CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

B. JOHNSON
R. MCCARTHY

A classification of degree *n* functors, II

Cahiers de topologie et géométrie différentielle catégoriques, tome 44, nº 3 (2003), p. 163-216
<http://www.numdam.org/item?id=CTGDC_2003__44_3_163_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLIV-3 (2003)

A CLASSIFICATION OF DEGREE *n* FUNCTORS, II

by B. JOHNSON and R. McCARTHY

RESUME. Utilisant une théorie de calcul pour des foncteurs de catégories pointées vers des catégories abéliennes qu'ils ont développée précédemment, les auteurs prouvent que les foncteurs de degré n peuvent être classifiés en termes de modules sur une algèbre graduée différentielle $P_{n\times n}(C)$. Ils montrent de plus que les foncteurs homogènes de degré n ont des classifications naturelles en termes de 3 catégories de modules différentes. Ils utilisent les structures développées pour ces théorèmes de classification pour montrer que tous les foncteurs de degré n se factorisent par une certaine catégorie $P_n(C)$, étendant un résultat de Pirashvili. Cet article dépend des résultats établis dans la Partie I.

The Taylor series of a function is a tremendously important tool in analysis. A similar theory, the calculus of homotopy functors developed by Tom Goodwillie ([G1], [G2], [G3]), has recently been used to prove several important results in K-theory and homotopy theory. In [J-M3], we defined and established the basic properties for a theory of calculus for functors from pointed categories to abelian categories. Given a functor $F : \mathcal{C} \to Ch\mathcal{A}$ where $\mathcal{C}$ is a pointed category and $\mathcal{A}$ is a cocomplete abelian category, we showed that by using a particular cotriple one could construct a tower of functors and natural transformations (see figure 1). For each n, the functor P_nF is a degree n functor in the sense that its $n+1$st cross effect as defined by Eilenberg and Mac Lane ([E-M2]) is acyclic.

In this paper and its predecessor [J-M4], we show that by using the models for P_n given in [J-M3], degree n functors can be classified in terms of modules over a differential graded algebra $P_{n\times n}(C)$. We

The second author was supported by National Science Foundation grant # 1-5-30943 and a Sloan Fellowship.

also show that homogeneous degree n functors, i.e., degree n functors G for which $P_{n-1}G \simeq *$, can be classified in terms of modules over three different differential graded algebras. One of these classifications was inspired by Goodwillie's classification of homogeneous degree n functors of spaces ([G3]). These classifications extend a classification of linear functors proved in [J-M1]. As part of the development of these classifications we also show that all degree n functors arise naturally as functors on a particular category $P_n\mathcal{C}$, following a similar result for strictly degree n functors due to Pirashvili [P]. (A strictly degree n functor is one whose $n+1$st cross effect is isomorphic, rather than quasi-isomorphic, to 0.) † In addition, we develop a "rank" filtration of F, i.e., we look at approximations to F that agree with F on a specified collection of objects.

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
 & & P_{n+1}F \\
 & {\scriptstyle p_{n+1}F} \nearrow & \downarrow {\scriptstyle q_{n+1}F} \\
F & \xrightarrow{p_nF} & P_nF \\
 & {\scriptstyle p_{n-1}F} \searrow & \downarrow {\scriptstyle q_nF} \\
 & & P_{n-1}F \\
 & & \vdots \\
 & & \downarrow \\
 & & P_0F = F(*).
\end{array}
$$

figure 1

† For those familiar with [J-M2], degree n functors in this paper correspond to homologically degree n functors in [J-M2], and strictly degree n functors correspond to degree n functors in [J-M2].

The papers are organized as follows. The first paper ([J-M4]) comprises sections 1, 2, and 3. Sections 4, 5, 6, and the appendix are in this paper. We begin in section 1 by reviewing the Taylor tower of [J-M3] and describing some natural transformations arising from the tower to be used in this work. We then start developing the framework needed to state and prove the classification results.

The sequence of results forming this framework were motivated and can be understood by considering a classification result for additive functors proved independently by Eilenberg and Watts:

Theorem ([E], [W]). *Let F be an additive, right continuous (preserves cokernels and filtered colimits) functor from the category of right R-modules to the category of right S-modules for some rings R and S. Let G be the functor given by*

$$G(-) = - \otimes_R F(R).$$

There is a natural transformation $\eta : G \to F$ that is an isomorphism on all R-modules. That is, additive, right continuous functors are characterized by $R - S$ bimodules $F(R)$.

To prove this result, one first establishes that $F(R)$ has the required bimodule structure and constructs the natural transformation η. The isomorphism is then proven in stages using various properties of the functors. The first stage is the observation that η is an isomorphism at R. Additivity of the functors then establishes the isomorphism at all finitely generated free R modules. From there, the fact that both functors preserve filtered colimits guarantees that η is an isomorphism on all free R modules. Finally, since every R module has a resolution by free R modules and the functors preserve cokernels, an isomorphism for all R modules is ensured. In essence, this proof depends upon two properties: the category of R modules has a generating object R and the functors behave well with respect to the operations needed to generate all R modules from R.

We will prove a similar result for degree n functors from a basepointed category $\mathcal{C}$ to $Ch\mathcal{A}$ for some abelian category $\mathcal{A}$. When considering this more general setting, one notices immediately that $\mathcal{C}$

lacks the generating object that was so useful for the classification of additive functors. This leads us to consider instead subcategories of $\mathcal{C}$ generated by objects C in $\mathcal{C}$. We call such subcategories "lines generated by C" and develop the notion of "functors defined along C" in parallel with the right continuous property used in the Eilenberg-Watts result. This material will be developed in section 2. The principal result will be the following.

Theorem 2.11. *Let $F, G : \mathcal{C} \to Ch\mathcal{A}$ be degree n functors defined along an object C in $\mathcal{C}$. A natural transformation $\eta : F \to G$ is an equivalence if and only if η is an equivalence at $\mathbf{n}_C = \vee_{i=1}^{n} C$.*

The theorem allows us to prove classification results by simply establishing equivalences at the object $\mathbf{n}_C$. In general, the class of functors that are determined by their value at $\mathbf{n}_C$ is strictly larger than the class of degree n functors defined along C. We refer to the functors that are determined by their value at $\mathbf{n}_C$ as rank n functors and explore the properties of such functors in section 3. In particular we show that any functor F from $\mathcal{C}$ to $Ch\mathcal{A}$ has a filtration of functors $\{\mathcal{L}_k F\}_{k\geq 0}$ of rank k, and show that degree n is a strictly stronger condition than rank n.

We will classify degree n functors defined along an object C by showing that any such functor F is equivalent to the functor

$$\perp_C^* (P_n F(\mathbf{n}_C) \hat{\otimes}_{P_{n\times n}(C)} P_n(C, -))$$

where $P_n(C, -) = P_n \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_C, -)]$ and $P_{n\times n}(C)$ is the differential graded algebra $P_n \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}_C, \mathbf{n}_C)]$. (The symbol $\perp_C^*$ indicates the resolution of a functor along C and is defined in section 2.) Constructing such a functor requires that $P_n(C, -)$ be given certain differential graded algebra and module structures. The properties underlying these structures are developed in section 4, although the actual algebra and module structures are not specified until section 5. In section 4, we use the properties that must be established for the algebra and module structures to construct a category $P_n\mathcal{C}$ through which all degree n functors must factor. This extends a result due to Pirashvili ([P]) for strictly degree n functors.

In section 5 we state and prove our classification theorems using the results of the previous sections. We present four classification theorems, one for degree n functors defined along C, and three for homogeneous degree n functors defined along C:

1) degree n functors defined along C are classified by modules over the DGA $P_{n\times n}(C)$
2) homogeneous degree n functors defined along C are classified by modules over a DGA $D_{n\times n}(C)$, modules over a DGA $D_1(C)$, and modules over a wreath product $D_{1\times 1}(C) \int \Sigma_n$.

In section 6, we consider various natural operations developed in [J-M3] that change the degree of a functor and determine their effect on the classification results of section 5. In particular, we look at differentiation, the structure maps in the Taylor tower, composition, and the inclusions from degree n to higher degree functors and from homogeneous degree n to degree n functors. We also include an appendix explaining the relationship between the three different classifications of homogeneous degree n functors.

4. The category $P_n\mathcal{C}$

Pirashvili has shown that strictly degree n functors from a basepointed category $\mathcal{C}$ with finite coproducts to an abelian category $\mathcal{A}$ are naturally isomorphic to linear functors from a certain category $p_n\mathcal{C}$ to $\mathcal{A}$ ([P]). Our objective in this section is to show how this characterization can be extended to degree n functors. Doing so will involve defining a new category $P_n\mathcal{C}$ and showing that every degree n functor can be obtained via a functor from $P_n\mathcal{C}$ to $Ch\mathcal{A}$. Many of the results proved in this section will be used in section 5 to define the algebra and module structures for the classification theorems. Throughout the section we will assume that $\mathcal{C}$ is a basepointed category with finite coproducts and $\mathcal{A}$ is a full and faithful subcategory of left A modules for some ring A.

We note that one should perhaps use the language of a category enriched in a monoidal category for the remainder of the paper as some definitions and constructions would become easier. However, as this translation is easily made for those already familiar with the language of monoidal categories we have decided not to add an additional layer of abstraction for those who are not yet familiar with these useful concepts.

We begin the section by reviewing Pirashvili's definition of $p_n\mathcal{C}$ and the isomorphism between strictly degree n functors and linear functors on $p_n\mathcal{C}$ ([P]). We will be using the following type of "linear" functors.

Definition 4.1. *For a ring A, a category $\mathcal{E}$ is* A-linear *provided that for any objects X and Y in $\mathcal{E}$, $\mathrm{Hom}_{\mathcal{E}}(X,Y)$ is an A-module and composition is bilinear with respect to this module structure. Thus a $\mathbb{Z}$-linear category is what is also called a preadditive category or a ringoid. A functor $F : \mathcal{E} \to \mathcal{E}'$ between A-linear categories $\mathcal{E}$ and $\mathcal{E}'$ is an* A-linear functor *if $\mathrm{Hom}_{\mathcal{E}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{E}'}(F(X),F(Y))$ is an A-module homomorphism. A category $\mathcal{B}$ is said to be a* differential graded category *with respect to the ring A, or DG-category, if for all objects X, Y, and Z in $\mathcal{B}$, $\mathrm{Hom}_{\mathcal{B}}(X,Y)$, $\mathrm{Hom}_{\mathcal{B}}(Y,Z)$, and $\mathrm{Hom}_{\mathcal{B}}(X,Z)$ are differential graded A modules and the composition rule makes the following diagram commute*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{B}}(X,Y) \otimes_A \mathrm{Hom}_{\mathcal{B}}(Y,Z) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathcal{B}}(X,Z) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
\mathrm{Hom}_{\mathcal{B}}(X,Y) \otimes_A \mathrm{Hom}_{\mathcal{B}}(Y,Z) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathcal{B}}(X,Z)
\end{array}$$

where $\otimes_A$ indicates the usual tensor product of differential graded modules. A functor $G : \mathcal{A} \to \mathcal{A}'$ between DG-categories $\mathcal{A}$ and $\mathcal{A}'$ is DG-linear provided that $\mathrm{Hom}_{\mathcal{A}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}'}(G(X),G(Y))$ is a chain map for all objects X and Y in $\mathcal{A}$.

We will let $\mathbf{A} - \mathrm{Func}(\mathcal{E},\mathcal{E}')$ denote the category of $\mathbf{A}$-linear functors from $\mathcal{E}$ to $\mathcal{E}'$ and $\mathrm{DG} - \mathrm{Func}(\mathcal{A},\mathcal{A}')$ denote the category of DG-linear functors from $\mathcal{A}$ to $\mathcal{A}'$. In this section, we will be working with $\mathbb{Z}$-linear categories and functors. Note that every additive category is $\mathbb{Z}$ -linear, though not all $\mathbb{Z}$ -linear categories are additive.

Remark 4.2. Pirashvili's characterization is based on modifying the following classical situation for categories and functors. Given a category $\mathcal{C}$, $\mathbb{Z}[\mathcal{C}]$ is the category with the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{\mathbb{Z}[\mathcal{C}]}(C, C') = \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(C, C')].$$

Composition is given by the composite:

$$\begin{array}{c}
\mathrm{Hom}_{\mathbb{Z}[\mathcal{C}]}(C, C') \times \mathrm{Hom}_{\mathbb{Z}[\mathcal{C}]}(C', C'') \\
\downarrow \\
\mathrm{Hom}_{\mathbb{Z}[\mathcal{C}]}(C, C') \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}[\mathcal{C}]}(C', C'') \\
\downarrow \cong \\
\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{C}}(C', C'')] \\
\downarrow \mathbb{Z}[\circ] \\
\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(C, C'')] = \mathrm{Hom}_{\mathbb{Z}[\mathcal{C}]}(C, C'').
\end{array}$$

Let h be the functor from $\mathcal{C}$ to $\mathbb{Z}[\mathcal{C}]$ that is the identity on objects and takes a morphism α to $1 \cdot [\alpha]$. Since $Ch\mathcal{A}$ is additive, any functor F from $\mathcal{C}$ to $Ch\mathcal{A}$ factors through h as $F = \mathbb{Z}[F] \circ h$ where $\mathbb{Z}[F](\Sigma_{i=1}^{n} z_i[\alpha_i]) = \Sigma_{i=1}^{n} z_i F(\alpha_i)$. In other words, the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & Ch\mathcal{A} \\
\downarrow h & \nearrow & \mathbb{Z}[F] \\
\mathbb{Z}[\mathcal{C}] & &
\end{array}$$

commutes. Since $\mathbb{Z}[F]$ is always a $\mathbb{Z}$-linear functor, the map

$$\mathbb{Z} - \mathrm{Func}(\mathbb{Z}[\mathcal{C}], Ch\mathcal{A}) \xrightarrow{h^*} \mathrm{Func}(\mathcal{C}, Ch\mathcal{A})$$

is a split surjection, i.e., all functors from $\mathcal{C}$ to $Ch\mathcal{A}$ are determined by a $\mathbb{Z}$-linear functor from $\mathbb{Z}[\mathcal{C}]$ to $Ch\mathcal{A}$.

In [P], Pirashvili developed a similar factorization for strictly degree n functors. He does so by defining a category $p_n\mathcal{C}$ whose objects are those of $\mathcal{C}$ and whose morphisms are given by

$$\mathrm{Hom}_{p_n\mathcal{C}}(C, C') = p_n(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(C, \star)])(C')$$

where $p_n = H_0P_n$ (see 1.10) and $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(C,\star)]$ is treated as a functor from $\mathcal{C}$ to abelian groups. The composition rule for these morphisms will be defined later in the section (theorem 4.8). We will also use $p_n : \mathbb{Z}[\mathcal{C}] \longrightarrow p_n\mathcal{C}$ to denote the functor that is the identity on objects and is defined on morphism sets as H_0p_n where in this case p_n refers to the natural transformation from a functor to the nth term in its Taylor tower (defined in 1.10). Pirashvili showed that a functor F from $\mathcal{C}$ to $\mathcal{A}$ is strictly degree n if and only if $F = G \circ p_n \circ h$ for some $\mathbb{Z}$ -linear functor G. That is,

$$\mathbb{Z} - \mathrm{Func}(p_n\mathcal{C}, \mathcal{A}) \overset{(p_n \circ h)^*}{\longrightarrow} n - \mathrm{Func}(\mathcal{C}, \mathcal{A})$$

is an isomorphism, where $n - \mathrm{Func}(\mathcal{C}, \mathcal{A})$ is the category of strictly degree n functors from $\mathcal{C}$ to $\mathcal{A}$.

To extend Pirashvili's factorization to degree n functors, we will construct a new category, $P_n\mathcal{C}$. The objects of this category will be the same as those of $\mathcal{C}$, and for objects C and C' in $\mathcal{C}$, the morphism set is defined by

$$\mathrm{Hom}_{P_n\mathcal{C}}(C, C') = P_n\mathbb{Z}[\mathrm{Hom}(C,\star)](C').$$

We will show that $P_n\mathcal{C}$ forms a DG-category. To define the composition of morphisms in the category $P_n\mathcal{C}$ and to produce a functor from $\mathrm{Func}(\mathcal{C}, Ch\mathcal{A})$ to the category of DG-linear functors from $P_n\mathcal{C}$ to $Ch\mathcal{A}$, we use the following.

Remark 4.3. Let H be a bifunctor from $\mathcal{C}^{op} \times \mathcal{C}$ to $Ch(\mathbb{Z}-Mod)$ and F and G be functors from $\mathcal{C}$ to $Ch\mathcal{A}$. Given a natural transformation of bifunctors $\gamma : H \longrightarrow \mathrm{Hom}_{Ch\mathcal{A}}(P_nF, P_nG)$, we can construct natural transformations

$$P_n\hat{\gamma}(X,Y) : P_nF(X) \otimes P_nH(X,\star)(Y) \longrightarrow P_nG(Y)$$

$$P_n\gamma(X,Y) : P_nH(X,\star)(Y) \longrightarrow \mathrm{Hom}_{Ch\mathcal{A}}(P_nF(X), P_nG(Y))$$

as follows.

By adjunction, a transformation $\gamma : H \longrightarrow \mathrm{Hom}_{Ch\mathcal{A}}(P_nF, P_nG)$ produces a natural transformation of bifunctors (where the tensor product of chain complexes is obtained by using $Tot^{\oplus}$):

$$P_nF \otimes H \xrightarrow{\hat{\gamma}} P_nG.$$

Fixing the first variable yields a natural transformation of functors, $\hat{\gamma}_X$. Since $P_nF(X) \otimes \star$ is a degree one endofunctor of $Ch\mathcal{A}$, we have a natural transformation given by the composite

$$\begin{array}{rcl} P_n\hat{\gamma}_X : P_nF(X) \otimes P_nH(X, \star) & \cong & P_n(P_nF(X) \otimes H(X, \star)) \\ & & \downarrow {\scriptstyle P_n(\hat{\gamma}_X)} \\ & & P_nP_nG(\star) \\ & & \downarrow {\scriptstyle \Sigma} \\ & & P_nG(\star) \end{array}$$

where Σ is the natural transformation defined in 1.16. This extends to a natural transformation of bifunctors

$$P_n\hat{\gamma}(X,Y) : P_nF(X) \otimes (P_nH(X, \star))(Y) \longrightarrow P_nG(Y)$$

that by adjunction produces a natural transformation:

$$P_n\gamma(X,Y) : P_nH(X, \star)(Y) \longrightarrow \mathrm{Hom}_{Ch\mathcal{A}}(P_nF(X), P_nG(Y)).$$

Using $P_n\gamma$ and $P_n\hat{\gamma}$, we can produce the natural transformation needed to define composition in $P_n\mathcal{C}$.

Example 4.4. Let $\mathbb{Z}\mathrm{H}$ be the bifunctor from $\mathcal{C}^{op} \times \mathcal{C}$ to $\mathbb{Z} - Mod$ defined by

$$\mathbb{Z}\mathrm{H}(X,Y) := \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X,Y)].$$

For $C \in \mathcal{C}$, let $\mathbb{Z}^C$ be the functor from $\mathcal{C}$ to $\mathbb{Z} - Mod$ defined by:

$$\mathbb{Z}^C(\star) := \mathbb{Z}\mathrm{H}(C, \star) = \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(C, \star)].$$

Composition produces a natural transformation

$$\circ : \mathbb{Z}\mathrm{H} \xrightarrow{\circ} \mathrm{Hom}_{\mathbb{Z}-Mod}(\mathbb{Z}^C, \mathbb{Z}^C) \xrightarrow{P_n} \mathrm{Hom}_{Ch\mathcal{A}}(P_n\mathbb{Z}^C, P_n\mathbb{Z}^C)$$

and so by remark 4.3 we have a natural transformation

$$\begin{array}{rcl}(P_n\hat{\circ})_C : P_n\mathbb{Z}^X(C) \otimes P_n\mathbb{Z}^C(Y) & = & P_n\mathbb{Z}^X(C) \otimes (P_n\mathbb{Z}\mathrm{H}(C,\star))(Y) \\ & & \downarrow \\ & & P_n\mathbb{Z}^X(Y).\end{array}$$

We use the following generalization of example 4.4 to produce a functor from $\mathrm{Func}(\mathcal{C}, Ch\mathcal{A})$ to $\mathrm{DG}-\mathrm{Func}(P_n\mathcal{C}, Ch\mathcal{A})$.

Example 4.5. Let F be a functor from $\mathcal{C}$ to $Ch\mathcal{A}$. Let γ be the natural transformation from $\mathbb{Z}\mathrm{H}$ to $\mathrm{Hom}_{Ch\mathcal{A}}(P_nF, P_nF)$ given by $\mathbb{Z}[P_nF]$ where $\mathbb{Z}[P_nF]$ is the functor defined in remark 4.2. By 4.3 we obtain a natural transformation

$$P_n\gamma : P_n\mathbb{Z}^X(Y) \longrightarrow \mathrm{Hom}_{Ch\mathcal{A}}(P_nF(X), P_nF(Y)).$$

To conclude that the transformation of example 4.4 makes $P_n\mathcal{C}$ a category and that the transformation of example 4.5 yields the desired functor of functor categories, we use the next lemma.

Lemma 4.6. *For a functor $F : \mathcal{C} \longrightarrow Ch\mathcal{A}$ and objects C, D, and E, the diagram below commutes:*

$$\begin{array}{ccc}P_nF(C) \otimes P_n\mathbb{Z}^C(D) \otimes P_n\mathbb{Z}^D(E) & \xrightarrow{P_n\hat{\gamma}\otimes\mathrm{id}} & P_nF(D) \otimes P_n\mathbb{Z}^D(E) \\ \downarrow{\scriptstyle \mathrm{id}\otimes P_n\hat{\circ}} & & \downarrow{\scriptstyle P_n\hat{\gamma}} \\ P_nF(C) \otimes P_n\mathbb{Z}^C(E) & \xrightarrow{P_n\hat{\gamma}} & P_nF(E).\end{array}$$

Proof. To save space we write $F(C)$ as F_C and consider the pair of composable diagrams (4.7):

$$\begin{array}{ccc}
P_nF_C \otimes P_n\mathbb{Z}^C(D) \otimes P_n\mathbb{Z}^D(E) & \xrightarrow{P_n(\hat\gamma)} & P_nP_nF_D \otimes P_n\mathbb{Z}^D(E) \\
\downarrow{\scriptstyle P_n(\hat o)} & (A) & \downarrow{\scriptstyle P_nP_n(\hat\gamma)} \\
P_nF_C \otimes P_nP_n\mathbb{Z}^C(E) & \xrightarrow{P_nP_n(\hat\gamma)} & P_nP_nP_nF_E \\
\downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle P_n(\Sigma)} \\
P_nF_C \otimes P_n\mathbb{Z}^C(E) & \xrightarrow{P_n(\hat\gamma)} & P_nP_nF_E.
\end{array}$$

$$\begin{array}{ccc}
P_nP_nF_D \otimes P_n\mathbb{Z}^D(E) & \xrightarrow{\Sigma} & P_nF_D \otimes P_n\mathbb{Z}^D(E) \\
\downarrow{\scriptstyle P_nP_n(\hat\gamma)} & & \downarrow{\scriptstyle P_n(\hat\gamma)} \\
P_nP_nP_nF_E & \xrightarrow{\Sigma(P_n)} & P_nP_nF_E \\
\downarrow{\scriptstyle P_n(\Sigma)} & & \downarrow{\scriptstyle \Sigma} \\
P_nP_nF_E & \xrightarrow{\Sigma} & P_nF_E.
\end{array}$$

We see that the upper left square (A) in the diagram commutes by considering the following. The diagram

$$\begin{array}{ccc}
P_nF(C) \otimes \mathbb{Z}^C(D) \otimes \mathbb{Z}^D(E) & \xrightarrow{\hat\gamma \otimes 1} & P_nF(D) \otimes \mathbb{Z}^D(E) \\
\downarrow{\scriptstyle 1 \otimes \hat o} & & \downarrow{\scriptstyle \hat\gamma} \\
P_nF(C) \otimes \mathbb{Z}^C(E) & \xrightarrow{\hat\gamma} & P_nF(E)
\end{array}$$

commutes by the definitions of $\hat\gamma$ and $\hat o$ in examples 4.4 and 4.5. One way of interpreting this diagram is to say that $\hat\gamma$ is a natural transformation from the functor $P_nF(C) \otimes \mathbb{Z}^C(-)$ to the functor P_nF since the vertical maps involve composing with elements of $\mathrm{Hom}_{\mathcal{C}}(D, E)$. Applying the functor P_n to the diagram (where we consider $P_nF(C) \otimes \mathbb{Z}^C(-)$ as a functor) we get the following commuting diagram:

$$\begin{array}{ccc}
P_nF(C) \otimes P_n\mathbb{Z}^C(D) \otimes \mathbb{Z}^D(E) & \xrightarrow{P_n(\hat\gamma) \otimes 1} & P_nP_nF(D) \otimes \mathbb{Z}^D(E) \\
\downarrow{\scriptstyle 1 \otimes P_n(\hat o)} & & \downarrow{\scriptstyle P_n(\hat\gamma)} \\
P_nF(C) \otimes P_n\mathbb{Z}^C(E) & \xrightarrow{P_n(\hat\gamma)} & P_nP_nF(E).
\end{array}$$

Now considering everything as a functor of E and noting that P_n is a functor, we see that the diagram

$$\begin{tikzcd}
 & P_n[P_nF(C) \otimes P_n\mathbb{Z}^C(D) \otimes \mathbb{Z}^D(-)](E) \arrow[dl, "P_n(1 \otimes P_n(\hat{o})) \circ"'] \arrow[dr, "P_n(P_n(\hat{\gamma}) \otimes 1)"] & \\
P_n[P_nF(C) \otimes P_n\mathbb{Z}^C(-)](E) \arrow[dr, "P_nP_n(\hat{\gamma})"'] & & P_n[P_n[P_nF(D) \otimes \mathbb{Z}^D(-)]](E) \arrow[dl, "P_nP_n(\hat{\gamma})"] \\
 & P_n[P_n[P_nF(-)]](E) &
\end{tikzcd}$$

commutes. This is equivalent to the upper left square (A) in diagrams (4.7). The proof is completed by noting that the upper right and lower left squares in (4.7) commute by naturality, and the lower right square commutes by the associativity of Σ.

Theorem 4.8. *There is a well defined DG-category $P_n\mathcal{C}$ whose objects are the objects of $\mathcal{C}$ and whose morphisms are given by*

$$\mathrm{Hom}_{P_n\mathcal{C}}(X, Y) = P_n\mathbb{Z}^X(Y) = P_n(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X, \star))(Y)$$

for objects X and Y in $\mathcal{C}$. Composition is given by $P_n\hat{o} : P_n\mathbb{Z}^X(Y) \otimes P_n\mathbb{Z}^Y(W) \to P_n\mathbb{Z}^X(W)$ and the identity morphism for an object C is defined by

$$\mathbf{1}_C = p_n(1 \cdot [\mathrm{id}_C]) \in \mathrm{Hom}_{P_n\mathcal{C}}(C, C)_0.$$

Proof. The only property left to check is that composition is associative. This follows from lemma 4.6 if we let $F = \mathbb{Z}^B$ for an object B in $\mathcal{C}$.

Theorem 4.9. *Given a functor F from $\mathcal{C}$ to $Ch\mathcal{A}$, we can define a DG-linear functor of DG-categories $P_n[F] : P_n\mathcal{C} \longrightarrow Ch\mathcal{A}$ by setting $P_n[F](C) = P_nF(C)$ and*

$$P_n[F] : \mathrm{Hom}_{P_n\mathcal{C}}(X,Y) = P_n\mathbb{Z}^X(Y) \xrightarrow{P_n\hat{\gamma}} \mathrm{Hom}_{Ch\mathcal{A}}(P_nF(X), P_nF(Y)).$$

Proof. It is an immediate consequence of the definitions that $P_n[F]$ preserves identity morphisms. To see that $P_n[F]$ preserves composition, note that for $c \otimes \alpha \otimes \beta \in P_nF(C) \otimes (P_n\mathbb{Z}^C)(D) \otimes (P_n\mathbb{Z}^D)(E)$,

$$P_n\hat{\gamma} \circ (P_n\hat{\gamma} \otimes \mathrm{id})(c \otimes \alpha \otimes \beta) = P_n[F](\beta)(P_n[F](\alpha)(c))$$

and

$$P_n\hat{\gamma} \circ (\mathrm{id} \otimes P_n\hat{o})(c \otimes \alpha \otimes \beta) = P_n[F](\beta \circ \alpha)(c).$$

Then by lemma 4.6, $P_n[F]$ preserves composition.

To see that $P_n[F]$ is a DG-linear functor we must show that $P_n\gamma$ is a chain map. Consider the map

$$\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X,Y)] \xrightarrow{\mathbb{Z}[P_nF]} \mathrm{Hom}_{Ch\mathcal{A}}(P_nF(X), P_nF(Y)).$$

This is a chain map (where objects in $\mathbb{Z}[\mathrm{Hom}(X,Y)]$ are considered to be chain complexes concentrated in degree 0) because $\mathbb{Z}[\ ]$ is $\mathbb{Z}$-linear and P_nF takes chain maps to chain maps. The map $P_n[F]$ is obtained from $\mathbb{Z}[P_nF]$ by applying P_n and composing with the plus map. That is, $P_n[F]$ is the following composite:

$$\begin{tikzcd}
P_n\mathbb{Z}[\mathrm{Hom}(X,Y)] \arrow[r, "P_n(\mathbb{Z}[P_nF])"] \arrow[dr] & P_n\mathrm{Hom}_{Ch\mathcal{A}}(P_nF(X), P_nF(Y)) \arrow[d, "\cong"] \\
& \mathrm{Hom}_{Ch\mathcal{A}}(P_nF(X), P_nP_nF(Y)) \arrow[d, "\Sigma"] \\
& \mathrm{Hom}_{Ch\mathcal{A}}(P_nF(X), P_nF(Y))
\end{tikzcd}$$

(the isomorphism is by considering $\mathrm{Hom}_{Ch\mathcal{A}}(P_nF(X), P_nF(Y))$ as a functor in the variable Y). Since Σ is a chain map and P_n takes chain maps to chain maps, it follows that $P_n[F]$ is a map of chain complexes. Hence $P_n[F]$ is a DG-linear functor.

Remark 4.10. By theorem 4.9 we have a functor from $\mathrm{Func}(\mathcal{C}, Ch\mathcal{A})$ to $\mathrm{DG}-\mathrm{Func}(P_n\mathcal{C}, Ch\mathcal{A})$. This factors the functor P_n in the following way. Let $p_n : \mathbb{Z}[\mathcal{C}] \longrightarrow P_n\mathcal{C}$ be the functor that is the identity on objects and on morphisms is given by the natural transformation

$$p_n : \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X,\star)](Y) \to P_n\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X,\star)](Y)$$

defined in definition 1.10. For any functor $F : \mathcal{C} \to Ch\mathcal{A}$, it follows from the definitions that $P_nF = P_n[F] \circ p_n \circ h$. Hence the diagram

$$\begin{array}{ccc} \mathrm{Func}(\mathcal{C}, Ch\mathcal{A}) & \xrightarrow{P_n} & \mathrm{Func}(\mathcal{C}, Ch\mathcal{A}) \\ {\scriptstyle P_n[\;]}\searrow & & \nearrow{\scriptstyle (p_n \circ h)^*} \\ & \mathrm{DG}-\mathrm{Func}(P_n\mathcal{C}, Ch\mathcal{A}) & \end{array}$$

commutes and so P_nF factors through the category $P_n\mathcal{C}$. We will see in theorem 4.13 that every degree n functor is obtained (up to natural quasi-isomorphism) via $(p_n \circ h)^*$ from a DG-linear functor from $P_n\mathcal{C}$ to $Ch\mathcal{A}$.

The factorization of P_n is related to Pirashvili's factorization of p_n.

Remark 4.11. The $0th$ homology functor, H_0, provides us with a functor from $P_n\mathcal{C}$ to $p_n\mathcal{C}$ that takes an object to itself and the morphism set $P_n\mathbb{Z}[\mathrm{Hom}(X,\star)](Y)$ to $H_0(P_n\mathbb{Z}[\mathrm{Hom}(X,\star)])(Y)$ (which is simply $p_n\mathbb{Z}[\mathrm{Hom}(X,\star)](Y)$ again). Thus $P_n[\;]$ passes to a functor from $\mathrm{Func}(\mathcal{C}, Ch\mathcal{A})$ to $\mathbb{Z}-\mathrm{Func}(p_n\mathcal{C}, \mathcal{A})$ and we have the commutative diagram below:

$$\begin{array}{ccc} \mathrm{Func}(\mathcal{C}, Ch\mathcal{A}) & \xrightarrow{p_n} & \mathrm{Func}(\mathcal{C}, \mathcal{A}) \\ {\scriptstyle H_0P_n[\;]}\searrow & & \nearrow{\scriptstyle (H_0(p_n)\circ h)^*} \\ & \mathbb{Z}-\mathrm{Func}(p_n\mathcal{C}, \mathcal{A}) & \end{array}$$

The next lemma will show that the target of $(H_0(p_n) \circ h)^*$ is the category of strictly degree n functors from $\mathcal{C}$ to $\mathcal{A}$. Moreover, since $p_nF = F$ for any strictly degree n functor, it will follow that

$$\mathbb{Z}-\mathrm{Func}(p_n\mathcal{C}, \mathcal{A}) \xrightarrow{(H_0(p_n)\circ h)^*} \mathrm{Func}(\mathcal{C}, \mathcal{A})$$

is a split surjection. Pirashvili ([P]) showed that this map is an isomorphism.

Lemma 4.12. *For a DG-linear functor $G : P_n\mathcal{C} \to Ch\mathcal{A}$, the composition $G \circ p_n \circ h : \mathcal{C} \to Ch\mathcal{A}$ is degree n.*

Proof. To show that $G \circ p_n \circ h$ is degree n, we must show that $cr_{n+1}(G \circ p_n \circ h)$ vanishes in homology. We will do so by using an alternative definition of the cross effect. Let $\mathbf{n+1} = \{*, 1, \ldots, n+1\}$. For a pointed subset U of $\mathbf{n+1}$, let $\pi_U : \mathbf{n+1} \longrightarrow U$ be the surjective pointed set map determined by

$$\pi_U(t) = \begin{cases} t & \text{if } t \in U \\ * & \text{if } t \notin U. \end{cases}$$

Let i_U be the inclusion from U to $\mathbf{n+1}$, and for an object C in $\mathcal{C}$, let ψ_U be the self map of $C \wedge \mathbf{n+1} = * \vee \bigvee_{i=1}^{n+1} C$ given by the composite:

$$\psi_U : \; C \wedge \mathbf{n+1} \overset{C\wedge\pi_U}{\longrightarrow} C \wedge U \overset{C\wedge i_U}{\longrightarrow} C \wedge \mathbf{n+1}.$$

It is straightforward to show that

$$cr_{n+1}F(C, \ldots, C) = \text{Image} \sum_{U \subseteq \mathbf{n+1}} (-1)^{|U|} \, F(\psi_U).$$

In fact, this was the definition of the $n+1$-st cross effect originally given in [E-M2].

Now consider $cr_{n+1}(G \circ p_n \circ h)$. By the above definition of cross effect, $cr_{n+1}(G \circ p_n \circ h)$ is the image in $(G \circ p_n \circ h)(C \wedge \mathbf{n+1})$ of the morphism

$$\sum_{U \subseteq \mathbf{n+1}} (-1)^{|U|} (G \circ p_n \circ h)(\psi_U).$$

But since G and p_n are DG-linear,

$$\sum_{U \subseteq \mathbf{n+1}} (-1)^{|U|} (G \circ p_n \circ h)(\psi_U) = G \circ p_n\Big(\sum_{U \subseteq \mathbf{n+1}} (-1)^{|U|} [\psi_U] \Big).$$

However,

$$\sum_{U \subseteq \mathbf{n+1}} (-1)^{|U|} [\psi_U] = \text{Image} \left(\sum_{U \subseteq \mathbf{n+1}} (-1)^{|U|} \mathbb{Z}^{C \wedge \mathbf{n+1}}(\psi_U) \right)$$

in $\mathbb{Z}^{C\wedge \mathbf{n+1}}(C\wedge \mathbf{n+1})$ and hence is in $cr_{n+1}\mathbb{Z}^{C\wedge\mathbf{n+1}}(C,\ldots,C)$. Since p_n preserves cross effects, it follows that $p_n(\sum_{U\subseteq \mathbf{n+1}}(-1)^{|U|}[\psi_U])$ is part of $cr_{n+1}P_n\mathbb{Z}^{C\wedge\mathbf{n+1}}(C,\ldots,C)$, and as a consequence, vanishes in homology. Since G is linear, it follows that $G(p_n\sum_{U\subseteq\mathbf{n+1}}(-1)^{|U|}[\psi_U])$ vanishes in homology as well. The result follows.

Theorem 4.13. *Let* $\mathrm{Func}_{hdeg\leq n}(\mathcal{C},Ch\mathcal{A})$ *be the full subcategory of* $\mathrm{Func}(\mathcal{C},Ch\mathcal{A})$ *consisting of degree n functors. The functor*

$$\mathrm{DG}-\mathrm{Func}(P_n\mathcal{C},Ch\mathcal{A})\overset{(p_n\circ h)^*}{\longrightarrow}\mathrm{Func}_{hdeg\leq n}(\mathcal{C},Ch\mathcal{A})$$

is a split surjection up to natural quasi-isomorphism.

Proof. By the above lemma, $(p_n\circ h)^*$ takes values in degree n functors. Since $P_nF=P_n[F]\circ(p_n\circ h)$ and $F\overset{\simeq}{\longrightarrow}P_nF$ if F is degree n we see that $(p_n\circ h)^*\circ P_n[\;]$ is naturally quasi-isomorphic to the identity on degree n functors.

Remark 4.14. The split surjection of theorem 4.13 can probably be made into an isomorphism by passing to derived categories and localizing the category $\mathrm{DG}-\mathrm{Func}(P_n\mathcal{C},Ch\mathcal{A})$ in an appropriate manner. Since we will have no need for this generalization in the present paper we leave this possible refinement (and the extra formalism it requires) to someone more interested in its details than we are.

For each $n\geq 0$, we now have a category, $P_n\mathcal{C}$, that determines degree n functors via the relationship established in the previous theorem. This category will be of further use to us when we classify degree n functors. We will also be interested in the relationship between functors of different degrees. One way to study this is to use the natural transformation $q_k:P_k\longrightarrow P_{k-1}$ defined in remark 1.12. We finish this section by defining, for any $t\geq n$, functors $q(t,n):P_t\mathcal{C}\to P_n\mathcal{C}$.

Definition 4.15. *Let* $q_k:\perp_{k+1}\to\perp_k$ *be the map of cotriples used in 1.12 to define* $q_k:P_k\to P_{k-1}$. *For* $t>n$, *we set*

$$q(t,n)=q_{n+2}\circ q_{n+1}\circ\cdots\circ q_t:\perp_{t+1}\longrightarrow\perp_{n+1}$$

and for $t=n$ *we set* $q(n,n)=\mathrm{id}_{\perp_{n+1}}$.

We will also write $q(t,n)$ for the map of complexes from P_t to P_n induced by the map of cotriples. We note that

$$q(n,m) \circ q(t,n) = q(t,m)$$

Lemma 4.16.*For $t \geq n$, the natural transformation $q(t,n) : P_t \longrightarrow P_n$ produces a functor $q[t,n]$ from $P_t\mathcal{C}$ to $P_n\mathcal{C}$ such that*

a) $q[n,m] \circ q[t,n] = q[t,m]$

b) $q[t,t] = \mathrm{id}_{P_t\mathcal{C}}$

c) For $F \in \mathrm{Func}(\mathcal{C}, Ch\mathcal{A})$, $P_n[F] \circ q[t,n] = q(t,n) \circ P_t[F]$.

Proof. We define $q[t,n]$ to be the identity on objects of $P_t\mathcal{C}$ and, for morphisms,

$$\begin{array}{rl} q[t,n] = q(t,n) : \mathrm{Hom}_{P_t\mathcal{C}}(X,Y) = & P_t Z_X(-)(Y) \\ & \downarrow \\ P_n Z_X(-)(Y) = & \mathrm{Hom}_{P_n\mathcal{C}}(X,Y). \end{array}$$

It follows immediately that $q[t,n]$ is well defined on objects and preserves identity morphisms. To show that $q[t,n]$ is preserves composition and satisfies conditions a) - c), we will show that for any functor $F : \mathcal{C} \to Ch\mathcal{A}$, the following diagram commutes:

$$\begin{array}{ccc} P_t\mathbb{Z}^X(\star) & \xrightarrow{P_t[F]} & \mathrm{Hom}(P_tF(X), P_tF(\star)) \\ \Big\downarrow{\scriptstyle q(t,n)} & & \Big\downarrow{\scriptstyle \Sigma(n,t)\circ(P_n(p_t))^*\circ P_n} \\ P_n\mathbb{Z}^X(\star) & \xrightarrow{P_n[F]} & \mathrm{Hom}(P_nF(X), P_nF(\star)). \end{array}$$

To do so, we first consider diagram (4.17) below.

$$\begin{array}{ccc}
P_t\mathbb{Z}^X(\star) & \overset{q(t,n)}{\longrightarrow} & P_n\mathbb{Z}^X(\star) \\
\Big\downarrow {\scriptstyle P_t(P_n[F]\circ p_n)} & & \Big\downarrow {\scriptstyle P_n(P_n[F]\circ p_n)} \\
P_t\mathrm{Hom}(P_nF(X),P_nF(\star)) & \overset{q(t,n)}{\longrightarrow} & P_n\mathrm{Hom}(P_nF(X),P_nF(\star)) \\
\Big\downarrow {\scriptstyle \cong} & & \Big\downarrow {\scriptstyle \cong} \\
\mathrm{Hom}(P_nF(X),P_nP_tF(\star)) & \overset{P_n(q(t,n))_*}{\longrightarrow} & \mathrm{Hom}(P_nF(X),P_nP_nF(\star)) \\
\Big\downarrow {\scriptstyle \Sigma(n,t)_*} & & \Big\downarrow {\scriptstyle \Sigma(n,n)_*} \\
\mathrm{Hom}(P_nF(X),P_nF(\star)) & \overset{\mathrm{id}}{\longrightarrow} & \mathrm{Hom}(P_nF(X),P_nF(\star)).
\end{array}$$

The upper two squares commute by naturality, and the lower square can be seen to commute by using 1.17. We also claim that the diagram

$$\begin{array}{ccc}
P_t\mathbb{Z}^X(\star) & \overset{P_t(P_t[F])}{\longrightarrow} & P_t\mathrm{Hom}(P_tF(X),P_tF(\star)) \\
& {\scriptstyle P_t(P_n[F])} \searrow & \Big\downarrow {\scriptstyle P_t(\Sigma(n,t)\circ(P_np_t)^*\circ P_n)} \\
& & P_t\mathrm{Hom}(P_nF(X),P_nF(\star))
\end{array}$$

commutes since $\mathbb{Z}^X(\star) \overset{P_t[F]\circ p_t}{\longrightarrow} \mathrm{Hom}(P_tF(X),P_tF(\star))$ is determined by sending α to $P_n[F](\alpha)$ and so by 1.17 the diagram below commutes:

$$\begin{array}{ccc}
\mathbb{Z}^X(\star) & \overset{=}{\longrightarrow} & \mathbb{Z}^X(\star) \\
\Big\downarrow {\scriptstyle P_t[F]\circ p_t} & & \Big\downarrow {\scriptstyle P_n[F]\circ p_n} \\
\mathrm{Hom}(P_tF(X),P_tF(\star)) & \overset{\Sigma(n,t)\circ P_n(p_t)^*\circ P_n}{\longrightarrow} & \mathrm{Hom}(P_nF(X),P_nF(\star)).
\end{array}$$

Applying the functor P_t, we see that (4.18) commutes. Using (1.17) one can see that the square (A) below commutes and hence that the entire diagram below commutes:

$$
(4.19)\qquad
\begin{array}{ccc}
P_t\mathbb{Z}^X(\star) & \xrightarrow{=} & P_t\mathbb{Z}^X(\star) \\
\Big\downarrow{\scriptstyle P_t(P_t[F])} & & \Big\downarrow{\scriptstyle P_t(P_n[F])} \\
P_t\mathrm{Hom}(P_tF(X), P_tF(\star)) & \xrightarrow{\mathbf{t}} & P_t\mathrm{Hom}(P_nF(X), P_nF(\star)) \\
\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} \\
\mathrm{Hom}(P_tF(X), P_tP_tF(\star)) & \xrightarrow{\mathbf{u}} & \mathrm{Hom}(P_nF(X), P_nP_tF(\star)) \\
\Big\downarrow{\scriptstyle \Sigma(t,t)_*} & (A) & \Big\downarrow{\scriptstyle \Sigma(n,t)_*} \\
\mathrm{Hom}(P_tF(X), P_tF(\star)) & \xrightarrow{\mathbf{v}} & \mathrm{Hom}(P_nF(X), P_nF(\star))
\end{array}
$$

where

$$
\begin{aligned}
\mathbf{t} &= P_t(\Sigma(n,t) \circ (P_n(p_t))^* \circ P_n) \\
\mathbf{u} &= (\Sigma(n,t) \circ (P_n(p_t))^* \circ P_n) \\
\mathbf{v} &= (\Sigma(n,t) \circ (P_n(p_t))^* \circ P_n).
\end{aligned}
$$

The diagrams (4.17) and (4.19) imply that

$$
\begin{array}{ccc}
P_t\mathbb{Z}^X(\star) & \longrightarrow & \mathrm{Hom}(P_tF(X), P_tF(\star)) \\
\Big\downarrow{\scriptstyle q(t,n)} & & \Big\downarrow{\scriptstyle \Sigma(n,t)\circ(P_n(p_t))^*\circ P_n} \\
P_n\mathbb{Z}^X(\star) & \longrightarrow & \mathrm{Hom}(P_nF(X), P_nF(\star))
\end{array}
$$

commutes, and hence that (c) holds. When $F = \mathbb{Z}^Y$, it follows by adjunction that

$$
\begin{array}{ccc}
P_t\mathbb{Z}^Y(X) \otimes P_t\mathbb{Z}^X(\star) & \xrightarrow{\circ} & P_t\mathbb{Z}^Y(\star) \\
\Big\downarrow{\scriptstyle q(t,n)\otimes q(t,n)} & & \Big\downarrow{\scriptstyle q(t,n)} \\
P_n\mathbb{Z}^Y(X) \otimes P_n\mathbb{Z}^X(\star) & \xrightarrow{\circ} & P_n\mathbb{Z}^Y(\star)
\end{array}
$$

commutes. Thus $q[t,n]$ preserves composition.

We conclude this section by noting that by using the functor D_n and the natural transformation $\Sigma_D(n) : D_nD_n \longrightarrow D_n$ of (1.22) we can construct a category $D_n\mathcal{C}$ in the same way we constructed $P_n\mathcal{C}$.

In addition, the natural transformation $d_n : D_n \longrightarrow P_n$ produces a functor

$$d_n : D_n\mathcal{C} \longrightarrow P_n\mathcal{C}.$$

One can prove that d_n is a functor by using the diagrams (1.19) and (1.20) in a proof similar to that of lemma 4.16. Moreover, there is a functor, $D_n[\ ]$, from $\mathrm{Func}(\mathcal{C}, Ch\mathcal{A})$ to $\mathrm{DG} - \mathrm{Func}(D_n\mathcal{C}, Ch\mathcal{A})$ that takes a functor F to the functor $D_n[F]$. The new functor $D_n[F]$ is defined in a manner similar to $P_n[F]$ and the details involved in establishing that $D_n[F]$ is a functor are the same as those for $P_n[F]$.

5. A classification of degree n functors defined along C

In this section we will show that degree n functors defined along an object C can be characterized by modules over a differential graded algebra, $P_{n\times n}(C)$, and refine this classification for homogeneous degree n functors in a manner similar to Goodwillie's classification of homogeneous degree n functors of spaces. Throughout this section F will be a functor from $\mathcal{C}$ to $Ch\mathcal{A}$ where $\mathcal{A}$ is a cocomplete, full and faithful subcategory of the category of left modules over a ring A. We will also assume that F is defined along C for some object C in $\mathcal{C}$.

Our classification will rely on endowing $P_nF(\mathbf{n}_C)$ with compatible module structures over A and $P_{n\times n}(C)$. To do so, we first establish some definitions and conventions.

Remark 5.1. For any $\mathbb{Z}$-linear category $\mathcal{E}$ and any object E in $\mathcal{E}$, $\mathrm{Hom}_{\mathcal{E}}(E, E)$ is a ring by composition. We will use the convention that $f * g = g \circ f$ for $f, g \in \mathrm{Hom}_{\mathcal{E}}(E, E)$. A similar result holds for any DG-category, $\mathcal{D}$. For objects X, Y, and Z in $\mathcal{D}$, recall that $\mathrm{Hom}_{\mathcal{D}}(X, Y)$, $\mathrm{Hom}_{\mathcal{D}}(Y, Z)$, and $\mathrm{Hom}_{\mathcal{D}}(X, Z)$ are differential graded objects and that the composition rule makes the diagram below commute:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{D}}(Y,Z) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathcal{D}}(X,Z) \\
\Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial} \\
\mathrm{Hom}_{\mathcal{D}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{D}}(Y,Z) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathcal{D}}(X,Z).
\end{array}$$

Using composition as above makes $\mathrm{Hom}_{\mathcal{D}}(X, X)$ a differential graded algebra (DGA). Moreover, for any objects Y, Z in $\mathcal{D}$, the composition rule in $\mathcal{D}$ makes $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ a left module and $\mathrm{Hom}_{\mathcal{D}}(Z, X)$ a right module over the DGA $\mathrm{Hom}_{\mathcal{D}}(X, X)$.

By the remark above, and theorem 4.8, we obtain the following.

Lemma 5.2. *For any objects X, Y, and Z in the DG-category P_nC, $\mathrm{Hom}_{P_nC}(X, X)$ is a DGA, and $\mathrm{Hom}_{P_nC}(X, Y)$ and $\mathrm{Hom}_{P_nC}(Z, X)$ are left and right modules, respectively, over $\mathrm{Hom}_{P_nC}(X, X)$.*

We can also give $P_nF(X)$ a right module structure over the DG-algebra $\mathrm{Hom}_{P_nC}(X, X)$ that is compatible with its structure as a chain complex of modules over the ring A. To do so, we use the following.

Remark 5.3. Let R be a DGA, $\mathcal{D}$ be a DG-category, and X be an object in $\mathcal{D}$. Given a map of DGA's, $R \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, X)$, adjunction produces a pairing $X \otimes R \xrightarrow{\mu} X$ that gives X the structure of a right module over the DGA R. If $\mathcal{D} = Ch\mathcal{A}$ for the ring A, then the right module structure of X over R is compatible with the left A-module chain complex structure of X in that the diagram below commutes:

$$\begin{array}{ccc} A \otimes X \otimes R & \xrightarrow{\mathrm{id}_A \otimes \mu} & A \otimes X \\ \big\downarrow{\scriptstyle \omega \otimes \mathrm{id}_R} & & \big\downarrow{\scriptstyle \omega} \\ X \otimes R & \xrightarrow{\mu} & X \end{array}$$

where $\omega : A \otimes X \to X$ is the left A-module structure map for X.

Definition 5.4. *Recall that $\mathcal{A}$ is a subcategory of $A -$ Mod for some ring A. For a DGA R, a $Ch\mathcal{A} - R$* bimodule *is an object $M \in Ch\mathcal{A}$ that is equipped with a DGA map $R \longrightarrow \mathrm{Hom}_{Ch\mathcal{A}}(M, M)$. We use $\mathcal{A} -$ Mod $- R$ to denote the category of $Ch\mathcal{A} - R$ bimodules. A morphism in $\mathcal{A} -$ Mod $- R$ is a morphism $\alpha \in \mathrm{Hom}_{Ch\mathcal{A}}(M, M')$ for which $\alpha \circ \mu = \mu' \circ (\alpha \otimes \mathrm{id}_R)$ where μ and μ' are the R-module structure maps for M and M' respectively.*

As a direct consequence of example 4.5 and the previous definitions and remarks, we obtain the next lemma.

Lemma 5.5. *Let X be an object in $\mathcal{C}$ and F and F' be functors from $\mathcal{C}$ to $Ch\mathcal{A}$. Then for any $n \geq 1$, $P_nF(X)$ and $P_nF'(X)$ are $Ch\mathcal{A} - \mathrm{Hom}_{P_n\mathcal{C}}(X,X)$ bimodules. Moreover, any natural transformation $\eta : F \to F'$ produces a map of $Ch\mathcal{A} - \mathrm{Hom}_{P_n\mathcal{C}}(X,X)$ bimodules, $P_n\eta_X : P_nF(X) \longrightarrow P_nF'(X)$.*

Using the DGA and module structures established above, we will construct a collection of degree n functors defined along $\mathcal{C}$ to which all other such functors are equivalent. The construction of these functors will involve tensor products over the DGA $\mathrm{Hom}_{P_n\mathcal{C}}(\mathbf{n}_\mathcal{C}, \mathbf{n}_\mathcal{C})$. We make the following functorial choice for the derived tensor product of modules over a DGA.

Definition 5.6. *Let R be a DGA and M and N be left and right modules over R, respectively. Then $M\hat{\otimes}_R N$ is the simplicial chain complex that in simplicial degree p is*

$$(M\hat{\otimes}_R N)_{[p]} = M \otimes R^{\otimes p} \otimes N$$

(where unlabeled tensors are over the commutative ground ring k) with simplicial operators given by

$$d_i(m, r_1, \ldots, r_p, n) = \begin{cases} (m \cdot r_1, r_2, \ldots, r_p, n) & \text{if } i = 0 \\ (m, \ldots, r_i \cdot r_{i+1}, \ldots, n) & \text{if } 1 \leq i \leq p-1 \\ (m, r_1, \ldots, r_{p-1}, r_p \cdot n) & \text{if } i = p \end{cases}$$

$$s_j(m, r_1, \ldots, r_p, n) = (m, r_1, \ldots, r_{j-1}, 1, r_j, \ldots, r_p, n).$$

The final ingredient for our classification theorem is some notation for the DGA's, modules, and derived categories involved.

Definition 5.7. *We define $P_n(C,\star)$ to be the functor:*

$$P_n(C,\star) := \mathrm{Hom}_{P_n\mathcal{C}}(\mathbf{n}_C,\star)$$

and we define $P_{n\times n}(C)$ to be the DGA:

$$P_{n\times n}(C) := \mathrm{Hom}_{P_n\mathcal{C}}(\mathbf{n}_C,\mathbf{n}_C) = P_n(C,\mathbf{n}_C).$$

Definition 5.8. *We will let* $\mathrm{Func}_{C,n}(\mathcal{C},Ch\mathcal{A}) \subseteq \mathrm{Func}(\mathcal{C},Ch\mathcal{A})$ *be the full subcategory of all degree n functors defined along C.*

We will write $[\mathcal{C},Ch\mathcal{A}]_{C,n}$ for its associated derived category, where a weak equivalence is a natural transformation that is a quasi-isomorphism for all objects of $\mathcal{C}$. We will write $[\mathcal{A}-\mathrm{Mod}-P_{n\times n}(C)]$ for the associated derived category of $\mathcal{A}-\mathrm{Mod}-P_{n\times n}(C)$ where a weak equivalence is a map $f : M \longrightarrow M'$ of bimodules that is a quasi-isomorphism as a map of chain complexes.

Theorem 5.9. *There is a natural isomorphism:*

$$[\mathcal{C},Ch\mathcal{A}]_{C,n} \;\cong\; [\mathcal{A}-Mod-P_{n\times n}(C)]$$

given by sending $F \in \mathrm{Func}_{C,n}(\mathcal{C},Ch\mathcal{A})$ to $P_nF(\mathbf{n}_C)$.

Proof. By lemma 5.5, $P_nF(\mathbf{n}_C)$ is a $Ch\mathcal{A}-P_{n\times n}(C)$ module for any functor $F:\mathcal{C}\longrightarrow Ch\mathcal{A}$. Hence we can define $\phi:\mathrm{Func}_{C,n}(\mathcal{C},Ch\mathcal{A})\longrightarrow \mathcal{A}-\mathrm{Mod}-P_{n\times n}(C)$ by $\phi(F) = P_nF(\mathbf{n}_C)$. If F is degree n, then $P_nF(\mathbf{n}_C) \simeq F(\mathbf{n}_C)$.

On the other hand, given a $Ch\mathcal{A} - P_{n\times n}(C)$ bimodule M, it is straightforward to show that $M\hat{\otimes}_{P_{n\times n}(C)}P_n(C,-)$ is a degree n functor. By lemma 2.9, the functor χ_M, defined by

$$\chi_M(-) = \perp_C^* (M\hat{\otimes}_{P_{n\times n}(C)}P_n(C,-))$$

is defined along C and degree n. This produces a natural transformation $\chi : \mathcal{A} - \mathrm{Mod} - P_{n\times n}(C) \longrightarrow \mathrm{Func}_{C,n}(\mathcal{C}, Ch\mathcal{A})$ that takes the module M to the functor χ_M.

For a $Ch\mathcal{A} - P_{n\times n}(C)$ bimodule M,

$$\begin{aligned}(\phi \circ \chi)(M) &= P_n(\perp_C^* (M \hat{\otimes}_{P_{n\times n}(C)} P_n(C,-))(\mathbf{n}_C) \\ &\simeq \perp_C^* (M \hat{\otimes}_{P_{n\times n}(C)} P_n(C,-))(\mathbf{n}_C) \\ &\simeq M \hat{\otimes}_{P_{n\times n}(C)} P_n(C, \mathbf{n}_C) \\ &= M \hat{\otimes}_{P_{n\times n}(C)} P_{n\times n}(C),\end{aligned}$$

where the last equivalence follows by lemma 2.4. But, M is equivalent to $M \hat{\otimes}_{P_{n\times n}(C)} P_{n\times n}(C)$ via the $\mathcal{A} - \mathrm{Mod} - P_{n\times n}(C)$ map that takes $(m, r_1, \ldots, r_n, r)$ to $m \cdot r_1 \cdots r_n \cdot r$. This map is an equivalence by the contracting homotopy that takes $(m, r_1, \ldots, r_n, r)$ to $(m, r_1, \ldots, r_n, r, 1)$. Thus, $(\phi \circ \chi)$ is the identity on the homotopy category $[\mathcal{A} - \mathrm{Mod} - P_{n\times n}(C)]$.

Similarly, for $F : \mathcal{C} \to Ch\mathcal{A}$, a degree n functor defined along C, we compute $(\chi \circ \phi)(F)(\mathbf{n}_C)$ to be:

$$\perp_C^* (P_n F(\mathbf{n}_C) \hat{\otimes}_{P_{n\times n}(C)} P_n(C,-))(\mathbf{n}_C) \simeq P_n F(\mathbf{n}_C) \simeq F(\mathbf{n}_C).$$

But, F and $(\chi \circ \phi)(F)$ are both degree n functors defined along C. Since they agree at $\mathbf{n}_C$, it follows by theorem 2.11 that $(\chi \circ \phi)(F) \simeq F$ for all objects in $\mathcal{C}$. Therefore $(\chi \circ \phi)$ is the identity on $[\mathcal{C}, Ch\mathcal{A}]_{C,n}$ and the result follows.

In the second part of this section we provide three classifications of homogeneous degree n functors defined along a line C. Recall that a functor F is homogeneous degree n if and only if $F \xrightarrow{\simeq} D_n F$. The classifications are motivated by three different properties of a homogeneous degree n functor F and produce the following classifying modules:

I) $F(\mathbf{n})$,

II) $F(\mathbf{1})$,

III) $D_1^{(n)} cr_n F(C, \ldots, C)$.

The first classification is a translation of theorem 5.9 using the category $D_n\mathcal{C}$ in place of $P_n\mathcal{C}$. The second classification exploits the fact that homogeneous degree n functors are also rank 1 functors. We then use the equivalence between D_nF and $(D_1^{(n)}cr_nF)_{h\Sigma_n}$ proved in section 3 of [J-M3] to obtain a third classification of n-homogeneous functors in terms of a wreath product ring. This third classification is motivated by a similar result due to Goodwillie ([G3]).

For our first two classification results, we will use the category $D_n\mathcal{C}$ described at the end of section 4. As was the case with $P_n\mathcal{C}$, the DG-category structure of $D_n\mathcal{C}$ allows us to treat $\mathrm{Hom}_{D_n\mathcal{C}}(X,X)$ as a DGA and $\mathrm{Hom}_{D_n\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{D_n\mathcal{C}}(Z,X)$ as left and right modules, respectively, over $\mathrm{Hom}_{D_n\mathcal{C}}(X,X)$ for objects X, Y, and Z in $\mathcal{C}$. Moreover, for any functor F, $D_nF(X)$ is a $Ch\mathcal{A} - \mathrm{Hom}_{D_n\mathcal{C}}(X,X)$ bimodule. We single out the following DGA's and functors for the classification theorems.

Definition 5.10. *We define $D_n(C,\star)$ to be the functor*

$$D_n(C,\star) := \mathrm{Hom}_{D_n\mathcal{C}}(\mathbf{n}_C,\star),$$

and $D_{n\times n}(C)$ to be the DGA

$$D_{n\times n}(C) := \mathrm{Hom}_{D_n\mathcal{C}}(\mathbf{n}_C,\mathbf{n}_C) = D_n(C,\mathbf{n}_C).$$

The classification results will be expressed in terms of the following categories and derived categories.

Definition 5.11.*Let* $\mathrm{Func}_{C,[n]}(\mathcal{C},CH\mathcal{A}) \subseteq \mathrm{Func}(\mathcal{C},CH\mathcal{A})$ *be the full subcategory determined by all homogeneous degree n functors defined along C.*

We will write $[\mathcal{C},CH\mathcal{A}]_{C,[n]}$ for its associated derived category (where a weak equivalence is a natural transformation that is a quasi-isomorphism for all objects in $\mathcal{C}$). We will use $[\mathcal{A}-\mathrm{Mod}-D_{n\times n}(C)]$ to denote the derived category associated to the category $\mathcal{A}-\mathrm{Mod}-D_{n\times n}(C)$ where a weak equivalence is a bimodule map that is a quasi-isomorphism as a map of chain complexes.

Theorem 5.12. (CLASSIFICATION I) *There is a natural isomorphism*

$$[\mathcal{C}, Ch\mathcal{A}]_{C,[n]} \cong [\mathcal{A} - \mathrm{Mod} - D_{n\times n}(C)]$$

given by sending $F \in \mathrm{Func}_{C,[n]}(\mathcal{C}, Ch\mathcal{A})$ *to* $D_nF(\mathbf{n}_C)$.

Proof. Since $D_nF(\mathbf{n}_C)$ is a $Ch\mathcal{A} - D_{n\times n}(C)$ bimodule for any functor $F : \mathcal{C} \longrightarrow Ch\mathcal{A}$, we can define $\phi : \mathrm{Func}_{C,[n]}(\mathcal{C}, Ch\mathcal{A}) \longrightarrow \mathcal{A} - \mathrm{Mod} - D_{n\times n}(C)$ by $\phi(F) = D_nF(\mathbf{n}_C)$. If we assume F is homogeneous degree n, then $D_nF(\mathbf{n}_C) \simeq F(\mathbf{n}_C)$.

On the other hand, given a $Ch\mathcal{A} - D_{n\times n}(C)$ bimodule M, it is straightforward to show that $M\hat{\otimes}_{D_{n\times n}(C)}D_n(C,-)$ is a homogeneous degree n functor. By lemma 2.9, the functor χ_M, defined by

$$\chi_M(-) = \perp_C^* (M\hat{\otimes}_{D_{n\times n}(C)}D_n(C,-))$$

is defined along C and homogeneous degree n. This produces a natural transformation $\chi : \mathcal{A} - \mathrm{Mod} - D_{n\times n}(C) \longrightarrow \mathrm{Func}_{C,[n]}(\mathcal{C}, Ch\mathcal{A})$ that takes the module M to the functor χ_M. One can then use arguments similar to those used in the proof of theorem 5.9 to show that χ_M and ϕ yield the desired isomorphisms of derived categories.

Theorem 5.13. (CLASSIFICATION II) *Write* $D_n(1)(C)$ *for the DG-algebra* $\mathrm{Hom}_{D_n\mathcal{C}}(\mathbf{1}_C, \mathbf{1}_C)$. *There is a natural isomorphism:*

$$[\mathcal{C}, Ch\mathcal{A}]_{C,[n]} \cong [\mathcal{A} - \mathrm{Mod} - D_n(1)(C)]$$

given by sending $F \in \mathrm{Func}_{C,[n]}(\mathcal{C}, Ch\mathcal{A})$ *to* $D_nF(\mathbf{1}_C)$.

Proof. As in the proofs of Theorems 5.9 and 5.12, $D_nF(\mathbf{1}_C)$ is a $Ch\mathcal{A} - D_n(1)(C)$ bimodule for any $F : \mathcal{C} \longrightarrow Ch\mathcal{A}$, and we define $\phi(F) = D_nF(\mathbf{1}_C)$. Given a $Ch\mathcal{A} - D_n(1)(C)$-bimodule M, we define

$$\chi_M(-) = \perp_C^* (M\hat{\otimes}_{D_n(1)(C)}\mathrm{Hom}_{D_n\mathcal{C}}(\mathbf{1}_C,-))$$

and note that χ_M is a homogeneous degree n functor defined along C by lemma 2.9. Hence, we can define χ from $\mathcal{A} - \mathrm{Mod} - D_n(1)(C)$

to $\mathrm{Func}_{C,[n]}(\mathcal{C}, Ch\mathcal{A})$ by $\chi(M) = \chi_M$. The proofs that $\chi \circ \phi$ and $\phi \circ \chi$ are equivalent to the identity functors on their respective domain categories is similar to the proofs used for theorems 5.9 and 5.12 with the exception that the equivalence $(\chi \circ \phi)(F) \simeq F$ is obtained by showing that $(\chi \circ \phi)(F)(\mathbf{1}_C) \simeq F(\mathbf{1}_C)$ and then using the facts that F is a rank 1 functor (corollary 3.16) and rank 1 functors defined along C are determined by their values at $\mathbf{1}_C$ (lemma 3.5).

Our third classification of homogeneous degree n functors parallels the characterization of D_nF given in section 3 of [J-M3]. That is, we will begin by classifying n-multilinear functors, and then extend the classification to homogeneous degree n functors by taking advantage of the fact that $D_nF \simeq (D_1^{(n)} cr_n F)_{h\Sigma_n}$ for any functor F (proposition 3.10 of [J-M3]). The classification of n-multilinear functors uses the next result.

Lemma 5.14. *Let G be an n-multilinear functor, i.e., $G : \mathcal{C}^{\times n} \to \mathcal{A}$ and G is linear in each variable. Then for objects $X_1, X_2, \dots, X_n$ in $\mathcal{C}$, $G(X_1, \dots, X_n)$ is equivalent to*

$$G(C, C, \dots, C) \hat{\otimes}_{D_1(C)^{\otimes n}} \left(\bigotimes_{i=1}^{n} \perp_C^* D_1(C, X_i) \right)$$

where unlabeled tensors are over $\mathbb{Z}$.

Proof. By fixing the objects $X_1, X_2, \dots, X_{n-1}$, we may consider the linear functor of one variable $G(X_1, X_2, \dots, X_{n-1}, -)$. It then follows from the proof of Theorem 5.12 that

$$G(X_1, \dots, X_{n-1}, -) \simeq G(X_1, \dots, X_{n-1}, C) \hat{\otimes}_{D_{1\times 1}(C)} \perp_C^* D_1(C, -).$$

Applying the same idea to each variable of G in turn shows that $G(X_1, \dots, X_n)$ is equivalent to (tensors are over $D_{1\times 1}(C)$)

$$(G) \qquad G(C, \dots, C) \hat{\otimes} \perp_C^* D_1(C, X_1) \hat{\otimes} \cdots \hat{\otimes} \perp_C^* D_1(C, X_n).$$

Recall that the n-fold tensor product of a DGA yields another DGA (see [M], p.181) so that $D_{1\times 1}(C)^{\otimes n}$ is a DGA. Similarly, $\bigotimes_{i=1}^{n} \perp_C^* D_1(C, X_i)$ is a module over $D_{1\times 1}(C)^{\otimes n}$. Then we can construct

$$G(C, C, \dots, C)\hat{\otimes}_{D_{1\times 1}(C)^{\otimes n}}\Big(\bigotimes_{i=1}^{n} \perp_C^* D_1(C, X_i)\Big).$$

But, this is the diagonal of (G) and so the result follows by the Eilenberg-Zilber theorem.

We let $\mathrm{Func}_{C,[1]^{\times n}}(\mathcal{C}, Ch\mathcal{A}) \subseteq \mathrm{Func}(\mathcal{C}^{\times n}, Ch\mathcal{A})$ be the full subcategory of all n–multilinear functors defined along C in each variable and $[\mathcal{C}^{\times n}, Ch\mathcal{A}]_{C,[1]^{\times n}}$ be its associated derived category. The functors in $\mathrm{Func}_{C,[1]^{\times n}}(\mathcal{C}^{\times n}, Ch\mathcal{A})$ can be classified as follows.

Corollary 5.15. *There is a natural isomorphism:*

$$[\mathcal{C}^{\times n}, Ch\mathcal{A}]_{C,[1]^{\times n}} \;\cong\; [\mathcal{A} - \mathrm{Mod} - D_{1\times 1}(C)^{\otimes n}].$$

Proof. Given a $Ch\mathcal{A} - D_{1\times 1}(C)^{\otimes n}$ bimodule M, we construct an n-multilinear functor χ_M that is defined along C in each variable:

$$\chi_M = M\hat{\otimes}_{D_{1\times 1}(C)^{\otimes n}} \bigotimes_{i=1}^{n} \perp_C^* D_{1\times 1}(C, -).$$

For an object G in $\mathrm{Func}_{(C,[1]^{\times n}}, Ch\mathcal{A})$ we produce a $Ch\mathcal{A}{-}D_{1\times 1}(C)^{\otimes n}$ bimodule $\phi(G)$ by

$$\phi(G) = D_1^{(n)} G(C, \dots, C).$$

It then follows by arguments similar to those used in the proofs of theorems 5.9, 5.12, and 5.13 that $\phi \circ \chi$ is the identity on $[\mathcal{A} - \mathrm{Mod} - D_{1\times 1}(C)^{\otimes n}]$. Using lemma 5.14 one can show that $\chi \circ \phi$ is the identity on $[\mathcal{C}^{\times n}, CH\mathcal{A}]_{C,[1]^{\times n}}$.

Following the lead of section 3 in [J-M3], we bring homotopy orbits into the picture to classify homogeneous degree n functors in terms of their multilinearized cross effects. But, the homotopy orbits we need will be equivalent to derived tensor products over the DGA defined below.

Definition 5.16. *Let $\Sigma_n = \mathrm{Aut}(\{1,\ldots,n\})$ be the permutation group with multiplication given by composition ($\sigma \cdot \tau = \tau \circ \sigma$) and let R be a DGA. We let $R \int \Sigma_n$ denote the additive wreath product of R. As a DG module, $R \int \Sigma_n = R^{\otimes n} \otimes \mathbb{Z}[\Sigma_n]$ with multiplication determined for $a = (a_1 \otimes \ldots \otimes a_n \otimes [\sigma])$ and $b = (b_1 \otimes \ldots \otimes b_n \otimes [\tau])$ by*

$$ab = (-1)^{\mathrm{sgn}(a,b)}(a_1 \cdot b_{\sigma(n)} \otimes \cdots \otimes a_n \cdot b_{\sigma(n)} \otimes [\sigma\tau]),$$

where

$$\mathrm{sgn}(a,b) = \sum_{i<j} |a_j||b_{\sigma(i)}| + \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |b_{\sigma(i)}||b_{\sigma(j)}|.$$

Modules over the additive wreath product $R \int \Sigma_n$ are also supplied with a natural $R^{\otimes n}$-module structure and Σ_n action. In particular, the map

$$(a_1 \otimes \cdots \otimes a_n) \mapsto (a_1 \otimes \cdots \otimes a_n \otimes [1])$$

determines a map of DGA's from $R^{\otimes n}$ to $R \int \Sigma_n$ and hence an $R^{\otimes n}$-module structure. In addition, the map $\sigma \mapsto (1 \otimes \cdots \otimes 1 \otimes [\sigma])$ from Σ_n to $R \int \Sigma_n$ gives every left (right) $R \int \Sigma_n$ module a natural left (right) Σ_n action as chain complexes.

The derived tensor product over an additive wreath product is equivalent to the homotopy orbits of a particular complex, as follows.

Lemma 5.17. *If M is a right $R\int\Sigma_n$ module and N is a left $R\int\Sigma_n$ module then*

$$M\hat{\otimes}_{R\int\Sigma_n}N \simeq (M\hat{\otimes}_{R^{\otimes n}}N)_{h\Sigma_n}$$

where an action of Σ_n on the simplicial object $M\hat{\otimes}_{R^{\otimes n}}N$ is determined by:

$$\sigma * (m\otimes r_1\otimes\cdots\otimes r_t\otimes n) \;=\; (m\sigma\otimes\sigma^{-1}r_1\sigma\otimes\cdots\otimes\sigma^{-1}r_t\sigma\otimes\sigma n).$$

Proof. There is a simplicial isomorphism from $M\hat{\otimes}_{R\int\Sigma_n}N$ to the diagonal of the bisimplicial object $(M\hat{\otimes}_{R^{\otimes n}}N)_{h\Sigma_n}$ determined by

$$(m\otimes(r_1\times\sigma_1)\otimes\cdots\otimes(r_n\times\sigma_t)\otimes n)$$

$$\Big\downarrow$$

$$(m(\sigma_{1\cdots n})\otimes(\sigma_{1\cdots n})^{-1}r_1(\sigma_{1\cdots n})\otimes(\sigma_{2\cdots n})^{-1}r_2(\sigma_{2\cdots n})\otimes\cdots\otimes\sigma_n^{-1}r_n\sigma_n\otimes n).$$

where $\sigma_{k\cdots n}=\sigma_k\cdot\sigma_{k+1}\cdots\sigma_n$. Applying the Eilenberg-Zilber theorem then yields the result.

The last classification of homogeneous degree n functors will be in terms of modules over the DGA $D_{1\times 1}(C)\int\Sigma_n$. Before stating the result, we need to know that $D_1^{(n)}cr_nF(C)$ can be given the structure of a right $D_{1\times 1}(C)\int\Sigma_n$ module, and $\bigotimes_{i=1}^n \perp_C^* D_1(C,\star)$ that of a left $D_{1\times 1}(C)\int\Sigma_n$ module. In the case of $D_1^{(n)}cr_nF(C)$, the module structure is derived from Σ_n and $D_{1\times 1}(C)^{\otimes n}$ actions on $D_1^{(n)}cr_nF(C)$. The Σ_n action on $D_1^{(n)}cr_nF(C)$ is the action induced by the isomorphism $cr_nF(X_1,\ldots,X_n)\xrightarrow{\cong}cr_nF(X_{\sigma(1)},\ldots,X_{\sigma(n)})$. The isomorphism induces a left Σ_n action on $D_1^{(n)}cr_nF(C)$ (by sending the object C in the ith position to that in the $\sigma^{-1}(i)$th position) and a right Σ_n action by the convention $x\sigma=\sigma^{-1}x$ for $x\in D_1^{(n)}cr_nF(C)$ and $\sigma\in\Sigma_n$.

The $D_{1\times 1}(C)^{\otimes n}$ action on $D_1^{(n)} cr_n F(C)$ comes from the fact that $D_1^{(n)} cr_n F(C)$ is a right $D_{1\times 1}(C)$ module in each variable separately. By making an appropriate choice of sign, the Σ_n and $D_{1\times 1}(C)^{\otimes n}$ actions can be combined to give $D_1^{(n)} cr_n F(C)$ a right $D_{1\times 1}(C) \int \Sigma_n$ action. Moreover, this $D_{1\times 1}(C) \int \Sigma_n$ action is compatible with the $Ch\mathcal{A}$ structure, so that $D_1^{(n)} cr_n F(C)$ is a $Ch\mathcal{A} - D_{1\times 1}(C) \int \Sigma_n$ bimodule. Similarly, by the comments in the proof of lemma 5.14, we know that $\bigotimes_{i=1}^{n} \perp_C^* D_1(C, \star)$ is a left $D_{1\times 1}(C)^{\otimes n}$ module. There is a left Σ_n action on $\bigotimes_{i=1}^{n} \perp_C^* D_1(C, \star)$ given by permuting the coordinates. Again, with the appropriate choice of sign, we obtain a left $D_{1\times 1}(C) \int \Sigma_n$ module structure on $\bigotimes_{i=1}^{n} \perp_C^* D_1(C, \star)$.

We let $[\mathcal{A} - \mathrm{Mod} - D_{1\times 1}(C) \int \Sigma_n\ ]_{h\Sigma_n}$ denote the derived category of $\mathcal{A} - \mathrm{Mod} - D_{1\times 1}(C) \int \Sigma_n$ where a map $f : X \longrightarrow Y$ of complexes is an equivalence if $f_{h\Sigma_n} : X_{h\Sigma_n} \longrightarrow Y_{h\Sigma_n}$ is an equivalence. Note that this condition holds whenever f is a quasi-isomorphism so that we are inverting more maps in $[\mathcal{A} - \mathrm{Mod} - D_{1\times 1}(C) \int \Sigma_n\ ]_{h\Sigma_n}$ than in $[\mathcal{A} - \mathrm{Mod} - D_{1\times 1}(C) \int \Sigma_n\ ]$.

Using the ideas above, we obtain our last classification.

Proposition 5.18. (Classification III) *There is a natural isomorphism:*

$$[\mathcal{C}, Ch\mathcal{A}]_{C,[n]} \;\cong\; [\mathcal{A} - \mathrm{Mod} - D_{1\times 1}(C) \int \Sigma_n\ ]_{h\Sigma_n}.$$

Proof. The proof of this result is again similar to those of the previous classification results. We define $\chi : \mathcal{A} - \mathrm{Mod} - D_{1\times 1}(C) \int \Sigma_n \longrightarrow \mathrm{Func}_{C,[n]}(\mathcal{C}, Ch\mathcal{A})$ by

$$\chi(M) = M \hat{\otimes}_{D_{1\times 1}(C) \int \Sigma_n} \bigotimes{}_{\mathbb{Z}}^{n} \perp_C^* D_1(C, \star),$$

and $\phi : \mathrm{Func}_{C,[n]}(\mathcal{C}, Ch\mathcal{A}) \longrightarrow \mathcal{A} - \mathrm{Mod} - D_{1\times 1}(C) \int \Sigma_n$ by $\phi(F) = D_1^{(n)} cr_n F(C, \ldots, C)$.

The facts that $\chi \circ \phi$ is the identity on $[\mathcal{C}, Ch\mathcal{A}]_{(C,[n])}$ and $\phi \circ \chi$ is the identity on $[\mathcal{A}-Mod-D_{1\times 1}(C) \int \Sigma_n]_{h\Sigma_n}$ follow from lemma 5.17, the proof of corollary 5.15, proposition 3.10 of [J-M3], and arguments similar to those in the other classification results.

6. Translation of natural functor operations to modules

In this section we explore the way in which the classification results of section 5 behave with respect to operations that change the degree of a functor. In particular, we will look at the effects of composition, the structure map q, and differentiation. We will also consider how the homogeneous degree n classification is related to the nonhomogeneous degree n classification, and how the classification of degree $n-1$ functors is related to that of degree n functors, since every degree $n-1$ functors is also degree n.

Many of the results in this section will be expressed in terms of the following objects. For an object C in $\mathcal{C}$ recall that

$$\mathbf{n} = \mathbf{n}_C = C \wedge \mathbf{n}.$$

We will use $P_t(\mathbf{m}, \mathbf{n})$ to denote the objects:

$$P_t(\mathbf{m}, \mathbf{n}) = \mathrm{Hom}_{P_t\mathcal{C}}(\mathbf{m}, \mathbf{n}).$$

These objects are equipped with the natural actions described below.

Convention for Module Actions 6.1. Let $n \geq 0$ and X, Y, and Z be objects in $\mathcal{C}$. From theorem 4.8, we know that $\mathrm{Hom}_{P_n\mathcal{C}}(X, X)$ is a DGA and that $\mathrm{Hom}_{P_n\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{P_n\mathcal{C}}(Z, X)$ are left and right modules, respectively, over $\mathrm{Hom}_{P_n\mathcal{C}}(X, X)$. For $t \geq n$, we have a functor $q[t, n] : P_t\mathcal{C} \longrightarrow P_n\mathcal{C}$ (lemma 4.16). For any $X \in \mathcal{C}$, this functor yields a natural map of DGA's $\mathrm{Hom}_{P_t\mathcal{C}}(X, X) \xrightarrow{q(t,n)} \mathrm{Hom}_{P_n\mathcal{C}}(X, X)$. Thus, for any $s, t \geq n$ and $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{P_n\mathcal{C}}(X, Y)$ is naturally a left $\mathrm{Hom}_{P_t\mathcal{C}}(X, X)$ and right $\mathrm{Hom}_{P_s\mathcal{C}}(Y, Y)$ bimodule. Using lemma 4.6 we see that these actions are compatible with one another as s and t vary. We will assume throughout this section that $\mathrm{Hom}_{P_n\mathcal{C}}(X, Y)$ has these bimodule structures and use them without further comment.

STRUCTURE MAP q

We first consider the effect of the natural transformation $q : P_n \to P_{n-1}F$ on the classification modules. Since every degree n functor F is naturally equivalent to P_nF, q produces a natural transformation from the derived category of degree n functors defined along C to that of degree $(n-1)$ functors defined along C. The corresponding transformation on the classification modules is given by the following.

Proposition 6.2. *The following diagram commutes:*

$$\begin{tikzcd}
{[\mathcal{C}, Ch\mathcal{A}]_{C,n}} \arrow[r, phantom, "\cong"] \arrow[d, "q"] & {[\mathcal{A} - \mathrm{Mod} - P_{n\times n}(C)]} \arrow[d, "\hat{\otimes}_{P_{n\times n}(C)} P_{n-1}(\mathbf{n},\mathbf{n-1})"] \\
{[\mathcal{C}, Ch\mathcal{A}]_{C,(n-1)}} \arrow[r, phantom, "\cong"] & {[\mathcal{A} - \mathrm{Mod} - P_{(n-1)\times(n-1)}(C)].}
\end{tikzcd}$$

Proof. Let X be an $\mathcal{A} - P_{n\times n}(C)$ bimodule and consider the following natural commuting diagram:

$$\begin{tikzcd}
X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})} P_n(\mathbf{n},\star) \arrow[d, "\simeq"] & \\
P_n(X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})} P_n(\mathbf{n},\star)) \arrow[r, "q"] \arrow[d, "\cong"] & P_{n-1}(X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})} P_n(\mathbf{n},\star)) \arrow[d, "\cong"] \\
X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})} P_n(P_n(\mathbf{n},\star)) \arrow[r, "q"] & X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})} P_{n-1}(P_n(\mathbf{n},\star)) \\
& X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})} P_{n-1}(\mathbf{n},\star) \arrow[u, "\simeq"]
\end{tikzcd}$$

From the diagram it follows that

$$q(\perp_C^* X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})} P_n(C,-)) \simeq \perp_C^* (X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})} P_{n-1}(\mathbf{n},\star))$$

where the left hand side is q applied to the degree n functor corresponding to the module X. The right hand side of the equivalence is a functor that agrees at the object $\mathbf{n-1}$ with the functor

$$\perp_C^* (X\hat{\otimes}_{P_{n-1}(\mathbf{n-1},\mathbf{n-1})} P_{n-1}(\mathbf{n},\mathbf{n-1})\hat{\otimes}_{P_{n-1}(\mathbf{n-1},\mathbf{n-1})} P_{n-1}(\mathbf{n-1},-))$$

via the natural transformation given by composition. Since both functors are degree n and defined along C, they are equivalent by theorem 2.11. Hence $q(\perp_C^* X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})}P_n(C,-))$ is equivalent to

$$\perp_C^* (X\hat{\otimes}_{P_{n-1}(\mathbf{n-1},\mathbf{n-1})}P_{n-1}(\mathbf{n},\mathbf{n-1})\hat{\otimes}_{P_{n-1}(\mathbf{n-1},\mathbf{n-1})}P_{n-1}(\mathbf{n-1},-))$$

and so the diagram commutes.

INCLUSION

Since every degree n functor is also a degree k functor we have an inclusion functor from $\mathrm{Func}(\mathcal{C},Ch\mathcal{A})_{C,n}$ to $\mathrm{Func}(\mathcal{C},Ch\mathcal{A})_{C,k}$ when $k>n$. Let

$$I(n,k)=P_n(\mathbf{n},\mathbf{k})$$

and note that $I(n)$ is a left $P_{n\times n}(C)$/right $P_{k\times k}(C)$-bimodule. To understand the effect of the inclusion functor on the classifying modules, we first consider the case where $k=n+1$.

Proposition 6.3. *For an object C in $\mathcal{C}$, the following diagram commutes:*

$$\begin{array}{ccc}
[\mathcal{C},Ch\mathcal{A}]_{C,n} & \cong & [\mathcal{A}-\mathrm{Mod}-P_{n\times n}(C)] \\
\downarrow \subseteq & & \downarrow \hat{\otimes}_{P_{n\times n}(C)}I(n,n+1) \\
[\mathcal{C},Ch\mathcal{A}]_{C,(n+1)} & \cong & [\mathcal{A}-\mathrm{Mod}-P_{(n+1)\times(n+1)}(C)]\,.
\end{array}$$

Proof. For an $\mathcal{A}-P_{n\times n}(C)$ bimodule M, we must show that the functor

$$\perp_C^* M\hat{\otimes}_{P_{n\times n}(C)}P_n(C,\star)$$

is naturally equivalent to

$$\perp_C^* (M\hat{\otimes}_{P_{n\times n}(C)}I(n,n+1)\hat{\otimes}_{P_{(n+1)\times(n+1)}(C)}P_{n+1}(C,\star)).$$

Using the structure map $q : P_{n+1} \to P_n$ and composition we obtain a natural transformation:

$$
\begin{array}{c}
\perp_C^* \, (M\hat{\otimes}_{P_{n\times n}(C)} I_n(n, n+1) \hat{\otimes}_{P_{(n+1)\times(n+1)}(C)} P_{n+1}(C, \star)) \\
\Big\downarrow{=} \\
\perp_C^* \, (M\hat{\otimes}_{P_{n\times n}(C)} P_n(\mathbf{n}, \mathbf{n}+\mathbf{1}) \hat{\otimes}_{P_{(n+1)}(\mathbf{n+1}, \mathbf{n+1})} \, P_{(n+1)}(\mathbf{n}+\mathbf{1}, \star)) \\
\Big\downarrow{q} \\
\perp_C^* \, (M\hat{\otimes}_{P_{n\times n}(C)} P_n(\mathbf{n}, \mathbf{n}+\mathbf{1}) \hat{\otimes}_{P_n(\mathbf{n+1}, \mathbf{n+1})} \, P_n(\mathbf{n}+\mathbf{1}, \star)) \\
\Big\downarrow{\circ} \\
\perp_C^* \, (M\hat{\otimes}_{P_{n\times n}(C)} P_n(\mathbf{n}, \star)).
\end{array}
$$

Clearly, this map is an equivalence at $\mathbf{n}+\mathbf{1}$. Since both functors are degree $n+1$ and defined along C, the result follows by theorem 2.11.

Proposition 6.4. *For $k > n$, the following diagram commutes:*

$$
\begin{array}{ccc}
[\mathcal{C}, Ch\mathcal{A}]_{C,n} & \cong & [\mathcal{A} - Mod - P_{n\times n}(C)] \\
\Big\downarrow{\subseteq} & & \Big\downarrow{\hat{\otimes} I(n,k)} \\
[\mathcal{C}, Ch\mathcal{A}]_{C,k} & \cong & [\mathcal{A} - Mod - P_{k\times k}(C)] \, .
\end{array}
$$

Proof. As in the proof of proposition 6.3 it suffices to show that for $t > n$, the natural transformation

$$
\begin{array}{c}
\perp_C^* \, (M\hat{\otimes}_{P_{n\times n}(C)} P_n(\mathbf{n}, \mathbf{t}) \hat{\otimes}_{P_t(\mathbf{t},\mathbf{t})} P_t(\mathbf{t}, \star)) \\
\Big\downarrow{q[n,t]} \\
\perp_C^* \, (M\hat{\otimes}_{P_{n\times n}(C)} (P_n(\mathbf{n}, \mathbf{t}) \hat{\otimes}_{P_n(\mathbf{t},\mathbf{t})} P_n(\mathbf{t}, \star)) \\
\Big\downarrow{\circ} \\
\perp_C^* \, (M\hat{\otimes}_{P_{n\times n}(C)} P_n(\mathbf{n}, \star)
\end{array}
$$

is an equivalence. Since the map is trivially an equivalence at $\mathbf{t}$ and this is a natural transformation of degree t functors defined along C, it is an equivalence at all objects by theorem 2.11.

Composition

Recall from 2.10 of [J-M3] that the composition of a degree m functor with a degree n functor is a degree mn functor. We now examine the way in which composition behaves with respect to the classification result. For this discussion, we assume that $\mathcal{A}$ has arbitrary sums, and $\mathcal{B}$ is an abelian category. We further assume that the functors $G : \mathcal{C} \to Ch\mathcal{A}$ and $F : Ch\mathcal{A} \to Ch\mathcal{B}$ are defined along the objects $C \in \mathcal{C}$ and $A \in Ch\mathcal{A}$ respectively. Since our classification results are dependent upon a functor being defined along an object C we must restrict our attention to compositions of functors that behave well with respect to this property. This depends on A and C satisfying the next property.

Definition 6.5. *We say that* $(A, C) \in Ch\mathcal{A} \times \mathcal{C}$ *is a* composable pair *if* $\perp_A (\perp_C^* G) \xrightarrow{\simeq} \perp_A (G)$ *is a* simplicial homotopy equivalence *for all functors* $G : \mathcal{C} \to Ch\mathcal{A}$ *that are defined along* C.

If $\mathcal{A}$ is the category of R-modules and A is any free R-module then (A, C) is a composable triple for any $C \in \mathcal{C}$. Note that if (A, C) is a composable pair, then for any functor $F : Ch\mathcal{A} \to Ch\mathcal{B}$, $\perp_A F(\perp_C^* G) \xrightarrow{\simeq} \perp_A F(G)$ since simplicial homotopy equivalences are preserved by functors. This is enough to guarantee that composition preserves the property of defined along C, as proven below.

Lemma 6.6. *If* (A, C) *is a composable pair and* $G : \mathcal{C} \to Ch\mathcal{A}$ *is defined along* C*, then* $F \circ G : \mathcal{C} \longrightarrow Ch\mathcal{B}$ *is defined along* C *for any functor* $F : Ch\mathcal{A} \to Ch\mathcal{B}$ *that is defined along* A.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} \perp_A^* F(\perp_C^* G) & \longrightarrow & \perp_A^* F(G) \\ \downarrow \simeq & & \downarrow \simeq \\ \perp_C^* (F \circ G) = F(\perp_C^* G) & \longrightarrow & F(G). \end{array}$$

Since (A, C) is a composable pair the upper map is an equivalence as it is a map of simplicial chains which is an equivalence degree-wise. Hence the lower map is an equivalence as well.

To describe the effect of composition on classifying modules, we use the following functors and pairings.

Definition 6.7. *Let $\tau_{A,C}$ be the functor from $\mathcal{A}-P_{m\times m}(C)$ bimodules to right $P_{(m\cdot n)\times(m\cdot n)}(C)$ /left $P_{n\times n}(A)$-bimodules given by*

$$\tau_{A,C}(M) = \mathrm{Hom}_{P_n Ch\mathcal{A}}(A\wedge \mathbf{n}, M\hat{\otimes}_{P_{m\times m}(C)}P_{m\times m}(C\wedge\mathbf{m}, C\wedge\mathbf{m}\cdot\mathbf{n})).$$

We define a pairing

$$\begin{array}{c} [\mathcal{A}-\mathrm{Mod}-P_{m\times m}(C)]\times[\mathcal{B}-\mathrm{Mod}-P_{n\times n}(A)] \\ \Big\downarrow{\scriptstyle \mu_{A,C}} \\ [\mathcal{A}-\mathrm{Mod}-P_{(m\cdot n)}(C)] \end{array}$$

by $\mu_{A,C}(M\times N) = N\hat{\otimes}_{P_{n\times n}(A)}\tau_{A,C}(M)$.

One can show that $\mu_{A,C}$ is well defined on the derived categories by directly using the conditions placed on the pair (A, C) or, alternatively, by using the next lemma.

Lemma 6.8. *For a composable pair (A, C) the following diagram commutes:*

$$\begin{array}{ccc} [\mathcal{C}, Ch\mathcal{A}]_{C,m} & \cong & [\mathcal{A}-\mathrm{Mod}-P_{m\times m}(C)] \\ \times & & \times \\ [Ch\mathcal{A}, Ch\mathcal{B}]_{A,n} & \cong & [\mathcal{B}-\mathrm{Mod}-P_{n\times n}(A)] \\ \Big\downarrow{\scriptstyle\circ} & & \Big\downarrow{\scriptstyle\mu_{A,C}} \\ [\mathcal{C}, Ch\mathcal{B}]_{C,(m\cdot n)} & \cong & [\mathcal{A}-\mathrm{Mod}-P_{(m\cdot n)\times(m\cdot n)}(C)]. \end{array}$$

Proof. Let M be an $\mathcal{A}-P_{m\times m}(C)$ bimodule and N be a $\mathcal{B}-P_{n\times n}(A)$ bimodule. It follows directly from the definitions involved that the composition of the functors corresponding to M and N is equivalent to the functor corresponding to $\mu_{(A,C)}(M, N)$.

HOMOGENEOUS FUNCTORS

Since homogeneous degree n functors are also degree n functors, there should be a nice relationship between the homogeneous degree n classifications and the degree n classification. We will concentrate on determining this relationship for classification I of homogeneous degree n functors. Once this is established, one can determine the relationships for classifications II and III by determining the relationship between the module categories used in the three classifications for homogeneous degree n functors. This is done in the appendix.

Proposition 6.9. *For an object C in $\mathcal{C}$, the following diagram commutes:*

$$\begin{array}{ccc} [\mathcal{C}, Ch\mathcal{A}]_{C,[n]} & \cong & [Ch\mathcal{A} - \text{Mod} - D_{n\times n}(C)] \\ \big\downarrow{\scriptstyle inc} & & \big\downarrow{\scriptstyle \hat{\otimes}_{D_{n\times n}(C)} D_{n\times n}(C)} \\ [\mathcal{C}, Ch\mathcal{A}]_{C,n} & \cong & [Ch\mathcal{A} - \text{Mod} - P_{n\times n}(C)] \\ \big\downarrow{\scriptstyle D_n} & & \big\downarrow{\scriptstyle \hat{\otimes}_{P_{n\times n}(C)} D_{n\times n}(C)} \\ [\mathcal{C}, Ch\mathcal{A}]_{C,[n]} & \cong & [Ch\mathcal{A} - \text{Mod} - D_{n\times n}(C)]. \end{array}$$

Proof. We first note that for the upper square to make sense, it must be the case that $D_{n\times n}(C)$ is a right $P_{n\times n}(C)$ module. This follows from the fact that $D_n(\mathbf{n}, \star)$ is a degree n functor. To see that the upper square commutes, let M be an $\mathcal{A} - \text{Mod} - D_{n\times n}(C)$ bimodule. It is straightforward to show that the homogeneous degree n functor corresponding to M under classification I and the degree n functor corresponding to $M\hat{\otimes}_{D_{n\times n}(C)} D_{n\times n}(C)$ under the degree n classification theorem agree at the object $\mathbf{n}$. The result then follows by theorem 2.11.

To see that the lower square commutes, first note that the natural map $\Sigma_D(n,n) : D_nP_n \xrightarrow{\simeq} D_n$ and the left $P_{n\times n}(C)$ module structure of $P_n(C, \star)$ give $D_n(\mathbf{n}, \star)$ the structure of a left $P_{n\times n}(C)$-module. Let

X be a $Ch\mathcal{A} \otimes P_{n\times n}(C)$ bimodule. Then we have

$$\begin{aligned} D_n\left(X\hat{\otimes}_{P_{n\times n}(C)}P_n(C,\star)\right) &= D_n(X\hat{\otimes}_{P_{n\times n}(C)}P_n(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]) \\ &= X\hat{\otimes}_{P_{n\times n}(C)}D_n(P_n(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]) \\ &\xrightarrow{\simeq} X\hat{\otimes}_{P_{n\times n}(C)}D_n(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]) \end{aligned}$$

where the equivalence comes from the natural map $\Sigma_D(n,n)$ of (1.22). This is naturally equivalent to

$$X\hat{\otimes}_{P_{n\times n}(C)}D_{n\times n}(C)\hat{\otimes}_{D_{n\times n}(C)}D_n(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)])$$

and by applying $\perp_C^*$ to this sequence of equivalences, we see that the lower square commutes.

Differentiation

Recall from remark 2.14.b that differentiation along an object C takes degree n functors defined along C to degree $(n-1)$ functors defined along C. That is, $\frac{d}{dC}$ is a functor from $\mathrm{Func}_{C,n}(\mathcal{C}, Ch\mathcal{A})$ to $\mathrm{Func}_{C,n-1}(\mathcal{C}, Ch\mathcal{A})$. The effect of $\frac{d}{dC}$ on the classifying modules for $\mathrm{Func}_{C,n}(\mathcal{C}, Ch\mathcal{A})$ is equivalent to tensoring with

$$D(n) = \frac{d}{dC}(P_n(\mathbf{n},\star))(\mathbf{n-1}),$$

which we prove below. Note first that by lemma 5.5 and the naturality of $\frac{d}{dC}$, $D(n)$ has a natural left $P_{n\times n}(C)$ action. In addition, $D(n)$ is a right $P_{(n-1)\times(n-1)}(C)$ by lemma 5.5 and remark 2.14.b.

Lemma 6.10. *For an object C in $\mathcal{C}$, the following diagram commutes:*

$$\begin{tikzcd} {[\mathcal{C}, Ch\mathcal{A}]_{C,n}} \arrow[r, phantom, "\cong"] \arrow[d, "\frac{d}{dC}"] & {[\mathcal{A}-\mathrm{Mod}-P_{n\times n}(C)]} \arrow[d, "\hat{\otimes}_{P_{n\times n}(C)}D(n)"] \\ {[\mathcal{C}, Ch\mathcal{A}]_{C,(n-1)}} \arrow[r, phantom, "\cong"] & {[\mathcal{A}-\mathrm{Mod}-P_{(n-1)\times(n-1)}(C)]}. \end{tikzcd}$$

Proof. Let X be an $\mathcal{A}-P_{n\times n}(C)$ bimodule. Since $\frac{d}{dC}$ commutes with linear functors, and the functor $X\hat{\otimes}_{P_n(\mathbf{n},\mathbf{n})}\star$ is linear, the derivative along C of the degree n functor corresponding to X is the functor given by

$$\perp_C^* (X\hat{\otimes}_{P_{n\times n}(C)}\frac{d}{dC}P_n(\mathbf{n},-)).$$

However, the functor corresponding to the $\mathcal{A}-P_{(n-1)\times(n-1)}(C)$ bimodule $X\hat{\otimes}_{P_{n\times n}(C)}D_n(C)$ is

$$\perp_C^* (X\hat{\otimes}_{P_{n\times n}(C)}D_n(C)\hat{\otimes}_{P_{n-1\times n-1}(C)}P_{n-1}(\mathbf{n-1},\star)).$$

Since the two functors are degree $(n-1)$ and defined along C, and agree at $\mathbf{n-1}$, the result follows by theorem 2.11.

We have not, in general, found a better characterization of $D(n)$ than as the derivative of another functor. However, using the product rule we can express $D(n)$ in terms of $P_{n-1}(\mathbf{n},\star)$ and derivatives of the functor $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]$. We will use this to determine a formula for $D_n(C)$ in the case where $\mathcal{C}$ is an additive category. Doing so necessitates presenting some new notation to describe actions of $P_t(\mathbf{1},\mathbf{1})$ on $P_t(\mathbf{n},\mathbf{n})$.

Definition 6.11. *For $1\le j\le n$, let i_j be the pointed set map from $\{*,1\}$ to $\{*,1,2,\dots,n\}$ given by sending 1 to j. For an object C in $\mathcal{C}$, we define*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}\wedge C,\mathbf{1}\wedge C)\xrightarrow{i(j)}\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}\wedge C,\mathbf{n}\wedge C)\cong\prod_{t=1}^{n}\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}\wedge C,\mathbf{n}\wedge C)$$

to be the homomorphism that takes α to $i(j)(\alpha)$, where

$$i(j)(\alpha)_t=\begin{cases}(i_j\wedge C)\circ\alpha & \text{if } t=j\\ (i_t\wedge C)\circ\mathrm{id} & \text{if } t\neq j.\end{cases}$$

The map $i(j)$ produces an algebra map from $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]$ to $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\mathbf{n})]$ and a left $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]$ module action on the algebra

$\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\mathbf{n})]$ which we will indicate by $\mathbb{Z}^{(j)}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\mathbf{n})]$. These algebra maps and module actions can be extended to $P_t(\mathbf{1},\mathbf{1})$ and $P_t(\mathbf{n},\mathbf{n})$ for all t. We will use similar notation for the actions of $P_t(\mathbf{1},\mathbf{1})$ on $P_t(\mathbf{n},\mathbf{n})$.

Definition 6.12. *For an object C in $\mathbf{C}$, $\mathrm{Mat}_{\mathcal{C}}(C)$ is the following subcategory of $\mathcal{C}$. We set $\mathrm{Obj}(\mathrm{Mat}_{\mathcal{C}}(C)) = \{\mathbf{n}_C \mid n \in \mathbb{N}\}$ and*

$$\mathrm{Hom}_{\mathrm{Mat}_{\mathcal{C}}(C)}(\mathbf{m},\mathbf{n}) = \prod^{m}\bigvee^{n}\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}).$$

In doing so, we are identifying $\prod^m\bigvee^n\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})$ with the subset of $\mathrm{Hom}_{\mathcal{C}}(\mathbf{m},\mathbf{n}) \cong \prod^m\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{n})$ determined by the image of the injective set map

$$\bigvee^{n}\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\bigvee^{n}\mathbf{1}) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{n})$$

obtained from the structure maps of the sum. Composition in the category $\mathrm{Mat}_{\mathcal{C}}(C)$ resembles matrix multiplication.

Proposition 6.13. *For an object C in $\mathcal{C}$, there are isomorphisms of functors:*

$$\frac{d}{dC}\left(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]\right)$$

$$\cong$$

$$\bigoplus_{j=1}^{n}\left(\frac{d}{dC}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]\hat{\otimes}_{\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]}\mathbb{Z}^{(j)}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]\right)$$

$$\frac{d}{dC}\left(P_t\left(\mathbf{n},\star\right)\right) \cong \bigoplus_{j=1}^{n}\left(\frac{d}{dC}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]\hat{\otimes}_{\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]}P_{t-1}^{(j)}(\mathbf{n},\star)\right).$$

These isomorphisms are natural in $\mathbf{n}$ with respect to $\mathrm{Mat}_{\mathcal{C}}(C)$, but not, in general, with respect to $\mathcal{C}$.

Proof. The strategy is to rewrite the functor $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]$ as a tensor product of functors to which we can apply the product rule of [J-M3], proposition 5.5. We begin by noting that $\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)$ and $\prod_{j=1}^{n}\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\star)$ are isomorphic as functors to pointed sets. Using this decomposition requires restricting the result to $\mathrm{Mat}_{\mathcal{C}}(C)$ since the decomposition is not natural with respect to morphisms in $\mathcal{C}$. The reason for this is that a morphism α from $\mathbf{m}$ to $\mathbf{n}\cong\bigvee^{n}\mathbf{1}$ is not necessarily determined by its values at $\pi_j\circ\alpha$ for $1\le j\le n$. For example, a map to a wedge of two circles may not be determined uniquely (even up to based homotopy) by its restriction to each circle separately. However, the decomposition is natural with respect to morphisms in $\mathrm{Mat}_{\mathcal{C}}(C)$.

Since the functor $\mathbb{Z}[\ \ ]$ takes products to tensor products, we see that $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]\cong\bigotimes_{j=1}^{n}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\star)]$. Applying the product rule ([J-M3], proposition 5.5) yields

$$\begin{array}{c}\frac{d}{dC}\left(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]\right)\\ \simeq \\ \bigoplus_{j=1}^{n}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\star)]^{\otimes_{\mathbb{Z}}(j-1)}\\ \otimes_{\mathbb{Z}} \\ \frac{d}{dC}\left(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\star)]\right)\otimes_{\mathbb{Z}}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\star)]^{\otimes_{\mathbb{Z}}(n-j)}.\end{array}\tag{1}$$

However, using remark 3.19 and lemma 3.5 we see that

$$\begin{array}{c}\frac{d}{dC}\left(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]\right)\hat{\otimes}_{\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\star)]\\ \wr\Vert \\ \frac{d}{dC}\left(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\star)]\right)\end{array}\tag{2}$$

since the functors agree at the value $\mathbf{1}$. Substituting (2) into (1) and reordering produces the first isomorphism to be proved.

To prove the second isomorphism, recall from remark 2.14.b that $\frac{d}{dC}P_t \cong P_{t-1}\frac{d}{dC}$. Since $M\hat{\otimes}_R\star$ is a linear functor for any DGA R and right R module M, it commutes with P_{t-1}. Thus,

$$\begin{gathered}
\tfrac{d}{dC}\left(P_t\left(\mathbf{n},\star\right)\right) \\
\wr\| \\
P_{t-1}\left(\tfrac{d}{dC}\left(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]\right)\right) \\
\wr\| \\
P_{t-1}\left(\bigoplus_{j=1}^{n}\left(\tfrac{d}{dC}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]\right)\hat{\otimes}_{\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]}\mathbb{Z}^{(j)}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\star)]\right) \\
\wr\| \\
\bigoplus_{j=1}^{n}\left(\tfrac{d}{dC}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]\right)\hat{\otimes}_{\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]}P_{t-1}^{(j)}(\mathbf{n},\star).
\end{gathered}$$

Note that the subcategory $\mathrm{Mat}_{\mathcal{C}}(C)$ is a full subcategory whenever $\mathcal{C}$ is an additive category. Hence, we have the following.

Corollary 6.14. *If $\mathcal{C}$ is an additive category then there is a natural isomorphism (as left $P_{n\times n}(C)$/right $P_{(n-1)\times(n-1)}(C)$ bimodules),*

$$D(n) \cong \bigoplus_{j=1}^{n}\left(\frac{d}{dC}\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]\hat{\otimes}_{\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]}P_{n-1}^{(j)}(\mathbf{n},\mathbf{n}-\mathbf{1})\right).$$

We now restrict our attention to the case where $\mathcal{C}$ is an additive category $\mathcal{R}$ and functors are defined along the object R in $\mathbb{R}$. Using the previous proposition and corollary, we can express $D(n)$ in terms of a sum of copies of $D_1(\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)])(R)\otimes_{\mathbb{Z}}P_{n-1}(\mathbf{n},\mathbf{n}-\mathbf{1})$. To do so we revisit the derivatives of $\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)]$.

Proposition 6.15.*For $\mathcal{R}$ an additive category and $R\in\mathcal{R}$,*

$$\frac{d}{dR}\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)] \;\cong\; D_1(\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)])(R)\;\otimes_{\mathbb{Z}}\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)].$$

Proof. We will write $\exp(R,\star)$ for $\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)]$. (This choice of notation is explained in the remark 6.17.) Since $\mathcal{R}$ is additive,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{R}}(R,X\oplus Y) &\cong \mathrm{Hom}_{\mathcal{R}}(R,X)\oplus \mathrm{Hom}_{\mathcal{R}}(R,Y)\\ &\cong \mathrm{Hom}_{\mathcal{R}}(R,X)\times \mathrm{Hom}_{\mathcal{R}}(R,Y).\end{aligned}$$

It follows that $\exp(R,X\oplus Y)\cong \exp(R,X)\otimes_{\mathbb{Z}}\exp(R,Y)$ and, as a consequence, that

$$cr_2\exp(R,-)(X,Y)\cong \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{R}}(R,X)]\otimes_{\mathbb{Z}}\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{R}}(R,Y)]$$

where for a pointed set U, $\tilde{\mathbb{Z}}[\star]=\mathbb{Z}[U]/\mathbb{Z}[*]$. Recall from definition 2.13 that computing $\frac{d}{dR}\exp(R,\star)$ entails determining

$$cr_2\exp(R;\ )(X,Y)\oplus cr_1\exp(R;\ )(X).$$

But this is given by

$$\begin{aligned}&\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{R}}(R,X)]\otimes_{\mathbb{Z}}\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{R}}(R,Y)]\oplus\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{R}}(R,X)]\\ &\cong \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{R}}(R,X)]\ \otimes_{\mathbb{Z}}\langle\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{R}}(R,Y)]\oplus\mathbb{Z}\rangle\\ &\cong \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{R}}(R,X)]\otimes_{\mathbb{Z}}\exp(R;Y).\end{aligned}$$

Hence, $\nabla\exp(R;\ )\cong D_1\exp(R;\ )\otimes_{\mathbb{Z}}\exp(R;\ )$ which is the result we were after.

Corollary 6.16. *For an additive category $\mathcal{R}$ and $R\in\mathcal{R}$,*

$$D(n)\simeq\bigoplus_{j=1}^{n}D_1(\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)])(R)\otimes_{\mathbb{Z}}P_{n-1}^{(j)}(\mathbf{n},\mathbf{n-1})$$

Proof. By the previous proposition and corollary 6.14, this result follows by applying P_n to the equivalence

$$\begin{array}{c} D_1(\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)])(R) \\ \otimes_{\mathbb{Z}} \\ \mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,R)]\hat{\otimes}_{\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,R)]}\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)] \\ \Big\downarrow \simeq \\ D_1(\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)])(R) \ \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)]. \end{array}$$

Remark 6.17. Suppose $\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)]$ is defined along R (for example, if R is a finitely generated projective module in $\mathcal{R}$ when $\mathcal{R}$ is a module category). Then the degree 1 approximation to the functor along R is $P_1(\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)])(-)$ and its linear part is given by $D_1(\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)])(-)$. If we were working with a degree 1 approximation to a real valued function f, the best linear approximation to f expanded about 0 would be of the form $g(x) = a + bx$ where a is a constant and $x \mapsto bx$ is linear. The coefficient b would be given by $b = f'(0)$. Hence, in the functor setting, $D_1(\mathbb{Z}[\mathrm{Hom}_{\mathcal{R}}(R,\star)])(R)$ plays the role of the derivative of the functor at 0. With this convention we can rewrite proposition 6.15 to read:

$$\exp'(R;X) = \exp'(R;0) \otimes_{\mathbb{Z}} \exp(R;X).$$

and so $\exp(R;\ )$ satisfies the differential equation $f' = f'(0)\cdot f$.

Appendix:
Relations between $D_{n\times n}(C)$, $D_n(1)(C)$ and $D_{1\times 1}(C)\int\Sigma_n$

By the existence of our three different classification results for n homogeneous functors (defined along C) one knows that there are direct equivalences between the module categories involved. We establish these here.

Lemma A.1. *The left $D_{n\times n}(C)$/right $D_n(1)$ module $D_n(\mathbf{n},\mathbf{1})$ is such that the following diagram commutes:*

$$\begin{tikzcd} & {[\mathcal{A}-Mod-D_{n\times n}(C)]} \arrow[d, "\hat{\otimes}_{D_{n\times n}(C)}D_n(\mathbf{n},\mathbf{1})"] \\ {[\mathcal{C},CH\mathcal{A}]_{C,[t]}} \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & \\ & {[\mathcal{A}-Mod-D_n(1)].} \end{tikzcd}$$

The inverse to $\hat{\otimes}_{D_{n\times n}(C)}D_n(\mathbf{n},\mathbf{1})$ is given by $\hat{\otimes}_{D_n(1)}D_n(\mathbf{1},\mathbf{n})$.

Proof. Let M be a $CH\mathcal{A}\otimes D_{n\times n}(C)$ bimodule. The associated homogeneous degree n functor defined along C is given by $\chi(M) =\perp_C^*$ $(M\hat{\otimes}_{D_{n\times n}(C)}D_n(C,\star))$. The associated $D_n(1)$ module is given by $\chi(M)(\mathbf{1})$ which is

$$\perp_C^* (M\hat{\otimes}_{D_{n\times n}(C)}D_n(C,\mathbf{1})) \xrightarrow{\cong} M\hat{\otimes}_{D_{n\times n}(C)}D_n(C,\mathbf{1})$$

and hence the result. The inverse statement is proved similarly.

To study the relationship between the modules of $D_{n\times n}(C)$ and $D_{1\times 1}(C) \int \Sigma_n$ we wish to establish some auxiliary DGA's and maps between these. Though this can be done directly, it is really no more effort to work at the level of categories which we now construct.

Definition A.2. *We define $D_1(\mathcal{C}) \int \Sigma_n$ to be the DG-category with the same objects as $\mathcal{C}$ and*

$$\mathrm{Hom}_{D_1(\mathcal{C})\int\Sigma_n}(X,Y) = \mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}} \mathrm{Hom}_{D_1(\mathcal{C})}(X,Y)^{\otimes_{\mathbb{Z}} n}.$$

Composition is defined so that for all $X \in \mathcal{C}$ we have an isomorphism of DGA's

$$\mathrm{Hom}_{D_1(\mathcal{C})\int\Sigma_n}(X,X) = \mathrm{Hom}_{D_1(\mathcal{C})}(X,X) \int \mathbb{Z}[\Sigma_n].$$

Using $*$ to denote the ring multiplication, we determine that composition is defined by:

$$\begin{array}{c}([\tau]\otimes b_1\otimes\cdots\otimes b_n)*([\sigma]\otimes a_1\otimes\cdots\otimes a_n)\\ \| \\ (-1)^\star([\tau\circ\sigma]\otimes b_{\sigma(1)}\circ a_1\otimes\cdots\otimes b_{\sigma(n)}\circ a_n)\end{array}$$

where $(-1)^\star$ is determined as in definition 5.16. We write the elements of the permutation group on the left in our presentation for the wreath product category because composition seems more natural to us when presented this way.

We define $\tilde{\mathbb{Z}}[\mathcal{C}]$ to be the $\mathbb{Z}$-linear category whose objects are those of $\mathcal{C}$ and whose morphisms are

$$\mathrm{Hom}_{\tilde{\mathbb{Z}}[\mathcal{C}]}(X,Y)=\frac{\mathrm{Hom}_{\mathbb{Z}[\mathcal{C}]}(X,Y)}{\mathbb{Z}[0]}=\frac{\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X,Y)]}{\mathbb{Z}[0]}.$$

If F from $\mathbb{Z}[\mathcal{C}]$ to some $\mathbb{Z}$-linear category $\mathcal{A}$ is reduced so that F of the morphism 0 between any two objects in $\mathcal{C}$ is sent to the 0 morphism in $\mathcal{A}$ then F factors through $\tilde{\mathbb{Z}}[\mathcal{C}]$. In particular, the morphism r from $\mathbb{Z}[\mathcal{C}]$ to $\mathbb{Z}[\mathcal{C}]$ which is the identity on objects and takes $[\alpha]$ to $[\alpha]-[0]$ is a reduced functor and hence produces a functor $r:\tilde{\mathbb{Z}}[\mathcal{C}]\longrightarrow\mathbb{Z}[\mathcal{C}]$ which is a retract to the natural projection functor.

We construct $\mathbb{Z}[\mathcal{C}]\int\Sigma_n$ and $\tilde{\mathbb{Z}}[\mathcal{C}]\int\Sigma_n$ from $\mathbb{Z}[\mathcal{C}]$ and $\tilde{\mathbb{Z}}[\mathcal{C}]$ as we constructed $D_1(\mathcal{C})\int\Sigma_n$ from $D_1(\mathcal{C})$. The natural transformation p_1 from $\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(X,\star)]$ to $P_1\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(X,\star)]$ produces a functor from $\tilde{\mathbb{Z}}[\mathcal{C}]$ to $D_1(\mathcal{C})$ and hence we have functors

$$\mathbb{Z}[\mathcal{C}]\int\Sigma_n\xleftarrow{r}\tilde{\mathbb{Z}}[\mathcal{C}]\int\Sigma_n\xrightarrow{d}D_1(\mathcal{C})\int\Sigma_n.$$

Our next objective is to attempt to construct a functor from $D_1(\mathcal{C})\int\Sigma_n$ to $D_n(\mathcal{C})$. We will not, however, succeed. We will instead construct a sequence of functors connecting these two categories which will be sufficient for our purposes. We first construct a functor from $\tilde{\mathbb{Z}}[\mathcal{C}]\int\Sigma_n$ to $\mathbb{Z}[\mathcal{C}]$. For $\sigma\in\Sigma_n$, we let

$$[\sigma]_Y=\sigma\wedge Y\in\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}\wedge Y,\mathbf{n}\wedge Y)=\mathrm{Hom}_{\mathcal{C}}(\bigvee^n Y,\bigvee^n Y).$$

We define ϕ to be the functor $\mathbb{Z}[\mathcal{C}] \int \Sigma_n \xrightarrow{\phi} \mathbb{Z}[\mathcal{C}]$ which takes X to $\mathbf{n} \wedge X$ and is determined on Hom groups by:

$$\mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}} \bigotimes_{i=1}^{n} \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X,Y)] \xrightarrow{\phi} \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\bigvee^{n} X, \bigvee^{n} Y)]$$

$$\sigma \times (\alpha_1 \times \cdots \times \alpha_n) \mapsto [\sigma]_Y \circ \left(\bigvee_{i=1}^{n} \alpha_i \right).$$

Composing with r we obtain a functor:

$$\tilde{\mathbb{Z}}[\mathcal{C}] \int \Sigma_n \xrightarrow{\phi \circ r} \mathbb{Z}[\mathcal{C}].$$

Consider for the moment the natural transformation $\phi \circ r$ on Hom as a natural transformation of functors (with X fixed):

$$\mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}} \bigotimes_{i=1}^{n} \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(X,\star)] \xrightarrow{\phi \circ r} \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n} \wedge X, \mathbf{n} \wedge \star)].$$

Since the functor on the left is n–multireduced, by section 3 of [J-M3] we know that D_n of the natural transformation $\phi \circ r$ is weakly equivalent to a natural transformation of the form we seek from $D_1(\mathcal{C}) \int \Sigma_n$ to $D_n(\mathcal{C})$. That is:

$$\begin{aligned}
\mathrm{Hom}_{D_1(\mathcal{C}) \int \Sigma_n}(X,\star) &= \mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}} \bigotimes_{i=1}^{n} \mathrm{Hom}_{D_1\mathcal{C}}(X,\star) \\
&= \mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}} \bigotimes_{i=1}^{n} D_1\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(X,\star)] \\
\text{(section 3)} \qquad &\xrightarrow{\simeq} D_n \left(\mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}} \bigotimes_{i=1}^{n} D_1\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(X,\star)] \right) \\
\text{(section 3)} \qquad &\xleftarrow{\simeq} D_n \left(\mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}} \bigotimes_{i=1}^{n} \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(X,\star)] \right) \\
&\xrightarrow{D_n(\phi \circ r)} D_n \left(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n} \wedge X, \mathbf{n} \wedge \star)] \right) \\
&= \mathrm{Hom}_{D_n(\mathcal{C})}(\mathbf{n} \wedge X, \mathbf{n} \wedge \star).
\end{aligned}$$

We do not know how to invert the two equivalences we obtain from section 3 of [J-M3], and so we instead use these to construct a sequence of functors. We first observe that we can define new DG-categories $D_n(D_1(\mathcal{C}) \int \Sigma_n)$ and $D_n(\mathbb{Z}[\mathcal{C}] \int \Sigma_n)$ just as we constructed $D_n\mathcal{C}$ from the linear category $\mathbb{Z}[\mathcal{C}]$. We will call a functor of DG categories a weak equivalence if it is an isomorphism on objects and a quasi-isomorphism on Hom complexes. Thus, we have established a sequence of functors of DG-categories:

$$D_1(\mathcal{C}) \int \Sigma_n \xrightarrow{\simeq} D_n(D_1(\mathcal{C}) \int \Sigma_n) \xleftarrow{\simeq} D_n(\tilde{\mathbb{Z}}[\mathcal{C}] \int \Sigma_n) \overset{D_n(\phi \circ r)}{\longrightarrow} D_n(\mathcal{C}).$$

Observation A.3. The map of DGA's $D_n(\phi \circ r)(\mathbf{1}, \mathbf{1})$:

$$\begin{array}{c} D_n\left(\mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}} \bigotimes_{i=1}^n \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \star)]\right)(\mathbf{1}) \\ \Big\downarrow {\scriptstyle D_n(\phi \circ r)} \\ D_n\left(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}, \mathbf{n} \wedge \star)]\right)(\mathbf{1}) \end{array}$$

is <u>not</u> an equivalence. To see this, we first observe that $\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}, \star) = \mathrm{Hom}_{\mathcal{C}}(\bigvee^n \mathbf{1}, \star) \cong \prod^n \mathrm{Hom}(\mathbf{1}, \star)$ and hence

$$\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}, \star)] \cong \bigotimes_{\mathbb{Z}}^{n} \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \star)].$$

This isomorphism is NOT as left $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}, \mathbf{n})]$ modules since the decomposition we used to obtain it is not natural. However, since the map $D_n(\phi \circ r)(\mathbf{1}, \mathbf{1})$ is already a map of DGA's, we only wish to study the map as a map of chain complexes. Since $\otimes_{\mathbb{Z}}^n$ is an n–homogeneous functor of $\mathbb{Z}$-modules we see that

$$D_n \bigotimes_{\mathbb{Z}}^{n} \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \star)] \cong \bigotimes_{\mathbb{Z}}^{n} D_1 \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \star)] = \bigotimes_{\mathbb{Z}}^{n} D_1 \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \star)].$$

If we evaluate at $\mathbf{n} = \bigvee^n \mathbf{1}$, since D_1 is linear, we get:

$$\begin{aligned} D_n(\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{n}, \mathbf{n} \wedge \star)])(\mathbf{1}) &\simeq \bigotimes_{\mathbb{Z}}^{n} D_1 \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{n})] \\ &\simeq \bigotimes_{\mathbb{Z}}^{n} \bigoplus^{n} D_1 \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})] \end{aligned}$$

$$\begin{gathered} \wr\| \\ \mathbb{Z}[\mathrm{Hom}_{Sets}(\{n\},\{n\})] \otimes_{\mathbb{Z}} \bigotimes_{\mathbb{Z}}^{n} D_1\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})] \end{gathered}$$

where $\{n\} = \{1, 2, \ldots, n\}$. On the other hand we have that

$$\begin{gathered} D_n\left(\mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}} \bigotimes_{i=1}^{n} \tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\star)]\right)(\mathbf{1}) \\ \simeq \\ \mathbb{Z}[\mathrm{Aut}_{Sets}(\{n\},\{n\})] \otimes_{\mathbb{Z}} \bigotimes_{\mathbb{Z}}^{n} D_1\tilde{\mathbb{Z}}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})]. \end{gathered}$$

and if one traces through the equivalences carefully they will find that the map $D_n(\phi \circ r)(\mathbf{1},\mathbf{1})$ becomes the map induced by including the pointed monoid of automorphisms of $\{n\}$ into the pointed monoid of all endomorphisms of $\{n\}$.

We let $D_n(\otimes^n \mathbb{Z}[\mathcal{C}])$ be the faithful subcategory of $D_n(\mathbb{Z}[\mathcal{C}] \int \Sigma_n)$ determined by morphisms whose $\mathbb{Z}[\Sigma_n]$ components are the identity. In other words, $D_n(\otimes^n \mathbb{Z}[\mathcal{C}])$ is simply the n-fold tensor category of the linear category $\mathbb{Z}[\mathcal{C}]$ viewed as a subcategory of $D_n(\mathbb{Z}[\mathcal{C}] \int \Sigma_n)$.

Lemma A.4. *The object $D_n(\mathbf{n},\mathbf{1})$ is naturally a left $D_{n\times n}(C)$/right $\mathrm{Hom}_{D_n(\otimes^n \mathbb{Z}[\mathcal{C}])}(\mathbf{1},\mathbf{1})$ bimodule.*

Proof. We observe:

$$\begin{aligned} \mathrm{Hom}_{D_n(\otimes^n \mathbb{Z}[\mathcal{C}])}(\mathbf{1},\mathbf{1}) &= D_n\left(\bigotimes_{\mathbb{Z}}^{n} \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\star)]\right)(\mathbf{1}) \\ &\cong D_n\left(\mathbb{Z}[\mathrm{Hom}(\mathbf{n},\star)]\right)(\mathbf{1}) \\ &= \mathrm{Hom}_{D_n(\mathcal{C})}(\mathbf{n},\mathbf{1}) \\ &= D_n(\mathbf{n},\mathbf{1}). \end{aligned}$$

Thus, $D_n(\mathbf{n},\mathbf{1})$ is a DGA and as such admits a right action of the DGA $\mathrm{Hom}_{D_n(\otimes^n \mathbb{Z}[\mathcal{C}])}(\mathbf{1},\mathbf{1})$. In order to check that we have determined a bimodule, we still need to see that the right action we have defined is compatible with the left action. Since D_n is a functor, it suffices to work at the level of the categories before taking D_n (for this purpose). The pairing from $\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\mathbf{1}) \times \mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\mathbf{1})$ to $\mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\mathbf{1})$ which we are using takes $\alpha \times \beta$ to $\alpha * \beta$ which is given by the composite:

$$\alpha * \beta := \left(\mathbf{n} \xrightarrow{\delta} (\mathbf{n} \times \mathbf{n}) \cong \mathbf{n} \wedge \mathbf{n} \xrightarrow{\mathbf{n}\wedge\alpha} \mathbf{n} \wedge \mathbf{1} \cong \mathbf{n} \xrightarrow{\beta} \mathbf{n}\right)$$

where $\delta(i) = (i \times i)$. Given $\gamma \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{n},\mathbf{n})$ we compute that

$$\gamma * (\alpha * \beta) = \left(\mathbf{n} \xrightarrow{\gamma} \mathbf{n} \xrightarrow{\delta} (\mathbf{n} \times \mathbf{n}) \cong \mathbf{n} \wedge \mathbf{n} \xrightarrow{\mathbf{n}\wedge\alpha} \mathbf{n} \wedge \mathbf{1} \cong \mathbf{n} \xrightarrow{\beta} \mathbf{n}\right)$$

$$(\gamma * \alpha) * \beta = \left(\mathbf{n} \xrightarrow{\delta} (\mathbf{n} \times \mathbf{n}) \cong \mathbf{n} \wedge \mathbf{n} \xrightarrow{\mathbf{n}\wedge\alpha\circ\gamma} \mathbf{n} \wedge \mathbf{1} \cong \mathbf{n} \xrightarrow{\beta} \mathbf{n}\right)$$

which are clearly equal and hence we have a well defined bimodule $D_n(\mathbf{n},\mathbf{1})$.

Proposition A.5. The left $D_{n\times n}(C)$/right $D_{1\times 1}(C) \int \Sigma_n$ bimodule

$$D(n,\Sigma_n) = D_n(\mathbf{n},\mathbf{1})\hat{\otimes}_{\mathrm{Hom}_{D_n(\otimes^n \mathbb{Z}[C])}(\mathbf{1},\mathbf{1})} D_n(D_{1\times 1}(C) \int \Sigma_n)$$

is such that the following diagram commutes:

$$\begin{array}{ccc} & \overset{\cong}{\nearrow} & [\mathcal{A} - Mod - D_{n\times n}(C)] \\ [\mathcal{C}, CH\mathcal{A}]_{C,[t]} & & \Big\downarrow \hat{\otimes}_{D_{n\times n}(C)} D_n(n,\Sigma_n) \\ & \overset{\cong}{\searrow} & [\mathcal{A} - Mod - D_{1\times 1}(C) \int \Sigma_n]_{h\Sigma_n}. \end{array}$$

The inverse to $\hat{\otimes}_{D_{n\times n}(C)} D_n(n,\Sigma_n)$ is given by $\hat{\otimes}_{D_{1\times 1}(C) \int \Sigma_n} D_n(\Sigma_n, n)$ where

$$D_n(\Sigma_n, n) = D_n(D_{1\times 1}(C) \int \Sigma_n)\hat{\otimes}_{\mathrm{Hom}_{D_n(\otimes^n \mathbb{Z}[C])}(\mathbf{1},\mathbf{1})} D_n(\mathbf{1},\mathbf{n}).$$

Proof. Let M be a $CH\mathcal{A}\otimes D_{n\times n}(C)$ bimodule. The associated homogeneous degree n functor (defined along C) is given by: $\chi(M) = \perp_C^*$ $(M\hat{\otimes}_{D_{n\times n}(C)}D_n(C,\star))$. Going the other way around the diagram, the associated functor is $\perp_C^*$ of:

$$M\hat{\otimes}_{D_{n\times n}(C)}D_n(\mathbf{n},\mathbf{1}) \hat{\otimes}_{\mathrm{Hom}_{D_n(\otimes^n\mathbb{Z}[C])}(\mathbf{1},\mathbf{1})} D_n(D_{1\times 1}(C)\wr\Sigma_n)\hat{\otimes}_{D_{1\times 1}(C)\wr\Sigma_n}\otimes^n_{\mathbb{Z}} D_{1\times 1}(\mathbf{1},\star).$$

The result now follows from the diagram of natural maps (the labels correspond to justifications given below):

$$\begin{array}{c}
M\hat{\otimes}_{D_{n\times n}(C)}D_n(\mathbf{n},\mathbf{1}) \hat{\otimes}_{\mathrm{Hom}_{D_n(\otimes^n\mathbb{Z}[C])}(\mathbf{1},\mathbf{1})} D_n(D_{1\times 1}(C)\wr\Sigma_n)\hat{\otimes}_{D_{1\times 1}(C)\wr\Sigma_n}\otimes^n_{\mathbb{Z}} D_{1\times 1}(\mathbf{1},\star) \\
(A)\Big\downarrow\simeq \\
\cdots\cdots D_n(D_{1\times 1}(C)\wr\Sigma_n)\hat{\otimes}_{D_{1\times 1}(C)\wr\Sigma_n}D_n\left(\bigotimes_{\mathbb{Z}}^{n} D_1(\mathbf{1},\star)\right) \\
(B)\Big\downarrow\simeq \\
\cdots\cdots D_n(D_{1\times 1}(C)\wr\Sigma_n)\hat{\otimes}_{D_n(D_{1\times 1}(C)\wr\Sigma_n)}D_n\left(\bigotimes_{\mathbb{Z}}^{n} D_1(\mathbf{1},\star)\right) \\
(C)\Big\downarrow\simeq \\
M\hat{\otimes}_{D_{n\times n}(C)}D_n(\mathbf{n},\mathbf{1})\hat{\otimes}_{\mathrm{Hom}_{D_n(\otimes^n\mathbb{Z}[C])}(\mathbf{1},\mathbf{1})}D_n\left(\bigotimes_{\mathbb{Z}}^{n} D_1(\mathbf{1},\star)\right) \\
(D)\Big\downarrow \\
M\hat{\otimes}_{D_{n\times n}(C)}D_n(\mathbf{n},\star)
\end{array}$$

The map (A) is determined by the equivalence from $\bigotimes^n D_1(\mathbf{1},\star)$ to $D_n\left(\bigotimes^n D_1(\mathbf{1},\star)\right)$ of section 3 of [J-M3]. The map (B) is given by the map of DGA's $D_{1\times 1}(C) \int \Sigma_n \xrightarrow{\simeq} D_n\left(D_{1\times 1}(C) \int \Sigma_n\right)$ which produces an equivalence on the derived tensor products. The map (C) is given by the collapsing equivalence

$$D_n(D_{1\times 1}(C) \int \Sigma_n)\hat{\otimes}_{D_n(D_{1\times 1}(C) \int \Sigma_n)}\star \xrightarrow{\simeq} \star.$$

The map (D) is determined by composition and is an equivalence at $\mathbf{1}$. Thus, $\perp_C^*$ of the map (D) is an equivalence since it is a map of homogeneous degree n functors defined along C (which are rank 1) and an equivalence at $\mathbf{1}$. The inverse statement is proved similarly.

References

[E] S. Eilenberg, *Abstract description of some basic functors,* J. Indian Math. Soc. **24** (1960), 231-234.

[E-M1] S. Eilenberg and S. Mac Lane, *Homology theories for multiplicative systems,* Trans. Amer. Math. Soc. **71** (1951), 294-330.

[E-M2] S. Eilenberg and S. Mac Lane, *On the groups $H(\pi, n)$, II,* Ann. of Math. **60** (1954), 49-139.

[G1] T.G. Goodwillie, *Calculus I: The first derivative of pseudoisotopy theory,* K-Theory **4** (1990), 1-27.

[G2] T.G. Goodwillie, *Calculus II: Analytic functors,* K-Theory **5** (1992), 295-332.

[G3] T.G. Goodwillie, *Calculus III:The Taylor series of a homotopy functor,* in preparation.

[J-M1] B. Johnson and R. McCarthy, *Linearization, Dold-Puppe stabilization, and Mac Lane's Q-construction*, Trans. Amer. Math. Soc. **350** (1998), 1555-1593.

[J-M2] B. Johnson and R. McCarthy, *Taylor towers for functors of additive categories,* J. Pure Appl. Algebra **137** (1999), 253-284.

[J-M3] B. Johnson and R. McCarthy, *Deriving calculus with cotriples*, preprint.

[J-M4] B. Johnson and R. McCarthy, *A classification of degree n functors, I*, to appear in Cahiers Topologie Géom. Différentielle Catég.

[M] S. Mac Lane, *Homology,* Springer-Verlag, Berlin, 1975.

[P] T. Pirashvili, *Polynomial approximation of Ext and Tor groups in functor categories*, Comm. in Algebra **21** (1993), 1705–1719.

[W] C. E. Watts, *Intrinsic characterizations of some additive functors,* Proc. Amer. Math. Soc. **11** (1960), 5-8.

[We] C. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.

Brenda Johnson
Department of Mathematics
Union College

Schenectady, NY 12308

johnsonb@union.edu

Randy McCarthy
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green St.
Urbana, IL 61801

randy@math.uiuc.edu