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LUCIANO STRAMACCIA

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SHAPE AND STRONG SHAPE EQUIVALENCES

by Luciano STRAMACCIA

RESUME. Les concepts d'équivalences de forme (shape) et de forme forte ont leur propre intérêt indépendamment de la Théorie de la Forme elle-même. Ils peuvent être définis dans le cadre abstrait d'une paire (**C**,**K**) de categories, où **C** est équippée d'un foncteur cylindre engendrant. Lié à leur étude est le problème de caractériser les épimorphismes et monomorphismes d'homotopie dans **C**. Pour le résoudre, l'auteur utilise la construction de l'application cylindre double et il introduit une propriété d'extension d'homotopie forte. Il étudie leurs connexions avec les concepts précédents.

0 INTRODUCTION

Shape and Strong Shape can be viewed as theories that allow to approximate general topological spaces by means of systems of spaces that have nice homotopical properties, such as absolute neighborhood retracts (ANR-spaces). The shape category **Sh(Top)** [12] has objects all topological spaces, while a shape morphism $\sigma : X \to Y$ can be represented as a natural transformation $\sigma : [Y,-] \to [X,-]$, where $[X,-] : \mathbf{ANR} \to \mathbf{Set}$ is the functor which associates with every ANR-space P, the set $[X,P]$ of homotopy classes of maps $X \to P$. A shape equivalence is a continuous map inducing an isomorphism in the shape category. Since the concept of strong shape is more geometrical in nature, the construction of the strong shape category **SSh(Top)** is much more involved than the previous one and, consequently, it gives rise to a more interesting concept of strong shape equivalence.
In this note we deal with a pair of $(\mathbf{C},\mathbf{K})$ of abstract categories, where $\mathbf{C}$ is endowed with a generating cylinder functor [9] and $\mathbf{K}$ is the subcategory of

models, which plays the role of the category **ANR** in the topological case. In such a context it is possible to define shape and strong shape equivalences internally, without going through a construction of a shape or strong shape category (see definitions 1.1 and 1.5). This is done in section 1, where we also introduce a strong version of the homotopy extension property (definition 1.6), which turns out to be useful in characterizing strong shape equivalences.

The strong shape category of compact metrizable spaces **SSh(CM)** can be obtained by localizing **CM** at the class of strong shape equivalences, while the localization of **CM** at ordinary shape equivalences is not equivalent to the ordinary shape category **Sh(CM)** ([4], see also [3]). More recently in [13], the strong shape category **SSh(pro Top)**, for the category of inverse systems of topological spaces, has been defined localizing **pro Top** at its class of strong shape equivalences between inverse systems of spaces, as defined in [11]. Although **SSh(pro Top)** contains **SSh(Top)** as a full subcategory, the arguments in [13] do not seem to give a representation of **SSh(Top)** as the localization of **Top** at strong shape equivalences. We feel that the material contained in this paper could be of help in studying such kind of problems.

In the second section we consider the double mapping cylinder construction in order to obtain further characterizations of strong shape equivalences, also generalizing some results from [6] and [7]. The abstract homotopy structure given by a cylinder, or cocylinder, allows one to study homotopy analogues of notions such as epimorphisms and monomorphisms. For instance [8] introduced the notion of homotopy epimorphism : $f : X \to Y$ is a homotopy epimorphism if, given $h_0, h_1 : Y \to P$, $h_0 f \simeq h_1 f$ implies that $h_0 \simeq h_1$. This corresponds to $f_P^* : [Y, P] \to [X, P]$ being 1-1. Here for us P will be from a subcategory $\mathbf{K}$ of the basic category $\mathbf{C}$, e.g. $\mathbf{C} = \mathbf{Top}$ with $\mathbf{K} =$ polyhedra. Then this notion of homotopy epimorphism will depend on the subcategory $\mathbf{K}$ being used. Since a shape equivalence for $(\mathbf{C}, \mathbf{K})$ has to be a homotopy epimorphism with respect to the subcategory of models $\mathbf{K}$, then we are able to shed some light on this problem, also considering the dual case of homotopy monomorphism and coshape equivalences.

1 PRELIMINARY RESULTS

In what follows $\mathbf{C}$ will denote a finitely cocomplete category, endowed with a cylinder functor (I, e_0, e_1, σ). $I : \mathbf{C} \to \mathbf{C}$ is a functor written $I(X) = X \times I$ and $I(f : X \to Y) = f \times 1 : X \times I \to Y \times I$. $e_0, e_1 : 1_{\mathbf{C}} \to I$ and $\sigma : I \to 1_{\mathbf{C}}$

are natural transformations such that $\sigma e_0 = \sigma e_1 = id$.
We will also assume that the cylinder functor is generating in the sense of [7].

Two morphisms $f, g : X \to Y$ in $\mathbf{C}$ are said to be homotopic, written $f \simeq g$, if there is a morphism $H : X \times I \to Y$, such that $He_0^X = f$ and $He_1^X = g$. In such a case we say that H is a homotopy connecting f and g. The equivalence class of a morphism f, generated by the homotopy relation, will be denoted by $[f]$, while the set of homotopy classes of morphisms from X to Y will be denoted by $[X, Y]$. Consequently, one can define a notion of homotopy equivalence in $\mathbf{C}$. In particular, it turns out that, for every $X \in \mathbf{C}$, the morphisms e_0^X and e_1^X are homotopy equivalences.

Let now fix a full subcategory $\mathbf{K}$ of $\mathbf{C}$. Then :

- a morphism $f : X \to Y$ has the homotopy extension property (HEP) with respect to $\mathbf{K}$, whenever the diagram

$$\begin{array}{ccc} X & \xrightarrow{e_0^X} & X \times I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1} \\ Y & \xrightarrow{e_0^Y} & Y \times I \end{array}$$

is a weak pushout w.r. to $\mathbf{K}$. This means that, for every $\phi : Y \to P$, $P \in \mathbf{K}$, and homotopy $F : X \times I \to P$ such that $Fe_0^X = \phi f$, there exists a homotopy $G : Y \times I \to P$ with $Ge_0^Y = \phi$ and $G(f \times 1) = F$.
A morphism f is called a **cofibration** when it has the HEP w.r. to $\mathbf{C}$.

- the **mapping cylinder** $M(f)$ of a morphism $f : X \to Y$ is given by the following pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{e_0^X} & X \times I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Pi_f} \\ Y & \xrightarrow{j_f} & M(f) \end{array}$$

There is an induced morphism $f_1 : M(f) \to Y$, such that $f_1 j_f = 1_Y$ and $f_1 \Pi_f = f\sigma^X$. It is easily shown [9] that f can be decomposed both as

$f = f_1 f_0$ and $f = f_1 f_0'$, where f_1 is a homotopy equivalence and $f_0 = \Pi_f e_0^X$ and $f_0' = \Pi_f e_1^X$ are both cofibrations.

- for every $X \in \mathbf{C}$, let $[X,-] : \mathbf{K} \to \mathbf{Set}$ be the functor that assigns to every $P \in \mathbf{K}$ the set $[X,P]$. Given $f : X \to Y$, there is an induced natural transformation $f^* : [Y,-] \to [X,-]$, defined by composition with f.

Definition 1.1 A morphism $f : X \to Y$ in $\mathbf{C}$ is said to be a **shape equivalence** for the pair $(\mathbf{C}, \mathbf{K})$, if f^* is a natural isomorphism.

It follows that f is a shape equivalence whenever, for every $P \in \mathbf{K}$, the following two properties hold :

(se.1) $f_P^* : [Y,P] \to [X,P]$ is onto, that is, for every $g : X \to P$ there exists an $h : Y \to P$ such that $hf \simeq g$.

(se.2) $f_P^* : [Y,P] \to [X,P]$ is 1-1, that is, given $h_0, h_1 : Y \to P$ with $h_0 f \simeq h_1 f$, then $h_0 \simeq h_1$.

In the sequel, we shall reserve the name of **extensor** for a morphism $f : X \to Y$ satisfying condition (se.1). In [6] such morphisms were called semiequivalences. Condition (se.2) states that f is a **homotopy epimorphism** [8]. We point out that both the notions are relative to the pair $(\mathbf{C}, \mathbf{K})$, although we shall often omit to indicate it explicitly.

Proposition 1.2 The class of extensors for $(\mathbf{C}, \mathbf{K})$ has the following properties :

1. Contains all homotopy equivalences.
2. It is closed under composition.
3. If a composition gf is an extensor, then f is a extensor.
4. If $f : X \to Y$ is an extensor and $f' \simeq f$, then $f' : X \to Y$ is also an extensor.
5. f is an extensor iff f_0 is.
6. $f \times 1$ is an extensor iff f is.
7. If $f : X \to Y$ is an extensor and has the HEP w.r. to $\mathbf{K}$ then, for every $g : X \to P$, $P \in \mathbf{K}$, there exists an $h : Y \to P$ such that $hf = g$.

Proof. (1) is obvious. Both assertions (2) and (3) follow from the observation that, given $f : X \to Y$ and $g : Y \to Z$, then $(gf)^* = f^* g^*$. (4) For any $g : X \to P$, $P \in \mathbf{K}$, there is an $h : Y \to P$ such that $hf \simeq g$. Since $hf \simeq hf'$, the assertion follows. (5). Since $f = f_1 f_0$ and f_1 is a homotopy equivalence, the assertion follows from the previous ones. (6). Let us observe that, for every $P \in \mathbf{K}$, there is a commutative diagram

$$\begin{array}{ccc}
[Y \times I, P] & \xrightarrow{(f \times 1)^*_P} & [X \times I, P] \\
{\scriptstyle (e_0^Y)^*_P} \downarrow & & \downarrow {\scriptstyle (e_0^X)^*_P} \\
[Y, P] & \xrightarrow{f^*_P} & [X, P]
\end{array}$$

Since both $(e_0^X)^*_P$ and $(e_0^Y)^*_P$ are bijections, it follows that $(f \times 1)^*_P$ is onto iff f^*_P is onto. (7). See, e.g., [9], pag.11.

Proposition 1.3 The class of homotopy epimorphisms for $(\mathbf{C}, \mathbf{K})$ has the following properties :

1. Contains all homotopy equivalences.
2. It is closed under composition.
3. If a composition gf is a homotopy epimorphism, then g is a homotopy epimorphism.
4. If $f : X \to Y$ is a homotopy epimorphism and $f' \simeq f$, then $f' : X \to Y$ is also a homotopy epimorphism.
5. f is a homotopy epimorphism iff f_0 is.
6. $f \times 1$ is a homotopy epimorphism iff f is.

Proof. Let us show part (5). Let f be a homotopy epimorphisms and let $v_0, v_1 : M(f) \to P$, $P \in \mathbf{K}$, be such that $v_0 f_0 = v_1 f_0$. Then, $v_0 \tilde{f}_1 f_1 f_0 \simeq v_1 \tilde{f}_1 f_1 f_0$, being $\tilde{f}_1$ the homotopy inverse of f_1. Hence, $v_0 \tilde{f}_1 \simeq v_1 \tilde{f}_1$, from which it follows $v_0 \simeq v_1$. Conversely, let f_0 be a homotopy epimorphisms and let $h_0, h_1 : Y \to P$, $P \in \mathbf{K}$, be such that $h_0 f \simeq h_1 f$. Then, $h_0 f_1 f_0 \simeq h_1 f_1 f_0$ implies $h_0 f_1 \simeq h_1 f_1$, hence $h_0 \simeq h_1$, being f_1 a homotopy equivalence.

From the two propositions above we obtain the following

Theorem 1.4 The class of shape equivalences for $(\mathbf{C}, \mathbf{K})$ has the following properties:

1. Contains all homotopy equivalences.
2. It is closed under composition.

3. If two of the maps f, g, gf, are shape equivalences, so is the third.
4. If $f : X \to Y$ is a shape equivalence and $f' \simeq f$, then $f' : X \to Y$ is also a shape equivalence.
5. f is a shape equivalence iff f_0 is a shape equivalence.
6. $f \times 1$ is a shape equivalence iff f is such.

Proof. We only show (3). Let $f : X \to Y$ and $g : Y \to Z$. It follows from the two preceding propositions that shape equivalences are closed under composition. Let gf and f be shape equivalences. It is clear that g is then a homotopy epimorphism. Let us show that it is also an extensor. Given $t : Y \to P$, there is a $v : Z \to P$ such that $vg\ f \simeq tf$. Since f is a homotopy epimorphism, the assertion follows. Let now gf and g be shape equivalences and let $h_0, h_1 : Y \to P$, $P \in \mathbf{K}$ be such that $h_0 f \simeq h_1 f$. There are morphisms $t_0, t_1 : Z \to P$ with $t_0 gf \simeq h_0 f$ and $t_1 gf \simeq h_1 f$. Since then $t_0 \simeq t_1$ and g is a shape equivalence, it follows that $h_0 \simeq h_1$.

We remark that a morphism f is a shape equivalence iff f_0 (and f_0') is also a shape equivalence. Hence, in studying shape equivalences we may restrict ourselves to consider morphisms that are cofibrations. Moreover, in such a context, condition (se.1) can be substituted by the stronger one expressed by 1.2(7).

The definition of strong shape equivalence is more involved and we need some informations in order to give it.
For objects $X \in \mathbf{C}$ and $P \in \mathbf{K}$, we shall denote by πP^X, the fundamental groupoid of P under X, as defined, e.g., in [2]. Let us recall that the objects of πP^X are the morphisms $X \to P$ in $\mathbf{C}$, while a morphism $\alpha = \{G\} \in \pi P^X(h_0, h_1)$, written $\alpha : h_0 \Rightarrow h_1$ is a track from h_0 to h_1. This means that α is the equivalence class of $G : X \times I \to P$, with respect to the following equivalence relation : $G \simeq G'$ iff there is a homotopy $\Gamma : Y \times I \times I \to P$ (rel end maps), that is having the following properties:

1. $\Gamma e_0^{Y \times I} = G$.
2. $\Gamma e_1^{Y \times I} = G'$.
3. $\Gamma\ (e_0^Y \times 1) = h_0 \sigma^Y$.
4. $\Gamma(e_1^Y \times 1) = h_1 \sigma^Y$.

Conditions (3) and (4) say, respectively, that $\Gamma(e_0^Y \times 1)$ is the constant homotopy at h_0 and that $\Gamma(e_1^Y \times 1)$ is the constant homotopy at h_1.

Let us note that, for every $f : X \to Y$ in $\mathbf{C}$, there is an induced morphism (functor) of groupoids $f_P^\sharp : \pi P^Y \to \pi P^X$, defined by $h \mapsto hf$ and $\{G\} \mapsto \{G(f \times 1)\}$. Furthermore, homotopic morphisms $f \simeq f' : X \to Y$ induce morphisms of groupoids $f_P^\sharp$ and $f_P'^\sharp$, which are naturally isomorphic as functors ([2], pag. 240).

Definition 1.5 A morphism $f : X \to Y$ is called a strong shape equivalence for $(\mathbf{C}, \mathbf{K})$ if it is an extensor and, moreover, $f_P^\sharp : \pi P^Y \to \pi P^X$ is full, for every $P \in \mathbf{K}$.

The condition that $f_P^\sharp$ is full, for every $P \in \mathbf{K}$, is equivalent to f being a **strong homotopy epimorphism** in the following sense: given in $\mathbf{C}$ morphisms h_0, $h_1 : Y \to P$, $P \in \mathbf{K}$, and a homotopy $G : X \times I \to P$, $G : h_0 f \simeq h_1 f$, there exists a homotopy $H : Y \times I \to P$, $H : h_0 \simeq h_1$, such that $H(f \times 1)$ and G are homotopic rel end maps. In case $\mathbf{C}$ is the category **Top** of topological spaces and f is a cofibration then, by ([2], 7.2.5), the above is equivalent to say that $H(f \times 1) = G$.
Such considerations lead us to give the following

Definition 1.6 A morphism $f : X \to Y$ in $\mathbf{C}$ is said to have the strong homotopy extension property (SHEP) (w.r.t. to $\mathbf{K}$), if the following diagram is a weak colimit (w.r.t. to $\mathbf{K}$)

$$\begin{array}{ccc} X & \overset{e_0^X}{\underset{e_1^X}{\rightrightarrows}} & X \times I \\ f \downarrow & & \downarrow f \times 1 \\ Y & \overset{e_0^Y}{\underset{e_1^Y}{\rightrightarrows}} & Y \times I \end{array}$$

It follows that in **Top** a cofibration is a strong homotopy epimorphism iff it has the SHEP. Hence a map which is both a cofibration and an extensor is a strong shape equivalence iff it has the SHEP.

Theorem 1.7 The class of strong shape equivalences for $(\mathbf{C}, \mathbf{K})$ has the following properties:

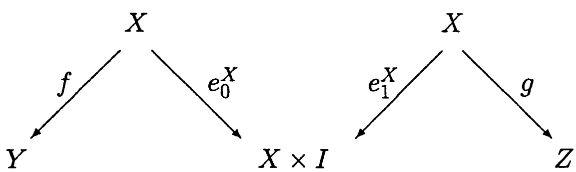
1. Contains all homotopy equivalences.
2. It is closed under composition.

3. If $f : X \to Y$ is a strong shape equivalence and $f' \simeq f$, then f' is also a strong shape equivalence.
4. f is a strong shape equivalence iff f_0 is.

Proof. It is clear that the composition of strong shape equivalences is a strong shape equivalence. Moreover, the Vogt's lemma [16] implies that every homotopy equivalence is a strong shape equivalence. (3). f' is an extensor by 1.2(4). Since $f^{\sharp}_P$ is full, for every $P \in \mathbf{K}$, and $f'^{\sharp}_P$ is naturally isomorphic to $f^{\sharp}_P$, it follows that it is full as well. (4). f_1 is a homotopy equivalence, hence a strong shape equivalence. If f_0 is also a strong shape equivalence, it follows that f is. Conversely, let f be a strong shape equivalence. From $f = f_1 f_0$, one has $f_0 \simeq \tilde{f}_1 f$, where $\tilde{f}_1$ is the homotopy inverse of f_1. The assertion then follows from (3) above and 1.2(5).

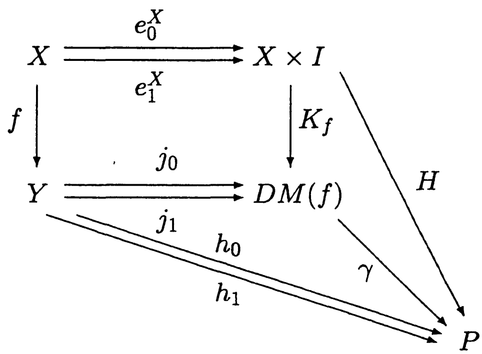
2 THE DOUBLE MAPPING CYLINDER CONSTRUCTION

With the same notations as in the previous section, let us consider the following diagram in $\mathbf{C}$



Its colimit is an object $DM(f,g)$ of $\mathbf{C}$, equipped with three morphisms $j_0 : Y \to DM(f,g)$, $j_1 : Z \to DM(f,g)$ and $K_{f,g} : X \times I \to DM(f,g)$, such that $K_{f,g} e_0^X = j_0 f$, $K_{f,g} e_1^X = j_1 g$, and having the suitable universal property [9]. It is called the double mapping cylinder of the pair (f,g). In case the two morphisms f and g coincide, then we speak of the double mapping cylinder of f and write simply $DM(f)$ for $DM(f,f)$ and K_f for $K_{f,f}$.
It is worth to point out that the triple $(DM(f,g), j_0, j_1)$ is actually the homotopy pushout of the diagram $Y \xleftarrow{f} X \xrightarrow{g} Z$ and $DM(f)$ is the case where the two morphisms involved are the same. The situation can be better

illustrated by the following diagram



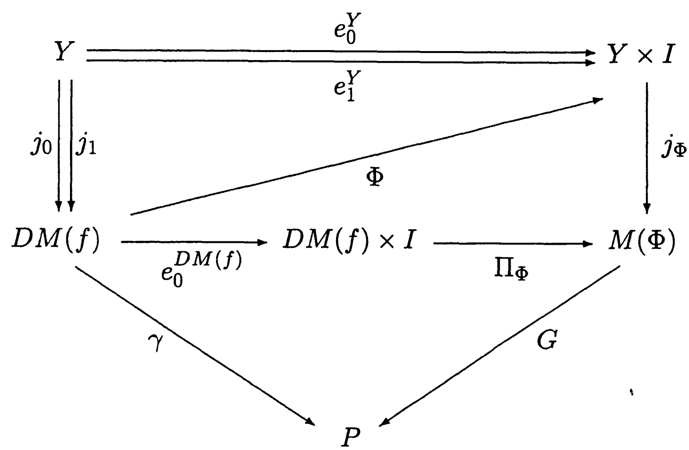
where the two squares are commutative and, for every triple $(h_0, h_1; H)$, $h_0, h_1 : Y \to P$, $P \in \mathbf{C}$, and $H : X \times I \to P$ a homotopy connecting $h_0 f$ with $h_1 f$, there exists a unique morphism $\gamma : DM(f) \to P$, such that $\gamma K = H$ and $\gamma j_0 = h_0$, $\gamma j_1 = h_1$.
Let us note that there is a unique morphism $\Phi : DM(f) \to Y \times I$ such that $\Phi K_f = f \times 1$ and $\Phi j_0 = e_0^Y$, $\Phi j_1 = e_1^Y$.

Remark 2.1 If $f : X \to Y$ is a continuous map, then $DM(f)$ may be described as the adjunction space $(X \times I) \cup_f (Y \times \partial I)$, where $\partial I = \{o, 1\}$. If f is an inclusion map, then $DM(f)$ is the subspace $(X \times I) \cup (Y \times \partial I)$ of $Y \times I$. In case f is a (closed) cofibration, the inclusion $\Phi : X \times I \cup Y \times \partial I \to Y \times I$ is also a closed cofibration ([14], Th.6). A discussion of the double mapping cylinder of a continuous map can be found in [5], [15] and, more recently, in [9], see also [6], [10] and [11].

A version of the following theorem was proved in [6], Thm. 5.2, for $\mathbf{C}$ the category of compact metric spaces and proper maps and $\mathbf{K} = \mathbf{ANR}$.

Theorem 2.2 If $\Phi : DM(f) \to Y \times I$ is an extensor, then f is a strong homotopy epimorphism.

Proof. Let $h_0, h_1 : Y \to P$, $P \in \mathbf{K}$, and let $H : X \times I \to P$ be a homotopy connecting $h_0 f$ and $h_1 f$. From the double mapping cylinder diagram one obtains a morphism $\gamma : DM(f) \to P$ such that $\gamma j_0 = h_0$, $\gamma j_1 = h_1$ and $\gamma K_f = H$. Since $\Phi = p_\Phi(\Pi_\Phi e_0^{DM(f)})$ is an extensor, then $\Pi_\Phi e_0^{DM(f)}$ is itself an extensor, by Prop. 1.2(3). Since it is also a cofibration, from Prop. 1.2(7), it follows that there exists a $G : M(\Phi) \to P$ such that $G\Pi_\Phi e_0^{DM(f)} = \gamma$. From the commutative diagram



it follows that Gj_Φ is a homotopy connecting h_0 and h_1. In fact :

$$Gj_\Phi e_0^Y = Gj_\Phi \Phi j_0 = G\ \Pi_\Phi e_0^{DM(f)} j_0 = \gamma j_0 = h_0.$$

$$Gj_\Phi e_1^Y = Gj_\Phi \Phi j_1 = G\Pi_\Phi e_0^{DM(f)} j_1 = \gamma j_1 = h_1.$$

Hence, f is a homotopy epimorphism. Finally, let us observe that $Gj_\Phi(f \times 1) = Gj_\Phi \Phi K_f = G\Pi_\Phi e_0^{DM(f)} K_f = \gamma K_f = H$, from which it follows that $f_P^\sharp$ is full, for every $P \in \mathbf{K}$.

Corollary 2.3 If $f : X \to Y$ and $\Phi : DM(f) \to Y \times I$ are extensors, then f is a strong shape equivalence.

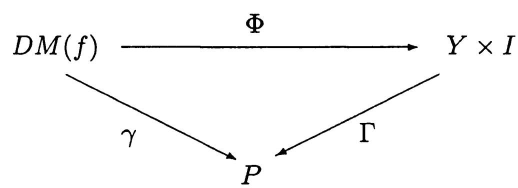
Proposition 2.4 Let $f : X \to Y$ be a morphism in $\mathbf{C}$. If f has the SHEP w.r. to $\mathbf{K}$, then Φ is an extensor for $(\mathbf{C}, \mathbf{K})$.

Proof. Let $\rho : DM(f) \to P$, $P \in \mathbf{K}$. Then, there is a morphism $V : Y \times I \to P$ such that $V(f \times 1) = \rho K_f$ and $Ve_i^Y = \rho j_i$, $i = 0, 1$. Since $V\Phi K_f = V(f \times 1) = \rho K_f$ and $V\Phi j_i = Ve_i^Y = \rho j_i$, $i = 0, 1$, by the universal property of the double mapping cylinder, it follows that $V\Phi = \rho$.

It is clear that an extensor $f : X \to Y$ is a strong shape equivalence whenever either Φ is an extensor or f has the SHEP w.r. to **K**. These last two facts turn out to be equivalent in the category of topological spaces, in fact, together with the above proposition, there the following theorem holds :

Theorem 2.5 Let $f : X \to Y$ be a cofibration in **Top**. If $\Phi : DM(f) \to Y \times I$ is an extensor for (**Top**, **K**), then f has the SHEP w.r. **K**.

Proof. Given maps $h_0, h_1 : Y \to P$, $P \in \mathbf{K}$, and a homotopy $H : X \times I \to P$, connecting $h_0 f$and $h_1 f$, there exists a map $\gamma : DM(f) \to P$, such that $\gamma K_f = H$ and $\gamma j_\lambda = h_\lambda$, $\lambda = 0, 1$. Since Φ has the HEP, by Prop. 1.2(5), there is a homotopy $\Gamma : Y \times I \to P$ such that $\Gamma\Phi = \gamma$.



We obtain the following relations :

1. $\Gamma(f \times 1) = \Gamma\Phi K_f = \gamma K_f = H$,
2. $\Gamma e_\lambda^Y = \Gamma\Phi j_\lambda = \gamma j_\lambda = h_\lambda$, $\lambda = 0, 1$,

from which it follows that Γ is a homotopy connecting h_0 and h_1.

Corollary 2.6 Let $f : X \to Y$ be a cofibration in **Top**. The following properties are equivalent:

1. f is a strong homotopy epimorphism.
2. f has the SHEP.
3. Φ is an extensor.

Remark 2.7 In view of 1.4(4) and 1.7(3), the assumption that f has to be a cofibration is almost harmless.

Proposition 2.8 Let $f : X \to Y$ be an extensor such that $f \times 1$ is a cofibration. Then f is a strong shape equivalence iff it has the SHEP w.r. to **K**.

Proof. Let $h_0, h_1 : Y \to P$, $P \in \mathbf{K}$, and let $H : X \times I \to P$ be a homotopy connecting $h_0 f$ with $h_1 f$. If f is a strong shape equivalence, there is a homotopy $G' : h_0 \simeq h_1$ such that $G'(f \times 1) \simeq H$. Since $f \times 1$ is an extensor and a cofibration, then there exists a homotopy $G \simeq G'$ such that $G(f \times 1) = H$. Finally, since f is a homotopy epimorphism, it follows that $G : h_0 \simeq h_1$.

Example 2.1 Let **Met** be the category of metrizable spaces and **Ar** be the full subcategory of absolute retracts for metrizable spaces. Every closed embedding $f : X \to Y$ in **Met** is a strong shape equivalence w.r. to **Ar**. In fact, every closed embedding of metrizable spaces is an extensor ([1], pag. 87) and has the HEP w.r.to **Ar** (this is the Borsuk's homotopy extension theorem, [1], pag. 94). Note that also $\Phi : DM(f) \to Y \times I$ is a closed embedding of metrizable spaces.

3 THE DUAL CASE

The results of the previous sections, concerning homotopy and strong homotopy epimorphisms, can be dualized in order to treat the problem of homotopy monomorphisms. We state here the appropriate dual concepts.

Starting again from a pair of categories $(\mathbf{C}, \mathbf{K})$, let us consider, for every $X \in \mathbf{C}$, the set-functor $[-, X] : \mathbf{K} \to \mathbf{Set}$, $P \mapsto [P, X]$. Every morphism $f : X \to Y$ induces a natural transformation $f_* : [-, X] \to [-, Y]$. The morphism f will be called here a **coshape equivalence** whenever f_* turns out to be a natural isomorphism. Then, f will be a coshape equivalence if

(cse.1) f is a **retractor**, that is, for every $g : P \to Y$, $P \in \mathbf{K}$, there is some $h : P \to X$ such that $fh \simeq g$.

(cse.2) f is a **homotopy monomorphism**, that is, for every $h_0, h_1 : P \to X$, the fact that $fh_0 \simeq fh_1$ implies that $h_0 \simeq h_1$.

Note that also these notions are to be considered with respect to $(\mathbf{C}, \mathbf{K})$.

We will assume that $\mathbf{C}$ is now finitely complete and that it is endowed with a generating **cocylinder functor** $((-)^I, \epsilon_0, \epsilon_1, s)$, as defined in [9]. Here $(-)^I : \mathbf{C} \to \mathbf{C}$, $X \mapsto X^I$, $f \mapsto f^I$, and $\epsilon_0, \epsilon_1 : (-)^I \to 1_{\mathbf{C}}$, $s : 1_{\mathbf{C}} \to (-)^I$, are natural transformations such that $\epsilon_0 s = \epsilon_1 s = id$. The homotopy relation used here is that induced by the cocylinder functor.

Let us recall that a morphism $f : X \to Y$ is said to have the homotopy lifting property (HLP) w.r. to $\mathbf{K}$, whenever the following diagram is a weak pullback w.r. to $\mathbf{K}$,

$$\begin{array}{ccc} X^I & \xrightarrow{f^I} & Y^I \\ \epsilon_0^X \downarrow & & \downarrow \epsilon_0^Y \\ X & \xrightarrow{f} & Y \end{array}$$

The **mapping cocylinder** of a morphism $f : X \to Y$ is the object $E(f)$ which appears in the following pullback diagram

$$\begin{array}{ccc} E(f) & \xrightarrow{\Gamma_f} & Y^I \\ i_f \downarrow & & \downarrow \epsilon_0^Y \\ X & \xrightarrow{f} & Y \end{array}$$

Every morphism $f : X \to Y$ in $\mathbf{C}$ has a mapping cocylinder decomposition $f = f^0 f^1$, where f^1 is a fibration, that is, it has has the HLP w.r. to $\mathbf{C}$, and f^0 is a homotopy equivalence.
We have now the notion of **double mapping cocylinder** of f, which is defined by the limit

$$X \underset{u_0}{\overset{u_1}{\leftleftarrows}} DE(f) \xrightarrow{v_f} Y^I$$

in $\mathbf{C}$ of the diagram

$$\begin{array}{ccccccc} & & Y & & Y & & \\ & \overset{f}{\nearrow} & & \overset{\epsilon_0^Y}{\nwarrow} \quad \overset{\epsilon_1^Y}{\nearrow} & & \overset{f}{\nwarrow} & \\ X & & & Y^I & & & X \end{array} .$$

Here again one should be careful that, in general, one defines the double

mapping cocylinder $DE(f,g)$ of a pair $X \xrightarrow{f} Y \xleftarrow{g} Z$ and that $DE(f)$ is just an abbreviation for the case where f and g coincide. Then $DE(f)$ is the homotopy pullback of $X \xrightarrow{f} Y \xleftarrow{f} X$ [9].
All the results of the previous section can be dualized for retractors, homotopy monomorphisms and coshape equivalences. For instance, let us note that there is a unique morphism $\Psi : X^I \to DE(f)$, having the property that $v_f\Psi = f^I$ and $u_0\Psi = \epsilon_0^Y$, $u_1\Psi = \epsilon_1^Y$. It is interesting to give the dual form of Theorem 2.2 and Cor. 2.6, as follows:

Theorem 3.1 If $\Psi : X^I \to DE(f))$ is a retractor, then f is a homotopy monomorphism.

Proposition 3.2 Let $f : X \to Y$ be a fibration in **Top**. The following are equivalent :

1. f is a strong homotopy monomorphism.
2. f has the SHLP.
3. Ψ is a retractor.

The definition of strong homotopy monomorphism and strong homotopy lifting property (SHLP) are the obvious ones.

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Dipartimento di Matematica e Informatica
Università di Perugia - Italia
stra@dipmat.unipg.it