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REPRESENTABLE POSETS AND THEIR ORDER COMPONENTS

by M.E. ADAMS and D. van der ZYPEN

RESUME. Un ensemble partiellement ordonné P est représentable s'il existe un (0;1)-treillis distributif dont l'ensemble ordonné des idéaux premiers est isomorphe à P. Dans cet article, nous voulons démontrer que, si toutes les composantes pour l'ordre de P sont représentables, P est aussi représentable. De plus, nous montrons que, bien que la topologie d'intervalle de chaque composante soit compacte, il existe un ensemble partiellement ordonné qui est représentable et qui possède une composante pour l'ordre non-représentable.[1]

1 Introduction

A poset is said to be *representable* if it is isomorphic to the poset of prime ideals of a bounded distributive lattice (that is a distributive lattice with a largest element 1 and a smallest element 0). The question of which posets are representable essentially dates back to Balbes [1] (see also, Balbes and Dwinger [2]) and has been considered by a number of authors since (see, for example, the expository article Priestley [6].)

In [5], Priestley proved that the category $\mathcal{D}$ of bounded distributive lattices with $(0,1)$-preserving lattice homomorphisms and the category $\mathcal{P}$ of compact totally order-disconnected spaces (henceforth referred to as *Priestley spaces*) with order-preserving continuous maps are dually equivalent. (A compact *totally order-disconnected space* $(X;\tau,\leq)$ is a poset $(X;\leq)$

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endowed with a compact topology τ such that, for $x, y \in X$, whenever $x \not\geq y$, then there exists a clopen decreasing set U such that $x \in U$ and $y \notin U$.) The functor $D : \mathcal{D} \to \mathcal{P}$ assigns to each object L of $\mathcal{D}$ a Priestley space $(D(L); \tau(L), \subseteq)$, where $D(L)$ is the set of all prime ideals of L and $\tau(L)$ is a suitably defined topology (the details of which will not be required here). The functor $E : \mathcal{P} \to \mathcal{D}$ assigns to each Priestley space X the lattice $(E(X); \cup, \cap, \emptyset, X)$, where $E(X)$ is the set of all clopen decreasing sets of X. In particular, a poset $(X; \leq)$ is seen to be representable iff there exists a topology τ such that $(X; \tau, \leq)$ is a Priestley space.

Let $(X; \leq)$ be a poset. Then we define a relation R on X by setting $(x, y) \in R$ whenever $x \leq y$ or $y \leq x$. Let R' be the transitive closure of R. Then R' is an equivalence relation. An *order component* of X is an equivalence class $[x]_{R'}$ of the relation R' for some $x \in X$. Further, for any $Y \subseteq X$, let $(Y] = \{x \in X \mid x \leq y \text{ for some } y \in Y\}$ and $[Y) = \{x \in X \mid x \geq y \text{ for some } y \in Y\}$. Should $Y = \{y\}$ for some $y \in X$, then, for simplicity, we will denote $(Y]$ and $[Y)$ by $(y]$ and $[y)$, respectively. Finally, let $[x, y] = [x) \cap (y]$, $\mathcal{S}^- = \{X \setminus (x] \mid x \in X\}$, and $\mathcal{S}^+ = \{X \setminus [x) \mid x \in X\}$. Then $\mathcal{S} = \mathcal{S}^- \cup \mathcal{S}^+$ is an open subbase for the so called *interval topology* τ_i on X (sometimes, in the interest of clarity, τ_i will be denoted $\tau_i(X)$ when we wish to emphasize the poset concerned). It is well known that if $(X; \tau, \leq)$ is a Priestley space, then τ contains the interval topology τ_i.

Our principal result is the following:

THEOREM 1.1. *If the order components of a poset $(X; \leq)$ are representable, then so is X. However, even though each order component of a representable poset is compact under its interval topology, there exists a representable poset with an order component which is not representable.*

The proof of 1.1 will be given in §2, where we begin in 2.1 by showing that a poset is compact under its interval topology iff each order component is compact under its respective interval topology. As observed in 2.2, it follows readily from this that each order component of a representable poset is compact with respect to its interval topology. We then establish in 2.3 that if every order component of a poset is representable, then so too is the poset. Finally, we define a countably infinite poset which we show to be order-isomorphic to an order component of a representable poset in 2.5, but which, as we show in 2.6, is not itself representable.

For any undefined terms or additional background, we refer the reader to the texts Grätzer [3] and Kelley [4], with each of which our notation is consistent.

2 Proof of 1.1

LEMMA 2.1. *Let* $(X_k; \leq_k)_{k\in K}$ *be a family of pairwise disjoint nonempty posets. Then for* $(X; \leq)$ *where* $X = \bigcup_{k\in K} X_k$ *and* $\leq = \bigcup_{k\in K} \leq_k$*, the following are equivalent*:

(i) for each $k \in K$*, the space* $(X_k; \tau_i(X_k))$ *is compact*;

(ii) $(X; \tau_i(X))$ *is compact.*

Proof. Assume that (i) holds and let $\mathcal{U}$ be an open cover of $X = \bigcup_{k\in K} X_k$. By Alexander's subbase lemma, we may assume that

$$\mathcal{U} = \{X\backslash(a] \mid a \in A\} \cup \{X\backslash[b) \mid b \in B\}$$

for some subsets $A, B \subseteq X$. We distinguish two cases:

First, there is some $k \in K$ such that $A \cup B \subseteq X_k$. In which case, consider $\mathcal{U}_{X_k} = \{X_k\backslash(a] \mid a \in A\} \cup \{X_k\backslash[b) \mid b \in B\}$. Since $(X_k; \tau_i(X_k))$ is compact by assumption, $\mathcal{U}_{X_k}$ has a finite subcover

$$\{X_k\backslash(a_1], ..., X_k\backslash(a_r]\} \cup \{X_k\backslash[b_1), ..., X_k\backslash[b_s)\},$$

so $\{X\backslash(a_1], ..., X\backslash(a_r]\} \cup \{X\backslash[b_1), ..., X\backslash[b_s)\}$ is a finite subcover of $\mathcal{U}$. Second, there is no $k \in K$ such that $A \cup B \subseteq X_k$. In which case there are

$w_1, w_2 \in A \cup B$ such that $w_1 \in X_k$ and $w_2 \in X_{k'}$ for some $k \neq k' \in K$. If $w_1, w_2 \in A$, then $\{X \backslash (w_1], X \backslash (w_2]\}$ is a finite subcover of $\mathcal{U}$. If $w_1 \in A, w_2 \in B$, then $\{X \backslash (w_1], X \backslash [w_2)\}$ is a finite subcover of $\mathcal{U}$ (similarly for $w_1 \in B, w_2 \in A$). Finally if $w_1, w_2 \in B$, then $\{X \backslash [w_1), X \backslash [w_2)\}$ is a finite subcover of $\mathcal{U}$.

Thus, in any case, $(X; \tau_i(X))$ is compact.

Assume that (ii) holds and let $k \in K$. Assume that $\mathcal{U}$ is an open cover of X_k. By Alexander's subbase lemma we may assume that

$$\mathcal{U} = \{X_k \backslash (a] \mid a \in A\} \cup \{X_k \backslash [b) \mid b \in B\}$$

for some subsets $A, B \subseteq X_k$. Consider the following open cover of $X = \bigcup_{l \in K} X_l$

$$\mathcal{U}^* = \{X \backslash (a] \mid a \in A\} \cup \{X \backslash [b) \mid b \in B\}.$$

Then $\mathcal{U}^*$ has a finite subcover $\{X \backslash (a_1], ..., X \backslash (a_r]\} \cup \{X \backslash [b_1), ..., X \backslash [b_s)\}$ since X is compact with its interval topology. Thus $\{X_k \backslash (a_1], ..., X_k \backslash (a_r]\} \cup \{X_k \backslash [b_1), ..., X_k \backslash [b_s)\}$ is a finite subcover of X_k. $\square$

If $(X; \leq)$ is representable, then, for some topology τ, $(X; \tau, \leq)$ is a Priestley space. In particular, $(X; \tau)$ is a compact space and, as $\tau_i \subseteq \tau$, so too is $(X; \tau_i)$. Thus, the following is an immediate consequence of 2.1.

LEMMA 2.2. *Each order component of a representable poset is compact with respect to its interval topology.*

We now go on to show that if the order components of a poset are representable, then so is the poset.

LEMMA 2.3. *Let $(X_k, \leq_k)_{k \in K}$ be a family of pairwise disjoint nonempty representable posets. Then $(X; \leq)$ is representable, where $X = \bigcup_{k \in K} X_k$ and $\leq = \bigcup_{k \in K} \leq_k$.*

Proof. If K is empty or a singleton, the statement is trivial. So we may assume that K has more than one element. For any $k \in K$, let τ_k be a topology making $(X_k; \tau_k, \leq_k)$ a Priestley space. Fix $k \in K$ and $x \in X_k$. We now build a subbase for a topology on X in three steps. We set:

$\mathcal{S}_1 = \bigcup_{l \in K \setminus \{k\}} \tau_l$;

$\mathcal{S}_2 = \{U \in \tau_k \mid x \notin U\}$;

$\mathcal{S}_3 = \{U \subseteq X \mid x \in U$ and $U \cap X_k \in \tau_k$ and, for some $k' \in K \setminus \{k\}$, $U = [U \cap X_k] \cup [\bigcup_{l \in K \setminus \{k,k'\}} X_l]\}$.

Then let τ be the topology having $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ as a subbase. Using Alexander's subbase lemma we check easily that $(X; \tau)$ is compact using the fact that any subbase member containing x is, in some sense, large by virtue of the definition of $\mathcal{S}_3 \subseteq \mathcal{S}$. Moreover, an easy distinction by cases tells us that $(X; \tau, \leq)$ is totally order-disconnected. □

It remains to provide an example of a poset $(P; \leq)$ which is order isomorphic to an order component of a representable poset, but is not representable itself.

On the set

$$P = \{p\} \cup \{p_{i_0,\ldots,i_n} \mid 0 \leq n < \omega \text{ and } 0 \leq i_j < \omega \text{ for } 0 \leq j \leq n\},$$

inductively define an order relation $\leq$ as follows.

For $0 \leq j < i < \omega$,

$$p < p_i < p_j.$$

For $0 \leq i_0 < \omega$, $0 \leq k \leq i_0$, and $0 \leq i < j < \omega$,

$$p_{i_0,i} < p_{i_0,j} < p_k.$$

For $0 \leq i_0, i_1 < \omega$, $0 \leq k \leq i_1$, and $0 \leq j < i < \omega$,

$$p_{i_0,k} < p_{i_0,i_1,i} < p_{i_0,i_1,j}.$$

In general, let $0 < r < \omega$.

For $0 \leq i_0, i_1, \ldots, i_{2r} < \omega$, $0 \leq k \leq i_{2r}$, and $0 \leq i < j < \omega$,

$$p_{i_0,i_1,\ldots,i_{2r-1},i_{2r},i} < p_{i_0,i_1,\ldots,i_{2r-1},i_{2r},j} < p_{i_0,i_1,\ldots,i_{2r-1},k}.$$

For $0 \leq i_0, i_1, \ldots, i_{2r+1} < \omega$, $0 \leq k \leq i_{2r+1}$, and $0 \leq j < i < \omega$,

$$p_{i_0,i_1,\ldots,i_{2r},k} < p_{i_0,i_1,\ldots,i_{2r},i_{2r+1},i} < p_{i_0,i_1,\ldots,i_{2r},i_{2r+1},j}.$$

To see that $(P; \leq)$ is a poset, for $0 \leq n < \omega$, let

$$P(n) = \{p\} \cup \{p_{i_0,\ldots,i_m} \mid 0 \leq m \leq n \text{ and, for } 0 \leq j \leq m, 0 \leq i_j < \omega\}.$$

Thus, $P(0)$ and, for each $0 \leq n < \omega$, $P(n+1) \setminus P(n)$ are clearly antisymmetric and transitive. Further, $x \in P(n)$ is comparable with $y \in P \setminus P(n)$ only if $x \in P(n) \setminus P(n-1)$ and $y \in P(n+1) \setminus P(n)$, where it is the case that $x > y$ and $x < y$ depending on whether n is even or odd, respectively. In particular, $\leq$ is antisymmetric. Moreover, if n is even, say $n = 2r$, then $x = p_{i_0,\ldots,i_{2r-1},k}$ and $y = p_{i_0,\ldots,i_{2r},i}$ providing $0 \leq k \leq i_{2r}$ and $0 \leq i < \omega$, and if n is odd, say $n = 2r+1$, then $x = p_{i_0,\ldots,i_{2r},k}$ and $y = p_{i_0,\ldots,i_{2r+1},i}$ providing $0 \leq k \leq i_{2r+1}$ and $0 \leq i < \omega$. In particular, $\leq$ is transitive and, as claimed, $(P; \leq)$ is seen to be a countable connected poset. We also note in passing that, for $0 \leq i_0, \ldots, i_n < \omega$, $[p_{i_0,\ldots,i_n})$ and $(p_{i_0,\ldots,i_n}]$ are finite chains depending on whether n is even or odd, respectively, a fact that we will refer back to later.

In order to show that $(P; \leq)$ is order-isomorphic to an order component of a representable poset, we will define a suitable order $\preceq$ on a compact totally disconnected space $(C; \tau)$ which itself is homeomorphic to the *Stone space* of a countable atomless Boolean algebra. To do so, we will need an explicit description of $(C; \tau)$, which we now give.

Let $\mathbf{Q} = (Q; \leq)$ denote the rational interval $(0, 1)$. Then (A, B) is a *Dedekind cut* of Q providing that A and B are disjoint non-empty sets such that $Q = A \cup B$ and, for $a \in A$ and $b \in B$, $a < b$. For a Dedekind cut (A, B) of $\mathbf{Q}$, A is a *gap* providing A does not have a greatest element and B does not have a smallest element and, otherwise, it is a *jump*. Let $(C; \leq)$ denote the set of all decreasing subsets of the rational interval $(0, 1)$ ordered by inclusion. Thus, for $I \in C$, if $I \neq \emptyset$ or Q, then I is a jump precisely when $I = (0, r)$ or $(0, r]$ for some $r \in Q$. Intuitively, $(C; \leq)$ may be thought of as the real interval $[0, 1]$ where every rational element $0 < r < 1$ is replaced by a covering pair. The interval topology τ_i, denoted henceforth simply by τ, on $(C; \leq)$ has as a base the open intervals C, $[\emptyset, I) = \{J \in C : J \subset I\}$, $(I, Q] = \{J \in C : I \subset J\}$, and $(I, J) = \{K \in C : I \subset K \subset J\}$. It is well-known that $(C; \tau)$ is a compact totally disconnected space, whose clopen subsets are precisely the sets $\emptyset$, C, and finite unions of sets of the form $[I, J] = \{K \in C : I \subseteq K \subseteq J\}$ where $I = (0, r]$ and $J = (0, s)$ for r, $s \in Q$ with $r < s$.

Setting $Q = (s_i : 0 \leq i < \omega)$ to be some enumeration of Q, we now inductively define a new partial order on C as follows:

In C, choose gaps x and, for $0 \leq i < \omega$, x_i such that

$$x < x_i < x_j \text{ for } 0 \leq j < i < \omega,$$

where x is a member of the closure of $\{x_i \mid 0 \leq i < \omega\}$, denoted $cl(\{x_i \mid 0 \leq i < \omega\})$, and set

$$x \prec x_i \prec x_j.$$

Choose clopen intervals $(X_i : 0 \leq i < \omega)$ such that $x_i \in X_i$, $X_i \cap X_j = \emptyset$ whenever $i \neq j$, the length of X_i, denoted $ln(X_i)$, is $\leq \frac{1}{2}$ in the pseudometric obtained from the metric imposed on C by the real metric on $(0, 1)$, and $(0, s_0)$, $(0, s_0] \notin X_i$ for any $0 \leq i < \omega$.

For $0 \leq i_0 < \omega$, $0 \leq k \leq i_0$, and $0 \leq i < \omega$, choose gaps $x_{i_0,i} \in X_{i_0}$ such that

$$x_{i_0,i} < x_{i_0,j} < x_k \text{ for } 0 \leq i < j < \omega,$$

where $x_{i_0} \in cl(\{x_{i_0,i} \mid 0 \leq i < \omega\})$, and set

$$x_{i_0,i} \prec x_{i_0,j} \prec x_k.$$

Choose clopen intervals $(X_{i_0,i} : 0 \leq i < \omega)$ such that $x_{i_0,i} \in X_{i_0,i}$, $X_{i_0,i} \cap X_{i_0,j} = \emptyset$ for $i \neq j$, $X_{i_0,i} \subseteq X_{i_0}$, $ln(X_{i_0,i}) \leq \frac{1}{2^2}$, and $(0, s_1)$, $(0, s_1] \notin X_{i_0,i}$ for $0 \leq i < \omega$.

For $0 \leq i_0, i_1 < \omega$, $0 \leq k \leq i_1$, and $0 \leq i < \omega$, choose gaps $x_{i_0,i_1,i} \in X_{i_0,i_1}$ such that

$$x_{i_0,k} < x_{i_0,i_1,i} < x_{i_0,i_1,j} \text{ for } 0 \leq j < i < \omega,$$

where $x_{i_0,i_1} \in cl(\{x_{i_0,i_1,i} \mid 0 \leq i < \omega\})$, and set

$$x_{i_0,k} \prec x_{i_0,i_1,i} \prec x_{i_0,i_1,j}.$$

Choose clopen intervals $(X_{i_0,i_1,i} : 0 \leq i < \omega)$ such that $x_{i_0,i_1,i} \in X_{i_0,i_1,i}$, $X_{i_0,i_1,i} \cap X_{i_0,i_1,j} = \emptyset$ for $i \neq j$, $X_{i_0,i_1,i} \subseteq X_{i_0,i_1}$, $ln(X_{i_0,i_1,i}) \leq \frac{1}{2^3}$, and $(0, s_2)$, $(0, s_2] \notin X_{i_0,i_1,i}$ for $0 \leq i < \omega$.

In general, let $0 < r < \omega$.

For $0 \leq i_0, i_1, \ldots, i_{2r} < \omega$, $0 \leq k \leq i_{2r}$, and $0 \leq i < \omega$, choose gaps $x_{i_0,i_1,\ldots,i_{2r},i} \in X_{i_0,i_1,\ldots,i_{2r}}$ such that

$$x_{i_0,i_1,\ldots i_{2r-1},i_{2r},i} < x_{i_0,i_1,\ldots i_{2r-1},i_{2r},j} < x_{i_0,i_1,\ldots,i_{2r-1},k} \text{ for } 0 \leq j < i < \omega,$$

where $x_{i_0,\ldots,i_{2r}} \in cl(\{x_{i_0,\ldots,i_{2r},i} \mid 0 \leq i < \omega\})$, and set

$$x_{i_0,i_1,\ldots i_{2r-1},i_{2r},i} \prec x_{i_0,i_1,\ldots i_{2r-1},i_{2r},j} \prec x_{i_0,i_1,\ldots,i_{2r-1},k}.$$

Choose clopen intervals $(X_{i_0,i_1,\ldots,i_{2r},i} : 0 \leq i < \omega)$ such that $x_{i_0,i_1,\ldots,i_{2r},i} \in X_{i_0,i_1,\ldots,i_{2r},i}$, $X_{i_0,i_1,\ldots,i_{2r},i} \cap X_{i_0,i_1,\ldots,i_{2r},j} = \emptyset$ for $i \neq j$, $X_{i_0,i_1,\ldots,i_{2r},i} \subseteq X_{i_0,i_1,\ldots,i_{2r}}$, $ln(X_{i_0,i_1,\ldots,i_{2r},i}) \leq \frac{1}{2^{2r+1}}$, and $(0, s_{2r+1})$, $(0, s_{2r+1}] \notin X_{i_0,i_1,\ldots,i_{2r},i}$ for $0 \leq i < \omega$.

For $0 \leq i_0, i_1, \ldots, i_{2r+1} < \omega$, $0 \leq k \leq i_{2r+1}$, and $0 \leq i < \omega$, choose gaps $x_{i_0,i_1,\ldots,i_{2r+1},i} \in X_{i_0,i_1,\ldots,i_{2r+1}}$ such that

$$x_{i_0,i_1,\ldots,i_{2r},k} < x_{i_0,i_1,\ldots i_{2r},i_{2r+1},i} < x_{i_0,i_1,\ldots i_{2r},i_{2r+1},j} \text{ for } 0 \leq i < j < \omega,$$

where $x_{i_0,\ldots,i_{2r+1}} \in cl(\{x_{i_0,\ldots,i_{2r+1},i} \mid 0 \leq i < \omega\})$, and set

$$x_{i_0,i_1,\ldots,i_{2r},k} \prec x_{i_0,i_1,\ldots i_{2r},i_{2r+1},i} \prec x_{i_0,i_1,\ldots i_{2r},i_{2r+1},j}.$$

Choose clopen intervals $(X_{i_0,i_1,\ldots,i_{2r+1},i} : 0 \leq i < \omega)$ such that $x_{i_0,i_1,\ldots,i_{2r+1},i} \in X_{i_0,i_1,\ldots,i_{2r+1},i}$, $X_{i_0,i_1,\ldots,i_{2r+1},i} \cap X_{i_0,i_1,\ldots,i_{2r+1},j} = \emptyset$ for $i \neq j$, $X_{i_0,i_1,\ldots,i_{2r+1},i} \subseteq$

$X_{i_0,i_1,\dots,i_{2r+1}}$, $ln(X_{i_0,i_1,\dots,i_{2r+1},i}) \le \frac{1}{2^{2(r+1)}}$, and $(0, s_{2r+2})$, $(0, s_{2r+2}] \not\subseteq X_{i_0,i_1,\dots,i_{2r+1}}$, for $0 \le i < \omega$.

Elsewhere on C, let $\preceq$ be trivial. Thus, since $(X;\preceq)$ is order-isomorphic to $(P;\le)$, $(C;\preceq)$ is a poset whose order components consist precisely of $X = \{x\} \cup \{x_{i_0,\dots,i_n} \mid 0 \le n < \omega \text{ and } 0 \le i_j < \omega \text{ for } 0 \le j \le n\}$ and 2^ω singletons.

LEMMA 2.4. *$(C;\tau,\preceq)$ is a Priestley space.*

Proof. As $(C;\preceq)$ is a poset and $(C;\tau)$ is a compact totally disconnected space, it remains to show that, for $u, v \in C$, whenever $u \not\preceq v$ there exists a clopen decreasing set U such that $u \in U$ and $v \notin U$.

Since $(X;\preceq)$ is order-isomorphic to $(P;\le)$, we set, for $0 \le n < \omega$,

$$X(n) = \{x\} \cup \{x_{i_0,\dots,i_m} \mid 0 \le m \le n \text{ and, for } 0 \le j \le n,\ 0 \le i_j < \omega\}$$

and observe that, as $[x_{i_0,\dots,i_n})$ or $(x_{i_0,\dots,i_n}]$ is a finite chain depending on whether n is even or odd, respectively, it follows from the choice of elements in $X \setminus X(n)$ that, for $0 \le i_0, \dots, i_n < \omega$, $\bigcup(X_{i_0,\dots,i_{n-1},k} : 0 \le k \le i_n)$ is clopen increasing or decreasing, accordingly.

Consider $u, v \in C$ with $u \not\preceq v$. In each case we will exhibit a clopen decreasing set U such that $u \in U$ and $v \notin U$.

If $u < v$, then $u \le (0, s] < v$ for some $s \in Q$. Since $\preceq$ is compatible with $\le$, set $U = (\emptyset, (0, s]]$. Henceforth, we assume that $u > v$ and, in particular, u and v are incomparable under $\preceq$.

Suppose there is an infinite sequence $(i_k : 0 \le k < \omega)$ such that $u \in X_{i_0,\dots,i_k}$ for any $0 \le k < \omega$. Then, by choice, u is a gap and, since $ln(X_{i_0,\dots,i_n}) \le \frac{1}{2^n}$, $v \notin X_{i_0,\dots,i_n}$ for some $0 \le n < \omega$. Without loss of generality, we may assume that n is even. Set $U = \bigcup(X_{i_0,\dots,i_n,l} : 0 \le l \le i_{n+1})$. By the above observation, U is clopen decreasing, $u \in U$, and, since $U \subseteq X_{i_0,\dots,i_n}$, $v \notin U$.

Likewise, if there is an infinite sequence $(j_k : 0 \leq k < \omega)$ such that $v \in X_{j_0,\dots,j_k}$ for any $0 \leq k < \omega$, then v is a gap and, since $ln(X_{j_0,\dots,j_m}) \leq \frac{1}{2^m}$, $u \notin X_{j_0,\dots,j_m}$ for some $0 \leq m < \omega$. We may assume, again with no loss in generality, that m is odd. Set $U = C \setminus \bigcup(X_{j_0,\dots,j_m,l} : 0 \leq l \leq j_{m+1})$. Then, U is clopen decreasing, $v \notin U$, and, since $U \subseteq C \setminus X_{j_0,\dots,j_m}$, $u \in U$.

Suppose, for some finite sequence $(i_k : 0 \leq k \leq n)$, $u \in X_{i_0,\dots,i_n}$, but $u \notin X_{i_0,\dots,i_n,l}$ for any $0 \leq l < \omega$. Then, providing $u \neq x_{i_0,\dots,i_n}$, it is not hard to see that there exists a clopen set U such that $u \in U$, $v \notin U$, and each element of U is incomparable under $\preceq$ to any other element of $(C; \preceq)$, whereby U is decreasing. Were it the case that $u \notin X_l$ for any $0 \leq l < \omega$, then a similar set may be defined unless $u = x$.

Likewise, suppose it is the case that, for some finite sequence $(j_k : 0 \leq k \leq m)$, $v \in X_{j_0,\dots,j_m}$, but that $v \notin X_{j_0,\dots,j_m,l}$ for any $0 \leq l < \omega$. Then, providing $v \neq x_{j_0,\dots,j_m}$, there exists a clopen set V such that $v \in V$, $u \notin V$, and each element of V is incomparable under $\preceq$ to any other element of $(C; \preceq)$. In this case, set $U = C \setminus V$. Likewise, unless $v = x$, a similar set may be defined whenever $v \notin X_l$ for any $0 \leq l < \omega$.

Thus, it now remains to consider the eventuality that $u = x$ or $x_{i_0,\dots,i_n}$ for some $(i_k : 0 \leq k \leq n)$ and $v = x$ or $x_{j_0,\dots,j_m}$ for some $(j_k : 0 \leq k \leq m)$. Observe that, by hypothesis, since $v < u$, $u = x$ is impossible and, hence, we need only consider $u = x_{i_0,\dots,i_n}$ for some $(i_k : 0 \leq k \leq n)$. Further, if $v = x$, then, by hypothesis, $u = x_{i_0,\dots,i_n}$ for some $n > 0$. Since $v \notin X_{i_0}$ and $u \neq x_{i_0}$, $u \in U = \bigcup(X_{i_0,k} : 0 \leq k \leq i_1) \subseteq X_{i_0}$, which, as observed above, is clopen decreasing. Thus, in addition, we may assume that $v = x_{j_0,\dots,j_m}$ for some $(j_k : 0 \leq k \leq m)$.

A number of possibilities still remain to be considered.

Suppose first that $n \leq m$.

Consider $i_k = j_k$ for all $0 \leq k \leq n$. Then, by hypothesis, $m \geq n + 2$ and, since $u > v$, n is even. Thus, $V = \bigcup(X_{i_0,\dots,i_n,j_{n+1},l} : 0 \leq l \leq j_{n+2})$ is clopen increasing $v \in V$, and $u \notin V$. Set $U = C \setminus V$.

Suppose $i_k = j_k$ for all $0 \leq k < n$, but $i_n \neq j_n$. Then, by hypothesis, $m \geq n+1$. Suppose n is even. Were it the case that $i_n > j_n$, then it would follow that $u < v$, contrary to hypothesis. Thus, we may assume that $i_n < j_n$. But then it follows that $m \geq n+2$. Thus, $v \in V = \bigcup(X_{i_0,\ldots,i_{n-1},j_n,j_{n+1},l} : 0 \leq l \leq j_{n+2})$ which is clopen increasing and, since $V \subseteq X_{i_0,\ldots,i_{n-1},j_n}$, $u \notin V$. Suppose n is odd. Thus, $v \in V = \bigcup(X_{i_0,\ldots,i_{n-1},j_n,l} : 0 \leq l \leq j_{n+1})$, which is clopen increasing, and again, since $V \subseteq X_{i_0,\ldots,i_{n-1},j_n}$, $u \notin V$. In either case, set $U = C \setminus V$.

Consider, for some $0 \leq k \leq n-1$, $i_l = j_l$ for all $0 \leq l < k$, but $i_k \neq j_k$. If k is even, then $u \in U = \bigcup(X_{i_0,\ldots,i_{k-1},i_k,l} : 0 \leq l \leq i_{k+1})$ which is clopen decreasing and, since $U \subseteq X_{i_0,\ldots,i_{k-1},i_k}$ and $v \in X_{i_0,\ldots,i_{k-1},j_k}$, $v \notin U$. If k is odd, then $v \in V = \bigcup(X_{i_0,\ldots,i_{k-1},j_k,l} : 0 \leq l \leq j_{k+1})$ which is clopen increasing and, since $V \subseteq X_{i_0,\ldots,i_{k-1},j_k}$ and $u \notin X_{i_0,\ldots,i_{k-1},j_k}$, $u \notin V$. In this case, set $U = C \setminus V$.

It remains to consider $n > m$.

Suppose $i_k = j_k$ for all $0 \leq k \leq m$. Then, by hypothesis, $n \geq m+2$ and, since $u > v$, m is odd. Hence, $u \in U = \bigcup(X_{j_0,\ldots,j_m,i_{m+1},l} : 0 \leq l \leq i_{m+2})$ which is clopen decreasing, whilst $v \notin U$.

Consider $i_k = j_k$ for all $0 \leq k < m$, but $i_m \neq j_m$. By hypothesis, $n \geq m+1$. Suppose m is even. Then, $u \in U = \bigcup(X_{j_0,\ldots,j_{m-1},i_m,l} : 0 \leq l \leq i_{m+1})$ which is clopen decreasing, and, since $U \subseteq X_{j_0,\ldots,j_{m-1},i_m}$, $v \notin U$. Suppose m is odd. Were $i_m < j_m$, then it would follow that $u < v$, contrary to hypothesis. Thus, we may assume that $i_m > j_m$ and, so, $n \geq m+2$. Hence, $u \in U = \bigcup(X_{j_0,\ldots,j_{m-1},i_m,i_{m+1},l} : 0 \leq l \leq i_{m+2})$ which is clopen decreasing, and, since it is also the case that $U \subseteq X_{j_0,\ldots,j_{m-1},i_m}$, $v \notin U$.

Finally, it remains to consider the case that, for some $0 \leq k \leq m-1$, $i_l = j_l$ for all $0 \leq l < k$, but $i_k \neq j_k$. However, the same argument holds, word for word, as given in the analogous case when $n \leq m$. □

Since the order components of $(C; \tau, \preceq)$ consist of precisely $X = \{x\} \cup$

$\{x_{i_0,\ldots,i_n} \mid 0 \leq n < \omega$ and $0 \leq i_j < \omega$ for $0 \leq j \leq n\}$ and 2^ω singletons and, by choice, $(X; \preceq)$ is order-isomorphic to $(P; \leq)$, the following is an immediate consequence of 2.4.

LEMMA 2.5. *$(P; \leq)$ is order-isomorphic to an order component of a representable poset.*

The proof of 1.1 will be complete once we have established the following.

LEMMA 2.6. *$(P; \leq)$ is not representable.*

Proof. Suppose, contrary to hypothesis, that $(P; \leq)$ is representable and let $(P; \tau, \leq)$ be a Priestley space for some topology τ.

We claim that, for $x \in P$, there is a sequence $(x_i : 0 \leq i < \omega)$ such that either, for $0 \leq j < i < \omega$, $x_i < x_j$ and x is the greatest lower bound of $\{x_i \mid 0 \leq i < \omega\}$ or, for $0 \leq i < j < \omega$, $x_i < x_j$ and x is the least upper bound of $\{x_i \mid 0 \leq i < \omega\}$.

To justify the claim, we consider the various possibilities. If $x = p$, then setting $x_i = p_i$ yields, for $0 \leq j < i < \omega$, $p < p_i < p_j$. Moreover, for $y \in P \setminus P(0)$, $[y) \cap P(0)$ is finite. In particular, p is the greatest lower bound of $\{p_i \mid 0 \leq i < \omega\}$. Similarly, for $x = p_{i_0,\ldots,i_n}$, let $x_i = p_{i_0,\ldots,i_n,i}$ for $0 \leq i < \omega$. If n is even, then, for $0 \leq i < j < \omega$,

$$p_{i_0,\ldots,i_n,i} < p_{i_0,\ldots,i_n,j} < p_{i_0,\ldots,i_n}.$$

Since $p_{i_0,\ldots,i_n}$ is the greatest lower bound of $[p_{i_0,\ldots,i_n,i})$ and, for $y \in P\setminus P(n+1)$, $(y] \cap P(n+1)$ is finite, it follows that $p_{i_0,\ldots,i_n}$ is the least upper bound of $\{p_{i_0,\ldots,i_n,i} \mid 0 \leq i < \omega\}$. Likewise, if n is odd, then, for $0 \leq j < i < \omega$,

$$p_{i_0,\ldots,i_n} < p_{i_0,\ldots,i_n,i} < p_{i_0,\ldots,i_n,j}.$$

Since $p_{i_0,\ldots,i_n}$ is the least upper bound of $(p_{i_0,\ldots,i_n}]$ and, for every $y \in P \setminus P(n+1)$, $[y) \cap P(n+1)$ is finite, it follows that $p_{i_0,\ldots,i_n}$ is the greatest lower bound of $\{p_{i_0,\ldots,i_n,i} \mid 0 \leq i < \omega\}$.

Using the above claim, we now show that every $x \in P$ is an accumulation point.

To see this, say x is the greatest lower bound of $\{x_i \mid 0 \leq i < \omega\}$ where, for $0 \leq j < i < \omega$, $x_i < x_j$. For $0 \leq i < \omega$, there exists a clopen increasing set V_i such that $x_i \in V_i$ and $x_{i+1} \notin V_i$. Clearly, $\{V_i \mid 0 \leq i < \omega\}$ is an open cover of $S = \{x_i \mid 0 \leq i < \omega\}$ with no finite subcover. In particular, S is not closed. Choose $y \in cl(S) \setminus S$. If $y \not\geq x$, then there is a clopen decreasing set U with $y \in U$ and $x \notin U$, from which it follows that $U \cap S = \emptyset$, contradicting $y \in cl(S)$. If $y > x$, then y is not a lower bound of S, as x is the greatest. In particular, for some $0 \leq n < \omega$, $x_n \not\geq y$. It follows that there is a clopen decreasing set U with $x_n \in U$ and $y \notin U$. Thus, $S \subseteq \{x_0, \ldots, x_n\} \cup U$, which is a closed set. On the other hand, $y \in P \setminus (\{x_0, \ldots, x_n\} \cup U)$, contradicting the fact that $y \in cl(S)$. We conclude that $y = x$ and, in particular, that, as claimed, x is an accumulation point. As similar argument holds in the case that x is the least upper bound of $\{x_i \mid 0 \leq i < \omega\}$ where, for $0 \leq i < j < \omega$, $x_i < x_j$.

Suppose then that L is a bounded distributive lattice such that $(D(L); \tau(L), \subseteq)$ (recall the notation introduced in §1) is homeomorphic and order-isomorphic to $(P; \tau, \leq)$. For a, $b \in L$, there correspond clopen decreasing sets A, B, respectively. Suppose $a < b$. Then $A \subset B$ and it is possible to choose $x \in B \setminus A$. Since x is an accumulation point, there exists a distinct element $y \in B \setminus A$. Say, without loss of generality, $x \not\geq y$. Then there exists a clopen decreasing set U with $x \in U$ and $y \notin U$. Set $C = A \cup (B \cap U)$. Then C is a clopen decreasing set such that $A \subset C \subset B$. In particular, C corresponds to an element $c \in L$ such that $a < c < b$. We conclude that $(Q; \leq)$ the rational interval $(0, 1)$ is embeddable in L, that is, $(Q^+; \leq)$ the rational interval $[0, 1]$ is a $(0, 1)$-sublattice of L. If one such embedding is denoted by $f^+ : Q^+ \longrightarrow L$, then f corresponds to continuous order-preserving map $D(f) : D(L) \longrightarrow D(Q^+)$ which is also onto. That is, there is a mapping from P onto $D(Q^+)$. Since $D(Q^+)$ is uncountable and P is countable, this is impossible and, as required, we conclude that $(P; \leq)$ is not representable. □

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