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EXACT COUPLES IN A RAÏKOV SEMI-ABELIAN CATEGORY

by Yaroslav KOPYLOV

RESUME. Nous étudions des couples exacts dans des catégories semi-abéliennes de Raïkov, une classe de catégories additives qui comprend beaucoup de catégories non-abéliennes d'analyse fonctionnelle et d'algèbre. Utilisant l'approche d'Eckmann et Hilton à la suite spectrale dans une catégorie abélienne, on considère des couples exacts dans une catégorie semi-abélienne et montre la possibilité de dérivation si l'endomorphisme du couple exact est strict et, par conséquent, l'existence de la suite spectrale du couple (§2) si tous ses morphismes sont stricts. On montre qu'il est aussi possible de dériver un système de Rees (§3).

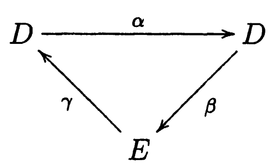
Introduction

Exact couples of abelian groups were introduced by Massey in [14] as a tool for unification of various spectral sequences of algebraic topology. Later Eilenberg and Moore in [7] and Eckmann and Hilton in [5] dealt with exact couples in an arbitrary abelian category. The approach adopted in [5] involves no additional axioms on the ambient abelian category for constructing the spectral sequence of an exact couple, the derivation being obtained by systematic use of pullbacks and pushouts.

In this paper, we apply Eckmann and Hilton's approach to the case of a semiabelian category in the sense of Raïkov [21]. Apart from all abelian categories, the class of Raïkov-semiabelian categories contains many nonabelian additive categories of functional analysis and topological algebra. The categories of (Hausdorff or all) topological abelian groups, topological vector spaces, Banach (or normed) spaces, filtered modules over filtered rings, and torsion-free abelian groups are typical examples of Raïkov-semiabelian categories. The main difference between the Raïkov-semiabelian and abelian categories lies in the fact that the standard diagram lemmas hold in semiabelian categories under some extra conditions which usually amount to the strictness of these morphisms. Raĭkov-semiabelian categories have been actively studied in the recent years (see [9, 11, 12, 18, 19, 20, 22, 23, 24]).

In the category $\mathcal{B}an$ of Banach spaces and the category $\mathcal{A}b\mathcal{T}op$ of topological abelian groups, the strictness of a morphism α means that the range of α is closed. In order to be able to construct the spectral sequence for a filtered complex of Banach spaces, one usually has to impose this condition on the differentials of the complexes in the grading (see, for example, the Lyndon-Hochschild-Serre spectral sequences for the bounded cohomology of discrete groups in [2, 16] and for the bouned countinuous cohomology of locally compact second countable topological groups in [3, 15]). The reason for this is that the exact couples that arise in this case are parts of the corresponding cohomology sequences which are in general not exact in a Raĭkov-semiabelian category [9, 12].

In this paper, we consider an exact couple



in a Raĭkov-semiabelian category and prove that if α is strict then we may pass to the derived couple in the sense of [5], which is in general only semiexact [26]. The derived couple is exact if all the morphisms of the exact couple are strict. We also demonstrate that if α^k is strict for each $k \leq n$ then the derivation of a semiexact couple can be iterated n times. These results are the contents of Section 2. In Section 1, we prove the semiabelian version of Theorem 2.19 of [5]. In Section 3, we discuss the possibility of the derivation of a Rees system [5].

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1 Raĭkov-Semiabelian Categories

We consider additive categories satisfying the following axiom.

Axiom 1. Each morphism has kernel and cokernel.

When it does not lead to confusion, we denote by $\ker\alpha$ ($\operatorname{coker}\alpha$) an arbitrary kernel (cokernel) of α and by $\operatorname{Ker}\alpha$ ($\operatorname{Coker}\alpha$) the corresponding object; the equality $a = \ker b$ ($a = \operatorname{coker} b$) means that a is a kernel of b (a is a cokernel of b).

In a category satisfying Axiom 1, every morphism α admits a canonical decomposition $\alpha = (\operatorname{im}\alpha)\overline{\alpha}(\operatorname{coim}\alpha) = (\operatorname{im}\alpha)\widetilde{\alpha}$, where $\operatorname{im}\alpha = \ker\operatorname{coker}\alpha$, $\operatorname{coim}\alpha = \operatorname{coker}\ker\alpha$. Two canonical decompositions of the same morphism are obviously naturally isomorphic. A morphism α is called *strict* if $\overline{\alpha}$ is an isomorphism.

We use the following notations of [13]:
O_c is the class of all strict morphisms,
M is the class of all monomorphisms,
M_c is the class of all strict monomorphisms,
P is the class of all epimorphisms,
P_c is the class of all strict epimorphisms.

Lemma 1 [4, 5, 13, 21]. *The following assertions hold in an additive category meeting Axiom* 1:

(1) $\ker\alpha \in M_c$ and $\operatorname{coker}\alpha \in P_c$ *for every* α;

(2) $\alpha \in M_c \Longleftrightarrow \alpha = \operatorname{im}\alpha$, $\alpha \in P_c \Longleftrightarrow \alpha = \operatorname{coim}\alpha$;

(3) *a morphism α is strict if and only if it is representable in the form $\alpha = \alpha_1\alpha_0$ with $\alpha_0 \in P_c$, $\alpha_1 \in M_c$; in every such representation,* $\alpha_0 = \operatorname{coim}\alpha$ *and* $\alpha_1 = \operatorname{im}\alpha$;

(4) *if some commutative square*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[\beta]{} & B \end{array} \qquad (1)$$

is a pullback then $\ker f = \alpha(\ker g)$ *and* $f = \ker \xi$ *implies* $g = \ker(\xi\beta)$; *in particular,* $f \in M \Longrightarrow g \in M$ *and* $f \in M_c \Longrightarrow g \in M_c$. *Dually, if* (1) *is a pushout then* $\operatorname{coker} g = (\operatorname{coker} f)\beta$ *and* $g = \operatorname{coker}\zeta$ *implies* $f = \operatorname{coker}(\alpha\zeta)$; *in particular,* $g \in P \Longrightarrow f \in P$ *and* $g \in P_c \Longrightarrow f \in P_c$.

An additive category meeting Axiom 1 is abelian if and only if $\overline{\alpha}$ is an isomorphism for every α. Consider the following axiom.[1]

Axiom 2. *For every morphism* α, $\overline{\alpha}$ *is a bimorphism, i.e., a monomorphism and an epimorphism.*

We write $\alpha \| \beta$ if the sequence $\cdot \xrightarrow{\alpha} \cdot \xrightarrow{\beta} \cdot$ is exact, that is, $\operatorname{im}\alpha = \ker\beta$ (which, in a category meeting Axioms 1 and 2, is equivalent to $\operatorname{coker}\alpha = \operatorname{coim}\beta$).

Lemma 2 [11]. *The following assertions hold in an additive category satisfying Axioms* 1 *and* 2:

(1) *if* $gf \in M_c$ *then* $f \in M_c$; *if* $gf \in P_c$ *then* $g \in P_c$;

(2) *if* $f, g \in M_c$ *and* fg *is defined then* $fg \in M_c$, *if* $f, g \in P_c$ *and* fg *is defined then* $fg \in P_c$;

(3) *if* $fg \in O_c$ *and* $f \in M$ *then* $g \in O_c$, *if* $fg \in O_c$ *and* $g \in P$ *then* $f \in O_c$.

It is well known (see, for example, [8] or [17]), that every abelian category satisfies the following two axioms dual to one another.

Axiom 3. *If* (1) *is a pullback then* $f \in P_c \Longrightarrow g \in P_c$.

Axiom 4. *If* (1) *is a pushout then* $g \in M_c \Longrightarrow f \in M_c$.

[1] In our earlier papers [11, 12], following [1], we called additive categories meeting Axioms 1 and 2 *preabelian*. This leads to confusion with the fact that "preabelian" is now the "official" name of additive categories with kernels and cokernels.

An additive category satisfying Axioms 1, 3, and 4, is called *Raĭkov-semiabelian* (or simply *semiabelian*).[2] As follows from Theorem 1 of [13], each semiabelian category meets Axiom 2.

Given an arbitrary commutative square (1), denote by $\widehat{g} : \operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\beta$ the morphism defined by the equality $g(\ker\alpha) = (\ker\beta)\widehat{g}$ and by $\widehat{f} : \operatorname{Coker}\alpha \longrightarrow \operatorname{Coker}\beta$ the morphism defined by the condition $\widehat{f}(\operatorname{coker}\alpha) = (\operatorname{coker}\beta)f$.

From now on, unless otherwise specified, the ambient category $\mathcal{A}$ will be assumed Raĭkov-semiabelian.

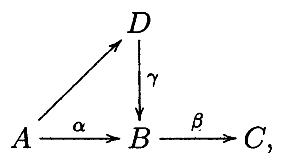
We unite Lemmas 5 and 6 of [11] into the following assertion.

Lemma 3 [11]. *Suppose that square* (1) *is a pullback. If* $\beta \in O_c$ *then* $\alpha \in O_c$ *and* $\widehat{f} \in M$.

Dually, if (1) *is a pushout and* $\alpha \in O_c$ *then* $\beta \in O_c$ *and* $\widehat{g} \in P$.

Kuz′minov and Cherevikin proved in [13] that an additive category meeting Axioms 1 and 2 is Raĭkov-semiabelian if and only if the following assertion holds therein.

Lemma 4. *If, in the commutative diagram*



$\alpha = \ker\beta$, $\beta = \operatorname{coker}\alpha$, *and* $\gamma \in M_c$, *then* $\beta\gamma \in O_c$.

We now prove the semiabelian versions of Propositions 2.1 and 2.2 of [5].

Lemma 5. *If* $\alpha = \operatorname{coker}\beta$ *and* $\beta \| \rho\alpha$ *then* $\rho \in M$.

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[2] Such categories are also known under the names of *quasi-abelian* [24, 25] and *almost abelian* [23]. It should be noted that the term "semiabelian category" is now used in a different context (see [10]). Therefore, we have added the prefix *Raĭkov-*.

Proof. Suppose that $\rho x = 0$. Consider the pullback

$$\begin{array}{ccc} \cdot & \xrightarrow{\;y\;} & \cdot \\ {\scriptstyle x'}\downarrow & & \downarrow{\scriptstyle x} \\ \cdot & \xrightarrow[\operatorname{coker}\beta]{} & \cdot \end{array}$$

We have $\rho(\operatorname{coker}\beta)x' = \rho xy = 0$. Since $\operatorname{im}\beta = \ker\bigl(\rho(\operatorname{coker}\beta)\bigr)$, it follows that there exists a unique morphism u with $x' = (\operatorname{im}\beta)u$. Hence $xy = (\operatorname{coker}\beta)x' = 0$. Furthermore, $y \in P_c$ because $\operatorname{coker}\beta \in P_c$. Therefore, $x = 0$ and thus $\rho \in M$. The lemma is proved.

Lemma 6. If $\rho\beta \| \alpha$ and $\rho \in M_c$ then $\beta \| \alpha\rho$.

Proof. We need to prove that $\operatorname{im}\beta$ is a kernel of $\alpha\rho$. Suppose that $\alpha\rho x = 0$. Since $\rho(\operatorname{im}\beta)\overline{\beta}(\operatorname{coim}\beta)$ is a canonical decomposition for $\rho\beta$, it follows that $\rho(\operatorname{im}\beta) = \operatorname{im}(\rho\beta) = \ker\alpha$. Consequently, there exists a unique morphism v such that $\rho x = \rho(\operatorname{im}\beta)v$, which implies $x = (\operatorname{im}\beta)v$. Thus $\operatorname{im}\beta = \ker(\alpha\rho)$. The lemma is proved.

Remark. As is seen from the proof, Lemma 6 requires only Axioms 1 and 2.

Now, following Eckmann and Hilton, consider the diagram

$$\begin{array}{ccccc} D_1 & & & & C_1 \\ {\scriptstyle \sigma}\uparrow & & & & \downarrow{\scriptstyle \rho} \\ D & \xrightarrow{\;\beta\;} & E & \xrightarrow{\;\gamma\;} & C, \end{array} \tag{2}$$

where $\rho \in M_c$ and $\sigma \in P_c$ are factors of $\gamma\beta$.

We use the notations of [5]: if (1) is a pullback then we write $(\alpha, g) = I(\beta, f)$; if (1) is a pushout then we use the notation $(\beta, f) = U(\alpha, g)$.

Consider the diagram

$$\begin{array}{ccccc} D_1 & \xrightarrow{\beta_1} & E_1 & \xrightarrow{\gamma_1} & C_1 \\ \uparrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle \sigma'} & & \| \\ D & \xrightarrow{\beta'} & E_\rho & \xrightarrow{\gamma'} & C_1 \\ \| & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} \\ D & \xrightarrow{\beta} & E & \xrightarrow{\gamma} & C \end{array} \tag{3}$$

in an additive category with kernels and cokernels. Here $(\gamma', \rho') = I(\gamma, \rho)$, $(\beta_1, \sigma') = U(\beta', \sigma)$, and the morphisms β' and γ_1 arise because ρ and σ are factors of $\gamma\beta$. By Lemma 1, $\rho' \in M_c$ and $\sigma' \in P_c$. Dually, one has the diagram

$$\begin{array}{ccccc} D_1 & \xrightarrow{\overline{\beta_1}} & \overline{E_1} & \xrightarrow{\overline{\gamma_1}} & C_1 \\ \| & & \downarrow{\scriptstyle \rho''} & & \downarrow{\scriptstyle \rho} \\ D_1 & \xrightarrow{\beta''} & E^\sigma & \xrightarrow{\gamma''} & C \\ \uparrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle \sigma''} & & \| \\ D & \xrightarrow{\beta} & E & \xrightarrow{\gamma} & C \end{array}$$

where $(\beta'', \sigma'') = U(\beta, \sigma)$ and $(\overline{\gamma_1}, \rho'') = I(\gamma'', \rho)$. Again, $\sigma'' \in P_c$ and $\rho'' \in M_c$. Furthermore, $\beta\gamma = \rho\overline{\gamma_1}\overline{\beta_1}\sigma = \rho\gamma_1\beta_1\sigma$, whence $\overline{\gamma_1}\overline{\beta_1} = \gamma_1\beta_1$.

Theorem 1. *There exists a unique canonical morphism* $\omega : E_1 \longrightarrow \overline{E_1}$ *such that*

(i) $\omega\beta_1 = \overline{\beta_1}$;

(ii) $\gamma_1 = \overline{\gamma_1}\omega$;

(iii) $\rho''\omega\sigma' = \sigma''\rho'$.

If the ambient category is Raĭkov-semiabelian then ω is an isomorphism.

Proof. Since $\sigma''\rho'\beta' = \beta''\sigma$, it follows that there exists a unique morphism $\varkappa : E_1 \longrightarrow E^\sigma$ with the properties $\varkappa\sigma' = \sigma''\rho'$ and $\varkappa\beta_1 = \beta''$. For this $\varkappa$,

$$\gamma''\varkappa\sigma' = \gamma''\sigma''\rho' = \gamma\rho' = \rho\gamma' = \rho\gamma_1\sigma',$$

whence we have $\gamma''\varkappa = \rho\gamma_1$. Consequently, there exists a unique morphism $\omega : E_1 \longrightarrow \overline{E_1}$ such that $\varkappa = \rho''\omega$, $\gamma_1 = \overline{\gamma_1}\omega$. Easily, $\omega\beta_1 = \overline{\beta_1}$ and $\rho''\omega\sigma' = \sigma''\rho'$. By duality, we have a unique morphism $\overline{\varkappa} : E_\rho \longrightarrow \overline{E_1}$ such that $\rho''\overline{\varkappa} = \sigma''\rho'$ and $\overline{\gamma_1}\,\overline{\varkappa} = \gamma'$. It is easy to see that $\overline{\varkappa}\beta' = \overline{\beta_1}\sigma$, which means that there exists a unique morphism $\overline{\omega} : E_1 \longrightarrow \overline{E_1}$ such that $\overline{\varkappa} = \overline{\omega}\sigma'$ and $\overline{\beta_1} = \overline{\omega}\beta_1$. We infer

$$\rho''\omega\sigma' = \sigma''\rho' = \rho''\overline{\varkappa} = \rho''\overline{\omega}\sigma'.$$

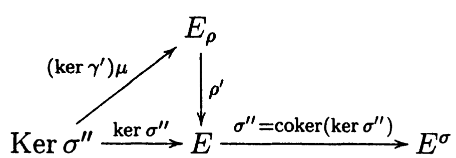
Since $\rho'' \in M$ and $\sigma' \in P$, it follows that $\omega = \overline{\omega}$.

Now, suppose that our category is Raĭkov-semiabelian.

Let $\tau = \ker\sigma$. Then, by Lemma 1(4), $\sigma' = \operatorname{coker}(\beta'\tau)$ and $\sigma'' = \operatorname{coker}(\beta\tau) = \operatorname{coker}(\rho'\beta'\tau)$. Since $\rho' \in M_c$, by Lemma 6, we have $\beta'\tau \| \sigma''\rho'$. Furthermore, $\sigma''\rho' = \rho''\omega\sigma'$ and $\rho'' \in M$; hence $\beta'\tau \| \omega\sigma'$. Involving the fact that $\sigma' = \operatorname{coker}(\beta'\tau)$, Lemma 5 gives $\omega \in M$. Duality yields $\omega \in P$. Thus, ω is a bimorphism.

It remains to prove that $\omega \in O_c$.

Since $\sigma'' = \operatorname{coker}(\beta\tau)$ and $\gamma\beta\tau = \gamma''\beta''\sigma\tau = 0$, it follows that $\gamma(\ker\sigma'') = \gamma(\operatorname{im}(\beta\tau)) = 0$. Consequently, there exists a unique morphism $\mu : \operatorname{Ker}\sigma'' \longrightarrow \operatorname{Ker}\gamma$ with $\ker\sigma'' = (\ker\gamma)\mu$. The fact that $\gamma\rho' = \rho\gamma'$ is a pullback and Lemma 1(4) imply that $\ker\gamma = \rho'(\ker\gamma')$. Therefore, $\ker\sigma'' = \rho'(\ker\gamma')\mu$. We have the commutative diagram



with $\rho' \in M_c$. By Lemma 4, $\sigma''\rho' \in O_c$. Considering the equality $\rho''\omega\sigma' = \sigma''\rho'$ and applying Lemma 2(3), we infer the strictness of ω.

Thus, ω is an isomorphism.

Theorem 1 is proved.

We use the notations of [5]:

$$(\beta', \gamma') = H_\rho(\beta, \gamma), \quad (\beta_1, \gamma_1) = H^\sigma(\beta', \gamma'),$$
$$(\beta'', \gamma'') = H^\sigma(\beta, \gamma), \quad (\overline{\beta_1}, \overline{\gamma_1}) = H_\rho(\beta'', \gamma'').$$

In this language, Theorem 1 asserts that

$$H^{\sigma} \circ H_{\rho} = H_{\rho} \circ H^{\sigma}.$$

Denote $H^{\sigma} \circ H_{\rho} = H_{\rho} \circ H^{\sigma}$ by H_{ρ}^{σ} and E_1 by $H_{\rho}^{\sigma}(E)$.

The following assertion holds in an additive category with kernels and cokernels.

Lemma 7 [5]. (i) *If* $(\alpha, f) = I(\beta, g)$ *and* $(\alpha', f') = I(\alpha, g')$ *then* $(\alpha', ff') = I(\beta, gg')$.

(ii) *If* $(\beta, g) = U(\alpha, f)$ *and* $(\alpha, g') = U(\alpha', f')$ *then* $(\beta, gg') = U(\alpha', ff')$.

Corollary. $H_{\rho_1}^{\sigma_1} \circ H_{\rho}^{\sigma} = H_{\rho\rho_1}^{\sigma_1\sigma}$.

2 Exact Couples

Let $\mathcal{A}$ be a Raĭkov-semiabelian category. A zero sequence

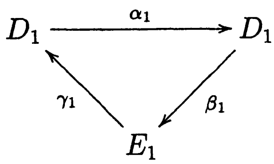
$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ {\scriptstyle\gamma}\nwarrow & & \swarrow{\scriptstyle\beta} \\ & E & \end{array} \qquad (4)$$

in $\mathcal{A}$ will be referred to as a *semiexact couple* [26]. A semiexact couple is called *exact* [5] if $\alpha\|\beta\|\gamma\|\alpha$. If α is strict then $\alpha = \rho\sigma$ with $\rho \in M_c$ and $\sigma \in P_c$; moreover, if (4) is an exact couple then $\rho = \ker\beta$ and $\sigma = \operatorname{coker}\gamma$. Thus we may consider diagram (2), where, in our case, $C = D$ and $C_1 = D_1 = \operatorname{Im}\alpha$ ($C_1 = D_1 = \operatorname{Ker}\beta = \operatorname{Coker}\gamma$ if sequence (4) is exact) and construct diagram (3) as above.

Theorem 2. *Let*

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ {\scriptstyle\gamma}\nwarrow & & \swarrow{\scriptstyle\beta} \\ & E & \end{array}$$

be a semiexact couple with $\alpha \in O_c$. *Put* $\alpha_1 = \sigma\rho$ *and* $(\beta_1, \gamma_1) = H_\rho^\sigma(\beta, \gamma)$ *with* C, D, C_1, *and* D_1 *as above. Then*



(5)

is a semiexact couple. Moreover, the following hold:

(i) *if* $\beta \| \gamma$ *then* $\beta_1 \| \gamma_1$*;*

(ii) *if* $\alpha \| \beta$ *and* $\beta \in O_c$ *then* $\alpha_1 \| \beta_1$*;*

(iii) *if* $\gamma \| \alpha$ *and* $\gamma \in O_c$ *then* $\gamma_1 \| \alpha_1$.

Proof. Since $0 = \gamma\beta = \rho\gamma_1\beta_1\sigma$, ρ is a monomorphism and σ is an epimorphism, it follows that $\gamma_1\beta_1 = 0$. Furthermore, the relations $\beta\alpha = \beta\rho\sigma = 0$ imply that $0 = \beta\rho = \rho'\beta'\rho$, and $\rho' \in M$ yields $\beta'\rho = 0$. Thus $\beta_1\alpha_1 = \beta_1\sigma\rho = \sigma'\beta'\rho = 0$. By duality, $\gamma_1\alpha_1 = 0$. Thus, sequence (5) is a semiexact couple, too.

Now, suppose that $\beta \| \gamma$. Then, by Lemma 6, from $\beta \| \gamma$ and $\beta = \rho'\beta'$ it follows that $\beta' \| \gamma\rho'$, i.e., $\beta' \| \rho\gamma'$. Hence, obviously, $\beta' \| \gamma'$. Furthermore, by Lemma 1(4), $\operatorname{coker} \beta' = (\operatorname{coker} \beta_1)\sigma'$. We also have $\gamma' = \gamma_1\sigma'$ and $\sigma' \in P_c$. Therefore, $\operatorname{coim} \gamma' = (\operatorname{coim} \gamma_1)\sigma'$. Involving the equality $\operatorname{coker} \beta' = \operatorname{coim} \gamma'$, we infer $\operatorname{coker} \beta_1 = \operatorname{coim} \gamma_1$, i.e., $\beta_1 \| \gamma_1$.

Assume that $\alpha \| \beta$ and $\beta \in O_c$. Since β is strict, from Lemma 3 it follows that the morphism $\widehat{\sigma}$ defined by the equality $\sigma(\ker \beta') = (\ker \beta_1)\widehat{\sigma}$ is an epimorphism. We have $\ker \beta' = \ker(\rho'\beta') = \ker \beta = \rho$ and hence $\alpha_1 = \sigma\rho = (\ker \beta_1)\widehat{\sigma}$. Therefore, $\ker \beta_1 = \operatorname{im} \alpha_1$, i.e., $\alpha_1 \| \beta_1$. Thus, (ii) holds.

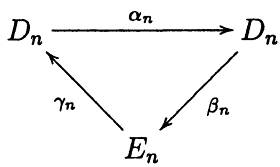
By duality, we have (iii).

Theorem 2 is proved.

By Theorem 2, for a semiexact couple (4) with strict endomorphism, like in the abelian case, we have the *derived semiexact couple* $\alpha_1 \| \beta_1 \| \gamma_1 \| \alpha_1$. By construction, if β and γ are strict then so are β_1 and γ_1.

Corollary. *For an exact couple* (4) *with strict* α, β, *and* γ, *the derived semiexact couple* (5) *is exact.*

Lemma 8. *If $\alpha^k \in O_c$ for $k \le n$ then the derivation of a semiexact couple* (4) *can be carried out n times. The nth derived couple of* (4) *is*



where

$$(\beta_n, \gamma_n) = H^{\sigma_{n-1}}_{\rho_{n-1}}(\beta_{n-1}, \gamma_{n-1}) = H^{\eta_n}_{\nu_n}(\beta, \gamma)$$

with $\eta_n = \sigma_{n-1} \dots \sigma_1 \sigma$ and $\nu_n = \rho\rho_1 \dots \rho_{n-1}$.

Proof. Using induction on k, we easily see that if $\alpha^k \in O_c$ for all $k \le n$ then, for each $k \le n-1$, the kth derived couple of (4) exists and $\alpha^n = \nu_k \alpha_k^{n-k} \eta_k$. In particular, α_{n-1} is defined and $\alpha^n = \nu_{n-1}\alpha_{n-1}\eta_{n-1}$. Since $\nu_{n-1} \in O_c$ and $\eta_{n-1} \in O_c$, it follows that $\alpha_{n-1} \in O_c$. Thus, by Theorem 2, the nth derived couple of (4) is also defined. Involving Lemma 7, we obtain that $(\beta_n, \gamma_n) = H^{\eta_n}_{\nu_n}(\beta, \gamma)$. The lemma is proved.

Recall that, for $A \in \mathrm{Ob}(\mathcal{A})$ and an endomorphism $\partial : A \longrightarrow A$ with $\partial^2 = 0$, the *(co)homology object* $H(A, \partial)$ is, by definition, $\mathrm{Coker}\,\iota$, where ι is the unique morphism such that $\partial = (\ker \partial)\iota$. As in [5], considering the case of $\partial = \beta\gamma : E \longrightarrow E$ and applying item 4 of Lemma 1, we have the following assertion.

Theorem 3. *For an exact couple* (4), $H(E, \partial) = E_1$.

Thus, if β, γ, and all the powers α^n are strict then we obtain the spectral sequence (E_n, ∂_n) with $\partial_n = \beta_n\gamma_n$ and $H(E_n, \partial_n) = E_{n+1}$. In particular, Lemma 2 yields the following assertion.

Corollary. *If $\beta, \gamma \in O_c$, and α is a kernel or a cokernel then the spectral sequence of an exact couple* (4) *is defined.*

In the category of (real or complex) Banach spaces and bounded linear operators, the strictness of an operator α means that its range $R(\alpha)$ is closed. Hence the corollary applies, for example, to injective and surjective bounded Fredholm operators α.

We remark that, generally speaking, the strictness of an endomorphism $\alpha : D \longrightarrow D$ in a Raĭkov-semiabelian category does not imply that α^2 is also strict. The following example was communicated to the author by Ya. Bazaĭkin.

Example. Suppose that we have a nonstrict continuous endomorphism f of a Banach space X. Define an endomorphism T of $X \oplus X$ by the formula $T(x, y) = (0, x + f(y))$. Then $R(T) = X \oplus X$. However, $T^2(x, y) = \big(0, f(x + f(y))\big)$. Hence $R(T^2) = 0 \oplus R(f)$ is not closed in $X \oplus X$ and thus T^2 is not strict.

Assuming that β, γ, and enough powers of α are strict, for $m, n \in \mathbb{Z}$, consider the diagram

$$\begin{array}{ccccc} D_m & \xrightarrow{\beta_{m,n}} & E_{m,n} & \xrightarrow{\gamma_{m,n}} & D_n \\ {\scriptstyle \eta_m}\uparrow & & & & \downarrow{\scriptstyle \nu_n} \\ D & \xrightarrow{\beta} & E & \xrightarrow{\gamma} & D \end{array},$$

where $(\beta_{m,n}, \gamma_{m,n}) = H^{\eta_m}_{\nu_n}(\beta, \gamma)$. Obviously, $E_n = E_{n,n} = H^{\eta_n}_{\nu_n}(\beta, \gamma)$. Easily, $I(\gamma_{m,n}, \rho_n) = (\gamma_{m,n+1}, \rho_{m,n})$ for some strict monomorphism $\rho_{m,n} : E_{m,n+1} \longrightarrow E_{m,n}$; dually, $U(\beta_{m,n}, \sigma_m) = (\beta_{m+1,n}, \sigma_{m,n})$ with $\sigma_{m,n} : E_{m,n} \longrightarrow E_{m+1,n}$ a strict epimorphism.

Theorem 4 (cf. Theorem 3.15 in [5]). *If enough powers of α are strict then the square*

$$\begin{array}{ccc} E_{m,n+1} & \xrightarrow{\sigma_{m,n+1}} & E_{m+1,n+1} \\ {\scriptstyle \rho_{m,n}}\downarrow & & \downarrow{\scriptstyle \rho_{m+1,n}} \\ E_{m,n} & \xrightarrow{\sigma_{m,n}} & E_{m+1,n} \end{array}$$

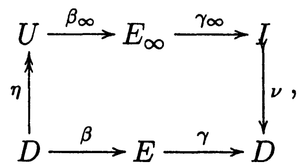
is a pullback and a pushout.

Proof. The argument of [5] holds in this situation.

Theorem 4 implies that all $\rho_{m,n}$ and $\sigma_{m,n}$ depend only on the spectral sequence.

Now, suppose that β, γ, and α^n for all n are strict and our Raĭkov-semiabelian category has countable direct sums and products. Assume that $(U, D \xrightarrow{\eta} U)$ is the direct limit of the family $\{\eta_n\}$ and $(I, I \xrightarrow{\nu} D)$

is the inverse limit of the family $\{\nu_n\}$. We have the diagram



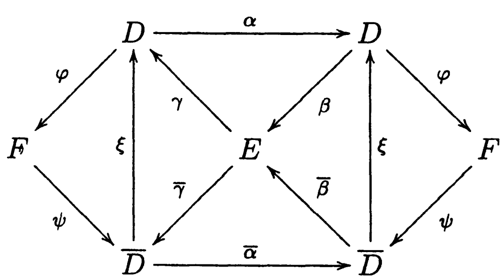
where $(\beta_\infty, \gamma_\infty) = H^\eta_\nu(\beta, \gamma)$. Then

$$E_\infty = \varinjlim_m \varprojlim_n E_{m,n} = \varprojlim_n \varinjlim_m E_{m,n} \tag{6}$$

(see §4 of [5] and [6]).

3 Spectral Sequence of a Rees System

Following [5], by a *Rees system* we mean a diagram



(7)

where

$$\alpha \| \beta \| \gamma \| \alpha \tag{8}$$

and

$$\overline{\alpha} \| \overline{\beta} \| \overline{\gamma} \| \overline{\alpha} \tag{9}$$

are exact couples and $\xi \| \varphi \| \psi \| \xi$; moreover, $\alpha\xi = \overline{\xi}\alpha$, $\beta\xi = \overline{\beta}$, and $\gamma = \xi\overline{\gamma}$.

Of course, the spectral sequences of (8) and (9) coincide if they exist. More exactly, if both (8) and (9) are derivable n times then they have the same term E_n in the nth derived couples (see [5]). We preserve the notations of Section 2 adding bar to all concerning (9).

Consider the diagram

$$\begin{array}{ccccc} D_1 & \xrightarrow{\varphi_1} & F_1 & \xrightarrow{\psi_1} & \overline{D}_1 \\ \uparrow{\scriptstyle\sigma} & & & & \downarrow{\scriptstyle\overline{\rho}} \\ D & \xrightarrow{\varphi} & F & \xrightarrow{\psi} & \overline{D} \end{array}$$

where $F_1 = H^{\sigma}_{\overline{\rho}}(F)$. One easily sees that $F_1 = F$, $\varphi_1\sigma = \varphi$, and $\overline{\rho}\psi_1 = \psi$. If α and $\bar{\alpha}$ are both strict then (8) and (9) are derivable as semiexact couples and we have an induced morphism $\xi_1 : \overline{D}_1 \longrightarrow D_1$ of the derived couples of (8) and (9), which is uniquely defined by $\xi_1\overline{\sigma} = \sigma\xi$ or $\rho\xi_1 = \xi\overline{\rho}$.

Theorem 5. *Suppose that $\alpha, \bar{\alpha}, \varphi, \psi$ are strict. Then $\xi_1 || \varphi_1 || \psi_1 || \xi_1$; if, moreover, $\xi \in O_c$ then $\xi_1 \in O_c$.*

Proof. The exactness is obtained using the argument of the proof of Theorem 2.

Suppose now that $\xi \in O_c$. Since $\overline{\beta}(\operatorname{im}\psi) = \beta\xi(\operatorname{im}\psi) = 0$, it follows that $\operatorname{im}\psi = \overline{\rho}a$ for some a. So, there is a commutative diagram

$$\begin{array}{ccccc} & & \operatorname{Ker}\overline{\beta} & & \\ & {\scriptstyle a}\nearrow & \downarrow{\scriptstyle \overline{\rho}\in M_c} & & \\ \operatorname{Im}\psi & \xrightarrow{\operatorname{im}\psi} & \overline{D} & \xrightarrow{\operatorname{coker}\psi} & \operatorname{Coker}\psi \end{array}$$

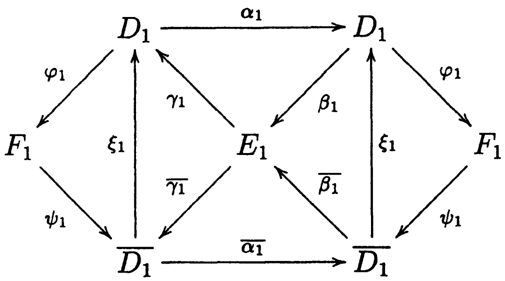
Lemma 4 now yields that $(\operatorname{coker}\psi)\overline{\rho} \in O_c$. Then

$$\rho\xi_1 = \xi\overline{\rho} = (\ker\varphi)(\operatorname{coker}\psi)\overline{\rho} \in O_c.$$

Hence $\xi_1 \in O_c$. Theorem 5 is proved.

Corollary. *Every Rees system* (7) *with $\alpha, \bar{\alpha}, \varphi, \psi \in O_c$ in a semia-*

belian category induces a derived Rees system



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