$$(3) \quad L[Z] = (S[Z], T[Z]): 1_{LA}^\bullet \rightarrow L(1_A^\bullet), \\ L[u, v] = (S[u, v], T[u, v]): Lu \otimes Lv \rightarrow L(u \otimes v).$$

Here, $Lu \otimes Lv$ and $L(u \otimes v)$ are the cells below

$$\begin{array}{c}
 \text{FSZ} = \text{FSZ} \xrightarrow{\alpha_Z} \text{RTZ} = \text{RTZ} \qquad \text{FSZ} \xrightarrow{\alpha_Z} \text{RTZ} \\
 \downarrow \qquad \text{FSu} \downarrow \qquad \alpha_u \qquad \downarrow \text{RTu} \qquad \downarrow \\
 (4) \quad \text{F[Su, Sv]} \quad \text{FSZ}' \longrightarrow \text{RTZ}' \quad \text{R[Tu, Tv]} \quad \downarrow \\
 \downarrow \qquad \text{FSv} \downarrow \qquad \alpha_v \qquad \downarrow \text{RTv} \qquad \downarrow \\
 \text{FSZ}'' = \text{FSZ}'' \xrightarrow{\alpha_Z''} \text{RTZ}'' = \text{RTZ}'' \qquad \text{FSZ}'' \xrightarrow{\alpha_Z''} \text{RTZ}''
 \end{array}$$

and the coherence condition on $L[u, v] = (S[u, v], T[u, v])$

$$(5) (Lu \otimes Lv \mid RT[u, v]) = (FS[u, v] \mid \alpha_w),$$

follows from the coherence condition of α as a cell in \mathbb{Dbl}

$$(6) (F[Su, Sv] \mid (\alpha_u \otimes \alpha_v) \mid (RT)[u, v]) = (FS[u, v] \mid \alpha_w).$$

Uniqueness is obvious. □

2.7. One-sided commas. (a) As a consequence of the previous theorem, the double comma $F \Downarrow B$ of a *colax* double functor $F: \mathbb{A} \rightarrow \mathbb{B}$ is the *tabulator* of F in \mathbb{Dbl} : it comes equipped with horizontal arrows P, Q and a cell π as below, so that any similar cell $\alpha: (1 \begin{smallmatrix} S \\ \top \\ T \end{smallmatrix} F)$ (with S, T lax) factors through it, by a unique lax double functor L

$$(1) \quad \begin{array}{ccc}
 Z & \xrightarrow{S} & A \\
 1 \downarrow & \alpha \swarrow & \downarrow F \\
 Z & \xrightarrow{T} & B
 \end{array} = \begin{array}{ccccc}
 Z & \xrightarrow{L} & F \Downarrow B & \xrightarrow{P} & A \\
 1 \downarrow & // & 1 \downarrow & \pi \swarrow & \downarrow F \\
 Z & \xrightarrow{L} & F \Downarrow B & \xrightarrow{Q} & B
 \end{array}$$

Indeed, the 1-dimensional universal property of tabulators follows directly from 2.6. Then one shows that tabulators in \mathbb{Dbl} are lax functorial (I.4.3), which implies the 2-dimensional property.

(b) Similarly, for a *lax* double functor $R: \mathbb{A} \rightarrow \mathbb{B}$, the double comma $B \Downarrow R$ is the tabulator of F in the transpose double category \mathbb{Dbl}^t .

3. Double adjunctions

The general case, called a 'colax/lax adjunction', is defined as an orthogonal adjunction in $\mathbb{D}bl$. See [Ke] for related work.

3.1. Colax/lax adjunctions. An orthogonal adjunction (F, R) in $\mathbb{D}bl$ (1.3) gives a notion of *adjunction* $(\eta, \epsilon): F \dashv R$ between pseudo double categories, which occurs naturally in various situations: the left adjoint $F: \mathbb{A} \rightarrow \mathbb{B}$ is *colax*, the right adjoint $R: \mathbb{B} \rightarrow \mathbb{A}$ is *lax*, and we have two $\mathbb{D}bl$ -cells η, ϵ

$$(1) \quad \begin{array}{ccc} \mathbb{A} & \xlongequal{\quad} & \mathbb{A} \\ F \downarrow & \eta \swarrow & \parallel \\ \mathbb{B} & \xrightarrow{\quad R \quad} & \mathbb{A} \end{array} \qquad \begin{array}{ccc} & R \cdot & \\ \mathbb{B} & \xrightarrow{\quad} & \mathbb{A} \\ \parallel & \epsilon \swarrow & \downarrow F \\ \mathbb{B} & \xlongequal{\quad} & \mathbb{B} \end{array}$$

satisfying the triangle equalities $\eta \circ \epsilon = 1_F$ and $\epsilon \circ \eta = 1_R$. (As in 2.2, the arrow of a colax double functor is marked with a dot *when displayed vertically*, in $\mathbb{D}bl$.)

This general adjunction will be said to be of *colax/lax type*. We speak of a *pseudo/lax* (resp. a *colax/pseudo*) adjunction when the left (resp. right) adjoint is pseudo, and of a *pseudo* adjunction when both adjoints are pseudo (replacing *pseudo* with *strict* when it is the case). For instance, an ordinary adjunction between abelian categories has a colax/lax extension to their double categories of relations, which is pseudo/lax (resp. colax/pseudo) when the left (resp. right) adjoint is exact (5.4). This and other examples will be examined in Section 5.

From general properties (1.3), we already know that the left adjoint of a lax double functor R is determined up to isomorphism (a special isocell between vertical arrows in $\mathbb{D}bl$) and that left adjoints compose, contravariantly. Similarly for right adjoints. As in 2.2, we may write $\eta: 1 \rightarrow RF$, by abuse of notation; *but one should recall that the coherence condition of such a transformation works through the interplay of the comparison cells of F and R* . Similarly for $\epsilon: FR \rightarrow 1$. Therefore, a *general colax/lax adjunction cannot be seen as an adjunction in some bicategory*; but we shall prove in the next section that this becomes legitimate in the *pseudo/lax* or *colax/pseudo* case.

3.2. Description. To make the previous definition explicit, a colax/lax adjunction $(\eta, \epsilon): F \dashv R$ between the *pseudo* double categories \mathbb{A}, \mathbb{B} consists of:

(a) a *colax* double functor $F: \mathbb{A} \rightarrow \mathbb{B}$, with comparison cells $\varphi_A: F(1_A^\bullet) \rightarrow 1_{FA}^\bullet$, $\varphi(u, u'): F(u \otimes u') \rightarrow Fu \otimes Fu'$,

(b) a *lax* double functor $R: \mathbb{B} \rightarrow \mathbb{A}$, with comparison cells $\rho_B: 1_{RB}^\bullet \rightarrow R(1_B^\bullet)$, $\rho(v, v'): Rv \otimes Rv' \rightarrow R(v \otimes v')$,

(c) two ordinary adjunctions at the levels $i = 0, 1$ (cf. 1.1), respecting domain and codomain

$$\begin{aligned} \eta_i: 1 &\rightarrow R_i F_i: \mathbf{A}_i \rightarrow \mathbf{A}_i, & \varepsilon_i: F_i R_i &\rightarrow 1: \mathbf{B}_i \rightarrow \mathbf{B}_i \\ \varepsilon_i F_i \circ F_i \eta_i &= 1_{F_i}, & R_i \varepsilon_i \circ \eta_i R_i &= 1_{R_i}, \end{aligned}$$

which means that we are assigning:

- horizontal maps $\eta_A: A \rightarrow RFA$ and cells $\eta u: (u \xrightarrow{\eta_A} RFu)$ in \mathbb{A} ,
- horizontal maps $\varepsilon_B: FRB \rightarrow B$ and cells $\varepsilon v: (FRv \xrightarrow{\varepsilon_B} v)$ in \mathbb{B} ,

satisfying the naturality conditions and the triangle identities (which we state at level 1, since this also implies level 0), on $a: (u \xrightarrow{f} u')$ in \mathbb{A} and $b: (v \xrightarrow{g} v')$ in \mathbb{B}

$$\begin{aligned} (c1) \quad (a \mid \eta u') &= (\eta u \mid RFa), & (FRb \mid \varepsilon v') &= (\varepsilon v \mid b), \\ (c2) \quad (F\eta u \mid \varepsilon Fu) &= 1_{Fu}, & (\eta Rv \mid R\varepsilon v) &= 1_{Rv}; \end{aligned}$$

(d) finally, the following conditions of coherence with the vertical operations are required (in terms of the comparison cells of F and R):

(d1) (*coherence of η and ε with identities*) for A in \mathbb{A} and B in \mathbb{B}

$$\begin{aligned} (1) \quad (\eta 1_A^\bullet \mid R\varphi_A) &= (1_{\eta_A}^\bullet \mid \rho FA) & (\eta 1_A^\bullet = 1_{\eta_A}^\bullet, & \text{if } F \text{ and } R \text{ are unitary}), \\ (2) \quad (F\rho_B \mid \varepsilon 1_B^\bullet) &= (\varphi RB \mid 1_{\varepsilon_B}^\bullet) & (\varepsilon 1_B^\bullet = 1_{\varepsilon_B}^\bullet, & \text{if } F \text{ and } R \text{ are unitary}); \end{aligned}$$

(d2') (*coherence of η with vertical composition*) for $u'' = u \otimes u'$ in \mathbb{A}

$$(3) \quad (\eta u'' \mid R\varphi(u, u')) = ((\eta u \otimes \eta u') \mid \rho(Fu, Fu')),$$

$$\begin{array}{c} \mathbf{A} \longrightarrow \mathbf{RFA} \equiv \mathbf{RFA} \\ \downarrow \eta u'' \quad \downarrow R\varphi \\ \mathbf{A}'' \longrightarrow \mathbf{RFA}'' \equiv \mathbf{RFA}'' \end{array} = \begin{array}{c} \mathbf{A} \longrightarrow \mathbf{RFA} \equiv \mathbf{RFA} \\ \downarrow \eta u \quad \downarrow R\varphi \quad \downarrow \\ \mathbf{A}' \longrightarrow \mathbf{RFA}' \xrightarrow{\rho F} \mathbf{R(Fu \otimes Fu')} \\ \downarrow \eta u' \quad \downarrow R\varphi' \quad \downarrow \\ \mathbf{A}'' \longrightarrow \mathbf{RFA}'' \equiv \mathbf{RFA}'' \end{array}$$

(d2'') (coherence of ε with vertical composition) for $v'' = v \otimes v'$ in \mathbb{B}

$$(4) \quad (F\rho(v, v') \mid \varepsilon v'') = (\varphi(Rv, Rv') \mid (\varepsilon v \otimes \varepsilon v')),$$

$$\begin{array}{c}
 \text{FRB} \quad \equiv \quad \text{FRB} \quad \longrightarrow \quad \text{B} \\
 \downarrow F(Rv \otimes Rv') \quad \downarrow F\rho \quad \downarrow FRv'' \quad \downarrow \varepsilon v'' \quad \downarrow v'' \\
 \text{FRB}'' \quad \equiv \quad \text{FRB}'' \quad \longrightarrow \quad \text{B}''
 \end{array}
 =
 \begin{array}{c}
 \text{FRB} \quad \equiv \quad \text{FRB} \quad \longrightarrow \quad \text{B} \\
 \downarrow \quad \downarrow FRv \quad \downarrow \varepsilon v \quad \downarrow v \\
 \downarrow \quad \varphi R \quad \text{FRB}' \quad \longrightarrow \quad \text{B}' \\
 \downarrow \quad \downarrow FRv' \quad \downarrow \varepsilon v' \quad \downarrow v' \\
 \text{FRB}'' \quad \equiv \quad \text{FRB}'' \quad \longrightarrow \quad \text{B}''
 \end{array}$$

3.3. Remarks. (a) In this colax/lax adjunction, the comparison cells of R , together with the unit η , determine the comparison cells of F . In fact, the first equation in (d1) says that the adjoint cell of φA , i.e. $(\varphi A)' = (\eta 1_A^* \mid R\varphi A)$, must be equal to $(1_{\eta A}^* \mid \rho FA)$. Similarly for $\varphi(u, u')$, from (d2').

(b) If the weak double category \mathbb{B} is *horizontally invariant* (1.5), as all our examples of real interest, Lemma I.2.5 proves that the colax adjoint $F: \mathbb{A} \rightarrow \mathbb{B}$ is also *vertically determined, up to vertical equivalence*, by R . In fact, a horizontal invertible transformation $\varphi: F \rightarrow F'$ produces a strong vertical transformation $\rho: F \rightarrow F'$, whose general component $\rho A: FA \rightarrow F'A$ is a vertical equivalence associated to the horizontal isomorphism $\varphi: FA \cong F'A$ (I.2.3), and determined as such up to a special isocell.

3.4. Theorem (Characterisation by hom-sets). *An adjunction $(\eta, \varepsilon): F \rightarrow R$ can equivalently be given by a colax double functor $F: \mathbb{A} \rightarrow \mathbb{B}$, a lax double functor $R: \mathbb{B} \rightarrow \mathbb{A}$, two functorial isomorphisms H_0 and H_1*

$$(1) \quad H_i: \mathbf{B}_i(F_i^-, \cdot) \rightarrow \mathbf{A}_i(-, R_i \cdot): \mathbf{A}_i^{op} \times \mathbf{B}_i \rightarrow \mathbf{Set} \quad (i = 0, 1),$$

whose components are consistent with domain, codomain and the vertical structure (through the comparison cells of F and R), i.e. satisfy the following conditions (cf. 1.1)

$$(ad.0) \quad H_0 \text{ has components } H(A, B): \mathbf{B}_0(FA, B) \rightarrow \mathbf{A}_0(A, RB),$$

$$(ad.1) \quad H_1 \text{ has components } H(u, v): \mathbf{B}_1(Fu, v) \rightarrow \mathbf{A}_1(u, Rv), \text{ which take a cell } b: (Fu \xrightarrow{g} v) \text{ to } Hb: (u \xrightarrow{Hg} Rv),$$

$$(ad.2) \quad H(\varphi A \mid 1_g^*) = (1_{Hg}^* \mid \rho B), \quad H(\varphi(u, u') \mid b \otimes b') = (Hb \otimes Hb' \mid \rho(v, v')),$$

$$\begin{array}{ccccccc}
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 F1_A^\bullet \downarrow & \varphi_A & \downarrow 1_{FA}^\bullet & 1_g^\bullet & \downarrow 1_B^\bullet & & 1_A^\bullet \downarrow & 1_{Hg}^\bullet & \downarrow 1_{RB}^\bullet & \rho_B & \downarrow R1_B^\bullet \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 F(u \otimes u') \downarrow & \varphi(u, u') & \downarrow Fu & b & \downarrow v & & u \downarrow & Hb & Rv \downarrow & \rho(v, v') & \downarrow \\
 & & \downarrow Fu' & b' & \downarrow v' & & u' \downarrow & Hb' & Rv' \downarrow & & \downarrow \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 & & & & & & & & & & R(v \otimes v') \\
 & & & & & & & & & & \downarrow
 \end{array}$$

Proof. We have only to verify the equivalence of 3.2.1-4 with the conditions above. To show, for instance, that 3.2.3 implies the second identity of (ad.2), consider that H is defined by the unit η as

$$(2) \quad H(A, B)(g) = Rg \circ \eta_A: FA \rightarrow B, \quad H(u, v)(b) = (\eta u \mid Rb): (u \xrightarrow{Hg} Rv),$$

whence (applying (2), 3.2.3 and 2.1(ii)):

$$\begin{aligned}
 (3) \quad H(\varphi(u, u') \mid b \otimes b') &= (\eta(u \otimes u') \mid R\varphi(u, u') \mid R(b \otimes b')) \\
 &= (\eta u \otimes \eta u' \mid \rho(Fu, Fu') \mid R(b \otimes b')) \\
 &= (\eta u \otimes \eta u' \mid Rb \otimes Rb' \mid \rho(v, v')) = (Hb \otimes Hb' \mid \rho(v, v')). \quad \square
 \end{aligned}$$

3.5. Corollary (Characterisation by commas). *An adjunction amounts to an isomorphism of pseudo double categories $H: F \parallel \mathbb{B} \rightarrow \mathbb{A} \parallel R$, over $\mathbb{A} \times \mathbb{B}$*

$$(1) \quad \begin{array}{ccc}
 & H & \\
 F \parallel \mathbb{B} & \longrightarrow & \mathbb{A} \parallel R \\
 & \searrow & \swarrow \\
 & \mathbb{A} \times \mathbb{B} &
 \end{array}$$

Proof. It is a straightforward consequence of the previous theorem. □

3.6. Theorem (Right adjoint by universal properties). *Given a colax double functor $F: \mathbb{A} \rightarrow \mathbb{B}$, the existence (and choice) of a right adjoint lax double functor R amounts to two conditions (rad.0-1) (including two choices):*

(rad.0) *for every object B in \mathbb{B} there is a universal arrow $(RB, \varepsilon_B: F(RB) \rightarrow B)$ from the functor F_0 to the object B (and we choose one),*

- explicitly, the universal property means that, for each A in \mathbb{A} and $g: FA \rightarrow B$ there is a unique $f: A \rightarrow RB$ such that $g = \varepsilon_B \circ Ff: FA \rightarrow F(RB) \rightarrow B$;

(rad.1) for every vertical map $v: B \rightarrow B'$ in \mathbb{B} there is a universal arrow (Rv, ε_v) from the functor F_1 to the object v of \mathbb{B}_1 (and we choose one); moreover, $\varepsilon_v: (FRv \begin{smallmatrix} \varepsilon_B \\ \varepsilon_{B'} \end{smallmatrix} v)$,

- explicitly, for each vertical arrow $u: A \rightarrow A'$ in \mathbb{A} and each cell $b: (Fu \begin{smallmatrix} g \\ g' \end{smallmatrix} v)$ in \mathbb{B} , there is a unique cell $a: (u \begin{smallmatrix} f \\ f' \end{smallmatrix} Rv)$ in \mathbb{A} such that $b = (Fa \mid \varepsilon_v)$.

The comparison cells $\rho_B: 1_{RB}^{\circ} \rightarrow R(1_B^{\circ})$ and $\rho(v, v'): Rv \otimes Rv' \rightarrow R(v \otimes v')$ of R are provided by the universal properties of ε , as the unique solution of the equations 3.2.2, 3.2.4, respectively; and R is pseudo if and only if all such cells are (special) isocells.

Proof. Write the comparison cells of F as usual: $\varphi_A, \varphi(u, u')$. The conditions (rad.0-1) are plainly necessary.

Conversely, (rad.0) provides an ordinary adjunction $(\eta_0, \varepsilon_0): F_0 \dashv R_0$ for the categories $\mathbb{A}_0, \mathbb{B}_0$, so that R, η and ε are correctly defined – as far as objects, horizontal arrows, their horizontal composition and horizontal identities are concerned.

Adding (rad.1), R, η and ε are (correctly) defined as far as objects, arrows, cells, horizontal composition and identities are concerned. In particular, the cells $Rb, \eta u$ are defined as the unique cells of \mathbb{A} satisfying the equations (1) and (2), respectively

$$\begin{array}{c}
 \begin{array}{ccccccc}
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 (1) \quad FRv \downarrow & FRb & \downarrow FRv' & \varepsilon v' & \downarrow v' & = & FRv \downarrow & \varepsilon v & \downarrow v & b & v' \downarrow \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet
 \end{array} \\
 \\
 \begin{array}{ccccccc}
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \bullet & \longrightarrow & \bullet \\
 (2) \quad Fu \downarrow & F\eta u & \downarrow FRFu & \varepsilon Fu & \downarrow Fu & = & Fu \downarrow & 1 & \downarrow Fu \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \bullet & \longrightarrow & \bullet
 \end{array}
 \end{array}$$

Now, define the R -comparison cells ρ as specified in the statement, so that the coherence properties of ε are satisfied (3.2.2, 3.2.4). One verifies easily, for such cells, the axioms of naturality and coherence (2.1).

Finally, we have to prove that $\eta: 1 \rightarrow RF$ satisfies the coherence property 3.2.3, with respect to $u \circ u'$ in \mathbb{A} (and similarly 3.2.1). By the universal property

of ε , it will suffice to show that $(F- | \varepsilon(Fu \otimes Fu'))$ takes the same value on both terms of 3.2.3. In fact, on the left-hand term $(\eta u'' | R\varphi(u, u'))$ we get $\varphi(u, u')$

$$(3) (F\eta u'' | FR\varphi(u, u') | \varepsilon(Fu \otimes Fu')) = (F\eta u'' | \varepsilon Fu'' | \varphi(u, u')) = \varphi(u, u');$$

but we get the same on the right-hand term $((\eta u \otimes \eta u') | \rho(Fu, Fu'))$, using 3.2.4, the naturality of φ , the middle four interchange in \mathbb{B} and a triangle identity

$$(4) (F(\eta u \otimes \eta u') | F\rho(Fu, Fu') | \varepsilon(Fu \otimes Fu')) \\ = (F(\eta u \otimes \eta u') | \varphi(RFu, RFu') | (\varepsilon Fu \otimes \varepsilon Fu')) \\ = (\varphi(u, u') | (F\eta u \otimes F\eta u') | (\varepsilon Fu \otimes \varepsilon Fu')) = \varphi(u, u'),$$

$$\begin{array}{ccccccc} FA & \longrightarrow & FRFA & \equiv & FRFA & \longrightarrow & FA \\ Fu'' \downarrow & & F(\eta u \otimes \eta u') \downarrow & & FRFu \downarrow & \varepsilon Fu \downarrow & Fu \downarrow \\ & & F(RFu \otimes RFu') \downarrow & \varphi RF & FRFA' & \longrightarrow & FA' \\ & & \downarrow & & FRFu' \downarrow & \varepsilon Fu' \downarrow & Fu' \downarrow \\ FA'' & \longrightarrow & FRFA'' & \equiv & FRFA'' & \longrightarrow & FA'' \end{array} =$$

$$\begin{array}{ccccccc} FA & \equiv & FA & \longrightarrow & FRFA & \longrightarrow & FA \\ Fu'' \downarrow & & Fu \downarrow & & F\eta u \downarrow & FRFu \downarrow & \varepsilon Fu \downarrow \\ & & \varphi & & FA' & \longrightarrow & FRFA' & \longrightarrow & FA' & = & \varphi(u, u'). \\ & & Fu' \downarrow & & F\eta u' \downarrow & FRFu' \downarrow & \varepsilon Fu' \downarrow & Fu' \downarrow \\ FA'' & \equiv & FA'' & \longrightarrow & FRFA'' & \longrightarrow & FA'' \end{array} \quad \square$$

3.7. Theorem (Factorisation of adjunctions). *Let $F \dashv R$ be a colax/lax adjunction between \mathbb{A} and \mathbb{B} . Then, using the isomorphism of double categories $H: F \parallel \mathbb{B} \rightarrow \mathbb{A} \parallel R$ (Corollary 3.5), we can factor it as*

$$(1) \quad \mathbb{A} \begin{array}{c} \xrightarrow{F'} \\ \xleftarrow{P} \end{array} F \parallel \mathbb{B} \begin{array}{c} \xrightarrow{H} \\ \xleftarrow{H^{-1}} \end{array} \mathbb{A} \parallel R \begin{array}{c} \xrightarrow{Q} \\ \xleftarrow{R'} \end{array} \mathbb{B} \quad F = QHF', \quad R = PH^{-1}R'.$$

- a coreflective colax/strict adjunction $F' \dashv P$ (with unit $PF' = 1$),
- an isomorphism $H \dashv H^{-1}$,
- a reflective strict/lax adjunction $Q \dashv R'$ (with counit $QR' = 1$),

where the comma projections P and Q are strict double functors.

Proof. We define the lax double functor $R': \mathbb{B} \rightarrow \mathbb{A} \Downarrow R$ by the strong universal property of commas (2.6a), applied to $R: \mathbb{B} \rightarrow \mathbb{A}$, $1: \mathbb{B} \rightarrow \mathbb{B}$ and $\alpha = 1_{\mathbb{R}}$, as in the diagram below

$$(2) \quad R'(\mathbb{B}) = (\mathbb{R}\mathbb{B}, \mathbb{B}; 1: \mathbb{R}\mathbb{B} \rightarrow \mathbb{R}\mathbb{B}), \quad R'(v) = (\mathbb{R}v, v; 1_{\mathbb{R}v}),$$

$$\rho'(v, v') = (\rho(v, v'), 1): (\mathbb{R}v \otimes \mathbb{R}v', v \otimes v'; \rho(v, v')) \rightarrow (\mathbb{R}(v \otimes v'), v \otimes v'; 1),$$

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\quad R \quad} & \mathbb{A} \\ 1 \downarrow & & \alpha \swarrow \downarrow 1 \\ \mathbb{B} & \xrightarrow{\quad T \quad} \mathbb{B} \xrightarrow{\quad R \quad} & \mathbb{A} \end{array} = \begin{array}{ccc} \mathbb{B} & \xrightarrow{\quad R \quad} & \mathbb{A} \\ -R' \rightarrow 1 \Downarrow R -P \rightarrow & & \mathbb{A} \\ 1 \downarrow \quad \beta \swarrow Q \downarrow \quad \pi \swarrow \downarrow 1 & & \\ \mathbb{B} & \xrightarrow{\quad T \quad} \mathbb{B} \xrightarrow{\quad R \quad} & \mathbb{A} \end{array}$$

Similarly, we define the colax $F': \mathbb{A} \rightarrow F \Downarrow \mathbb{B}$ by the dual result (2.6b)

$$(3) \quad F'(\mathbb{A}) = (\mathbb{A}, F\mathbb{A}; 1: F\mathbb{A} \rightarrow F\mathbb{A}), \quad F'(u) = (u, Fu; 1_{Fu}),$$

$$\varphi'(u, u') = (1, \varphi(u, u')): (u \otimes u', F(u \otimes u')); 1 \rightarrow (u \otimes u', Fu \otimes Fu'; \varphi(u, u')).$$

The coreflective adjunction $F' \dashv P$ is obvious

$$(4) \quad \eta'A = 1_A: A \rightarrow PF'A,$$

$$\epsilon'(A, B; g: FA \rightarrow B) = (1_A, g): (A, FA; 1_{FA}) \rightarrow (A, B; g: FA \rightarrow B),$$

as well as the reflective adjunction $Q \dashv R'$ and the factorisation above. □

4. Adjunctions and pseudo double functors

We consider now adjunctions where the left or right adjoint is pseudo. Adjoint equivalences of pseudo double categories are introduced.

4.1. Double adjunctions and 2-adjunctions. Let us recall, from 3.1, that a *pseudo/lax* adjunction $F \dashv R$ is a colax/lax adjunction between pseudo double categories where the left adjoint F is pseudo. Then, the comparison cells of F are horizontally invertible and the composites RF and FR are lax double functors; it follows (from the definition, 2.2) that the unit and counit are horizontal transformations of such functors. Therefore, a *pseudo/lax adjunction gives an adjunction in the 2-category $LxDbI$* of pseudo double categories, lax double functors and horizontal

transformations (2.2); and we shall prove that these two facts are actually equivalent (Theorem 4.3).

Dually, a *colax/pseudo* adjunction, where the right adjoint R is pseudo, will amount to an adjunction within the 2-category \mathbf{CxDbf} of pseudo double categories, colax double functors and horizontal transformations. Finally, a *pseudo* adjunction, where both F and R are pseudo, will be the same as an adjunction in the 2-category \mathbf{PsDbf} , whose vertical arrows are the pseudo double functors.

4.2. Theorem (Companions in \mathbb{Dbl}). *A lax double functor R has an orthogonal companion F in \mathbb{Dbl} if and only if it is pseudo; then one can define $F = R_*$ as the colax double functor which coincides with R except for comparison cells $\varphi = \rho^{-1}$, horizontally inverse to the ones of R .*

Proof. If R is pseudo, it is obvious that R_* , as defined above, is an orthogonal companion.

Conversely, suppose that $R: \mathbb{A} \rightarrow \mathbb{B}$ (lax) has an orthogonal companion F (colax). There are thus two cells η, ε in \mathbb{Dbl}

$$(1) \quad \begin{array}{ccc} \mathbb{A} & \xlongequal{\quad} & \mathbb{A} \\ 1 \downarrow & \eta \swarrow & \downarrow F \\ \mathbb{A} & \xrightarrow{\quad R \quad} & \mathbb{B} \end{array} \qquad \begin{array}{ccc} & \xrightarrow{\quad R \quad} & \mathbb{B} \\ \mathbb{A} & & \downarrow 1 \\ F \downarrow & \varepsilon \swarrow & \downarrow 1 \\ \mathbb{B} & \xlongequal{\quad} & \mathbb{B} \end{array}$$

which satisfy the identities $\eta|\varepsilon = 1_R^*$, $\eta \otimes \varepsilon = 1_F$. This means two 'horizontal transformations' $\eta: F \rightarrow R$, $\varepsilon: R \rightarrow F$, as defined in 2.2; for all $u: A \rightarrow A'$ in \mathbb{A} , we have cells ηu and εu in \mathbb{B} which are horizontally inverse

$$(2) \quad \begin{array}{ccc} FA & \xrightarrow{\eta A} & RA \\ Fu \downarrow & \eta u & \downarrow Ru \\ FA' & \xrightarrow{\eta A'} & RA' \end{array} \qquad \begin{array}{ccc} RA & \xrightarrow{\varepsilon A} & FA \\ Ru \downarrow & \varepsilon u & \downarrow Fu \\ RA' & \xrightarrow{\varepsilon A'} & FA' \end{array}$$

because of the previous identities (cf. 2.2.4 for pastings)

$$(3) \quad 1_{Ru} = (\eta | \varepsilon)(u) = (\varepsilon u | \eta u), \qquad 1_{Fu} = (\eta \otimes \varepsilon)(u) = (\eta u | \varepsilon u).$$

Applying now the coherence condition (c3) (in 2.2), for the transformations η, ε (and $w = u \otimes v$ in \mathbb{A}), we find

$$(4) \quad \eta w = (\varphi(u, v) \mid (\eta u \otimes \eta v) \mid \rho(u, v)), \quad (\rho(u, v) \mid \varepsilon w \mid \varphi(u, v)) = \varepsilon u \otimes \varepsilon v.$$

Since all η -cells and ε -cells are horizontally invertible, this proves that $\rho(u, v)$ has a left and right horizontal inverse: R is pseudo (and F is horizontally isomorphic to R_*).

4.3. Theorem. (a) (Pseudo/lax adjunctions). *Any adjunction $F \dashv R$ in the 2-category \mathbf{LxDbl} has F pseudo and is a pseudo/lax adjunction (cf. 3.1 or 4.1).*

(b) (Colax/pseudo adjunctions). *Any adjunction $F \dashv R$ in the 2-category \mathbf{CxDbI} has R pseudo and is a colax/pseudo adjunction (3.1, 4.1).*

More formally, (a) can be rewritten saying that, in \mathbf{DbI} , if the horizontal arrow R has a 'horizontal left adjoint' F (within the horizontal 2-category $\mathbf{HDbI} = \mathbf{LxDbl}$), then it also has an orthogonal adjoint G (colax). (Then, applying 1.4, it would follow that F and G are companions, whence F is pseudo, by 4.2, and isomorphic to G .)

Proof. Let the lax structures of F and R be given by the following comparison cells, where $u'' = u \otimes u'$ and $v'' = v \otimes v'$ (and similarly for vertical identities)

$$(1) \quad \lambda(u, u'): Fu \otimes Fu' \rightarrow F(u \otimes u'), \quad \rho(v, v'): Rv \otimes Rv' \rightarrow R(v \otimes v'),$$

$$(2) \quad \eta u'' = ((\eta u \otimes \eta u') \mid \rho(Fu, Fu') \mid R\lambda(u, u')): (u'' \overset{\eta A}{\eta A''} RFu''),$$

$$\varepsilon v \otimes \varepsilon v' = (\lambda(Rv, Rv') \mid F\rho(v, v') \mid \varepsilon v''): (FRv \otimes FRv' \overset{\varepsilon B}{\varepsilon B''} v'').$$

We construct now a colax structure φ for F

$$(3) \quad \varphi(u, u') = (F(\eta u \otimes \eta u') \mid F\rho(Fu, Fu') \mid \varepsilon(Fu \otimes Fu')): (Fu'' \overset{FA}{FA''} Fu \otimes Fu').$$

and prove that $\varphi(u, u')$ and $\lambda(u, u')$ are horizontally inverse:

$$(4) \quad (\varphi(u, u') \mid \lambda(u, u')) = (F(\eta u \otimes \eta u') \mid F\rho(Fu, Fu') \mid \varepsilon(Fu \otimes Fu') \mid \lambda(u, u'))$$

$$= (F(\eta u \otimes \eta u') \mid F\rho(Fu, Fu') \mid FR\lambda(u, u') \mid \varepsilon Fu'') \quad (\text{by naturality of } \varepsilon, 3.2),$$

$$= (F(\eta u'') \mid \varepsilon Fu'') = 1_{Fu''} \quad (\text{by (2) and a triangle identity});$$

$$(5) \quad (\lambda(u, u') \mid \varphi(u, u')) = (\lambda(u, u') \mid F(\eta u \otimes \eta u') \mid F\rho(Fu, Fu') \mid \varepsilon(Fu \otimes Fu'))$$

$$= (F\eta u \otimes F\eta u' \mid \lambda(RFu, RFu') \mid F\rho(Fu, Fu') \mid \varepsilon(Fu \otimes Fu')) \quad (\text{by naturality}),$$

$$= ((F\eta u \otimes F\eta u') \mid (\varepsilon Fu \otimes \varepsilon Fu')) \quad (\text{by (1)}),$$

$$= (F\eta u \mid \varepsilon Fu) \otimes (F\eta u' \mid \varepsilon Fu') \quad (\text{by middle-four interchange}),$$

$$= 1_{Fu} \otimes 1_{Fu'} = 1_{Fu \otimes Fu'}. \quad (\text{by a triangle identity and unitality of } \mathbb{B}). \quad \square$$

4.4. Equivalences of pseudo double categories. An *adjoint equivalence* between two pseudo double categories \mathbb{A} and \mathbb{B} will be a pseudo adjunction $(\eta, \varepsilon): F \rightarrow R$ where the horizontal transformations $\eta: 1_{\mathbb{A}} \rightarrow RF$ and $\varepsilon: FR \rightarrow 1_{\mathbb{B}}$ are invertible.

The following properties of a pseudo double functor $F: \mathbb{A} \rightarrow \mathbb{B}$ will allow us to characterise this fact in the usual way (under a mild restriction, cf. 4.5):

(a) We say that F is *faithful* if the ordinary functor $F_1: \mathbf{A}_1 \rightarrow \mathbf{B}_1$ (between the categories of vertical arrows and double cells) is such: given two double cells $a, a': u \rightarrow u'$ of \mathbb{A} between the same vertical arrows, $F(a) = F(a')$ implies $a = a'$. Plainly, this implies that also the functor $F_0: \mathbf{A}_0 \rightarrow \mathbf{B}_0$ (between the categories of objects and horizontal arrows) is faithful.

(b) Similarly, we say that F is *full* if $F_1: \mathbf{A}_1 \rightarrow \mathbf{B}_1$ is: for every double cell $b: F(u) \rightarrow F(u')$ in \mathbb{B} , there is a cell $a: u \rightarrow u'$ in \mathbb{A} such that $F(a) = b$. Again, it follows that also $F_0: \mathbf{A}_0 \rightarrow \mathbf{B}_0$ is full: for a horizontal map $g: F(A) \rightarrow F(A')$, there is a cell $a: 1_A^{\circ} \rightarrow 1_{A'}^{\circ}$ such that $F(a) = 1_g^{\circ}$ and both its horizontal arrows $f_i: A \rightarrow A'$ satisfy $F(f_i) = g$.

(c) Finally, we say that F is *representative* (or *essentially surjective on vertical arrows*) if F_1 is: for every vertical arrow $v: B \rightarrow B'$ in \mathbb{B} , there is some vertical arrow $u: A \rightarrow A'$ in \mathbb{A} and some cell $\beta: F(u) \cong v$, horizontally invertible in \mathbb{B} . Again, this implies that also F_0 is essentially surjective on objects.

4.5. Theorem (Characterisations of equivalences). *Let $F: \mathbb{A} \rightarrow \mathbb{B}$ be a pseudo double functor between two horizontally invariant pseudo double categories (1.5). The following conditions are equivalent:*

- (i) $F: \mathbb{A} \rightarrow \mathbb{B}$ is (belongs to) an equivalence of pseudo double categories;
- (ii) F is faithful, full and representative (4.4);
- (iii) the ordinary functor $F_1: \mathbf{A}_1 \rightarrow \mathbf{B}_1$ (between the categories of vertical arrows and double cells) is an equivalence of categories.

Proof. By our previous definitions (4.4a-c), (ii) and (iii) only concern the ordinary functor F_1 , and are well-known to be equivalent. Moreover, if F belongs to an adjoint equivalence $(\eta, \varepsilon): F \rightarrow R$, then F_1 is obviously an equivalence of categories. Conversely, let us assume that F_1 is an equivalence of ordinary categories and let us extend the pseudo double functor F to an adjoint equivalence.

First, also F_0 is an equivalence of categories (by 4.4) and we begin by constructing an adjoint quasi-inverse $R_0: \mathbf{B}_0 \rightarrow \mathbf{A}_0$ in the usual way: choose for

every B some $R(B)$ and some isomorphism $\varepsilon_B: FR(B) \rightarrow B$; then a horizontal morphism $g: B \rightarrow B'$ in \mathbb{B} is sent to the unique \mathbb{A} -map $R(g): R(B) \rightarrow R(B')$ coherent with the previous choices (since F_0 is full and faithful), while the isomorphism $\eta_A: A \rightarrow RF(A)$ is determined by the triangle equations.

We proceed similarly for the next level, taking care that the new choices be consistent with the previous ones: for every $v: B \rightarrow B'$ in \mathbb{B} we want to choose some $R(v): R(B) \rightarrow R(B')$ in \mathbb{A} and some cell $\varepsilon_v: FR(v) \cong v$, horizontally invertible in \mathbb{B} . In fact, we can choose some $u: A \rightarrow A'$ and some $\beta: F(u) \cong v$; but then $F(A) \cong B \cong FR(B)$ and there is some isomorphism $f: R(B) \cong A$, as well as $g: R(B') \cong A'$; now, by *horizontal invariance* of \mathbb{A} , we can choose a horizontally invertible cell λ as in the left square below, and we define $R(v)$ to be its left vertical arrow

$$(1) \quad \begin{array}{ccc} RB & \xrightarrow{f} & A \\ Rv \downarrow & \lambda & \downarrow u \\ RB' & \xrightarrow{g} & A' \end{array} \quad \begin{array}{ccccc} FRB & \xrightarrow{Ff} & FA & \longrightarrow & B \\ FRv \downarrow & F\lambda & \downarrow Fu & \beta & \downarrow v \\ FRB' & \xrightarrow{Fg} & FA' & \longrightarrow & D \end{array}$$

finally, we define $\varepsilon_v = (F\lambda \mid \beta): FR(v) \cong v$, as in the right diagram above.

Now, a cell $b: v \rightarrow v'$ in \mathbb{B} is sent to the unique \mathbb{A} -cell $R(b): R(v) \rightarrow R(v')$ satisfying the naturality condition for ε : $(FRb \mid \varepsilon v') = (\varepsilon v \mid b)$ (since F_1 is full and faithful), and the isocell $\eta_u: u \rightarrow RF(u)$ is determined by the triangle equations.

We still have to define the comparison cells ρ of R , for vertical identities and composition. These are uniquely determined by their coherence conditions (3.2)

$$(2) \quad (F\rho_B \mid \varepsilon 1_B^\bullet) = (\varphi RB \mid 1_{\varepsilon_B}^\bullet), \quad (F\rho(v, v') \mid \varepsilon v'') = (\varphi(Rv, Rv') \mid (\varepsilon v \otimes \varepsilon v')).$$

One ends by proving that R is indeed a pseudo double functor and that also ε is coherent with the comparison cells of F and R :

$$(3) \quad (\eta 1_A^\bullet \mid R\varphi A) = (1_{\eta_A}^\bullet \mid \rho FA), \quad (\eta u'' \mid R\varphi(u, u')) = ((\eta u \otimes \eta u') \mid \rho(Fu, Fu')).$$

□

5. Examples

Various double adjunctions show the role and necessity of (co)lax comparisons.

5.1. Posets and metric spaces. We know that the double category $\mathbb{P}os = \mathbf{2}\text{-Cat}$ of preordered sets, monotone mappings and poset-profunctors has a canonical embedding $M: \mathbb{P}os \rightarrow \mathbf{M}tr$ in the double category $\mathbf{M}tr = \mathbf{R}_+\text{-Cat}$ of (generalised) metric spaces [La], weak contractions and profunctors, identifying a preordered set with a metric space having distance in $\{0, +\infty\}$, and similarly for profunctors (I.3.3). It is easy to show that this embedding is *reflective and lax coreflective*, i.e. has a left adjoint (reflector) P and a lax right adjoint (lax coreflector) Q

$$(1) \quad M: \mathbb{P}os \rightleftarrows \mathbf{M}tr : P, Q, \quad P \dashv M \dashv Q.$$

In fact, the functors M, P, Q are produced by a reflective and coreflective embedding, at the level of the bases $\mathbf{2}$ and (\mathbf{R}_+, \geq) , realised by strictly monoidal functors $p \dashv m \dashv q$

$$(2) \quad \begin{array}{ll} m: \mathbf{2} \rightarrow \mathbf{R}_+, & 0 \mapsto +\infty, \quad 1 \mapsto 0, \\ p: \mathbf{R}_+ \rightarrow \mathbf{2}, & \lambda \mapsto 1, \quad +\infty \mapsto 0 \quad (\text{for } \lambda < +\infty), \\ q: \mathbf{R}_+ \rightarrow \mathbf{2}, & 0 \mapsto 1, \quad \lambda \mapsto 0 \quad (\text{for } \lambda > 0), \\ pm = qm = 1, & mq \geq 1 \geq mp: \mathbf{R}_+ \rightarrow \mathbf{R}_+. \end{array}$$

The 'functions' M, P, Q act likewise on objects and similarly on profunctors (whereas they 'do not modify' the horizontal arrows)

$$(3) \quad \begin{array}{l} d_{MX}(x, x') = 0 \iff x \prec x', \\ y \prec_{PY} y' \iff d(y, y') < +\infty, \quad y \prec_{QY} y' \iff d(y, y') = 0, \end{array}$$

but M and P preserve (also) the vertical composition and are double functors (since p and m preserve colimits, hence coends), while Q is unitary lax.

5.2. Sets and categories. At the level of 1-categories, there is a chain of ordinary adjunctions between \mathbf{Cat} and \mathbf{Set}

$$(1) \quad F: \mathbf{Cat} \rightarrow \mathbf{Set}, \quad \pi_0 \dashv D \dashv F \dashv C,$$

where $F = \text{Ob}$ is the forgetful functor of *objects*, D and C , respectively, associate to a set X its *discrete* category DX (one identity arrow 1_x , for each $x \in X$) and its *codiscrete* category CX (having precisely one morphism $x \rightarrow x'$ for each pair of elements of X); π_0 takes a category A to its set of connected components.

Extending this chain to the pseudo double categories \mathbf{Cat} (categories, functors and profunctors) and \mathbf{Set} (sets, mappings and spans), we find a colax/strict adjunction $\pi_0 \dashv D$ and a strict/lax adjunction $D \dashv F$. First – viewing a span as a

profunctor of discrete categories – the discrete embedding $D: \mathbf{Set} \rightarrow \mathbf{Cat}$ is a strict double functor.

To define its (unitary) colax left adjoint π_0 on profunctors, the set $(\pi_0 u)(\bar{a}, \bar{b})$ is a quotient of $\Sigma u(a, b)$, under the equivalence relation generated by identifying a formal arrow $\lambda: a \rightarrow b$ of u with all composites $\beta\lambda\alpha: a' \rightarrow a \rightarrow b \rightarrow b'$. Then, the lax right adjoint $F = \text{Ob}: \mathbf{Cat} \rightarrow \mathbf{Set}$ of D assigns to a profunctor $u: A \rightarrow B$ its restriction $Fu: \text{Ob}A \rightarrow \text{Ob}B$; F is *not unitary*, as the vertical unit u of a category A is taken to the set-profunctor $Fu: \text{Ob}A \rightarrow \text{Ob}A$ having $Fu(x, y) = A(x, y)$, with a comparison cell $\varphi_A: 1_{FA}^\bullet \rightarrow F1_A^\bullet$, which is invertible if and only if A is discrete. (This fact will appear, in 6.3, to be linked with the fact that D does not preserve cotabulators: in fact, cotabulators in \mathbf{Set} are quotients of the corresponding ones in \mathbf{Cat} ; cf. I.6.4). Finally, F cannot have a (lax) right adjoint because it is not pseudo (Theorem 4.3).

5.3. Spans and cospans. (a) The pseudo double categories $\mathbf{Set} = \mathbf{SpSet}$ and $\mathbf{CospSet}$ of spans and cospans over \mathbf{Set} (or any other category with pullbacks and pushouts) are linked by an obvious colax/lax adjunction, which is unitary

$$(1) F: \mathbf{SpSet} \rightleftarrows \mathbf{CospSet} : R, \quad \eta: 1 \rightarrow RF, \quad \varepsilon: FR \rightarrow 1.$$

At the level 0 (of sets and mappings), everything is an identity. At the level 1 (of vertical arrows and cells), F operates by pushout over spans and cells, and R by pullbacks; the special cells ηu and εv are obvious:

$$(2) \quad U \begin{array}{ccc} & \xrightarrow{u'} & \bullet \\ & \nearrow & \searrow \\ \bullet & \xrightarrow{-\eta u} & \bullet \\ & \searrow & \nearrow \\ & \xrightarrow{u''} & \bullet \end{array} \quad \begin{array}{ccc} & \xrightarrow{\quad} & \bullet \\ & \nearrow & \searrow \\ \bullet & \xrightarrow{\quad} & \bullet \\ & \searrow & \nearrow \\ & \xrightarrow{\quad} & \bullet \end{array} \quad \begin{array}{ccc} & \xrightarrow{\quad} & \bullet \\ & \nearrow & \searrow \\ \bullet & \xrightarrow{\quad} & \bullet \\ & \searrow & \nearrow \\ & \xrightarrow{\quad} & \bullet \end{array} \quad \begin{array}{ccc} & \xrightarrow{\quad} & \bullet \\ & \nearrow & \searrow \\ \bullet & \xrightarrow{\quad} & \bullet \\ & \searrow & \nearrow \\ & \xrightarrow{\quad} & \bullet \end{array} \quad \begin{array}{ccc} & \xrightarrow{v'} & \bullet \\ & \nearrow & \searrow \\ \bullet & \xrightarrow{-\varepsilon v} & \bullet \\ & \searrow & \nearrow \\ & \xrightarrow{v''} & \bullet \end{array} \quad V$$

Finally, at the level 2, it is easy to check that F is a colax double functor (dually, R is lax), with comparison cells $\varphi: F(u \circ v) \rightarrow Fu \otimes Fv$ for vertical composition given by the natural mapping from the pushout of $(u'z', v''z'')$, to the cospan $Fu \otimes Fv$.

The factorisation 3.7.1 can be realised replacing the two double commas with an (isomorphic) double category \mathbf{A}

$$(3) \text{ SpSet} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{P} \end{array} A \begin{array}{c} \xrightarrow{Q} \\ \xleftarrow{R'} \end{array} \text{ CospSet} \quad F = QF', \quad R = PR',$$

whose category of objects and horizontal morphisms is **Set**, while a vertical arrow is a commutative square of mappings.

(b) Now, let **C** be a 2-category with 2-pullbacks, 2-pushouts, comma and cocomma squares. **SpC** and **CospC** are defined as usual (and only depend on the 1-dimensional structure of **C**), but cocommas and commas provide a second colax/lax adjunction

$$(4) C : \text{SpC} \rightleftarrows \text{CospC} : K, \quad \eta : 1 \rightarrow KC, \quad \varepsilon : CK \rightarrow 1.$$

where *neither C nor K* are unitary.

5.4. Relations for abelian categories. Every (well powered) abelian category **A** has a (locally ordered) 2-category of relations **RelA**.

A relation has *binary factorisations* $u = ba^\# : A \twoheadrightarrow B$, where a and b are morphisms of **A**. Such a factorisation will be said to be *strict* if the pair (a, b) is jointly monic (corresponding to a subobject of $A \oplus B$). Dually, there are *cobinary factorisations* $u = b^\#a'$, *strict* if (a', b') is jointly epi (corresponding to a quotient of $A \oplus B$). Two cobinary factorisations yield the same relation u if and only if they have the same pullback, which is then a *strict* binary factorisation of u .

This produces a (flat) double category **RelA**, with **A**-morphisms as horizontal arrows and **A**-relations as vertical ones. A cell corresponds to an inequality $gu \leq vf$, as in the left diagram below (equivalently, $fu^\# \leq v^\#g$)

$$(1) \begin{array}{ccc} A & \xrightarrow{f} & A' \\ u \downarrow & \leq & \downarrow v \\ B & \xrightarrow{g} & B' \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ \uparrow & & \uparrow \\ \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & B' \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \\ \uparrow & & \uparrow \\ B & \xrightarrow{g} & B' \end{array}$$

and amounts to a commutative **A**-diagram for strict binary factorisations (as in the central diagram above), or equivalently, for strict cobinary ones (as at the right).

A *left* exact functor $R : A \rightarrow A'$ between abelian categories preserves all pullbacks, which become *bicommutative* squares in **RelA'**. Therefore, we can extend

it to relations by using, equivalently, a *strict binary* factorisation $u = ba^\#$ or an *arbitrary cobinary* factorisation $u = b^\#a$

$$(2) \quad R'(ba^\#) = (Rb)(Ra)^\#, \quad R'(b^\#a) = (Rb)^\#(Ra);$$

note that the right-hand parts above are *not* strict factorisations, generally. Thus, we get a unitary *lax* double functor $R' = \mathbb{R}el(R): \mathbb{R}elA \rightarrow \mathbb{R}elA'$. Dually, a *right exact* functor F is extended – equivalently – via *binary* or *strict cobinary* factorisations, and yields a *colax* double functor $\mathbb{R}el(F)$. An *exact* functor, extended via binary or cobinary factorisations, gives a *pseudo* double functor.

Now, given an arbitrary adjunction between abelian categories

$$(3) \quad F: A \rightleftarrows B : R, \quad \eta: 1 \rightarrow RF, \quad \varepsilon: FR \rightarrow 1,$$

we get a colax and a lax extension, respectively

$$(4) \quad \mathbb{R}el(F): \mathbb{R}elA \rightarrow \mathbb{R}elB \text{ (colax),} \quad \mathbb{R}el(R): \mathbb{R}elB \rightarrow \mathbb{R}elA \text{ (lax),}$$

forming a colax/lax adjunction (which is pseudo/lax if F is exact, and dually). The unit ηu is the pasting of two cells, depending on an arbitrary binary factorisation $u = ba^\#$

$$(5) \quad \begin{array}{ccccc} A & \xrightarrow{\eta^A} & RFA & \equiv & RFA & & u = ba^\#, \\ u \downarrow & \leq & \downarrow w & \leq & \downarrow R'Fu & & \\ B & \xrightarrow{\eta^B} & RFB & \equiv & RFB & & w = (RFb)(RFa)^\#, \end{array}$$

the right one coming from the colax-property of R' and the definition of F'

$$(6) \quad w = (RFb)(RFa)^\# \leq R'((Fb)(Fa)^\#) = R'Fu.$$

The counit is defined dually; the coherence relations follow from flatness.

It is easy to exhibit an adjunction where both our extensions are not pseudo

$$(7) \quad F: \mathbf{Ab} \rightleftarrows \mathbf{Ab} : R, \quad F = -\otimes \mathbb{Z}/2, \quad R = \text{Hom}(\mathbb{Z}/2, -).$$

Indeed, the canonical projection $p: \mathbb{Z} \rightarrow \mathbb{Z}/2$ gives $pp^\# = 1_{\mathbb{Z}/2}$ but $Rp = 0$, while the monomorphism $m = 2.-: \mathbb{Z} \rightarrow \mathbb{Z}$ has $m^\#m = 1_{\mathbb{Z}}$ and $Fm = 0$. We have also shown that one cannot compute R' using the binary, non-strict factorisation $1 = pp^\#$.

5.5. Monoidal closedness. We will only sketch the subject. Let us start considering an ordinary monoidal closed category A , with endofunctors $-\otimes Y \rightarrow$

$\text{Hom}(Y, -)$ for every object Y , and see what we get on some (pseudo) double categories constructed on \mathbb{A} .

(a) First, the double category $\mathbb{A} = \square \mathbb{A}$ of commutative squares of \mathbb{A} is also monoidal closed, in a *strict* sense. In fact, the endofunctors $-\otimes Y, \text{Hom}(Y, -): \mathbb{A} \rightarrow \mathbb{A}$ obviously extend to strict double functors $\mathbb{A} \rightarrow \mathbb{A}$, and the natural bijection $\mathbb{A}(X \otimes Y, Z) \rightarrow \mathbb{A}(X, \text{Hom}(Y, Z))$, $f \mapsto f'$ extends to a bijection between categories of vertical arrows

$$(1) \mathbb{A}_1(u \otimes Y, w) \rightarrow \mathbb{A}_1(u, \text{Hom}(Y, w)), \quad (f, g) \mapsto (f', g')$$

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{f} & Z \\ u \otimes Y \downarrow & & \downarrow w \\ X' \otimes Y & \xrightarrow{g} & Z' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & \text{Hom}(Y, Z) \\ u \downarrow & & \downarrow \text{Hom}(Y, w) \\ X' & \xrightarrow{g'} & \text{Hom}(Y, Z') \end{array}$$

since the left square commutes if and only if the right one does, by naturality.

(b) Second, if \mathbb{A} has pullbacks, the pseudo double category $\mathbb{A} = \text{Sp} \mathbb{A}$ of morphisms and spans inherits a family of *colax/pseudo* adjunctions $-\otimes Y \dashv \text{Hom}(Y, -): \mathbb{A} \rightarrow \mathbb{A}$. In fact, every $\text{Hom}(Y, -): \mathbb{A} \rightarrow \mathbb{A}$ preserves pullbacks (by adjunction) and extends to a pseudo double endofunctor of \mathbb{A} , while $-\otimes Y: \mathbb{A} \rightarrow \mathbb{A}$ need not preserve them and gives a colax double endofunctor. (A cartesian product preserves pullbacks, but the tensor product of abelian groups does not.) Finally, since a cell of spans is formed of a pair of commutative squares, the 'same' argument as in (1) applies.

(c) Dually, if the cartesian closed category \mathbb{A} has pushouts, then the pseudo double category $\mathbb{A} = \text{Cosp} \mathbb{A}$ of morphisms and cospans has a family of *pseudo/lax* adjunctions $-\otimes Y \dashv \text{Hom}(Y, -)$.

Now, take a pseudo double category \mathbb{A} , equipped with a *monoidal* structure, i.e. an identity object I , a *colax* tensor product $-\otimes -: \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$ and the usual coherence framework, formed of invertible horizontal transformations (formally, all this is a pseudo monoid in the strict 2-category CxDbf , 2.2). We say that \mathbb{A} is *weakly monoidal closed* if each colax double functor $-\otimes Y: \mathbb{A} \rightarrow \mathbb{A}$ has a lax right adjoint. (Note that, fixing one variable, we *do* get a colax double functor, by our unitarity assumption on \mathbb{A} .)

The cartesian case is necessarily simpler, since a cartesian product – as a limit – is automatically lax as soon as it works for vertical arrows. More precisely, let \mathbb{A} be a pseudo double category with a lax functorial choice of binary products (I.4.3-

4), as it happens for all of our profunctor based examples (I.6). We say that \mathbb{A} is *lax cartesian closed* if every lax double functor $- \times Y: \mathbb{A} \rightarrow \mathbb{A}$ has a lax right adjoint $(-)^Y: \mathbb{A} \rightarrow \mathbb{A}$, forming a pseudo/lax adjunction ($- \times Y$ is automatically pseudo).

6. Limits and adjoints

We deal here with the relations between double limits and double adjoints.

6.1. The importance of unitarity. Limits and colimits are well behaved with *unitary* adjunctions (Theorem 6.2). Loosely speaking, there is an evident motivation for this: in a pseudo double category \mathbb{A} , an object amounts to a (strict) double functor $A: \underline{1} \rightarrow \mathbb{A}$ defined on the 'singleton', and a vertical arrow to a double functor $u: \underline{2} \rightarrow \mathbb{A}$ defined on the 'formal vertical arrow' $0 \rightarrow 1$ (all other arrows of $\underline{2}$ being identities). Now, composing with a *unitary* (co)lax double functor $\mathbb{A} \rightarrow \mathbb{B}$ does preserve such things (while a general lax one would produce a vertical monad, on each object); therefore, *such compositions preserve cones and their vertical transformations*.

We shall see that unitarity is 'nearly' necessary, in order that a right adjoint preserve tabulators (6.3). And indeed, we have already encountered a strict double functor which does not preserve cotabulators, yet has a (non-unitary) lax right adjoint (5.2; see also 5.3b). Finally, the terminal object of a double category amounts to a *unitary* right adjoint of the terminal double functor $\mathbb{A} \rightarrow \underline{1}$ (6.4), and – at least for strict double categories – a lax functorial choice of *I-limits* in \mathbb{A} (I.4.3-4) amounts to a unitary lax double functor $L: \mathbb{A}^I \rightarrow \mathbb{A}$ right adjoint to the diagonal (6.5).

Before going on, it will be useful to reformulate the definition of double limits in terms of cells in \mathbb{Dbl} . Given a lax double functor $T: \mathbb{I} \rightarrow \mathbb{B}$, a cone (A, x) for it amounts to a cell x as in the left diagram

$$(1) \quad \begin{array}{ccccc} \mathbb{I} & \xrightarrow{Q} & \underline{1} & \equiv & \underline{1} \\ \parallel & \searrow x & \downarrow A & \swarrow t & \downarrow A' \\ \mathbb{I} & \xrightarrow{T} & A & \equiv & A \end{array} \qquad \begin{array}{ccccc} \mathbb{I} \times \underline{2} & \xrightarrow{Q} & \underline{2} & \equiv & \underline{2} \\ P \downarrow & \searrow \lambda & \downarrow & & \downarrow u \\ \mathbb{I} & \xrightarrow{\quad} & \underline{1} & & \downarrow \\ \parallel & \searrow x & \downarrow A & \swarrow \tau & \downarrow \\ \mathbb{I} & \xrightarrow{T} & A & \equiv & A \end{array}$$

(i.e., a horizontal transformation $x: AQ \rightarrow T$ of lax 'functors'). Now, (A, x) is a 1-dimensional limit of T if (i) holds, and the double limit if also (ii) does:

(i) for every object A' in \mathbb{A} , the mapping $t \mapsto x' = (x | t)$ (a horizontal composition of cells, in \mathbb{Dbl}) gives a bijection between horizontal maps $t: A' \rightarrow A$ in \mathbb{A} and cones (A', x') ;

(ii) for every vertical arrow $u: A' \rightarrow A''$ in \mathbb{A} , the mapping $\tau \mapsto (\lambda \otimes x | a)$ gives a bijection between \mathbb{A} -cells $\tau: (u \begin{smallmatrix} x' \\ x'' \end{smallmatrix} 1_A)$ and \mathbb{Dbl} -cells whose boundary is the outer perimeter of the right diagram above (λ being the obvious 'commutative cell').

6.2. Theorem (Preservation of limits). *Take a colax/lax adjunction $(\eta, \varepsilon): F \dashv R$.*

(a) *The lax functor $R: \mathbb{B} \rightarrow \mathbb{A}$ preserves all (existing) 1-dimensional double limits of horizontal lax double functors $T: \mathbb{I} \rightarrow \mathbb{B}$.*

(b) *If both functors are unitary, then $R: \mathbb{B} \rightarrow \mathbb{A}$ preserves all (existing) double limits of unitary lax double functors $T: \mathbb{I} \rightarrow \mathbb{B}$.*

Proof. (a). Obvious: our limit is just an ordinary limit in the category of objects and horizontal arrows of \mathbb{B} .

(b). To prove that, if $(B, y: BQ \rightarrow T)$ is a limit of T , then $R(B, y) = (RB, Ry)$ is a limit of RT , we follow the standard procedure, for the 1-dimensional property, as rewritten above (6.1i); the verification of the 2-dimensional property is similar.

Given a cone $(A, x: AQ \rightarrow RT)$ of RT , the pasting y' of \mathbb{Dbl} -cells displayed at the left (flipping R to its vertical adjoint F , in \mathbb{Dbl} ; cf. 1.6.)

$$(1) \quad \begin{array}{c} \mathbb{I} \xrightarrow{Q} \mathbb{1} \\ \parallel \\ \mathbb{I} \xrightarrow{T} \mathbb{B} \xrightarrow{R} \mathbb{A} \\ \parallel \quad \parallel \\ \mathbb{I} \xrightarrow{T} \mathbb{B} \end{array} \begin{array}{c} \xrightarrow{x} \downarrow A \\ \xrightarrow{\varepsilon} \downarrow F \end{array} = \begin{array}{c} \mathbb{I} \xrightarrow{Q} \mathbb{1} = \mathbb{1} \\ \parallel \quad \downarrow A \\ \mathbb{I} \xrightarrow{T} \mathbb{B} \downarrow \\ \parallel \quad \downarrow F \end{array}$$

gives a cone $(FA, y': F(A).Q \rightarrow T)$ of T (by the unitarity of F). Therefore, there is a unique $t': FA \rightarrow B$ such that $y' = (y | t')$: $F(A).Q \rightarrow T$. Now, the adjoint morphism $t = Rt'.\eta_A: A \rightarrow RFA \rightarrow RB$ is the unique horizontal arrow of \mathbb{A} such that $x = (Ry | t): AQ \rightarrow RT$ (by pasting the \mathbb{Dbl} -cell $\eta: (F_R^1 1)$ at the right of both diagrams above, to 'flip back' F to R). \square

6.3. Theorem (Adjoints and unitality). *Take a general colax/lax adjunction between pseudo double categories*

$$(1) \quad F : \mathbb{A} \rightleftarrows \mathbb{B} : R, \quad (\eta, \varepsilon): F \dashv R.$$

(a) *If F is pseudo unitary, then R preserves all the existing (1-dimensional) tabulators of \mathbb{B} : given a vertical arrow $v: B \rightarrow B'$ having tabulator $\pi: (\nu \underset{q}{\overset{p}{\dashv}} v)$, the object RV is the tabulator object of Rv in \mathbb{A} , via the obvious cell*

$$(2) \quad \tau = (R[V] | R\pi): (1_{RV}^{\bullet} \underset{q}{\overset{p}{\dashv}} Rv).$$

(b) *Take A in \mathbb{A} and assume that $v = F(1_A^{\bullet}): FA \rightarrow FA$ has a tabulator in \mathbb{B} , preserved by R ; then the colaxity cell $\varphi_A: F(1_A^{\bullet}) \rightarrow 1_{FA}^{\bullet}$ has a horizontal retraction $\zeta: 1_{FA}^{\bullet} \rightarrow F(1_A^{\bullet})$, giving $(\varphi | \zeta) = 1$.*

(There are examples of right adjoints which do not preserve tabulators, cf. 5.2; as well as of non-unitary right adjoints which do, cf. 6.4).

Proof. Write $\rho_B: 1_{RB}^{\bullet} \rightarrow R(1_B^{\bullet})$ and $\varphi_A: F(1_A^{\bullet}) \rightarrow 1_{FA}^{\bullet}$ the laxity and colaxity cells of R, F .

(a) Assume that φ_A has a horizontal inverse ζ . We want to prove that RV is the tabulator of Rv via τ . Take an \mathbb{A} -cell $\alpha: (A \underset{g}{\overset{f}{\dashv}} RV)$, and its adjoint cell $\alpha' = (F\alpha | \varepsilon v): (F1_{A'}^{\bullet} \underset{g'}{\overset{f'}{\dashv}} v)$. There is precisely one map $k: FA \rightarrow V$ such that $(k | \pi) = (\zeta | \alpha')$; its adjoint $h = Rk.\eta.A: A \rightarrow RV$

$$(3) \quad \begin{array}{ccccc} FA & \overset{k}{\dashrightarrow} & V & \xrightarrow{p} & B & & A & \overset{h}{\dashrightarrow} & RV & \equiv & RV & \xrightarrow{Rp} & RB \\ 1 \downarrow & & 1 \downarrow & & \pi & & 1 \downarrow & & 1 \downarrow & & \rho & \downarrow R1^{\bullet} & R\pi & \downarrow Rv \\ FA & \overset{k}{\dashrightarrow} & V & \xrightarrow{q} & B' & & A & \overset{h}{\dashrightarrow} & RV & \equiv & RV & \xrightarrow{Rq} & RB' \end{array}$$

satisfies the condition $(h | \tau) = \alpha$:

$$(4) \quad \begin{aligned} (h | \tau) &= (h | \rho V | R\pi) = (\eta A | Rk | \rho V | R\pi) = (\eta A | \rho FA | R1_k^{\bullet} | R\pi) \\ &= (\eta(1_A^{\bullet}) | R(\varphi A) | R\zeta | R\alpha') = (\eta(1_A^{\bullet}) | R\alpha') = \alpha. \end{aligned}$$

Conversely, if $(h | \tau) = \alpha$, then the adjoint $k = \varepsilon V.Fh: FA \rightarrow V$ has

$$(5) \quad \begin{aligned} (\varphi A | 1_k^{\bullet} | \pi) &= (\varphi A | 1_{Fh}^{\bullet} | 1_{\varepsilon V}^{\bullet} | \pi) = (F1_h^{\bullet} | \varphi R(V) | 1_{\varepsilon V}^{\bullet} | \pi) \\ &= (F1_h^{\bullet} | F\rho V | \varepsilon 1_v^{\bullet} | \pi) = (F1_h^{\bullet} | F\rho V | FR\pi | \varepsilon v) = (F1_h^{\bullet} | F\tau | \varepsilon v) \\ &= (F\alpha | \varepsilon v) = \alpha'. \end{aligned}$$

This means that the cell $(\varphi_A | 1_k^\bullet)$ is uniquely determined by α . Since φ_A is invertible, k is uniquely determined as well.

(b) Take any object A in \mathbb{A} . Recall that the colaxity cell $\varphi = \varphi_A: F(1_A^\bullet) \rightarrow 1_{FA}^\bullet$ corresponds to $\varphi' = (1_{\eta_A}^\bullet | \rho FA)$ (3.3a). In the diagram (3), let $B = B' = FA$ and $v = F(1_A^\bullet): FA \rightarrow FA$, with tabulator $V, \pi: (v \overset{p}{q} v)$ preserved by R : thus, RV is the tabulator of Rv via $\tau = (\rho V | R\pi)$.

The unit of the adjunction yields a cell $\alpha = (1_v)' = \eta(1_A^\bullet): (A \overset{\eta_A}{\eta_A} RF(1_A^\bullet))$, whence one map $h: A \rightarrow RV$ in \mathbb{A} such that $(1_h^\bullet | \tau) = \eta(1_A^\bullet)$. The map h corresponds to $k: FA \rightarrow V$ and it suffices to verify that the cell $\zeta = (1_k^\bullet | \pi): 1_{FA}^\bullet \rightarrow v$ is a retraction of φ :

$$\begin{aligned} (6) \quad (\varphi | k | \pi)' &= (\varphi' | R1_k^\bullet | R\pi) = (1_{\eta_A}^\bullet | \rho FA | R1_k^\bullet | R\pi) \\ &= (1_{\eta_A}^\bullet | 1_{Rk}^\bullet | \rho V | R\pi) = (1_h^\bullet | \tau) = \eta(1_A^\bullet) = (1_v)'. \quad \square \end{aligned}$$

6.4. Terminal object. Let us consider the existence of a right adjoint to the diagonal $D: \mathbb{A} \rightarrow \mathbb{A}^{\underline{Q}} = \underline{1}$ (where \underline{Q} is the empty double category).

A lax double functor $R: \underline{1} \rightarrow \mathbb{A}$ amounts to a monad in the vertical 2-category \mathbb{VA} : an object $Z = R(0)$, a vertical arrow $z = R(v_0): Z \rightarrow Z$ and two special cells $\eta: 1_Z^\bullet \rightarrow z, \mu: z \otimes z \rightarrow z$ satisfying the monad axioms.

According to 3.6, D has a (lax) right adjoint R if and only if

- (T.1) there is an object $Z (= R(0))$ such that, for each A in \mathbb{A} , there is a unique horizontal arrow $t: A \rightarrow Z$ (written also t_A);
- (T.2) there is a vertical arrow $z: Z \rightarrow Z (= R(v_0))$ such that, for each vertical arrow $u: A \rightarrow A'$ in \mathbb{A} , there is a unique cell $\tau: (u \overset{f}{f} z)$ (written also τ_A).

The special cells $\eta = R[0]: 1_Z^\bullet \rightarrow z$ and $\mu = R[v_0, v_0]: z \otimes z \rightarrow z$ are then provided by the universal property of z , taking u equal to 1_Z^\bullet or $z \otimes z$, respectively. Note also that the 2-dimensional property (T.2) implies (T.1): apply it to 1_A^\bullet .

We get a more general notion of *double terminal* than we used in Part I: the latter amounts to a *unitary* lax right adjoint ($z = 1_Z^\bullet, \eta = 1$).

In fact, there is an interesting case, the double category \mathbb{Rng} of unitary rings, homomorphisms and bimodules, where this *generalised initial object* $I: \underline{1} \rightarrow \mathbb{Rng}$ is not unitary: the left adjoint $I \dashv D$ is the ring of integers equipped with the zero bimodule $I(1_\ast^\bullet) = 0: \mathbb{Z} \rightarrow \mathbb{Z}$. Comparing with Theorem 6.3, the right adjoint $D: \mathbb{Rng} \rightarrow \underline{1}$ obviously preserves the existing tabulators, and in particular the tabulator $\mathbb{Z} \times \mathbb{Z}$ of $I(1_\ast^\bullet)$; this agrees with the fact that the unit comparison $I(1_\ast^\bullet) =$

$0 \rightarrow 1_{I*}^{\bullet} = 1_Z^{\bullet}$ has a retraction. More generally, the *generalised sum* in $\mathbb{R}ng$ of J copies (J is a small set) of the vertical identity of the ring of integers $\mathbf{Z}: \mathbf{Z} \rightarrow \mathbf{Z}$ is the free \mathbf{Z} -bimodule $\mathbf{Z}^{(J)}: \mathbf{Z} \rightarrow \mathbf{Z}$; this is never unital, unless J is a singleton.

Other examples come from monoidal categories. Take a monoidal category $\mathbf{V} = (\mathbf{V}, \otimes, I)$, consider it as a bicategory with one object, and then as a double category \mathbb{V} ; the latter has one object $*$, one horizontal arrow, its vertical arrows are the objects of \mathbf{V} and its cells are the maps of \mathbf{V} . Then \mathbb{V} has a terminal object if and only if \mathbf{V} has, which is unitary if and only if I is terminal.

6.5. Functorial limits as unitary adjoints. Let now \mathbb{A} and \mathbb{I} be strict double category, with diagonal double functor $D: \mathbb{A} \rightarrow \mathbb{A}^{\mathbb{I}}$ with values in the double category of double functors, their horizontal and vertical transformations and their modifications. Then, a *lax* (resp. *pseudo*, *strict*) *functorial choice of \mathbb{I} -limits* in \mathbb{A} (I.4.3-4) amounts to the choice of a unitary lax (resp. pseudo, ordinary) double functor $L: \mathbb{A}^{\mathbb{I}} \rightarrow \mathbb{A}$ right adjoint to D .

In fact, according to Theorem 3.6, D has a unitary lax right adjoint L precisely when the conditions of Lemma I.4.4 are satisfied:

(a) every double functor $F: \mathbb{I} \rightarrow \mathbb{A}$ has a 1-dimensional double limit $(LF, \varepsilon F: DLF \rightarrow F)$, satisfying (dl.1),

(b) every vertical transformation $U: F \rightarrow F'$ has a *limit*

$$(1) \quad LU: LF \rightarrow LF', \quad \varepsilon U: (DLU \xrightarrow{EF} EF' U).$$

with respect to the horizontal composition of modifications (as specified in I.4.3), so that $L1_F^{\bullet} = 1_{LF}^{\bullet}$.

If this is the case, every composite $U'' = U \otimes U': F \rightarrow F \rightarrow F''$ of vertical transformations gives a comparison special cell, determined by the universal property of (L, ε)

$$(2) \quad L[U, U']: LU \otimes LU' \rightarrow LU'', \quad (DL[U, U'] \mid \varepsilon U'') = \varepsilon U \otimes \varepsilon U',$$

and our choice L is *pseudo* (or *strict*) if and only if all such special cells are horizontally invertible (or identities, respectively).

7. Double monads

We only treat here the 'standard' cases: *colax* monads (in \mathbf{CxDbI}) have *Eilenberg-Moore* algebras and are linked with *colax/pseudo* adjunctions, while *lax* monads (in

LxDbl) have *Kleisli* algebras and are linked with *pseudo/lax* adjunctions. \mathbb{X} is always a unitary pseudo double category.

7.1. Colax monads and their algebras. A *colax monad* $T = (T, \eta, \mu)$ (on \mathbb{X}) will be a monad in the 2-category **CxDbl** (2.2). Thus, $T: \mathbb{X} \rightarrow \mathbb{X}$ is a *colax* double endofunctor (with comparison cells $T[X]: T1_X^\bullet \rightarrow 1_{TX}^\bullet$, $T[r, s]: T(r\otimes s) \rightarrow Tr\otimes Ts$) and $\eta: 1 \rightarrow T$, $\mu: T^2 \rightarrow T$ are horizontal transformations (I.7.3) satisfying the usual unit and associativity axioms: $\mu \cdot \eta T = 1 = \mu \cdot T\eta$, $\mu \cdot \mu T = \mu \cdot T\mu$.

For instance, any *colax/pseudo* adjunction $F \dashv U$, being precisely an adjunction in **CxDbl**, produces a colax monad $T = UF$ on the domain of F .

Given T , one can construct the pseudo double category \mathbb{X}^T of (*Eilenberg-Moore*) T -algebras:

- the 1-dimensional horizontal part is as usual; a morphism will be written as $(X, x: TX \rightarrow X)$, $f: (X, x) \rightarrow (X', x')$;
- an *algebraic arrow* $(r, \rho): (X, x) \dashv\dashv (Y, y)$ is a vertical arrow $r: X \dashv\dashv Y$ in \mathbb{X} together with a *structure* $\rho: (Tr \overset{x}{\underset{y}{\rceil}})$ satisfying the axioms $(\eta r \mid \rho) = 1: r \rightarrow r$ and $(\mu r \mid \rho) = (T\rho \mid \rho): T^2r \rightarrow r$;
- its vertical composition with $(s, \sigma): (Y, y) \dashv\dashv (Z, z)$ is $(r\otimes s, (T[r, s] \mid \rho\otimes\sigma))$, where the composite $\rho\otimes\sigma$ is 'corrected' by the comparison cell $T[r, s]$, as in the left diagram below

$$\begin{array}{ccc}
 TX & = & TX \longrightarrow X \\
 \downarrow T(r\otimes s) & & \begin{array}{ccc} Tr \downarrow & \rho & \downarrow r \\ TY & \longrightarrow & Y \\ Ts \downarrow & \sigma & \downarrow s \end{array} \\
 TZ & = & TZ \longrightarrow Z
 \end{array}
 \qquad
 \begin{array}{ccc}
 TX & = & TX \xrightarrow{x} X \\
 T1 \downarrow & T[X] \downarrow & 1 \downarrow \quad 1_X^\bullet \downarrow 1 \\
 TX & = & TX \xrightarrow{x} X
 \end{array}$$

similarly, the vertical identity of (X, x) is $(1_X^\bullet, (T[X] \mid 1_X^\bullet))$;

- a cell ξ with the boundary displayed at the left

$$\begin{array}{ccc}
 (X, x) \xrightarrow{f} (X', x') & TX \xrightarrow{x} X \xrightarrow{f} X' & TX \xrightarrow{Tf} TX' \xrightarrow{x'} X' \\
 \downarrow (r, \rho) \quad \xi \quad (r', \rho') \downarrow & Tr \downarrow \quad \rho \quad \downarrow r \quad \xi \quad \downarrow = \downarrow & T\xi \quad \downarrow \quad \rho' \quad \downarrow r' \\
 (Y, y) \xrightarrow{g} (Y', y') & TY \xrightarrow{y} Y \xrightarrow{g} Y' & TY \xrightarrow{Tg} TY' \xrightarrow{y'} Y'
 \end{array}$$

comes from a cell $\xi: (r \begin{smallmatrix} f \\ g \end{smallmatrix} r')$ of \mathbb{X} such that $(\rho | \xi) = (T\xi | \rho')$.

The horizontal and vertical composition of cells are defined by the same operations in \mathbb{X} . It will be useful to note that the cell above, in (2), is horizontally invertible in \mathbb{X}^T if and only if its underlying cell ξ is so in \mathbb{X} . In fact, in this case, $(\rho' | \xi^{-1}) = (T\xi^{-1} | \rho)$.

7.2. Colax monads and colax adjunctions. We have already noted that a colax/pseudo adjunction produces a colax monad, in the usual way.

On the other hand, a colax monad $T = (T, \eta, \mu)$ (on \mathbb{X}) produces a *colax/strict* adjunction

$$(1) \quad F^T: \mathbb{X} \rightleftarrows \mathbb{X}^T: U^T, \quad F^T \dashv U^T,$$

by extending the usual procedure to the vertical structure. U^T is the obvious projection, $U^T(X, x) = X$ etc. (a strict double functor). F^T is extended to vertical arrows by the cells $\mu(r)$, while its colaxity cells $F^T[r, s]$ come from the ones of T (the cells $F^T[X]$ are not written down)

$$(2) \quad F^T: \mathbb{X} \rightarrow \mathbb{X}^T,$$

$$\begin{aligned} F^T(X) &= (TX, \mu_X: T^2X \rightarrow TX), & F^T(f) &= Tf, \\ F^T(r: X \twoheadrightarrow Y) &= (Tr, \mu(r)), & \mu(r): (T^2r \begin{smallmatrix} \mu_X \\ \mu_Y \end{smallmatrix} Tr), \\ F^T(\xi) &= T\xi: ((Tr, \mu r) \begin{smallmatrix} T^2f \\ T^2g \end{smallmatrix} (Tr', \mu r')) & ((\mu r | T\xi) &= (T^2\xi | \mu r')), \\ F^T[r, s] &= T[r, s]: (T(w), \mu(w)) \rightarrow (Tr, \mu(r)) \otimes (Ts, \mu(s)) & (w = r \otimes s), \end{aligned}$$

$$(3) \quad \begin{array}{ccccccc} T^2X & \xrightarrow{\mu_X} & TX & = & TX & & T^2X & = & T^2X & = & T^2X & \xrightarrow{\mu_X} & TX \\ T^2w \downarrow & \mu_w \downarrow & T^2w \downarrow & T[r,s] \downarrow & = & \downarrow & T(T[r,s]) \downarrow & \downarrow & T(Tr, Ts) \downarrow & \downarrow & \mu_r \otimes \mu_s \downarrow & T^2w \downarrow & Tr \otimes Ts \\ T^2Z & \xrightarrow{\mu_Z} & TZ & = & TZ & & T^2Z & = & T^2Z & = & T^2Z & \xrightarrow{\mu_Z} & TZ \end{array}$$

The unit $\eta: 1 \rightarrow U^T F^T = T$ coincides with the unit of T ; the counit $\varepsilon: F^T U^T \rightarrow 1$ is defined as usual on objects, and similarly on $(r, \rho): (X, x) \twoheadrightarrow (Y, y)$

$$(4) \quad \varepsilon(r, \rho) = \rho: (Tr, \mu(r)) \rightarrow (r, \rho) \quad ((\mu r | \rho) = (T\rho | \rho)).$$

Now, given a colax/pseudo adjunction $F \dashv U$, the associated colax monad $T = UF$ and its pseudo double category of algebras, the comparison $K: \mathbb{A} \rightarrow \mathbb{X}^T$ is defined in the usual way on objects and horizontal arrows, and extended to the vertical structure. We obtain a *pseudo* double functor, with comparison isocells defined by the ones of U (again, $K[A]$ is understood)

$$(5) \quad K: \mathbf{A} \rightarrow \mathbb{X}^T, \quad K(A) = (UA, U\varepsilon_A: TUA \rightarrow UA), \quad K(f) = U(f),$$

$$K(u) = (Uu, U\varepsilon_u), \quad K(\alpha) = U\alpha,$$

$$K[u, v] = U[u, v]: K(u \otimes v) \rightarrow Ku \otimes Kv.$$

The comparison $K[u, v]$ is horizontally invertible, by the last remark in 7.1.

Say that the pseudo double functor $U: \mathbf{A} \rightarrow \mathbb{X}$ is *monadic, or algebraic*, if K is an equivalence of pseudo double categories (4.4–4.5): this implies that the underlying horizontal category $\mathbf{A} = \mathbf{HA}$ is monadic over $\mathbf{X} = \mathbf{HX}$.

In the monoidal case, given a strong monoidal functor $U: \mathbf{A} \rightarrow \mathbf{X}$ having a left adjoint F (automatically comonoidal), the comparison $K: \mathbf{A} \rightarrow \mathbf{X}^T$ is necessarily strong. Therefore, \mathbf{A} is 'monoidally monadic' on \mathbf{X} if and only if it is so in the ordinary sense (of the underlying categories, which sit vertically in the associated pseudo double categories).

7.3. Elementary examples. (a) (*Adjunctions and products*) The standard adjunction $F: \mathbf{Set} \rightleftarrows \mathbf{Ab}: U$ is colax/pseudo monoidal with respect to cartesian product (written \times and \oplus , respectively): the product is preserved by U , while F has an obvious comparison $F(X \times Y) \rightarrow FX \oplus FY$, corresponding to $\eta_X \times \eta_Y: X \times Y \rightarrow UFX \times UFY = U(FX \oplus FY)$. Thus the well-known monad $T = UF$, which describes abelian groups on \mathbf{Set} , is colax with respect to cartesian product, and the comparison

$$(1) \quad K: \mathbf{Ab} \rightarrow \mathbf{Set}^T, \quad K(A) = (|A|, |\varepsilon_A|: |F(|A|)| \rightarrow |A|),$$

preserves it (which is trivial, since K is an isomorphism). Similar facts hold for all categories with finite products, monadic over \mathbf{Set} ; or also for the adjunction $\mathbf{Gp} \rightleftarrows \mathbf{Ab}: U$.

(b) All this can be easily extended to the weak double categories \mathbf{SpSet} , \mathbf{SpAb} (or \mathbf{SpA} , for any category \mathbf{A} with finite limits, monadic over \mathbf{Set}).

7.4. Lax monads and Kleisli algebras. Similarly, a *lax monad* $T = (T, \eta, \mu)$ (on \mathbf{X}) will be a monad in the 2-category \mathbf{LxDbl} (2.2). Thus, $T: \mathbf{X} \rightarrow \mathbf{X}$ is a *lax double endofunctor*, with comparison cells $T[X]: 1_{TX}^\bullet \rightarrow T1_X^\bullet$, $T[r, s]: Tr \otimes Ts \rightarrow T(r \otimes s)$. For instance, any *pseudo/lax adjunction* $F \dashv U$ produces a lax monad $T = UF$ on the domain of F .

Given T , one can construct, in a standard way, the pseudo double category \mathbb{X}_T of *Kleisli T-algebras*:

- objects X and horizontal arrows $\hat{f}: X \rightarrow X'$ are as usual (defined by $f: X \rightarrow TX'$ in \mathbb{X}), as well as their horizontal composition: $\hat{f}' \cdot \hat{f} = (\mu X'' \cdot Tf \cdot f) \hat{}$;
- vertical arrows $r: X \rightarrow Y$ are as in \mathbb{X} , with the same identities and composition;
- a cell $\hat{\xi}: (r \hat{}_g r')$ is defined by a cell $\xi: (r \xrightarrow{f} Tr')$ of \mathbb{X} ;

$$(1) \quad \begin{array}{ccccccc} X & \xrightarrow{f} & TX' & & X & \xrightarrow{f} & TX' & \xrightarrow{Tr'} & T^2X'' & \xrightarrow{\mu X''} & TX'' \\ r \downarrow & \xi & \downarrow Tr' & & r \downarrow & \xi & \downarrow Tr' & T\xi' & \downarrow T^2r'' & \mu r'' & \downarrow Tr'' \\ Y & \xrightarrow{g} & TY' & & Y & \xrightarrow{g} & TY' & \xrightarrow{Tg'} & T^2Y'' & \xrightarrow{\mu Y''} & TY'' \end{array}$$

the horizontal composition of cells is the obvious extension of the one of maps (as in the right diagram above). The vertical one uses the laxity cells of T

$$(2) \quad \hat{\xi} \otimes \hat{\eta} = (\xi \otimes \eta \mid T[r', s']) \hat{}: r \otimes s \rightarrow Tr' \otimes Ts'$$

To verify the middle-four interchange, we get an (outer) diagram of vertical arrows and cells in \mathbb{X} (τ denotes the comparison cells of T)

$$(3) \quad \begin{array}{ccccccc} r \otimes s & \xrightarrow{\xi \otimes \eta} & Tr' \otimes Ts' & \xrightarrow{T\xi \otimes T\eta'} & T^2r'' \otimes T^2s'' & = & T^2r'' \otimes T^2s'' & \xrightarrow{\mu \otimes \mu} & Tr'' \otimes Ts'' \\ & & \tau \downarrow & & \downarrow \tau T & & \downarrow \tau' & & \downarrow \tau \\ & & T(r' \otimes s') & \xrightarrow{T(\xi \otimes \eta')} & T(Tr'' \otimes Ts'') & \xrightarrow{T\tau} & T^2(r'' \otimes s'') & \xrightarrow{\mu} & T(r'' \otimes s'') \end{array}$$

whose commutativity is proved by inserting two middle cells, $\tau T = T[Tr'', Ts'']$ and $\tau' = T^2[r'', s'']$. Then, the left square commutes by naturality of τ on ξ', η' ; the middle one by definition of the comparison cells of T^2 ; the right one by coherence of $\mu: T^2 \rightarrow T$ on r'', s'' .

7.5. Lax monads and lax adjunctions. As already observed, a pseudo/lax adjunction produces a lax monad. On the other hand, a lax monad $T = (T, \eta, \mu)$ (on \mathbb{X}) produces a *strict/lax* adjunction

$$(1) \quad F_T: \mathbb{X} \rightleftarrows \mathbb{X}_T: U_T, \quad F_T \dashv U_T,$$

by extending the usual procedure to the vertical structure. F_T is the obvious embedding, $F_T(X) = X$, $F_T(f) = \eta X' \cdot f$, $F_T(r) = F(r)$, etc. (a strict double functor). U_T is extended to cells by the cells μ , while its laxity cells $U_T[r, s]$ come from the ones of T (the cells $U_T[X]$ are similar)

$$\begin{aligned}
 (2) \quad U_T: \mathbb{X}_T &\rightarrow \mathbb{X}, \\
 U_T(X) &= TX, & U_T(\hat{f}) &= \mu X'. Tf, \\
 U_T(r: X \twoheadrightarrow Y) &= Tr: TX \twoheadrightarrow TY, & U_T(\hat{\xi}) &= (T\xi \mid \mu r')^\wedge: (Tr \begin{smallmatrix} \mu X'. Tf \\ \mu Y'. Tg \end{smallmatrix} Tr'), \\
 U_T[r, s] &= (T[r, s] \mid \eta w)^\wedge: Tr \otimes Ts \rightarrow Tw & & (w = r \otimes s),
 \end{aligned}$$

$$(3) \quad \begin{array}{ccccc}
 TX & \equiv & TX & \xrightarrow{\eta X} & T^2X \\
 Tr \otimes Ts \downarrow & & T[r, s] \downarrow & \eta w & \downarrow T^2w \\
 T^2Z & \equiv & TZ & \xrightarrow{hZ} & T^2Z
 \end{array}$$

The unit $\eta: 1 \rightarrow U_T F_T = T$ coincides with the unit of T ; the counit $\varepsilon: F_T U_T \rightarrow 1$ is defined in the usual way on objects, and similarly on a vertical arrow $r: X \twoheadrightarrow Y$ of \mathbb{X}_T

$$(4) \quad \varepsilon(r) = (1_{Tr})^\wedge: Tr \rightarrow r.$$

Now, given a pseudo/lax adjunction $F \dashv U$, the associated lax monad $T = UF$ and its pseudo double category of Kleisli algebras, the comparison $L: \mathbb{A} \rightarrow \mathbb{X}_T$ is defined in the usual way on objects and horizontal arrows, and extended to the vertical structure. We obtain a *lax* double functor, with comparison cells defined by the ones of U ($L[A]$ is similar)

$$\begin{aligned}
 (5) \quad L: \mathbb{A} &\rightarrow \mathbb{X}_T, & L(A) &= UA, & L(f) &= \eta UB.U(f), \\
 L(u) &= Uu, & L(\alpha) &= \eta Uv.U\alpha & & (\alpha: (u \begin{smallmatrix} f \\ g \end{smallmatrix} v) \text{ in } \mathbb{A}), \\
 L[u, v] &= (U[u, v] \mid \eta U(u \otimes v))^\wedge: Lu \otimes Lv \rightarrow L(u \otimes v).
 \end{aligned}$$

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Dipartimento di Matematica
 Università di Genova
 via Dodecaneso 35
 16146 Genova, Italy
 grandis@dima.unige.it

Department of Mathematics and Statistics
 Dalhousie University
 Halifax NS
 Canada B3H 3J5
 pare@mathstat.dal.ca