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FORBIDDEN FORESTS IN PRIESTLEY SPACES

by Richard N. BALL and Aleš PULTR

Dedicated to the memory of Japie Vermeulen

RESUME. Les auteurs présentent une formule du premier ordre caractérisant les treillis distributifs *L* dont les espaces de Priestley *P(L)* ne contiennent aucune copie d'une forêt finie *T*. Pour des algèbres de Heyting *L* le fait qu'il n'y ait pas d'ordre fini *T* dans *P(L)* est caractérisé par des équations ssi *T* est un arbre. Ils donnent une condition qui caractérise les treillis distributifs dont les espaces de Priestley ne contiennent aucune copie d'une forêt finie avec un seul point additionnel à la base.

1. Introduction

Priestley duality provides an important link between distributive lattices and (special) ordered topological spaces. Some properties of distributive lattices L are well-known to be expressed in forbidden configurations in the order structure of the corresponding spaces $\mathcal{P}(L)$. Thus, for instance, L is a Boolean algebra iff $\mathcal{P}(L)$ has just one layer, that is, if it contains no non-trivial chain. Or L is relatively normal iff $\mathcal{P}(L)$ is a forest, that is, if it contains no copy of the three element set of Figure 1; see Proposition 4.2. Or there are the distributive lattices of Adams and Beazer characterized by non-existence of an n-chain in $\mathcal{P}(L)$.

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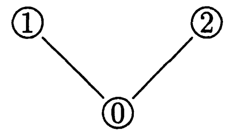

FIGURE 1. A poset is a forest iff it contans no copy of this poset.

The question naturally arises as to whether, given a finite poset T, the lattices L for which $\mathcal{P}(L)$ does not contain a copy of T can be characterized by a first order condition in the language of lattice theory. In this article we answer this question in the positive for trees and forests T, and for their duals. Moreover, for Heyting algebras L we show that trees are precisely the forbidden configurations which determine subvarieties (or, equivalently, subquasivarieties) of the variety of Heyting algebras.

The authors continue to investigate the general problem: to find "nice" conditions characterizing those lattices whose Priestley spaces admit no copy of a given poset. In this article, the techniques developed for forests are modified to treat forests with a single additional point at the bottom; this is Theorem 6.6. The condition that arises, however, is not first order on its face, and we do not know if a first order condition exists. Many more types of posets can be handled similarly, and in fact a solution to the general problem seems possible. But these results constitute another article in preparation.

2. Preliminaries

A *Priestley space* is an ordered compact space $(X, \tau, \leq)$ such that for any two $x, y \in X$ with $x \not\leq y$ there is a closed open increasing $U \subseteq X$ such that $x \in U$ and $y \notin U$. The category of Priestley spaces and monotone continuous maps will be denoted by

$$\mathbf{PSp}.$$

There is the famous *Priestley duality* (see, e.g., [9], [10]) between $\mathbf{PSp}$ and the category

$$\mathbf{DLat}$$

of bounded distributive lattices. The equivalence functors

$$\mathcal{P} : \mathbf{DLat} \to \mathbf{PSp}^{\mathrm{op}}, \quad \mathcal{D} : \mathbf{PSp} \to \mathbf{DLat}^{\mathrm{op}}$$

can be given as

$$\mathcal{P}(L) = \{x \mid x \text{ is a proper prime ideal on } L\}, \qquad \mathcal{P}(h)(x) = h^{-1}[x],$$
$$\mathcal{D}(X) = \{U \mid U \text{ clopen decreasing } \subseteq X\}, \qquad \mathcal{D}(f)(U) = f^{-1}[U],$$

$\mathcal{P}(L)$ is endowed with a suitable topology and partially ordered by inclusion; in this article we will be concerned solely with the partial order.

Finite Priestley spaces are the finite partially ordered sets (with the discrete topologies). Note that, however, the functor $\mathcal{D}$ associates with a finite $(X, \leq)$ the (lower) Alexandroff topology (not the discrete one) and that thus the restriction of $\mathcal{D}$ to the finite case coincides with the restriction of the open-set-lattice functor $\Omega : \mathbf{Top} \to \mathbf{Frm}$ to finite T_0-spaces. (**Frm** is the category of frames, that is, complete lattices with the distributivity $(\bigvee a_i) \wedge b = \bigvee (a_i \wedge b)$.)

The full subcategory of **DLat** consisting of the Heyting lattices, i.e., those lattices of **DLat** admitting the Heyting operation, will be denoted by

$$\mathbf{HLat}.$$

We will also consider the variety of Heyting algebras *with Heyting homomorphisms* and denote it by

$$\mathbf{Hey}.$$

The letters a, b, c and d, often decorated by subscripts or primes, will be reserved for the elements of the distributive lattices. The prime ideals on such lattices L (elements of $\mathcal{P}(L)$) will be typically denoted by x, y, z. For a subset $A \subseteq L$ we use

$$\mathrm{Idl}(A) = \{a \mid a \leq \bigvee A_0 \text{ for some finite } A_0 \subseteq A\} \text{ and}$$
$$\mathrm{Fltr}(A) = \{a \mid a \geq \bigwedge A_0 \text{ for some finite } A_0 \subseteq A\}$$

to designate the ideal and filter, respectively, generated by A. The set of all *proper* ideals on L (ordered by inclusion) will be denoted by

$$\mathfrak{J}(L).$$

Definition 2.1. If A is an ideal and B a filter on L we set

$$A \downarrow B \equiv \{c : \exists a \in A, b \in B \ (b \wedge c \leq a)\},$$
$$A \uparrow B \equiv \{c : \exists a \in A, b \in B \ (b \wedge c \leq a)\}.$$

Obviously, $A \downarrow B$ is an ideal and $A \uparrow B$ is a filter. For general subsets $A, B \subseteq L$ we write $A \downarrow B$ meaning $\mathrm{Idl}(A) \downarrow \mathrm{Fltr}(B)$. For elements $a, b \in L$ we write $a \downarrow B$ and $A \downarrow b$ for $\{a\} \downarrow B$ and $A \downarrow \{b\}$. Thus for instance $a \downarrow b = \{c \mid b \wedge c \leq a\}$. The dual conventions apply to $A \uparrow B$ and $a \uparrow b$.

Lemma 2.2. *Let $A, B \subseteq L$. Then the following, and their duals, hold.*

(1) $A \subseteq A \downarrow B$.
(2) $(A \downarrow B) \downarrow B = A \downarrow B$.
(3) $A \downarrow B$ *is proper iff* $A \cap B = \emptyset$.
(4) *For an ideal A and a filter B such that $A \downarrow B$ is proper (that is, $A \cap B = \emptyset$) there is an $x \in \mathcal{P}(L)$ such that $A \subseteq x$ and $x \cap B = \emptyset$.*

Proof. We leave the proofs of the dual statements to the reader. (1) $1 \wedge a \leq a$. (2) If $c \in (A \downarrow B) \downarrow B$ there is a $d \in A \downarrow B$ and a $b \in B$ such that $b \wedge c \leq d$. There is an $a \in A$ and a $b' \in B$ such that $b' \wedge d \leq a$; hence $(b \wedge b') \wedge c \leq a$. (3) $A \downarrow B$ is proper iff $1 \notin A \downarrow B$ iff $b \not\leq a$ for any $a \in A$ and $b \in B$.

(4) Using Zorn's lemma we obtain an ideal J maximal with respect to $J \supseteq A$ and $J \cap B = \emptyset$. J is prime, for if $c_1, c_2 \notin J$ it can only be because there exist $b_i \in \mathrm{Idl}\,(c_i, J) \cap B$, from which we get

$$b_1 \wedge b_2 \in \mathrm{Idl}\,(c_1, J) \cap \mathrm{Idl}\,(c_2, J) \cap B = \mathrm{Idl}\,(c_1 \wedge c_2, J) \cap B,$$

and this implies that $c_1 \wedge c_2 \notin J$. Put $x \equiv J$. □

The symbol

$$T$$

will designate a finite poset. This poset will play a central role in our considerations.

Definition 2.3. A map $m : T \to \mathfrak{J}(L)$ (resp. $m : T \to \mathcal{P}(L)$) is said to be *monotone* if

$$\tau \leq t \Longrightarrow m(\tau) \subseteq m(t),$$

and m is said to be a *copy of T* if it is monotone and

$$\tau \not\leq t \quad \Longrightarrow m(\tau) \not\subseteq m(t).$$

A mapping $a : T \to L$ is a *separator* of a monotone map $m : T \to \mathfrak{J}(L)$ (resp. $m : T \to \mathcal{P}(L)$) provided that for all $t, \tau \in T$,

$$a(\tau) \in m(t) \Longleftrightarrow t \not\leq \tau.$$

We say that a *separates* m.

We introduce a handy notation.

Definition 2.4. For a map $a : T \to L$ define an associated map $a' : T \to L$ by the rule

$$a'(t) \equiv \bigvee_{t \not\leq \tau} a(\tau), \quad t \in T.$$

Then a separates the monotone map m iff $a'(t) \in m(t)$ and $a(t) \notin m(t)$ for all $t \in T$.

Obviously a monotone map with a separator is a copy, whether in $\mathfrak{J}(L)$ or $\mathcal{P}(L)$. The converse holds for monotone maps into $\mathcal{P}(L)$.

Lemma 2.5. *Each copy $x : T \to \mathcal{P}(L)$ has a separator.*

Proof. Fix $t \in T$. For each $\tau \not\leq t$ choose a $b(\tau) \in x(\tau) \smallsetminus x(t)$ and set $a(t) = \bigwedge_{\tau \not\leq t} b(\tau)$. Since $x(t)$ is prime, $a(t) \notin x(t)$. Of course $a(t) \in x(\tau)$ for all the $\tau \not\leq t$, which gives $a'(t) \in x(t)$. □

3. Prohibiting forests

In a poset we will write

$$\tau \prec t$$

and say that τ *is a descendent of* t, or that t *is an ascendent of* τ, if $\tau < t$ and if for every s, $\tau \leq s \leq t$ implies that either $s = \tau$ or $s = t$. Denote by

- $\max(T)$ the set of all maximal elements of T, and by
- $\min(T)$ the set of all minimal elements of T.

A finite poset T is said to be a *forest* if each $t \in T$ has at most one ascendent. A forest T with exactly one maximal element is called a tree, and its maximal element will be denoted by e_T, or simply by e. Obviously, forests are precisely the disjoint unions of trees. *From this point until Theorem 5.1 we assume T to be a forest.* Without this assumption the next result, which is crucial for our purposes, does not hold.

Proposition 3.1. *Let $a : T \to L$ separate the copy $J : T \to \mathfrak{J}(L)$. If*

$$J(t) \downarrow a(t) = J(t), \quad t \in T,$$

then there is a copy $x : T \to \mathcal{P}(L)$, separated by the same a, such that $x(t) \supseteq J(t)$ for all t.

Proof. First, for maximal elements t of T choose $x(t) \in \mathcal{P}(L)$ such that $x(t) \supseteq J(t)$ and $x(t) \cap \mathrm{Fltr}(a(t)) = \emptyset$, by Lemma 2.2(4). Then $a(t) \notin x(t)$, and for any other maximal t' we have $a(t') \in J(t) \subseteq x(t)$.

Now suppose J has already been extended as desired on an increasing subset T' of T containing $\max(T)$. Let T'' be T' augmented by all the descendents τ of elements t minimal in T'. For $\tau \in T'' \smallsetminus T'$ with ascendent t we have

$$J(\tau) \cap \mathrm{Fltr}((L \smallsetminus x(t)) \cup \{a(\tau)\}) = \emptyset.$$

Indeed, otherwise we would have $b \wedge a(\tau) \in J(\tau)$ for some $b \notin x(t)$ and hence $b \in J(\tau) \downarrow a(\tau) = J(\tau) \subseteq J(t) \subseteq x(t)$, a contradiction. Choose by Lemma 2.2(4) an $x(\tau)$ in $\mathcal{P}(L)$ such that $x(\tau) \supseteq J(\tau)$ and $x(\tau) \cap \mathrm{Fltr}((L \smallsetminus x(t)) \cup \{a(\tau)\} = \emptyset$. In particular, $a(\tau) \notin x(\tau)$ and $a'(\tau) \in J(\tau) \subseteq x(\tau)$ by construction. When this process is complete, it is clear that we have a monotone map $x : T \to \mathcal{P}(L)$ separated by a. □

Definition 3.2. Let T be a forest and $a : T \to L$ any mapping. Define $I_a : T \to \mathfrak{J}(L)$ inductively by setting

$$\begin{aligned} I_a(t) &\equiv a'(t) \downarrow a(t), \quad t \in \min(T), \\ I_a(t) &\equiv \left(\bigcup_{\tau \prec t} I_a(\tau) \cup \{a'(t)\} \right) \downarrow a(t), \quad t \in T \setminus \min(T). \end{aligned}$$

Lemma 3.3. *If a separates a copy $x : T \to \mathcal{P}(L)$ then $I_a(t) \subseteq x(t)$ for all $t \in T$.*

Proof. We induct on T from the bottom up. If $t \in \min(T)$ then any $c \in I_a(t)$ satisfies $c \wedge a(t) \leq a'(t) \in x(t)$. Since $a(t) \notin x(t)$ and $x(t)$ is prime, $c \in x(t)$. Assume that $I_a(\tau) \subseteq x(\tau)$ for all $\tau \prec t$. If $c \in I_a(t)$ we have

$$c \wedge a(t) \leq \bigvee_{\tau \prec t} b_\tau \vee a'(t)$$

for some $b_\tau \in I_a(\tau) \subseteq x(\tau) \subseteq x(t)$, $\tau < t$. But then since $a'(t) \in x(t)$ we see that $c \wedge a(t) \in x(t)$, and because $x(t)$ is prime and $a(t) \notin x(t)$, c must therefore lie in $x(t)$. □

Lemma 3.4. *Let $a : T \to L$ be any map. Then the following hold for all $t, \tau \in T$.*

(1) *If $\tau \leq t$ then $I_a(\tau) \cup \{a'(\tau)\} \subseteq I_a(t)$.*
(2) *$I_a(t) \downarrow a(t) = I_a(t)$.*

Proof. (1) yields to a simple induction on T using Lemma 2.2(1), and (2) follows from Lemma 2.2(2). □

Theorem 3.5. *Let T be a finite forest and let L be a bounded distributive lattice. Then $\mathcal{P}(L)$ does not contain a copy of T iff for each $a : T \to L$ there is a $t \in \max(T)$ such that $I_a(t)$ is improper.*

Proof. Let $\mathcal{P}(L)$ contain a copy $x : T \to L$ separated by $a : T \to L$. Then by Lemma 3.3 each $I_a(t)$ lies in the proper ideal $x(t)$. On the other hand, if there is an $a : T \to L$ such that $I_a(t)$ is proper for all t then from Lemmas 2.2(3) and 3.4(2) we can conclude that for all $t \in T$, $a(t) \notin I_a(t)$ since otherwise $I_a(t)$ would be improper. Furthermore, Lemma 3.4(1) shows that $a'(t) \in I_a(t)$ for all t. Since I_a is monotone there is a copy $x : T \to \mathcal{P}(L)$ separated by a by Proposition 3.1. □

Our next objective is to show that the absence of a copy of T in $\mathcal{P}(L)$ can be characterized by the satisfaction in L of a specific first order sentence ψ_T in the language of lattice theory, Corollary 3.10. For that purpose let us rewrite the formulas of Definition 3.2 in a slightly more specific form. Recall that $a'(t)$ is the join of the $a(\sigma)$'s with $\sigma \not\geq t$. Now if $\sigma > \tau$ for some $\tau \prec t$ then $\sigma \geq t$ since T is a forest. Thus

$$a'(t) = \bigvee_{\tau \prec t} a'(\tau) \vee \bigvee_{\tau \prec t} a(\tau).$$

Now those $a'(\tau)$'s with $\tau \prec t$ are already in the $I_a(\tau)$'s by Lemma 2.2(1). Therefore the formulas of Definition 3.2 can be replaced by

$$\begin{aligned} I_a(t) &\equiv a'(t) \downarrow a(t), \quad t \in \min(T), \\ I_a(t) &\equiv \left(\bigcup_{\tau \prec t} I_a(\tau) \cup \left\{ \bigvee_{\tau \prec t} a(\tau) \right\} \right) \downarrow a(t), \quad t \in T \setminus \min(T). \end{aligned}$$

This observation motivates the following definition.

Definition 3.6. Let $a : T \to L$ be any function. A *T-supplement of a* is a function $c : T \to L$ such that for all $t \in T$,

$$\begin{aligned} a(t) \wedge c(t) &\leq a'(t), \quad t \in \min(T), \\ a(t) \wedge c(t) &\leq \bigvee_{\tau \prec t} c(\tau) \vee \bigvee_{\tau \prec t} a(\tau), \quad t \in T \setminus \min(T). \end{aligned}$$

A *T-complement of a* is a T-supplement c for which $c(t) = 1$ for some $t \in \max(T)$.

The remarks prior to Definition 3.6 make it clear that $c(t) \in I_a(t)$ for all $t \in T$ whenever c is a T-supplement of a. This allows us to reformulate Theorem 3.5 in terms of T-complements in Theorem 3.9.

Example 3.7. *Denote by* $\mathbf{n}$ *the chain* $\{0 < 1 < \cdots < n\}$*. An* $\mathbf{n}$*-complement of a system* $(a_0, a_1, \ldots, a_n)$ *is a* $(c_0, c_1, \ldots, c_n)$ *such that*

$$\begin{aligned} a_0 \wedge c_0 &= 0, \\ a_k \wedge c_k &\leq a_{k-1} \vee c_{k-1} \quad \text{for } 0 < k \leq n, \text{ and} \\ c_n &= 1. \end{aligned}$$

Thus, $(a, 1)$ *has exactly one* $\mathbf{1}$*-complement, namely the* $(c, 1)$ *with* c *the complement of* a *in* L*.*

Observe that if $(c_0, \ldots, c_{n-1}, c_n)$ *is an* $\mathbf{n}$*-complement of a system of the form* $(a_0, \ldots, a_{n-1}, 1)$ *then it is also an* $\mathbf{n}$*-complement of* $(a_0, \ldots, a_{n-1}, a_n)$ *for any* a_n*. Thus every system* $(a_0, \ldots, a_n)$ *has an* $\mathbf{n}$*-complement iff for every smaller system* $(a_0, \ldots a_{n-1})$ *there is some* $(c_0, \ldots, c_{n-1})$ *such that*

$$\begin{aligned} a_0 \wedge c_0 &= 0, \\ a_k \wedge c_k &\leq a_{k-1} \vee c_{k-1} \quad \text{for } 0 < k < n, \text{ and} \\ a_{n-1} \vee c_{n-1} &= 1. \end{aligned}$$

Example 3.8. *Let* T *be an antichain. Then* $a : T \to L$ *has a* T*-complement iff there is a* $t \in T$ *such that*

$$a(t) \leq \bigvee_{t \neq \tau} a(\tau).$$

Theorem 3.9. *Let* T *be a finite forest and* L *a bounded distributive lattice. Then* $\mathcal{P}(L)$ *does not contain a copy of* T *iff each mapping* $a : T \to L$ *has a* T*-complement. If* T *is a tree this in turn is equivalent to requiring that each* $a : T \to L$ *such that* $a(e_T) = 1$ *has a* T*-complement.*

Proof. To use Theorem 3.5 fix a map $a : T \to L$. We have already remarked that if c is a T-supplement of a then $c(t) \in I_a(t)$ for all $t \in T$. Since the $I_a(t)$'s are all proper, no $c(t)$ can be 1. On the other hand suppose that for some $t_0 \in T$ we have

$$1 \in I_a(t_0) = \left(\bigcup_{\tau \prec t_0} I_a(\tau) \cup \left\{ \bigvee_{\tau \prec t_0} a(\tau) \right\} \right) \downarrow a(t_0).$$

Without loss of generality we may assume that $t_0 \in \max(T)$. Set $c(t_0) \equiv 1$. Then there must be elements $c(\tau)$, $\tau \prec t_0$, such that

$$a(t_0) = a(t_0) \wedge c(t_0) \leq \bigvee_{\tau \prec t_0} c(\tau) \vee \bigvee_{\tau \prec t_0} a(\tau).$$

Now we can proceed inductively. If $c(t)$ has already been chosen in $I_a(t)$, we have $a(t) \wedge c(t) \leq \bigvee_{\tau \prec t} c(\tau) \vee \bigvee_{\tau \prec t} a(\tau)$ for some $c(\tau) \in I_a(\tau)$, $\tau \prec t$. Proceeding thus down to the minimum elements under t_0, and defining $c(t) \equiv 0$ for all t not below t_0, we obtain a T-complement of a. □

We remark in passing that it is not difficult to show that for any monotone map $a : T \to L$ and any $t \in T$, $I_a(t) = \{c(t) : c \ T\text{-supplement of } a\}$.

Corollary 3.10. *For any forest T there is a sentence ψ_T in the first order language of lattice theory such that for any bounded distributive lattice L, $\mathcal{P}(L)$ contains no copy of T iff ψ_T holds in L. Moreover, ψ_T is of the form $\forall\exists\phi$, where ϕ is quantifier-free and built up from atomic formulas by conjunction and disjunction. If T is a tree then ϕ is a conjunction of atomic formulas.*

Proof. Let x_t, y_t, $t \in T$, be syntactic variables. We think of maps $a, c : T \to L$ as assigning values $a(t)$ to variables x_t and $c(t)$ to variables y_t, $t \in T$. Furthermore, a look at Definition 3.6 reveals that most of the inequalities which make c a T-complement of a are combined by conjunction, with the only disjunction being

$$y_{t_1} = 1 \text{ or } y_{t_2} = 1 \text{ or} \ldots \text{ or } y_{t_n} = 1,$$

where $\max(T) = \{t_1, t_2, \ldots, t_n\}$. If T is a tree there is only a single disjunct, i.e., the formula is atomic. The result follows from Theorem 3.9. □

Example 3.11 (Example 3.7 revisited). *We obtain the characterization of Adams and Beazer* [1] *stating that $\mathcal{P}(L)$ contains no chain of length n iff for any $a_0, a_1, \ldots, a_{n-1}$ in L there are $c_0, c_1, \ldots, c_{n-1}$ in L such that*

$$\begin{aligned} a_0 \wedge c_0 &= 0, \\ a_k \wedge c_k &\leq a_{k-1} \vee c_{k-1} \quad \textit{for} \quad 0 < k < n, \quad \textit{and} \\ a_{n-1} \vee c_{n-1} &= 1. \end{aligned}$$

In particular, the order in $\mathcal{P}(L)$ is trivial iff L is a Boolean algebra.

Example 3.12 (Example 3.8 revisited). *Each antichain in $\mathcal{P}(L)$ has at most $n-1$ elements iff for any $a_1, \ldots, a_n$ in L there is a k such that*

$$a_k \leq \bigvee_{j \neq k} a_j.$$

Example 3.13. *No antichain with at least n elements, n fixed and ≥ 2, has an upper bound in $\mathcal{P}(L)$ iff for any $a_1, \ldots, a_n \in L$ there are $c'_1, \ldots, c'_n \in L$ such that for all k, $a_k \wedge c'_k \leq \bigvee_{j \neq k} a_j$, and $\bigvee_j a_j \vee \bigvee_j c'_j = 1$. Replacing the c'_k by $c_k = c'_k \vee \bigvee_{j \neq k} a_j$, we see that no antichain with at least n elements, n fixed and ≥ 2, has an upper bound in $\mathcal{P}(L)$ iff for any $a_1, \ldots, a_n \in L$ there are $c_1, \ldots, c_n \in L$ such that for all k,*

$$a_k \wedge c_k \leq \bigvee_{j \neq k} a_j, \quad \textit{and} \quad \bigvee_j c_j = 1.$$

Example 3.14. *$\mathcal{P}(L)$ has no independent system of n $\mathbf{1}$-chains iff for any $a_1, \ldots, a_n$ and $b_1 \ldots, b_n$ in L there are $c_1, \ldots, c_n$ in L and a k_0 such that*

$$\forall k, \; a_k \wedge c_k \leq \bigvee_{j \neq k} a_j \vee \bigvee_{j \neq k} b_j, \quad \textit{and} \quad b_{k_0} \leq a_{k_0} \vee c_{k_0}.$$

The characterizations in the theorems above can be easily modified for dual trees and dual forests. Denote by $|\mathcal{P}(L)|$ the poset structure of $\mathcal{P}(L)$. A prime ideal in L is a prime filter in L^{op}, and vice versa. Thus we have an anti-isomorphism

$$(J \mapsto L \smallsetminus J) : |\mathcal{P}(L)| \to |\mathcal{P}(L^{\mathrm{op}})|$$

and

$$|\mathcal{P}(L)|^{\mathrm{op}} \cong |\mathcal{P}(L^{\mathrm{op}})|.$$

Hence, T^{op} is forbidden in $\mathcal{P}(L)$ iff T is forbidden in $\mathcal{P}(L^{\mathrm{op}})$. For example, since T is isomorphic to T^{op} in Examples 3.11 and 3.12, the conditions that arise there must be equivalent to their duals. In fact, they are self-dual. This is obvious in the case of Exercise 3.11; to see that the condition of Example 3.12 is self dual, suppose that for any subset $\{a_i : 1 \leq i \leq n\} \subseteq L$ there is a k such that $a_k \leq \bigvee_{j \neq k} a_j$. Apply this condition to $\left\{\bigwedge_{i \neq j} a_i : 1 \leq i \leq n\right\}$ to get a k for which

$$\bigwedge_{i \neq k} a_i \leq \bigvee_{j \neq k} \bigwedge_{i \neq j} a_i = \bigvee_{j \neq k} \left(a_k \wedge \bigwedge_{i \neq j,k} a_i \right) = a_k \wedge \bigvee_{j \neq k} \bigwedge_{i \neq j,k} a_i,$$

which shows that $a_k \geq \bigwedge_{i \neq k} a_i$, i.e., L satisfies the dual condition.

4. Normal and relatively normal lattices

We digress for a brief discussion of normal and relatively normal lattices. Most of this material is well-known; we offer proofs only because the ideas are very close in spirit to those of the rest of this article.

Recall that a distributive lattice L is *normal* if for any two $a_1, a_2 \in L$ such that $a_1 \vee a_2 = 1$ there are $c_1, c_2 \in L$ such that $a_1 \vee c_1 \geq a_2$, $a_1 \vee c_2 \geq a_1$, and $c_1 \wedge c_2 = 0$. This is an extrapolation of the homonymous notion from topology: a space is normal iff the lattice of open sets is normal in the sense just defined. The following characterization of normal lattices in terms of their Priestley spaces is well known and is essentially due to Monteiro ([7], [8]), who proved it in the context of the open set lattice of a space.

Proposition 4.1. *The following are equivalent.*

(1) *L is normal.*
(2) *Every point of $\mathcal{P}(L)$ lies below a unique maximal point.*
(3) *In $\mathcal{P}(L)$, any pair of elements with a common lower bound must have a common upper bound.*

Proof. The equivalence of (2) and (3) is clear upon reflecting on a general fact about Priestley spaces: every point lies below a maximal point and above a minimal point. That is because the family of prime ideals of a distributive lattice is closed under both the union and intersection of chains.

Suppose that L is normal, and assume for the sake of argument that $\mathcal{P}(L)$ contains a point x_3 which lies below two distinct maximal points x_1 and x_2. Since $\mathrm{Idl}(x_1, x_2)$ is improper, there are $a_1 \in x_1 \setminus x_2$ and

$a_2 \in x_2 \setminus x_1$ such that $a_1 \vee a_2 = \top$. Find $c_1, c_2 \in L$ for which $a_1 \vee c_2 = c_1 \vee a_2 = 1$ and $c_1 \wedge c_2 = 0$. Then because x_3 is prime, it contains either c_1 or c_2. But if $c_1 \in x_3$ then $1 = c_1 \vee a_2 \in x_2$ and if $c_2 \in x_3$ then $1 = a_1 \vee c_2 \in x_1$, a contradiction in either case. We conclude that every point of $\mathcal{P}(L)$ lies below a unique maximal point.

Now suppose that L is not normal; this means we have elements a_1 and a_2 for which $a_1 \vee a_2 = 1$ but $\mathrm{Fltr}(a_1 \uparrow a_2, a_2 \uparrow a_1)$ is proper. Let $\mathcal{F}$ designate the set of pairs (F_1, F_2) of filters on L with the following properties.

- $a_1 \uparrow a_2 \subseteq F_1$ and $a_2 \uparrow a_1 \subseteq F_2$.
- $\mathrm{Fltr}(a_1 \uparrow F_1, a_2 \uparrow F_2)$ is proper.

Note that $\mathcal{F}$ is nonempty because it contains $(a_1 \uparrow a_2, a_2 \uparrow a_1)$ by hypothesis. Also note that, when ordered by the rule

$$(F_1, F_2) \leq (K_1, K_2) \iff F_1 \subseteq K_1 \text{ and } F_2 \subseteq K_2,$$

$\mathcal{F}$ is closed under joins of chains and so contains a maximal element (K_1, K_2). Observe that $K_1 = a_1 \uparrow K_1$ and $K_2 = a_2 \uparrow K_2$ by maximality, so that $a_1 \notin K_1$ and $a_2 \notin K_2$ lest the filters be improper. We claim that K_1 and K_2 are prime. To verify this claim consider $b, b' \notin K_1$. This implies that there are elements $k_1, k_1' \in K_1$ and $k_2, k_2' \in K_2$ such that

$$b \wedge k_1 \wedge k_2 = b' \wedge k_1' \wedge k_2' = 0.$$

But if we set $k_1'' \equiv k_1 \wedge k_1' \in K_1$ and $k_2'' \equiv k_2 \vee k_2' \in K_2$ we get

$$(b \vee b') \wedge k_1'' \wedge k_2'' = 0,$$

which implies $b \vee b' \notin K_1$. The proof that K_2 is prime is similar.

Let x_1, x_2 be maximal elements of $\mathcal{P}(L)$ containing the prime ideals $L \setminus K_1$ and $L \setminus K_2$, respectively. Then $a_1 \in x_1 \setminus x_2$ and $a_2 \in x_2 \setminus x_1$, so the maximal elements are distinct. Let x_2 be maximal among ideals disjoint form $\mathrm{Fltr}(K_1, K_2)$. Then x_3 is a common lower bound for x_1 and x_2 in $\mathcal{P}(L)$. □

A lattice L is *relatively normal* if for any two $a_1, a_2 \in L$ there are c_1, c_2 such that $a_1 \vee c_1 \geq a_2$, $a_2 \vee c_2 \geq a_1$, and $c_1 \wedge c_2 = 0$. In topology this corresponds to the requirement that each open subspace of the space in question is normal. Relative normality plays a fundamental, though sometimes unacknowledged, role in several areas of mathematics. For example, the lattice of cozero sets of a topological space is always relatively normal [6], and it is no accident that relative normality is the key ingredient in the construction of what are called Wallman covers of topological spaces [5]. A second example is the penetrating and beautiful structure theory of lattice-ordered groups (ℓ-groups for short), developed by Conrad and his students ([3] is the best general reference), based on the lattice of convex ℓ-subgroups. This lattice is algebraic, i.e., complete and generated by its compact elements, and the compact elements themselves form a relatively normal sublattice. The class of just such lattices, devoid of any group structure, was subsequently extensively investigated by Tsinakis and his students ([4], [11], [12]). Their work shows that a large part of the ℓ-group structure theory comes directly from the lattice theory, and in fact from the relative normality of the sublattice of compact elements of the lattice of convex ℓ-subgroups.

We content ourselves here with the observation that the relative normality of L is equivalent to $\mathcal{P}(L)$ being a forest. This is essentially due to Monteiro ([7], [8]), who proved it in the context of the open set lattice of a space.

Proposition 4.2. *L is relatively normal iff $\mathcal{P}(L)$ is a forest.*

Proof. The definition of relative normality is the dual of the condition of Example 3.13 for $n = 2$. Thus L is relatively normal iff no two unrelated elements of $\mathcal{P}(L)$ have a common lower bound, i.e., iff $\mathcal{P}(L)$ is a forest. $\square$

Thus Proposition 4.2 characterizes the relative normality of L by the prohibition of the poset of Figure 1 in $\mathcal{P}(L)$. The reader may naturally ask whether the normality of L can be likewise characterized. The answer is negative. Take a standard example of a normal space A with subspace B which is not normal, and let L and M be the topologies (sets of open sets) of A and B, respectively. Let $h : L \to M$ be the mapping which sends U to $U \cap B$. This is an onto lattice homomorphism, and it is easy to check that $\mathcal{P}(h) : \mathcal{P}(M) \to \mathcal{P}(L)$ is an order embedding. Now if normality of a lattice were characterized by the nonexistence of a copy of a finite poset in its Priestley space then such a copy would exist in $\mathcal{P}(M)$ but not in $\mathcal{P}(L)$, a contradiction.

5. Equations forbidding trees

We show that if L is a Heyting algebra then each map $a : T \to L$ has a largest T-supplement, which we will designate a^T. (We order maps $a, b : T \to L$ by declaring that $a \geq b$ if $a(t) \geq b(t)$ for all $t \in T$.) Thus any such map a will have a T-complement iff a^T is a T-complement. This fact eliminates the existential quantifier in the sentence ψ_T of Corollary 3.10. Consequently the Heyting algebra whose Priestley spaces contain no copy of a given fixed tree form a variety. The surprise is that only trees have this property; this is the content of Theorem 5.9.

For the rest of this section we assume that L is a Heyting lattice, i.e., that L is a bounded distributive lattice which admits the Heyting implication operation $\to$. When actually equipped with this operation, L becomes a Heyting algebra and the maps are required to preserve $\to$ as well as the lattice structure. We ask the reader to keep in mind the defining feature of $\to$, namely that for $a, b, c \in L$,

$$(*) \qquad a \leq b \to c \iff a \wedge c \leq b.$$

Definition 5.1. Let L be a Heyting algebra. For $a : T \to L$ define $a^T : T \to L$ inductively as follows.

$$\begin{aligned} a^T(t) &\equiv a(t) \to a'(t), \quad t \in \min(T), \\ a^T(t) &\equiv a(t) \to \left(\bigvee_{\tau \prec t} a^T(\tau) \vee \bigvee_{\tau \prec t} a(\tau)\right), \quad t \in T \setminus \min(T). \end{aligned}$$

Lemma 5.2. *Let L be a Heyting algebra. For any map $a : T \to L$, a^T is the largest T-supplement of a.*

Proof. Compare Definitions 3.6 and 5.1 in light of $(*)$. □

Corollary 5.3. *Let L be a Heyting algebra. Then a map $a : T \to L$ has a T-complement iff a^T is a T-complement.*

Corollary 5.4. *Let L be a Heyting algebra. For any map $a : T \to L$,*

$$I_a(t) = \mathrm{Idl}\left(a^T(t)\right)$$

for all $t \in T$.

Corollary 5.5. *Let L be a Heyting algebra. $\mathcal{P}(L)$ contains no copy of T iff for every map $a : T \to L$ we have $a^T(t) = 1$ for some $t \in \max(T)$.*

Corollary 5.6. *Let L be a Heyting algebra and T be a tree with top element e. Then $\mathcal{P}(L)$ contains no copy of T iff $a^T(e) = 1$ for every map $a : T \to L$.*

The condition of Corollary 5.6 can be captured by the satisfaction of a particular first-order formula in the language of Heyting algebras. For the $a^T(t)$'s of Definition 5.1 build up to

$$a^T(e) = p^T(a),$$

where p^T is a polynomial in $\vee$ and $\to$ into which the values of a are inserted. Thus we have the following.

Corollary 5.7. *For every tree T there is a polynomial p^T in $\vee$ and $\to$ with variables indexed by the nodes of T such that a Heyting algebra L satisfies the equation $p^T(a) = 1$ iff $\mathcal{P}(L)$ does not contain a copy of T.*

We pause to review the main features of Priestley duality in the context of Heyting algebras. For a subspace U of a Priestley space X, denote as usual

$$\downarrow U \equiv \{x : \exists u \in U \ (x \le u)\}, \quad \downarrow x \equiv \downarrow\{x\},$$
$$\uparrow U \equiv \{x : \exists u \in U \ (x \ge u)\}, \quad \uparrow x \equiv \uparrow\{x\}.$$

(The danger of confusing this notation with the use of $\uparrow$ and $\downarrow$ elsewhere in this article is minimal, since there they designated binary functions on subsets of the lattice, whereas here they designate unary functions on subsets of the Priestley space.)

Remark 5.8. *The following facts are well-known.*

(1) *$\mathcal{D}(X)$ is a Heyting algebra iff $\uparrow U$ is open for every open $U \subseteq X$.*
(2) *The Heyting operation in $\mathcal{D}(X)$, in terms of clopen downsets U and V, is then*

$$U \to V = X \setminus \uparrow(U \setminus V).$$

(3) *If $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ are Heyting algebras then a Priestley map $f : Y \to X$ yields a Heyting homomorphism $\mathcal{D}(f) : \mathcal{D}(X) \to \mathcal{D}(Y)$ iff for all $x \in X$,*

$$f(\downarrow x) = \downarrow f(x).$$

The last fact will be crucial in the remainder of this section.

For the rest of this article we suspend the convention that T is a forest, and assume henceforth only that T is a finite poset.

Theorem 5.9. *Let T be a finite poset, and let $\mathcal{X}$ be the class of Priestley spaces X such that the Priestley dual of X is a Heyting lattice, and such that X contains no copy of T. Let $\mathcal{V}$ be the class of Heyting algebras with Priestley spaces (of the underlying lattices) in $\mathcal{X}$. Then the following statements are equivalent.*

(1) *T is a tree.*
(2) *$\mathcal{V}$ is a subvariety of* **Hey** *determined by one extra equation in $\vee$ and $\rightarrow$.*
(3) *$\mathcal{V}$ is a subquasivariety of* **Hey**, *i.e., $\mathcal{V}$ is closed under products and subobjects.*

Proof. The implication from (1) to (2) is Proposition 5.7, and from (2) to (3) is trivial. So assume (3) to prove (1). Since $\mathcal{V}$ is closed under products, $\mathcal{X}$ is closed under sums and hence T has to be connected. Now define $\tilde{T}$ as the set of all words $w = t_1 t_2 \dots t_n$ in vertices t_i of T such that t_n is maximal and $t_i \prec t_{i+1}$ for all $i < n$. Order $\tilde{T}$ by declaring

$$w \leq w' \quad \text{iff} \quad w = w''w' \text{ for some } w''.$$

Define $f : \tilde{T} \rightarrow T$ by setting

$$f(t_1 \dots t_n) = t_1.$$

Then

(a) f is monotone and onto,
(b) $f(\downarrow w) = \downarrow f(w)$, and
(c) if $w \prec w'$ then w' is uniquely determined.

If $\tilde{L}$ and L are the Heyting algebras with Priestley spaces $\tilde{T}$ and T, we have L isomorphic to a subalgebra of $\tilde{L}$ by Remark 5.8(3). Hence $\tilde{L} \notin \mathcal{V}$ since otherwise $L \in \mathcal{V}$ and $T \in \mathcal{X}$. Thus $\tilde{T}$ contains a copy of T, and by (c) and the connectedness we see that T is a tree. □

Restricting ourselves to the finite case we similarly obtain the following.

Theorem 5.10. *Let T be a finite poset, let $\mathcal{X}$ be the class of finite posets containing no copy of T, and let $\mathcal{V}$ be the class of Heyting algebras with Priestley duals (of the underlying lattices) lying in $\mathcal{X}$. Then the following statements are equivalent.*

(1) *T is a tree.*

(2) *$\mathcal{V}$ is the class of finite algebras of a subvariety $\mathcal{V}$ of* **Hey** *determined by one extra equation in $\vee$ and $\rightarrow$.*
(3) *$\mathcal{V}$ is the class of finite algebras of a subquasivariety of* **Hey**.

6. Forests with a bottom point

The general problem we would like to solve is to find, for any finite poset T, a "nice" condition on a lattice L which is both necessary and sufficient to insure that $\mathcal{P}(L)$ admits no copy of T. In this section we make use of our techniques to provide a small step in the direction of the general problem by treating the case of a forest with one additional point at the bottom. The result is Theorem 6.6, and the reader may judge whether the condition on L which arises there is nice. However, this condition is not first order on its face, and we do not know if a first-order condition exists.

Throughout this section T designates a finite poset with least element t_0 such that $T' \equiv T \setminus \{t_0\}$ is a nonempty forest. We retain some of the notation used heretofore; for example, the map $a' : T \to L$ associated with the map $a : T \to L$ is still given by the rule

$$a'(t) = \bigvee_{t \not\leq \tau} a(\tau).$$

Note in particular that $a'(t_0) = 0$, and that as before, a separates a copy $m : T \to \mathfrak{I}(L)$ iff $a'(t) \in m(t)$ and $a(t) \notin m(t)$ for all $t \in T$. We must, however, modify slightly the notion of a T-supplement of a map $a : T \to L$.

Definition 6.1. Let $a : T \to L$ be any function. A *T-supplement of a* is a function $c : T \to L$ such that

$$\begin{aligned} a(t) \wedge c(t) &\leq c(t_0) \vee a'(t), \quad t \in \min(T'), \\ a(t) \wedge c(t) &\leq \bigvee_{\tau \prec t} c(\tau) \vee \bigvee_{\tau \prec t} a(t), \quad t \in T' \setminus \min(T'). \end{aligned}$$

A *T-complement of a* is a T-supplement c for which $c(t) = 1$ for some $t \in \max(T)$. We let

$$F(a) \equiv \mathrm{Fltr}\{c(t_0) \wedge a(t_0) : c \text{ is a } T\text{-complement of } a\}.$$

The distinction between the two notions of T-supplement, i.e., between Definitions 3.6 and 6.1, is subtle. The crucial difference is that, although $c(t_0)$ appears on the right-hand-side of some of the inequalities displayed in Definition 6.1, it never appears on the left-hand-side of such an inequality.

Lemma 6.2. *Let $x : T \to \mathcal{P}(L)$ be a copy and let $c : T \to L$ be a T-complement of a separator a of x. Then $c(t_0) \notin x(t_0)$.*

Proof. Put $S \equiv \{s \in T : c(s) \notin x(s)\}$. Since $c(t) = 1$ for some $t \in \max(T)$, we see that $t \in S \cap \max(T) \neq \emptyset$. And since for every $s \in S \cap T'$ we have $\bigvee_{\tau \prec s} c(\tau) \vee \bigvee_{\tau \prec s} a(\tau) \geq a(s) \wedge c(s) \notin x(s)$ and $\bigvee_{\tau \prec s} a(\tau) \in x(s)$, it follows that there is some $\tau \prec s$ such that $\tau \in S$. We conclude that $t_0 \in S$. □

Corollary 6.3. *If a separates a copy $x : T \to \mathcal{P}(L)$ then $F(a)$ is a proper filter.*

Proof. Since $c(t_0) \notin x(t_0)$ for any T-complement c of a by Lemma 6.6, and since $a(t_0) \notin x(t_0)$ by definition of separator, the generators of $F(a)$ all lie in the proper filter $L \setminus x(t_0)$. □

Proposition 6.4. *Suppose $a : T \to L$ is such that $F(a)$ is a proper filter. Then for any $I \in \mathfrak{I}(L)$ such that $I \cap F(a) = \emptyset$ there is a copy $x : T \to L$ separated by a such that $I \subseteq x(t_0)$.*

Proof. First use Lemma 2.2(4) to find a point y in $\mathcal{P}(L)$ such that $I \subseteq y$ and $y \cap F(a) = \emptyset$. Then define $J : T' \to \mathfrak{I}(L)$ inductively as follows.

- For $t \in \min(T')$, $J(t) \equiv (y, a'(t)) \downarrow a(t)$.
- For $t \in T' \setminus \min(T')$, $J(t) \equiv \left(\bigcup_{\tau \prec t} J(\tau) \cup \left\{\bigvee_{\tau \prec t} a(\tau)\right\}\right) \downarrow a(t)$.

We claim that $J(t)$ is proper for all $t \in T'$. For if not, then because J is monotone there are elements $t \in \max(T)$ for which $J(t)$ is improper. In this case we can define a T-complement c for a as follows. For $t \in \max(T)$ choose $c(t)$ to be any member of $J(t)$, subject only to the proviso that $c(t)$ is chosen to be 1 whenever $J(t)$ contains 1. Suppose now that $c(t)$ has been defined for some $t \in T' \setminus \min(T')$ in such a way that $c(t) \in J(t)$. Since $J(t) = \left(\bigcup_{\tau \prec t} J(\tau) \cup \left\{\bigvee_{\tau \prec t} a(\tau)\right\}\right) \downarrow a(t)$, there exist $c_\tau \in J(\tau)$, $\tau \prec t$, such that $\bigvee_{\tau \prec t} a(\tau) \vee \bigvee_{\tau \prec t} c_\tau \geq a(t) \wedge d(t)$. Put $c(\tau) \equiv c_\tau$ for $\tau \prec t$. Finally, suppose that $c(t)$ has been defined for all $t \in \min(T')$ in such a way that $c(t) \in J(t) = (y, a'(t)) \downarrow a(t)$, say $a(t) \wedge c(t) \leq b_t \vee a'(t)$ for $b_t \in y$. Define $c(t_0) \equiv \bigvee_{\min(T')} b_t \in y$. The resulting map $c : T \to L$ is clearly a T-complement for a such that $c(t_0) \in y$, contrary to the hypothesis that $y \cap F(a) = \emptyset$. This contradiction establishes the claim that $J(t)$ is proper for all $t \in T'$.

The claim shows that $a(t) \notin J(t)$ for any $t \in T'$, since otherwise $J(t) = J(t) \downarrow a(t)$ would be improper. But then a separates J, since $a'(t) \in J(t)$ for any $t \in T'$ by construction. Proposition 3.1 then supplies a copy $x : T' \to \mathcal{P}(L)$ of T' separated by the restriction of a to T' such that $J(t) \subseteq x(t)$ for all $t \in T'$. If we simply extend this map to T by defining $x(t_0) \equiv y$, we get a copy $x : T \to \mathcal{P}(L)$ of T separated by a such that $I \subseteq y \subseteq x(t_0)$. That a separates this extension of x is because $a'(t_0) = 0 \in y = x(t_0)$ and $a(t_0) \notin y = x(t_0)$. □

Corollary 6.5. *Let $a : T \to L$ be a map and I an ideal on L. Then there is a copy $x : T \to L$ separated by a such that $I \subseteq x(t_0)$ iff $I \cap F(a) = \emptyset$.*

We come to the major result of this section.

Theorem 6.6. *$\mathcal{P}(L)$ admits no copy of T iff $F(a)$ is improper for every map $a : T \to L$.*

The definition of T-complement of a map $a : T \to L$ is certainly first order in the constants $a(t)$, $t \in T$. But the condition that $F(a)$ be improper is not first order on its face because there seems to be no intrinsic bound on the number of generators b which this filter requires.

We close with a simple application of Theorem 6.6. Figure 2 shows the diamond $T = \{0, 1, 2, 3\}$.

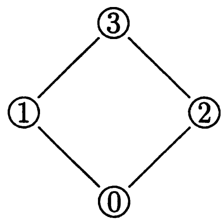

FIGURE 2. The diamond

Proposition 6.7. *$\mathcal{P}(L)$ contains no diamond iff*

$$F(a_1, a_2) \equiv$$
$$\equiv \mathrm{Fltr}\{b : \exists c_1, c_2 \ (c_1 \vee c_2 = 1,\ c_1 \wedge a_1 \le b \vee a_2,\ c_2 \wedge a_2 \le b \vee a_1)\}$$

is an improper filter for every pair $a_1, a_2 \in L$.

Proof. Suppose $F(a_1, a_2)$ is proper for some $a_1, a_2 \in L$. Define $a : T \to L$ by setting $a(1) \equiv a_1$, $a(2) \equiv a_2$, $a_0 \equiv a_1 \wedge a_2$, and $a(3) \equiv 1$. We claim that $F(a) \subseteq F(a_1, a_2)$. To establish this claim, consider an arbitrary T-complement c of a. By definition of T-complement we have these inequalities.

$$\begin{aligned} a(3) \wedge c(3) &\leq c(1) \vee c(2) \vee a'(3) = c(1) \vee c(2) \vee a_1 \vee a_2 \\ a(2) \wedge c(2) &\leq c(0) \vee a'(2) = c(0) \vee a_1 \\ a(1) \wedge c(1) &\leq c(0) \vee a'(1) = c(0) \vee a_2 \end{aligned}$$

If we set $c_1 \equiv c(1) \vee a_2$, $c_2 \equiv c(2) \vee a_1$, and $b \equiv c(0)$, we get the inequalities which define b as a generator of $F(a_1, a_2)$. This shows that $c(0) \in F(a_1, a_2)$. Since it is clear that $a_1 \in F(a_1, a_2)$ (set $c_1 = 1$ and $c_2 = 0$) and that $a_2 \in F(a_1, a_2)$ likewise, we see that $a(0) \in F(a_1, a_2)$ and hence that $F(a) \subseteq F(a_1, a_2)$. Theorem 6.6 then produces a copy of T in $\mathcal{P}(L)$.

Now suppose that $\mathcal{P}(L)$ admits a copy of T. Then by Theorem 6.6 there is some $a : T \to L$ for which $F(a)$ is proper. Put $a_1 \equiv a(0) \vee a(1)$ and $a_2 \equiv a(0) \vee a(2)$. We claim that $F(a_1, a_2) \subseteq F(a)$. To verify this claim consider a generator b of $F(a_1, a_2)$, say c_1 and c_2 satisfy $c_1 \vee c_2 = 1$, $c_1 \wedge a_1 \leq b \vee a_2$, and $c_2 \wedge a_2 \leq b \vee a_1$. Define $c : T \to L$ by setting $c(0) \equiv b$, $c(1) \equiv c_1$, $c(2) \equiv c_2$, and $c(3) \equiv 1$. Then it is routine to verify that c is a T-complement of a, so that $b \geq c(0) \wedge a(0) \in F(a)$ hence $b \in F(a)$. This proves the claim and the proposition. □

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