CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

ALFRED FRÖLICHER

Linear spaces and involutive duality functors

Cahiers de topologie et géométrie différentielle catégoriques, tome 43, nº 1 (2002), p. 35-48

<http://www.numdam.org/item?id=CTGDC_2002__43_1_35_0>

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Volume XLIII-1 (2002)*

LINEAR SPACES AND INVOLUTIVE DUALITY FUNCTORS

by Alfred FRÖLICHER

Résumé.
Selon [2] la catégorie des espaces localement convexes admet des sous-catégories pleines $\mathcal{A}$ avec les propriétés suivantes: $\mathcal{A}$ est complet et cocomplet; $\mathcal{A}$ admet des bifoncteurs L et $\otimes$ avec les propriétés usuelles de catégorie fermée, donc en particulier tels que $L(E, L(F, G)) \cong L(E \otimes F, G)$ et $E \otimes F \cong F \otimes E$; en plus on a: $\Delta := L(-, \mathbb{R}) : E \mapsto E' := L(E, \mathbb{R})$ est un foncteur involutif. Cela veut dire on a $\Delta \circ \Delta \cong \mathrm{Id}_{\mathcal{A}}$. Donc tout objet E est réflexif au sens $E \cong E''$. Ceci est remarquable car $\dim E = \infty$ en général. On donne des descriptions et des preuves explicites. Enfin on présente un foncteur de dualité involutif pour une catégorie de géométries projectives de dimension quelconque.

Introduction.

Many categories of linear spaces over $\mathbb{R}$ (or an other field $\mathbb{K}$) admit an internal lifting L of the Hom-functor and then $L(-, \mathbb{K}) : E \mapsto E' := L(E, \mathbb{K})$ is a natural duality functor. An object E is called reflexive (in the given category) if $E'' \cong E$ holds. Usually only finite dimensional objects are reflexive. We are interested in duality functors which are involutive, i.e. $E'' \cong E$ must hold for all E, including infinite dimensional ones. An example is given in [2] where M. Barr shows that the category $\underline{\mathcal{D}}$ of separated dualized vector spaces is *-automonous, i.e. one has the following properties: there exist functors $L : \underline{\mathcal{D}}^{op} \times \underline{\mathcal{D}} \to \underline{\mathcal{D}}$ and $\otimes : \underline{\mathcal{D}} \times \underline{\mathcal{D}} \to \underline{\mathcal{D}}$ with $L(E, L(F, G)) \cong L(E \otimes F, G)$ so that $\underline{\mathcal{D}}$ becomes monoidal closed; and that, moreover, the functor $L(-, \mathbb{K}) : E \mapsto E' := L(E, \mathbb{K})$ is involutive. In sections 1 to 3 we present a different approach to these results, including explicit descriptions of the spaces $L(E, F)$ and $E \otimes F$. In section 4 we show that the category $\underline{\mathcal{D}}$ is complete and co-complete. The proofs yield explicit constructions for the limit or colimit of any given diagram in $\underline{\mathcal{D}}$.

In [2], also two embeddings of $\underline{\mathcal{D}}$ into the category $\underline{LCS}$ of separated locally convex spaces are described. The respective full subcategories of $\underline{LCS}$ are also $\star$-autonomous. We discuss these examples in section 5 from an other point of view.

The category $\underline{\mathcal{D}}$ has a lot of nice properties. So one might believe that it would be useful for analysis. This however is not the case, due to the lack of an appropriate completeness of the objects. By adding this completeness and an other natural condition, one gets the category $\underline{Con}$ of the so-called convenient vector spaces. These give the best setting for a generalization of classical Banach space calculus; cf. [4] and [5]. Since $\underline{Con}$ is a subcategory of $\underline{\mathcal{D}}$, every convenient vector space is reflexive in $\underline{\mathcal{D}}$, but examples show that this does not imply their reflexivity in $\underline{Con}$; cf. [4].

In section 6 a quite different example is considered. The objects are not linear spaces, but *dualized projective geometries*; cf. [3]. With suitably defined morphisms, one gets the category $\underline{DPG}$ of dualized projective geometries. Though no lifting of the Hom-functor is available, one can describe an involutive duality functor. The property of Desargues is not supposed. Hence homogeneous coordinates are not available and one cannot use previous results on categories of linear spaces.

My sincere thanks go to Michael Barr who encouraged me to publish results of his in my version; and to Claude-Alain Faure who worked out the results of section 6.

1. The category $\underline{\mathcal{D}}$ of separated dualized vector spaces.

1.1 Definition.

1° *We denote by* $\underline{DVS}$ *the category having as objects the couples* (E, E') *where* E *is a vector space over a fixed field* $\mathbb{K}$ *and* E' *is a vector subspace of the algebraic dual* $E^* := \{l : E \longrightarrow \mathbb{K} \ / \ l \text{ is linear}\}$*; and as morphisms from* (E, E') *to* (F, F') *the linear maps* $f : E \longrightarrow F$ *which satisfy* $f^*(F') \subseteq E'$.

2° *By* $\underline{\mathcal{D}}$ *we denote the full subcategory of* $\underline{DVS}$ *whose objects are* **separated**, *i.e. such that* $\bigcap_{l \in E'} \ker l = \{0\}$ *(equivalently: the functions of* E' *separate points of* E*).*

3° Notation. Since we consider an object (E, E') of $\underline{DVS}$ as a vector space E with E' as additional structure, we usually write just E for (E, E').

1.2 Remarks.

1. If E is object of $\underline{\mathcal{D}}$ and $\dim(E) < \infty$, then necessarily $E' = E^*$.
2. If $E' = E^*$, then every linear map $E \to F$ is a morphism.
3. If $S \subseteq F'$ generates the vector space F', then a linear map $\varphi : E \to F$ is a morphism iff $\varphi^*(S) \subseteq E'$.
4. A linear map $\varphi : E \to F$ is an isomorphism iff it is bijective and $\varphi^*(F') = E'$.

1.3 Proposition. *Let E, F be objects of $\underline{\mathcal{D}}$. The set $\underline{\mathcal{D}}(E, F)$ has a natural structure of vector space. It becomes an object of $\underline{\mathcal{D}}$, noted $L(E, F)$, by defining*

$$L(E, F)' := < l \circ \mathrm{ev}_a \;/\; a \in E, l \in F' >.$$

Since the map $E \times F' \to L(E, F)^*$ defined by $(a, l) \mapsto l \circ \mathrm{ev}_a$ is bilinear, the image set $\{l \circ \mathrm{ev}_a \;/\; a \in E, l \in F'\}$ has not to be a vector subspace of $L(E, F)^*$; so we consider the generated vector space $<..>$ which is formed by all finite sums $\sum l_i \circ \mathrm{ev}_{a_i}$.
Proof. Let $0 \neq \varphi \in L(E, F)$. Then there exists $a \in E$ with $\varphi(a) \neq 0 \in F$ and hence there exists $l \in F'$ such that $l(\varphi(a)) \neq 0$. So $l \circ \mathrm{ev}_a \in L(E, F)'$ satisfies $(l \circ \mathrm{ev}_a)(\varphi) \neq 0$. □

1.4 Remark. Obviously the structure of $L(E, F)$ is initial with respect to the evaluation maps $\mathrm{ev}_a : L(E, F) \to F$, $a \in E$; i.e. a linear map $\varphi : G \to L(E, F)$ is a morphism iff for all $a \in E$ the map $\mathrm{ev}_a \circ \varphi : G \to F$ is a morphism.

1.5 Theorem. $(E, F) \mapsto L(E, F)$ *extends to a lifting of the Hom-functor of* $\underline{\mathcal{D}}$.

Proof. (a) Let $\psi : E_1 \to E_2$ be a morphism. One defines $\psi^* : L(E_2, F) \to L(E_1, F)$ by $\psi^*(\varphi) := \varphi \circ \psi$. For ψ^* to be a morphism, it is enough to show that one has $l \circ \mathrm{ev}_a \circ \psi^* \in L(E_2, F)'$ for all $l \in F'$ and $a \in E_1$. This holds since $l \circ \mathrm{ev}_a \circ \psi^* = l \circ \mathrm{ev}_{\psi a}$ as verified by evaluating on elements $\varphi \in L(E_2, F)$.
(b) Let $\psi : F_1 \to F_2$ be a morphism. One defines $\psi_* : L(E, F_1) \to L(E, F_2)$ by $\psi_*(\varphi) := \psi \circ \varphi$. For ψ_* to be a morphism, it is enough to show that

one has $l \circ \mathrm{ev}_a \circ \psi_* \in L(E, F_1)'$ for all $l \in F_2'$ and $a \in E$. One has $l \circ \mathrm{ev}_a \circ \psi_* = (l \circ \psi) \circ \mathrm{ev}_a$ as verified by evaluating on elements $\varphi \in L(E, F_1)$, and from $l \circ \psi = \psi^*(l) \in F_1'$ follows that $(l \circ \psi) \circ \mathrm{ev}_a$ belongs to $L(E, F_1)'$.

1.6 Proposition.

1. *The map* $(-)^* : L(E_1, E_2) \to L(L(E_2, F), L(E_1, F))$ *with* $\psi \mapsto \psi^*$ *is a morphism.*
2. *The map* $(-)_* : L(F_1, F_2) \to L(L(E, F_1), L(E, F_2))$ *with* $\psi \mapsto \psi_*$ *is a morphism.*

Proof. 1. Applying twice 1.4 shows that the claim is equivalent with the statement $\Psi := \mathrm{ev}_a \circ \mathrm{ev}_\varphi \circ (-)^* : L(E_1, E_2) \to F$ is a morphism for all $\varphi \in L(E_2, F)$ and all $a \in E_1$. This in fact holds since one has $\Psi = \mathrm{ev}_{\psi a}$, as easily verified by evaluating on elements $\varphi \in L(E_1, E_2)$.
2 is proved similarly. □

2. The Duality Functor of $\underline{\mathcal{D}}$

2.1 Lemma. *The underlying vector space of* $L(E, \mathbb{K})$ *is* E'.

Proof. Let $l \in L(E, \mathbb{K})$. Then $l^*(\mathbb{K}') \subseteq E'$. Since $\mathrm{Id}_{\mathbb{K}} \in K^* = \mathbb{K}'$ one gets $l = l^*(\mathrm{Id}_{\mathbb{K}}) \in E'$. Conversely, let $l \in E'$. Since $\mathbb{K}' = <\mathrm{Id}_{\mathbb{K}}>$ the equation $l = \mathrm{Id}_{\mathbb{K}} \circ l$ shows that $l \in L(E, \mathbb{K})$; cf. 3 of 1.2. □

According to this lemma the vector space E' of any object E becomes an object of $\underline{\mathcal{D}}$ and we can write $E' = L(E, \mathbb{K})$.

2.2 Definition. *The partial functor* $\Delta := L(-, \mathbb{K})$ *is called the* ***duality functor*** *of* $\underline{\mathcal{D}}$*. So* Δ *is a contravariant endofunctor of* $\mathcal{D}$*. It associates to an object* E *the object* $\Delta E = E'$ *and to a morphism* $\varphi : E \to F$ *the morphism (cf. 1.2)* $\Delta\varphi := \varphi^* : F' \to E'$.

2.3 Remarks.
1° One has $E'' = L(E, \mathbb{K})' = < l \circ \mathrm{ev}_a \; / \; a \in E, l \in \mathbb{K}' >$. The map l is multiplication by some $\lambda \in \mathbb{K}$; hence $l \circ \mathrm{ev}_a$ is itself an evaluation (at $\lambda \cdot a$) and all these form a vector space. Hence $E'' = \{\mathrm{ev}_a : E' \to \mathbb{K} \; / \; a \in E\}$.
2° By 1 of 1.6 the maps $\Delta : L(E, F) \to L(F', E')$ are morphisms.

2.4 Theorem. *One has natural* $\underline{\mathcal{D}}$*-isomorphisms* $j_E : E \to E''$.

Proof. We just remarked that $E'' = \{\mathrm{ev}_a \; / \; a \in E\}$. So we define $j = j_E$ by $j(a) := \mathrm{ev}_a$. This map $j : E \to E''$ is obviously linear and surjective. It is injective since E is separated. We will use the following equation:

$$j_E^* \circ j_{E'} = \mathrm{Id}_{E'}.$$

It is verified by evaluating on elements $l \in E'$ and $a \in E$. One obtains $((j_E^* \circ j_{E'})(l))(a) = (j_E^*(\mathrm{ev}_l))(a) = (\mathrm{ev}_l \circ j_E)(a) = \mathrm{ev}_l(\mathrm{ev}_a) = (\mathrm{ev}_a)(l) = l(a)$. Simplification by a and then by l yields the equation. Using it one gets $j_E^*(E''') = j_E^*(j_{E'}(E')) = E'$ and this shows that j_E is a $\mathcal{D}$-isomorphism, cf. 4 of 1.2. □

2.5 Lemma. *For any morphism $\varphi : E \to F$ the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "j_E"'] & F \arrow[d, "j_F"] \\ E'' \arrow[r, "\varphi^{**}"] & F'' \end{tikzcd}$$

This means that the morphisms j_E form a natural transformation and since they are $\mathcal{D}$-isomorphisms one has a natural isomorphism $\mathrm{Id} \cong \Delta^2$.

Proof. Let $a \in E$, $l \in F'$. Then $(\varphi^{**} \circ j_E)(a) = \varphi^{**}(\mathrm{ev}_a) = \mathrm{ev}_a \circ \varphi^*$, hence $((\varphi^{**} \circ j_E)(a))(l) = \mathrm{ev}_a(\varphi^*(l)) = \mathrm{ev}_a(l \circ \varphi) = l(\varphi(a)) = \mathrm{ev}_{\varphi(a)}(l)$. Together this gives $(\varphi^{**} \circ j_E)(a) = \mathrm{ev}_{\varphi(a)}$. On the other hand $(j_F \circ \varphi)(a) = j_F(\varphi(a)) = \mathrm{ev}_{\varphi(a)}$. □

2.6 Lemma. *Let $\Delta_{E,F} : L(E,F) \to L(F',E')$ be the map defined by $\varphi \mapsto \Delta\varphi = \varphi^*$. We put $i := (j_E)^{-1}$ and $j := j_F$. The following diagram commutes:*

$$\begin{tikzcd} L(E,F) \arrow[r, "\Delta_{E,F}"] \arrow[d, "i^*"'] & L(F',E') \arrow[d, "\Delta_{F',E'}"] \\ L(E'',F) \arrow[r, "j_*"] & L(E'',F'') \end{tikzcd}$$

Proof. Let $\varphi \in L(E,F)$. One trivially has $(\Delta_{F',E'} \circ \Delta_{E,F})(\varphi) = \varphi^{**}$. One shows that also $(j_* \circ i^*)(\varphi) = \varphi^{**}$. One gets $(j_* \circ i^*)(\varphi) = j \circ \varphi \circ i$. We evaluate on the elements of E''. These are of the form $j_E(a) = i^{-1}(a)$ for some $a \in E$. So $(j_* \circ i^*)(\varphi)(i^{-1}a) = (j \circ \varphi \circ i)(i^{-1}(a) = j_F(\varphi(a)) = (\varphi^{**}(j_E)(a)$; cf. 2.5. $\square$

2.7 Proposition. *The map $\Delta_{E,F} : L(E,F) \to L(F',E')$ defined by $\varphi \mapsto \varphi^*$ is a $\underline{\mathcal{D}}$-isomorphism.*

Proof. By the preceding lemma $\Delta_{F',E'} \circ \Delta_{E,F}$ is bijective. Hence $\Delta_{F',E'}$ is surjective and $\Delta_{E,F}$ is injective. Since this injectivity holds for any E, F one concludes that $\Delta_{F',E'}$ is also injective and hence bijective, and now the same follows for $\Delta_{E,F}$. By evaluating on the elements $\varphi \in L(E,F)$ one easily verifies that for $a \in E$ and $l \in F'$ one has $\Delta^*(j_E(a) \circ \mathrm{ev}_l) = l \circ \mathrm{ev}_a$. Since $j_E : E \to E''$ is bijective, the equation implies that Δ^* maps the generators of $L(F',E')'$ bijectively onto those of $L(E,F)'$. Hence $\Delta^*(L(F',E')') = L(E,F)'$ and Δ is a $\underline{\mathcal{D}}$-isomorphism, cf. 3 and 4 of 1.2. $\square$

3 The Tensor Product of $\underline{\mathcal{D}}$

3.1 Definition. *A bilinear map $\beta : E \times F \to G$ where E, F, G are objects of $\underline{\mathcal{D}}$ is called a* **bilinear morphism** *if for every $a \in E$ the map $\beta(a,-) : y \mapsto \beta(a,y)$ is a morphism $F \to G$ and for every $b \in F$ the map $\beta(-,b) : x \mapsto \beta(x,b)$ is a morphism $E \to G$. The bilinear morphisms $\beta : E \times F \to G$ form a vector space $B(E,F;G)$ which becomes an object of $\underline{\mathcal{D}}$ by defining*

$$B(E,F;G)' := < l \circ \mathrm{ev}_{a,b} \;/\; (a,b) \in E \times F; l \in G' >.$$

3.2 Proposition. *One has isomorphisms*

$$\Theta : L(E, L(F,G)) \to B(E,F;G)$$

described by $(\Theta f)(x,y) = (fx)(y)$.

Proof. Let $f \in L(E, L(F,G))$ and put $\beta := \Theta f$. Obviously β is bilinear. For $a \in E$ one has $\beta(a,-) = f(a) \in L(F,G)$. And for $b \in F$ one has $\beta(-,b) = \mathrm{ev}_b \circ f$. One concludes that $\beta \in B(E,F;G)$. So the map Θ is well defined.

Similarly, for $\beta \in B(E,F;G)$ one defines $((\Omega\beta)x)(y) = \beta(x,y)$. We put $f := \Omega\beta$. For $a \in E$ one has $f(a) = \beta(a,-)$ and hence $f(a) \in L(F,G)$. And $f : E \to L(F,G)$ is a morphism iff $\mathrm{ev}_b \circ f : E \to G$ is a morphism for all $b \in F$, and this holds since $\mathrm{ev}_b \circ f = \beta(-,b)$.

The maps Θ and Ω are obviouly linear and each others inverse. Remains to show that they are morphisms of $\underline{\mathcal{D}}$. The structure of $L(E, L(F, G))$ is initial with respect to the maps $\mathrm{ev}_a : (E, L(F, G)) \to L(F, G)$ and the one of $L(F, G)$ is initial with respect to the evaluations $\mathrm{ev}_b : L(F, G) \to G$. So together the maps $\mathrm{ev}_b \circ \mathrm{ev}_a : L(E, L(F, G)) \to G$ form an initial cone. The same holds for the maps $\mathrm{ev}_{a,b} : B(E, F; G) \to G$. Since $\mathrm{ev}_b \circ \mathrm{ev}_a = \mathrm{ev}_{a,b} \circ \Theta$, a linear map ξ from any dualized vector space into $L(E, L(F, G))$ is a morphism iff $\Theta \circ \xi$ is a morphism; in particular, Θ and Ω are morphisms. $\square$

3.3 Definition. *For any dualized vector spaces E, F we define their* ***tensor product*** *as the object*

$$E \otimes F := B(E, F; \mathbb{K})'$$

together with the bilinear morphism $\tau : E \times F \to E \otimes F$ *defined by*

$$\tau : (x, y) \mapsto \mathrm{ev}_{x,y}.$$

3.4 Theorem. *For every bilinear morphism* $\beta : E \times F \to G$ *there exists a unique (linear) morphism* $\varphi : E \otimes F \to G$ *such that* $\beta = \varphi \circ \tau$.

Proof. *Uniqueness.* This follows since $E \otimes F = B(E, F; \mathbb{K})'$ is generated by the elements of the form $\mathrm{ev}_{a,b} = \tau(a, b)$ for $(a, b) \in E \times F$.

Existence. In the special case $G = \mathbb{K}$ one verifies that $\beta = \mathrm{ev}_\beta \circ \tau$ and this shows that $\varphi := \mathrm{ev}_\beta$ is a solution. For the general case one considers for any $l \in G'$ the composite $\gamma(l) := l \circ \beta$. It belongs to $B(E, F; \mathbb{K})$ and hence $l \circ \beta = \mathrm{ev}_{l \circ \beta} \circ \tau$. One also checks easily that the map $\gamma : G' \to B(E, F; \mathbb{K})$ is a $\underline{\mathcal{D}}$-morphism. The same now follows for the map $\psi := j \circ \gamma : G' \to B(E, F; \mathbb{K})'' = (E \otimes F)'$, where j is the isomorphism from $B(E, F; \mathbb{K}) \to B(E, F; \mathbb{K})''$, cf. 2.4. By 2.7 there exists a unique morphism $\varphi : (E \otimes F)' \to G$ such that $\psi = \varphi^*$. Now one gets $l \circ \varphi = \varphi^*(l) = \psi(l) = \mathrm{ev}_{l \circ \beta}$ and furthermore $(l \circ \varphi \circ \tau)(a, b) = \mathrm{ev}_{l \circ \beta}(\tau(a, b)) = \tau(a, b)(l \circ \beta) = (l \circ \beta)(a, b)$. Simplifying by$(a, b)$ and then by l one gets $\varphi \circ \tau = \beta$. $\square$

Since $B(E, F; G) \cong B(F, E; G)$ one finally has

3.5 Theorem. *There are natural isomorphisms as follows*

$$L(F, L(E, G)) \cong L(E, L(F, G)) \cong B(E, F; G) \cong L(E \otimes F, G).$$

The following two propositions show how $E' \otimes F$ and $L(E,F)$ are related. Analogous results for $E \otimes F$ and $L(E',F)$ then follow.

3.6 Proposition.. *Let $\beta : E' \times F \to L(E,F)$ be the map defined by $\beta(l,b)(x) := l(x) \cdot b$. It is a bilinear morphism and hence factors as $\beta = \alpha \circ \tau$. The so obtained morphism $\alpha : E' \otimes F \to L(E,F)$ is injective.*

Proof. Let $z \in \ker\alpha$. One can write z in the form $z = \tau(l_1,b_1) + \ldots + \tau(l_n,b_n)$ and one can choose the elements $b_1,\ldots,b_n$ to be linearly independent. Then $\alpha(z) = \alpha(\tau(l_1,b_1)) + \ldots + \alpha(\tau(l_n,b_n))$ and $\alpha(z)(x) = \beta(l_1,b_1)(x) + \ldots + \beta(l_n,b_n)(x) = l_1(x) \cdot b_1 + \ldots + l_n(x) \cdot b_n$. Since by hypothesis one has $\alpha(z) = 0$ and $b_1,\ldots,b_n$ are linearly independent one gets $l_1(x) = \ldots = l_n(x) = 0 \;\; \forall x$ and therefore $z = 0$. □

3.7 Proposition. *The map $\alpha : E' \otimes F \to L(E,F)$ is an epimorphism; i.e. for morphisms $\varphi,\psi : L(E,F) \to G$ one has $\varphi \circ \alpha = \psi \circ \alpha \Rightarrow \varphi = \psi$.*

Proof. We may assume that $G = \mathbb{K}$, and it is enough to show that if $\varphi : L(E,F) \to G$ satisfies $\varphi \circ \alpha = 0$, then $\varphi = 0$. We know that φ is of the form $\varphi = \sum_{i=1}^n l_i \circ \mathrm{ev}_{a_i}$ with $a_i \in E$ and $l_i \in F'$, and we can choose the points a_i to be linearly independent (if they are dependent, one can reduce the number of terms in the sum). Since $\beta = \alpha \circ \tau$ one has $\varphi \circ \beta = 0$. So one has for every $(k,b) \in E' \times F$

$$0 = \varphi(\beta(k,b)) = \textstyle\sum_{i=1}^n (l_i \circ \mathrm{ev}_{a_i})\beta(k,b) = \sum_{i=1}^n l_i(k(a_i) \cdot b) =$$
$$\textstyle\sum_{i=1}^n k(a_i) \cdot l_i(b) = k\big(\sum_{i=1}^n l_i(b) \cdot a_i\big).$$

Since this holds for all $k \in E'$ one gets $\sum_{i=1}^n l_i(b) \cdot a_i = 0$, and this implies $l_i(b) = 0$ for all i. Having this for all $b \in F$ one deduces $l_i = 0 \in F'$ and $\varphi = 0$. □

4. Categorical completeness of $\underline{\mathcal{D}}$

4.1 Proposition. *The category $\underline{DVS}$ is complete and co-complete.*

Proof. It is well known that the category $\underline{VS}$ of vector spaces (over $\mathbb{K}$) is complete since it has products and equalizers and co-complete since it has coproducts and coequalizers.

One now considers the forgetful functor $U : \underline{DVS} \to \underline{VS}$ which associates to an object (E,E') the vector space E. With respect to U there exist all initial and final structures. We just describe them and leave the verification of the respective universal property as exercise.

Initial structures: Suppose given a family of linear maps $f_i : E \to E_i$ where $E_i = U(E_i, E_i')$. One defines $E' := <\bigcup_{i \in I} f_i^*(E_i')>$. Then one has the following universal property: If $g : F \to E$ is a linear map and $F = U(F, F')$, then g is a $\underline{DVS}$-morphism $(F, F') \to (E, E')$ iff $f_i \circ g$ is a $\underline{DVS}$-morphism $(F, F') \to (E_i, E_i')$ $\forall i$.
Final structures: Suppose given linear maps $f_i : E_i \to E$ with E, E_i as before. One defines $E' := \{l \in E^* \ / \ f_i^*(l) \in E_i' \forall i\}$. Then a linear map $g : E \to F$ where $F = U(F, F')$ is a $\underline{DVS}$-morphism $(E, E') \to (F, F')$ iff $g \circ f_i$ is a $\underline{DVS}$-morphism $(E, E') \to (F_i, F_i')$ $\forall i$.

The functor U has a left-adjoint and a right adjoint since any vector space E has a coarsest and a finest $\underline{DVS}$-structure. These are $E' := \{0\}$ respectively $E' := E^*$. Hence U commutes with limits and colimits. Limits (colimits) are obtained by taking the limit (colimit) of the underlying vector spaces in $\underline{VS}$, endowed with the respective initial (final) structure. □

4.2 Proposition. *The inclusion functor* $i : \underline{\mathcal{D}} \to \underline{DVS}$ *has a left adjoint retracting functor* $\rho : \underline{DVS} \to \underline{\mathcal{D}}$*, called separation functor. This means that* $\underline{\mathcal{D}}$ *is a reflective subcategory of* $\underline{DVS}$*.*

Proof. Let $(E, E') \in \underline{DVS}$. We put $N := \{x \in E \ / \ l(x) = 0 \forall l \in E'\}$, define ρE as the vector space E/N endowed with the final $\underline{DVS}$-structure induced by the canonical projection $\pi : E \to E/N$, i.e. $\rho(E, E') = (E/N, \{h \in (E/N)^* \ / \ h \circ \pi \in E'\})$. First one verifies that $\rho E \in \underline{\mathcal{D}}$. Let $\pi(x) \neq \pi(y) \in E/N$. Then $x - y \notin N$ and therefore $l(x) \neq l(y)$ for some $l \in E'$. The function l factors as $l = h \circ \pi$ for a unique function $h : E/N \to \mathbb{K}$, and h is linear. From the fact that $l = h \circ \pi$ is a morphism one deduces by the finality of π that h is a morphism. And since $h \circ \pi$ is a morphism, $h \in (\rho E)'$. □

4.3 Theorem. *The category* $\underline{\mathcal{D}}$ *is complete and co-complete. Limits in* $\underline{\mathcal{D}}$ *are the same as in* $\underline{DVS}$*; colimits in* $\underline{\mathcal{D}}$ *are obtained by applying the separation functor* σ *to the colimit taken in* $\underline{DVS}$*.*

Proof. By a well known categorical result 4.3 is a consequence of 4.1 and 4.2. □

5. Locally Convex Spaces with Involutive Duality Functor

5.1 Proposition. *Suppose we have a commutative diagram of functors*

$$\begin{tikzcd} \underline{A} \arrow[r, "S"] \arrow[d, "U"'] & \underline{B} \arrow[r, "R"] \arrow[d, "V"] & \underline{A} \arrow[d, "U"] \\ \underline{S} \arrow[r, "\mathrm{Id}"] & \underline{S} \arrow[r, "\mathrm{Id}"] & \underline{S} \end{tikzcd}$$

Suppose, furthermore, that the functor V is faithful and that $R \circ S = \mathrm{Id}$. Then S identifies $\underline{A}$ with the full subcategory $\underline{B_o}$ of $\underline{B}$ formed by the objects Y satisfying the condition $(S \circ R)(Y) = Y$. (Remark that also $U = V \circ S$ is faithful).

Proof. We first show that the functor $S : \underline{A} \to \underline{B}$ can be factorized over the full subcategory $\underline{B_o}$ of $\underline{B}$ formed by the objects Y of the form $Y = SX$ for some object of $\underline{A}$; one has: $S = i \circ S_o$, where i is the inclusion functor and $S_oX = SX$, $S_of = Sf$, for any object X and any morphism f of $\underline{A}$. We put $R_o := R \circ i : \underline{B_o} \to \underline{A}$ and show that S_o and R_o are each others inverse.

(a) One has $R_o \circ S_o = R \circ i \circ S_o = R \circ S = \mathrm{Id}$.

(b') For an object Y of $\underline{B_o}$ one has $Y = SX$ for some X of $\underline{A}$, hence $(S_o \circ R_o)(Y) = (SR)(Y) = (SRS)(X) = S(X) = Y$.

(b") Consider $(S_o \circ R_o)(g)$ for a morphism g of $\underline{B_o}$. One gets (using the commutative diagram): $(VS_oR_o)(g) = (VSR)(g) = (UR)(g) = V(g)$ and since V is a faithful functor this implies $(S_oR_o)(g) = g$. Now (a) and (b) show that $\underline{A} \cong \underline{B_o}$. □

We now consider separated locally convex spaces over $\mathbb{R}$ (or $\mathbb{C}$). By the theorem of Hahn-Banach the continuous dual F' separates points of F, i.e. $(F, F') \in \underline{\mathcal{D}}$. Since for a $\underline{LCS}$-morphism $f : F_1 \to F_2$ one has $f^*(F_2') \subseteq F_1'$ one gets a functor $\delta : \underline{LCS} \to \underline{\mathcal{D}}$ by defining $F \mapsto (F, F')$ and $f \mapsto f$.

It is well known that for any separated dualized vector space $E = (E, E')$ there exist separated locally convex topologies on E such that E' becomes the continuous dual. Among these topologies there exists a coarsest one, called the weak topology, and a finest one, called the Mackey topology (of

E with respect to E') Let σE respectively μE denote the locally convex space obtained by means of the weak respectively the Mackey topology. If $f : E \to F$ is a morphism of $\underline{\mathcal{D}}$, then $f : \sigma E \to \sigma F$ and $f : \mu E \to \mu F$ are continuous and hence one has functors $\sigma : \underline{\mathcal{D}} \to \underline{LCS}$ and $\mu : \underline{\mathcal{D}} \to \underline{LCS}$ which preserve the underlying vector spaces and maps. They have δ as left-inverse, i.e. $\delta \circ \sigma = \delta \circ \mu = \mathrm{Id}$. One easily verifies that σ is right adjoint and μ is left adjoint to δ.

5.2 Theorem. *Let $\underline{WLCS}$respectively $\underline{MLCS}$ be the full subcategory of $\underline{LCS}$ formed by the ojects having the weak topology respectively the Mackay topology. Each of these subcategories is isomorphic to to the category $\underline{\mathcal{D}}$.*

Proof of the first isomorphism.
Consider the diagram in 5.1 for $\underline{A} = \underline{\mathcal{D}}, \underline{B} = \underline{LCS}, \underline{S} = \underline{VS}$, U and V being the respective forgetful functor "underlying vector space", $S = \sigma$, $R = \delta$. All the considered functors preserve the underlying vector spaces and linear maps, hence the diagram commutes. Furthermore, $\delta \circ \sigma = \mathrm{Id}$ and $\underline{LCS_o}$ contains the locally convex spaces F satisfying $\sigma\delta\ F = F$, i.e. carrying the weak topology. By 5.1 we obtain $\underline{LCS_o} = \underline{WLCS}$ and and $\underline{\mathcal{D}} \cong \underline{WLCS}$.
The second isomorphism is proved in the same way. □

The fact that the categories $\underline{\mathcal{D}}$, $\underline{WLCS}$ and $\underline{MLCS}$ are isomorphic follows also from 2.1.9 and 2.1.11 in [4]. The results there are even without separation hypothesis.

5.3 Remark. The three categories $\underline{\mathcal{D}}$, $\underline{WLCS}$ and $\underline{MLCS}$ are concretely isomorphic categories over the base category $\underline{VS}$. This means that the objects are vector spaces with differently described structures yielding the same morphisms. Similarly one can use open sets, or closed sets, or neighborhoods, or closure operators for defining $\underline{Top}$. *"The differences between the various descriptions are regarded as inessential and we can in good conscience call each of them $\underline{Top}$"* (citation from remark 5.12 in [1]). In this sense we may write $\underline{\mathcal{D}} = \underline{WLCS} = \underline{MLCS}$. Another example of a concrete category over $\underline{VS}$ having numerous descriptions is $\underline{Con}$, cf. [4].

6. An Involutive Duality Functor for Dualized Projective Geometries

Since we do not know whether the category which will be considered admits a lifting of the Hom-functor, we cannot define a duality functor by means of an object which should represent the functor. One proceeds differently. It is well known that the set G^* of all hyperplanes of a projective geometry G is in natural way also a projective geometry, called the **dual geometry** of G. One can extend the duality to a contravariant endofunctor provided one uses appropriate morphisms. One has two kinds of morphisms. Both are ***partial maps*** between the respective point sets, noted $g : G_1 \dashrightarrow G_2$, i.e. maps $g : G_1 \setminus \operatorname{Ker} g \to G_2$ where $\operatorname{Ker} g$, called ***kernel*** of g, is a subset of G_1. The notation $g : G_1 \dashrightarrow G_2$ will remind that $g(x)$ is not defined for all $x \in G_1$ (unless $\operatorname{Ker} g = \emptyset$). Let $g : G_1 \dashrightarrow G_2$ be such a partial map.

6.1 Definition.
1° *g is called a* **morphism of projective geometries** *if for every subspace F of G_2 the set $g^\sharp(F) := g^{-1}(F) \cup \operatorname{Ker} g$ is a subspace of G_1.*

2° *g is called a* **homomorphism of projective geometries** *if for every hyperplane H of G_2 the set $g^\sharp(H)$ is either G_1 or a hyperplane of G_1.*

Since any subspace is an intersection of hyperplanes one deduces that homomorphisms are special morphisms. If $g : G_1 \dashrightarrow G_2$ is a homomorphism one can define $g^*(H) := g^\sharp(H)$ if $g^\sharp(H)$ is a hyperplane of G_1; and $H \in \operatorname{Ker} g^*$ otherwise, i.e. if $g^\sharp(H) = G_1$. One verifies that the so defined partial map is a homomorphism $g^* : {G_2}^* \dashrightarrow {G_1}^*$. We also recall that $a \mapsto J(a) := \{H \in G^* \ / \ a \in H\}$ defines a homomorphism, with empty kernel, noted $J : G \to G^{**}$. For details and proofs we refer to [3].

6.2 Definition. *The category $\underline{DPG}$ has as objects the* **dualized projective geometries**, *i.e. the couples (G, G') where G is a projective geometry and G' a subspace of the dual geometry G^* satisfying the* **separation condition:** *$\bigcap G' = \emptyset$.*

The $\underline{DPG}$-morphisms from (G_1, G_1') to (G_2, G_2') are the homomorphisms $g : G_1 \dashrightarrow G_2$ which satisfy the following **continuity condition:** *The associated homomorphism $g^* : {G_2}^* \dashrightarrow {G_1}^*$ restricts to the given subspaces. This means that there exists a partial map g' making commutative the following diagram:*

$$\begin{tikzcd}
{G_2}' \arrow[r, dashed, "g'"] \arrow[d, "i_2"'] & {G_1}' \arrow[d, "i_1"] \\
{G_2}^* \arrow[r, dashed, "g^*"] & {G_1}^*
\end{tikzcd}$$

Since i_2 and g^* are homomorphisms, the same holds for $i_1 \circ g'$. Subspace inclusions are obviously initial (with respect to the forgetful functor from the category of projective geometries and homomorphisms to the category of sets and partial maps). So one deduces that $g' : {G_2}' \dashrightarrow {G_1}'$ is also a homomorphism.

We shall provide G' with a structure of dualized projective geometry by choosing an appropriate subspace of $(G')^*$. The key is the following result.

6.3 Lemma. *The map j defined by $j(a) := \{H \in G' \ / \ a \in H\}$ is a homomorphism $j : G \longrightarrow (G')^*$, has empty kernel, and is injective.*

Proof. For any $\Omega \in G^{**}$ one has $\Omega \in \operatorname{Ker}(i^*) \Leftrightarrow i^{\natural}(\Omega) = G' \Leftrightarrow G' \cap \Omega = G' \Leftrightarrow G' \subseteq \Omega \Rightarrow \bigcap G' \supseteq \bigcap \Omega \Rightarrow \bigcap \Omega = \emptyset$. Since for $\Omega = J(a)$ for some $a \in G$ one has $a \in \bigcap \Omega$ it is impossible that $\Omega \in \operatorname{Ker} i^*$. So one sees that $i^* \circ J$ and j both have empty kernel. They even coincide since one has $i^*(J(a)) = i^{-1}(\{H \in G^* \ / \ a \in H\}) = \{H \in G^* \ / \ a \in H\} \cap G' = j(a)$ for all $a \in G$. Now $j = i^* \circ J$ shows that j is a homomorphism. The injectivity of j follows from the following lemma.

6.4 Lemma. *For $a \in G$ one has $\bigcap j(a) = \{a\}$.*

Proof. Trivially $a \in \bigcap j(a)$. Let also $b \in \bigcap j(a)$ for some $b \neq a$. Then the line generated by a and b, noted $a \star b$, intersects the hyperplane H_1 in a unique point $c := (a \star b) \cap H_1$. Let now H be an arbitrary element of G'. We claim that $c \in H$.

Case 1: $a \in H$. This means $H \in j(a)$ and hence $b \in H$. Now $c \in H$ follows.

Case 2: $a \notin H$. Since $c \in H_1$ the case $H = H_1$ is trivial. So let $H \neq H_1$. Then $H_2 := (H_1 \cap H) \vee a$ is a hyperplane of G and from $H, H_1 \in G'$ one gets $H_2 \in G'$ (the three hyperplanes are collinear in G^* and G' is a subspace of G^*). By construction $H_2 \in j(a)$. From $c \in H_2 = (H_1 \cap H) \vee a$ follows that there is a point $d \in H_1 \cap H$ such that $c \in d \star a$. If $c \neq d$ one obtains $a \in c \star d \subseteq H_1$ which is against the choice of H_1. Hence $c = d \in H$.

So we have obtained $c \in \bigcap G'$ in contradiction to $\bigcap G' = \emptyset$ and one concludes that $\bigcap j(a)$ cannot contain any point different from a. $\square$

We can now define on G' a structure of dualized geometry by putting $G'' := j(G)$. In fact, $j(G) \subseteq (G')^*$ and as image by a homomorphism it is a subspace. The separation condition holds, because if $H \in j(a)$ for all $a \in G$ one obtains $G \subseteq H$ which is absurd for a hyperplane $H \subseteq G$.

6.5 Proposition. *Let $g : G_1 \dashrightarrow G_2$ be a morphism of* <u>*DPG*</u>*. Then the homomorphism $g' : G_2{}' \dashrightarrow G_1{}'$ is a morphism of* <u>*DPG*</u> *and hence one has a contravariant endofunctor of the that category.*

Proof. One has to verify that g'^* restricts to the subspaces $j_\nu(G_\nu)$ of $(G_\nu{}')^*$. This means that $(g')^\sharp \circ j_1 = j_2 \circ g$. Using that $g^{**} \circ J_1 = J_2 \circ g$, cf. 11.4.7 in [3], one gets

$g'^* \circ j_1 = g'^* \circ i^* \circ J_1 = (i_1 \circ g')^* \circ J_1 = (g^* \circ i_2)^* \circ J_1 = i_2^* \circ g^{**} \circ J_1 = i_2^* \circ J_2 \circ g = j_2 \circ G.$ $\square$

6.6 Theorem. *For every dualized geometry G the map $j : G \to G''$ is an isomorphism of* <u>*DPG*</u>*.*

Proof. We saw that j is an injective homomorphism. Surjectivity holds according the definition of G''. From $j \circ j^{-1} = \mathrm{Id}$ follows that also j^{-1} is a homomorphism. Trivially j and j^{-1} satisfy the continuity condition since one has $E''' = j'(E')$. $\square$

References

[1] Adamek, J., Herrlich, H. and Strecker, G.: *Abstract and Concrete Categories*, J. Wiley and Sons, Chichester, 1990.

[2] Barr, M.: On $*$-Autonomous Categories of Topological Vector Spaces, *to appear.*

[3] Faure, C.-A. and Frölicher, A.: *Modern Projective Geometry*, Kluwer, Dordrecht, 2000.

[4] Frölicher, A. and Kriegl, A.: *Linear Spaces and Differentiation Theory*, J. Wiley and Sons, Chichester, 1988.

[5] Kriegl, A. and Michor, P.: *The Convenient Setting of Global Analysis*, Mathematical Surveys and Monographs Vol. 53, A. M.S. 1997.

Université de Genève, Section de Math., C. p. 124, CH-1211 Genève 24