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Cahiers de topologie et géométrie différentielle catégoriques, tome 42, nº 4 (2001), p. 285-316

<http://www.numdam.org/item?id=CTGDC_2001__42_4_285_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLII-4 (2001)

SYMMETRIC MONOIDAL CLOSED STRUCTURES IN PRAP

*by M. SIOEN**

RESUME. Le fait que la catégorie **PRTOP** (des espaces prétopologiques et applications continues) n'est pas cartésienne fermée est bien connu et par conséquent, la même conclusion est valide pour la catégorie **PRAP** des espaces de pré-approximation et des contractions, comme elle a été introduite par E. Lowen et R. Lowen dans [1].
Le but de cet article est de montrer que **PRAP** n'admet qu'une seule structure monoïdale fermée symétrique (à un isomorphisme naturel près), à savoir la structure monoïdale inductive canonique (étudiée dans le contexte des catégories topologiques ou initialement structurées par Wischnewsky et Činčura). On démontrera ce résultat en se basant sur la technique développée par J. Činčura [5] pour résoudre cette question dans **PRTOP**.

1 Introduction

It is well-known that the topological construct **PRTOP** of pre-topological spaces and continuous maps is extensional but that, unfortunately, it lacks another important so-called convenience property: the cartesian closedness, or equivalently, the existence of nice function spaces satisfying a nice exponential law. The numerification superconstruct **PRAP** of **PRTOP**, as intoduced by E. Lowen and R. Lowen in [11],

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was shown still to be extensional, but clearly fails to be cartesian closed, because it contains **PRTOP** as a full simultaneously concretely bireflectice and concretely bicoreflective subconstruct. Because a decent exponential law in a category is vital for making it useful for duality theory, categorical algebra and enriched category theory, many attempts can be undertaken to force this nice exponential behaviour if it is not present. A first approach is to look for those objects in our category which still have decent exponential behaviour, i.e. the so-called exponential objects. The exponential objects in e.g. **TOP**, **PRTOP** and **PRAP** have been identified to be resp. the core-compact topological, the finitely generated pre-topological and the ∞pqs-metric spaces (see e.g. E. Lowen-Colebunders and G. Sonck [14] and E. Lowen, R. Lowen and C. Verbeeck [13]). A second approach can be weakening the structure of the spaces we work with, in order to obtain a cartesian closed topological supercategory, which is preferably the smallest such one, i.e. looking for the cartesian closed topological hull (if it exists). It is well-known that for **TOP** resp. **PRTOP** and **PRAP**, this yields the category of Antoine spaces, resp. the categories **PSTOP** and **PSAP** (see [12]). A third thing we can do is retaining the category we work in and looking in there for an alternative, more 'algebraic' product satisfying a decent exponential law. Such a product, if it exists, is called a tensorproduct, by analogy to the algebraic tensorproduct of modules or vectorspaces. Tensorproducts or more formally, symmetric monoidal closed structures were introduced by S. Eilenberg and G. M. Kelly in [6] and have been extensively studied since then. In our familiar topological constructs, they even can be assumed to be in some standard form, relating very much to the cartesian case as can be found in (J. Činčura [5] and A. Logar and F. Rossi [9]). In the realm of topological constructs or more generally, of initially structured categories in the sense of L. D. Nel, there always exists a canonical tensorproduct, called the inductive one, which is studied in (M. B. Wischnewsky [23] and J. Činčura [5]). This type of tensorproduct even can be related to the algebraic one by means of a notion of bimorphisms, as follows from (B. Banaschewski and E. Nelson [2] and D. Pumplün [22]). It was shown by J. Činčura in [5] that this inductive tensorproduct, together with the function space structure of pointwise convergence, is (up to natural equivalence) the only possible structure of symmetric monoidal closed category which can be imposed on **PRTOP**. It is our aim in this paper, building on his technique, to show the same for **PRAP**.

2 Preliminaries

For notations, terminology or general information of categorical nature, we refer to (J. Adámek, H. Herrlich and G. Strecker [1], F. Borceux [3] and S. MacLane [20]). If $F : \mathbf{A} \longrightarrow \mathbf{B}$ and $G : \mathbf{B} \longrightarrow \mathbf{A}$ are functors such that the pair (F, G) is a pair of adjoint functors, we will briefly denote this by $F \dashv G$. To start, we briefly recall some material related to approach theory and we refer to (P. Brock and D. C. Kent [4], E. Lowen and R. Lowen [11], [12], E. Lowen, R. Lowen and C. Verbeeck [13], R. Lowen [16], [17] and R. Lowen, D. Vaughan and C. Verbeeck [18]). For every set X, we will write 2^X (resp. $2^{(X)}$) for the powerset (resp. the set of all finite subsets) of X. The first infinite ordinal is denoted by ω. Approach spaces (resp. pre-approach spaces) were introduced by R. Lowen in [16] (resp. by E. Lowen and R. Lowen in [11], [12]) to yield suitable topological numerified superconstructs **AP** (resp. **PRAP**) of the well-studied 'classical' topological constructs **TOP**, of topological spaces and continuous maps (resp. **PRTOP**, of pre-topological spaces and continuous maps). This means that **AP** (resp. **PRAP**) contains at the same time both of the constructs **TOP** and pq$\mathbf{MET}^\infty$, of ∞pq-metric spaces and non-expansive maps, (resp. **PRTOP** and pqs$\mathbf{MET}^\infty$, of ∞pqs-metric spaces and non-expansive maps,) as full subconstructs, the former concretely bireflectively and concretely bicoreflectively, the latter only concretely bicoreflectively. This relation between these categories is summarized in the scheme below where an r (resp. a c) indicates that the first construct is concretely bireflectively (resp. concretely bicoreflectively) embedded in the second one.

$$
\begin{tikzcd}
 & \mathbf{PRAP} & \\
\mathbf{PRTOP} \arrow[ur, hook, "r", "c"'] & & \mathrm{pqs}\mathbf{MET}^{\infty} \arrow[ul, hook', "c"'] \\
 & \mathbf{AP} \arrow[uu, hook', "r"] & \\
\mathbf{TOP} \arrow[uu, hook', "r"] \arrow[ur, hook, "r", "c"'] & & \mathrm{pq}\mathbf{MET}^{\infty} \arrow[ul, hook', "c"'] \arrow[uu, hook', "r"']
\end{tikzcd}
$$

The aim of these quantified supercategories is that in the setting of **AP** (resp. **PRAP**), products or more generally, initial structures of families of ∞pq-metric, (resp. ∞pqs-metric) objects can be formed on the numerical level, yielding canonical quantified information which is still compatible with the underlying topologies (resp. pre-topologies). On the other hand, initial or final lifts of sources or sinks consisting of topological (resp. pre-topological) objects, formed in **AP** (resp. **PRAP**) coincide with the corresponding liftings formed in **TOP** (resp. **PRTOP**). For any further information or notations concerning approach theory, we refer to [16, 17, 11, 12]. In what follows, we will heavily depend on the use of so-called ideals, palying the role of quantified counterparts of filters.

Definition 2.1 *If X is a set , then $\emptyset \neq \mathfrak{F} \subset [0,\infty]^X$ is called an ideal on X, if it satisfies the following conditions:*

(I1) $\infty \notin \mathfrak{F}$,

(I2) $\forall \varphi \in \mathfrak{F}, \forall \psi \in [0,\infty]^X : \psi \leq \varphi \Rightarrow \psi \in \mathfrak{F}$,

(I3) $\forall \varphi, \psi \in \mathfrak{F} : \varphi \vee \psi \in \mathfrak{F}$,

(I4) $\forall \psi \in [0,\infty]^X$:

$$(\forall \varepsilon > 0, \forall K < \infty : \exists \varphi_\varepsilon^K \in \mathfrak{F} : \psi \wedge K \leq \varphi_\varepsilon^K + \varepsilon) \Rightarrow \psi \in \mathfrak{F}.$$

The set of all ideals on X will be denoted by $\mathbf{I}(X)$.

We want to remark that condition (I4) is simply a saturation condition applying to subsets of $[0,\infty]^X$ which appears frequently in approach theory and for all $\mathfrak{G} \subset [0,\infty]^X$, we will write $\langle\mathfrak{G}\rangle$ for its saturation with respect to (I4).

Let X be a set and $\mathfrak{F} \in \mathbf{I}(X)$. In the sequel, the number

$$c(\mathfrak{F}) \doteq \sup_{\varphi\in\mathfrak{F}} \inf_{x\in X} \varphi(x)$$

will be called the level of $\mathfrak{F}$ and it is an easy consequence of (I1) and (I4) that every ideal has a finite level. The ideal $\mathfrak{F}$ will be called prime if satisfies the following supplementary condition

$$\text{(PI)}\ \forall\varphi,\psi \in [0,\infty]^X : \varphi \wedge \psi \in \mathfrak{F} \Rightarrow (\varphi \in \mathfrak{F} \text{ or } \psi \in \mathfrak{F}).$$

We will use the notation $\mathbf{P}(X)$ for the set of all prime ideals on X and we will also write

$$\mathbf{P}(\mathfrak{F}) \doteq \{\mathfrak{G} \in \mathbf{P}(X) \mid \mathfrak{G} \supset \mathfrak{F}\}$$

and

$$\mathbf{P}_{\mathrm{m}}(\mathfrak{F}) \doteq \{\mathfrak{G} \in \mathbf{P}(\mathfrak{F}) \mid \mathfrak{F} \text{ minimal w.r.t. } \subset\}.$$

Concerning (symmetric) monoidal (closed) structures, or more shortly (S)M(C) structures, as introduced by S. Eilenberg and M. Kelley in [6], we refer to F. Borceux [2], or S. Mac Lane [20], [19] for any further terminology, information and notations. We only recall that it follows from a general categorical result which can be found in J. Činčura [5], and A. Logar and F. Rossi [9] that, without loss of generality, we can assume SMC structures on our familiar topological constructs to be in some standard form, where the tensorproduct of two objects always has the cartesian product of the underlying sets as its underlying set, the adjoint inner hom-functor indeed is a structured hom-functor and the corresponding adjunction between the tensorproduct and the inner hom-fuctor is precisely the standard one, used in the cartesian closed case.

3 The Situation in PRAP

In the context of pre-topological spaces, the well-known inductive tensorproduct usually is given in terms of neighbourhood systems or through its (non-idempotent) closure operator. Also in the setting of topological constructs (see M. Wischnewski [23]) or in the even more general framework of initially structured categories (see J. Činčura [5]), an inductive tensorproduct has been defined and studied. We first define a symmetric monoidal structure on **PRAP** in terms of the local distances and then prove it to coincide with the inductive tensorproduct from [23] and [5].

Definition 3.1 *Let $X, Y \in |\mathbf{PRAP}|$. For every $x \in X, y \in Y, \varphi \in \mathcal{A}_X(x)$ and $\psi \in \mathcal{A}_Y(y)$ we write*

$$\Gamma_{xy}(\varphi, \psi) \doteqdot ((\theta_{\{x\}} \circ \mathrm{pr}_X) \vee (\psi \circ \mathrm{pr}_Y)) \wedge ((\varphi \circ \mathrm{pr}_X) \vee (\theta_{\{y\}} \circ \mathrm{pr}_Y)).$$

Then

$$(\{\, \Gamma_{xy}(\varphi, \psi) \mid \varphi \in \mathcal{A}_X(x), \psi \in \mathcal{A}_Y(y) \,\})_{(x,y) \in X \times Y}$$

is a base for a pre-approach system on $X \times Y$, denoted $\mathcal{A}_X \otimes \mathcal{A}_Y$. The pre-approach space

$$X \otimes Y \doteqdot (X \times Y, \mathcal{A}_X \otimes \mathcal{A}_Y)$$

is called the inductive tensorproduct of X and Y.

To see its relation to the inductive tensorproduct, we need the next lemma the obvious verification of which we omit.

Lemma 3.2 *For every $X, Y, Z \in |\mathbf{PRAP}|$ and for every function $f : X \times Y \longrightarrow Z$ the folllowing assertions are equivalent:*

(1) *$f : X \otimes Y \longrightarrow Z$ is a contraction,*

(2) *for every $(x, y) \in X \times Y$, both*

$$f(x, \cdot) : Y \longrightarrow Z \text{ and } f(\cdot, y) : X \longrightarrow Z \text{ are contractions.}$$

■

Corollary 3.3 *Let $X, Y \in |\mathbf{PRAP}|$ and put*

$$\mathcal{S}_{XY} \doteqdot \{\, f : X \times Y \longrightarrow Z \mid Z \in |\mathbf{PRAP}|, \forall (x, y) \in X \times Y : f(x, \cdot) \text{ and } f(\cdot, y) \text{ contractions} \,\},$$

then

$$(f : X \otimes Y \longrightarrow Z)_{f \in \mathcal{S}_{XY}}$$

*is initial in **PRAP**.* ■

This shows that the symmetric monoidal structure $\otimes$ defined here coincides with the canonical inductive tensorproduct defined and investigated in (J. Činčura [5]) in the setting of initially structured categories, whence is a tensorproduct, hereby justifying our terminology and the definitions below.

If $X, X', Y, Y' \in |\mathbf{PRAP}|$ and $f : X \longrightarrow X', g : Y \longrightarrow Y'$ are contractions, we put $f \otimes g : X \otimes Y \longrightarrow X' \otimes Y' : (x, y) \mapsto (f(x), g(y))$. Then $f \otimes g$ is a contraction and together with definition 3.1, this defines a bifunctor $- \otimes - : \mathbf{PRAP} \times \mathbf{PRAP} \longrightarrow \mathbf{PRAP}$, making $\mathbf{PRAP}$ into a symmetric monoidal category together with the standard natural isomorphisms and unit.

For all $X, Y \in |\mathbf{PRAP}|$, we define $[X, Y]$ to be the $\mathbf{PRAP}$ object with underlying set $\mathbf{PRAP}(X, Y)$, equipped with the pre-approach structure of pointwise convergence (i.e. the pre-approach structure $\mathbf{PRAP}(X, Y)$ inherits as a subspace of $\prod_{x \in X} Y$.) If $X, X', Y, Y' \in |\mathbf{PRAP}|$ and $f : X' \longrightarrow X, g : Y \longrightarrow Y'$ are contractions, then $[f, g] : [X, Y] \longrightarrow [X', Y'] : h \mapsto g \circ h \circ f$ is a contraction. This defines a bifunctor $[-, -] : \mathbf{PRAP}^{\mathrm{op}} \times \mathbf{PRAP} \longrightarrow \mathbf{PRAP}$.

Then it follows from categorical results proved in [5], that $(- \otimes -, [-, -])$ is an SMC-structure on $\mathbf{PRAP}$. Just as in the pre-topological case it turns out very practical to have a reformulation of the inductive tensorproduct in terms of closure operators, the next result which describes the inductive tensorproduct in $\mathbf{PRAP}$ by means of pre-hulls will prove to be usefull in the sequel.

Proposition 3.4 *For all $X, Y \in |\mathbf{PRAP}|$ and all $\gamma \in [0, \infty]^{X \times Y}$, we have that*

$$h_{X \otimes Y}(\gamma) = \left(\bigwedge_{x \in X} ((\theta_{\{x\}} \circ \mathrm{pr}_X) \vee (h_Y(\gamma(x, \cdot)) \circ \mathrm{pr}_Y)) \right) \wedge \left(\bigwedge_{y \in Y} ((h_X(\gamma(\cdot, y)) \circ \mathrm{pr}_X) \vee (\theta_{\{y\}} \circ \mathrm{pr}_Y)) \right)$$

Proof : Fix $\gamma \in [0,\infty]^{X\times Y}$, put

$$h(\gamma) \doteqdot \left(\bigwedge_{x\in X}((\theta_{\{x\}}\circ \mathrm{pr}_X)\vee(h_Y(\gamma(x,\cdot))\circ \mathrm{pr}_Y))\right)\wedge \left(\bigwedge_{y\in Y}((h_X(\gamma(\cdot,y))\circ\ \mathrm{pr}_X)\vee(\theta_{\{y\}}\circ\ \mathrm{pr}_Y))\right)$$

and pick $(s,t)\in X\times Y$. Then $h(\gamma)(s,t)=h_Y(\gamma(s,\cdot))(t)\wedge h_X(\gamma(\cdot,t))(s)$. On the other hand, we know that

$$h_{X\otimes Y}(\gamma)(s,t)=\sup_{\varphi\in\mathcal{A}_X(s)}\ \sup_{\psi\in\mathcal{A}_Y(t)}\ \inf_{(x,y)\in X\times Y}(\gamma(x,y)+\Gamma_{st}(\varphi,\psi)(x,y)).$$

We start by proving the inequality $h_{X\otimes Y}(\gamma)(s,t)\le h(\gamma)(s,t)$. If $h(\gamma)(s,t)=\infty$, we are done, so assume that $h(\gamma)(s,t)<\alpha$ with $\alpha\in[0,\infty[$, whence $h_X(\gamma(\cdot,t))(s)<\alpha$ or $h_Y(\gamma(s,\cdot))(t)<\alpha$. Assume, without loss of generality, that

$$h_X(\gamma(\cdot,t))(s)=\sup_{\varphi\in\mathcal{A}_X(s)}\ \inf_{x\in X}(\gamma(x,t)+\varphi(x))<\alpha.$$

This implies that for each $\varphi\in\mathcal{A}_X(s)$, we can find $x_\varphi\in X$ with $\gamma(x_\varphi,t)+\varphi(x_\varphi)<\alpha$, yielding that

$$\begin{aligned}&h_{X\otimes Y}(\gamma)(s,t)\\ &\le \sup_{\varphi\in\mathcal{A}_X(s)}\ \sup_{\psi\in\mathcal{A}_Y(t)}(\gamma(x_\varphi,t)+\Gamma_{st}(\varphi,\psi)(x_\varphi,t))\\ &\le \sup_{\varphi\in\mathcal{A}_X(s)}(\gamma(x_\varphi,t)+\varphi(x_\varphi))\le\alpha,\end{aligned}$$

so by arbitrariness of α, this part is completed. To show the converse inequality, note that there is nothing to do in case that $h(\gamma)(s,t)=0$, so without loss of generality, we can assume that

$$\alpha<h(\gamma)(s,t)=h_X(\gamma(\cdot,t))(s)\wedge h_Y(\gamma(s,\cdot))(t)$$

with $\alpha\in[0,\infty[$. By using the transition formula (pre-approach system $\rightarrow$ pre-hull) we find that there exist $\varphi_0\in\mathcal{A}_X(s)$ and $\psi_0\in\mathcal{A}_Y(t)$ with

$\inf_{x\in X}(\gamma(x,t)+\varphi_0(x)) > \alpha$ and $\inf_{y\in Y}(\gamma(s,y)+\psi_0(y)) > \alpha$. Therefore

$$\begin{aligned}
&h_{X\otimes Y}(\gamma)(s,t)\\
&\geq \inf_{(x,y)\in X\times Y}(\gamma(x,y)+\Gamma_{st}(\varphi_0,\psi_0)(x,y))\\
&= \left(\inf_{x\in X}(\gamma(x,t)+\varphi_0(x))\right) \wedge \left(\inf_{y\in Y}(\gamma(s,y)+\psi_0(y))\right) > \alpha,
\end{aligned}$$

so again, arbitrariness of α completes the proof. ∎

Practically, this means that if $X, Y \in |\mathbf{PRAP}|, \gamma \in [0,\infty]^{X\times Y}$ and $(x,y) \in X \times Y$, then

$$h_{X\otimes Y}(\gamma)(x,y) = h_X(\gamma(\cdot,y))(x) \wedge h_Y(\gamma(x,\cdot))(y).$$

If X is a set then $\mathfrak{F} \in \mathbf{I}(X)$ is called principal if

$$\mathfrak{F} = \{\varphi \in [0,\infty]^X | \varphi \leq \gamma\}$$

for some $\gamma \in [0,\infty]^X \setminus \{\infty\}$. We write

$$\mathbf{P}_{\mathbf{np}}(X) \doteqdot \{\mathfrak{F} \in \mathbf{P}(X) \,|\, \mathfrak{F} \text{ non-principal}\}.$$

If $\alpha \in [0,\infty[$ and $x \in X$, we put

$$\mathfrak{F}_x^\alpha \doteqdot \{\varphi \in [0,\infty]^X \,|\, \varphi(x) \leq \alpha\}$$

and $\mathfrak{F}_x \doteqdot \mathfrak{F}_x^0$. Then we have that

$$\mathbf{P}(X) \setminus \mathbf{P}_{\mathbf{np}}(X) = \{\mathfrak{F}_x^\alpha \,|\, x \in X, \alpha \in \mathbb{R}^+\}.$$

A key argument used in (J. Činčura [5]) to reduce the problem of classifying all SMC structures on **PRTOP** up to natural isomorphism, is that tensorproducts (in the standard form) are completely determined by their action on a finally dense class of pre-topological spaces. A suitable candidate in the pre-topological case, used in [5], is the class of all non-principal ultraspaces. We briefly recall that for an infinite set X, a point $\infty_X \notin X$ and a non-principal ultrafilter $\mathcal{U}$ on X, the *non-principal ultraspace* $X^{\mathcal{U}} \doteqdot (X, \mathcal{U}, \infty_X)$ is defined to be the topological space having $X \cup \{\infty_X\}$ as underlying set and where the neighbourhood system is given by

$$\mathcal{V}_{\mathcal{U}}(x) \doteq \begin{cases} \dot{x} & x \in X \\ \{U \cup \{\infty_X\} \mid U \in \mathcal{U}\} & x = \infty_X. \end{cases}$$

For every $x \in X$, $\dot{x}$ is the fixed ultrafilter at x on $X \cup \{\infty_X\}$. Our next step will be defining a suitable finally dense class in **PRAP** so that we can invoke the same reduction argument, in our approach setting. It turns out that what we will call *non-principal prime ideal spaces* constitute a good candidate.

Definition 3.5 *Let X be an infinite set, $\mathfrak{F} \in \mathbf{P}_{\mathrm{np}}(X)$ and $\infty_X \notin X$. For every $\varphi \in [0,\infty]^X$, we let $\varphi^* \in [0,\infty]^{X\cup\{\infty_X\}}$ be given by $\varphi^*(x) \doteq \varphi(x)$ for $x \in X$ and $\varphi^*(\infty_X) \doteq 0$. We also set*

$$\mathcal{A}_{\mathfrak{F}}(x) \doteq \begin{cases} \{\gamma \in [0,\infty]^{X\cup\{\infty_X\}} \mid \gamma(x) = 0\} & x \in X \\ \{\varphi^* \mid \varphi \in \mathfrak{F}\} & x = \infty_X. \end{cases}$$

Then

$$X^{\mathfrak{F}} \doteq (X \cup \{\infty_X\}, (\mathcal{A}_{\mathfrak{F}}(x))_{x \in X\cup\{\infty_X\}}) \in |\mathbf{AP}|$$

is called the non-principal prime ideal space (nppi-space) determined by $\mathfrak{F}$. The full subcategory of **PRAP** *formed by all nppi-spaces is denoted by* **NPPI**. *(Note that $\delta_{X^{\mathfrak{F}}}(\infty_X, X) = c(\mathfrak{F})$)*

Proof : We have to verify that $X^{\mathfrak{F}} \in |\mathbf{AP}|$. It is easy to see that for all $x \in X \cup \{\infty_X\}$, $\mathcal{A}_{\mathfrak{F}}(x) \in \mathbf{I}(X \cup \{\infty_X\})$ with $\mathcal{A}_{\mathfrak{F}}(x) \subset \mathfrak{F}_x$, so only the triangular axiom (A2) needs to be verified. Let $x \in X, \varphi \in \mathcal{A}_{\mathfrak{F}}(x)$ and $K < \infty$. If we now pick $\varphi_0 \in \mathfrak{F}$ and define $\varphi_y \doteq \theta_{\{y\}}$ for all $y \in X$ and $\varphi_{\infty_X} \doteq \varphi_0^*$, then $\varphi_y \in \mathcal{A}_{\mathfrak{F}}(y)$ for all $y \in X \cup \{\infty_X\}$ and it is clear that

$$\forall y, z \in X \cup \{\infty_X\} : \varphi(y) \wedge K \leq \varphi_x(z) + \varphi_z(y). \qquad (1)$$

If on the other hand $\varphi \in \mathfrak{F}$ and $K < \infty$ and if we define $\varphi_y \doteq \theta_{\{y\}}$ for all $y \in X$ and $\varphi_{\infty_X} \doteq \varphi^*$, then obviously $\varphi_y \in \mathcal{A}_{\mathfrak{F}}(y)$ for all $y \in X \cup \{\infty_X\}$ and (1) holds again and we are done. ∎

From now on we use the characterization of **PRAP** by means of ideal structures by R. Lowen, D. Vaughan and C. Verbeeck ([18]), where an ideal structure on a set X is a map

$$I : X \longrightarrow 2^{\mathbf{I}(X)}$$

subject to some axioms formulated in [18]. Again for a pre-approach space X, we write I_X for its ideal structure. We remind that if $X \in |\mathbf{PRAP}|$, the corresponding pre-approach system is determined by $\mathcal{A}_X(x) = \bigcap I_X(x)$, for each $x \in X$. If $Y \subset X$ and $\mathfrak{G} \subset [0,\infty]^X$, then

$$\mathfrak{G}|_Y \doteqdot \{\varphi|_Y \mid \varphi \in \mathfrak{G}\}.$$

For a filter $\mathcal{F}$ on X, we put $\omega(\mathcal{F}) \doteqdot \langle\{\theta_F \mid F \in \mathcal{F}\}\rangle$ and if $\mathfrak{F} \in \mathbf{I}(X)$, we let

$$\iota(\mathfrak{F}) \doteqdot \{\{\gamma < \infty\} \mid \gamma \in \mathfrak{F}\}.$$

We also recall the following results from [10] and [13]: 3.9 describes the structure of minimal prime ideals containing a given ideal, playing the role in convergence-approach theory of ultrafilters containing a given filter in convergence, while 3.10 helps us passing to such minimal prime ideals, maintaining at the same time control over the levels.

Proposition 3.6 *([18]) If X is a set, $\mathfrak{F} \in \mathbf{I}(X)$ and $\mathcal{U}$ is an ultrafilter on X for which $\mathcal{U} \supset \iota(\mathfrak{F})$ and*

$$\sup_{\gamma\in\mathfrak{F}} \sup_{U\in\mathcal{U}} \inf_{x\in U} \gamma(x) < \infty,$$

then

$$\mathfrak{F} \vee \omega(\mathcal{U}) \doteqdot \langle\{\gamma \vee \theta_U \mid \gamma \in \mathfrak{F}, U \in \mathcal{U}\}\rangle \in \mathbf{P}_{\mathrm{m}}(\mathfrak{F}).$$

■

Proposition 3.7 *(Proved as in [10]) If X is a set and $\mathfrak{F} \in \mathbf{I}(X)$, then there exists $\mathfrak{G} \in \mathbf{P}_{\mathrm{m}}(\mathfrak{F})$ with $c(\mathfrak{G}) = c(\mathfrak{F})$.*

Proof : The proof is mutatis mutandis the same as the proof of theorem 2.3 in [10]. ■

For the description of the formation of final structures in **PRAP** we refer to E. Lowen, R. Lowen and C. Verbeeck [13].

In the proof of the final density of **NPPI** in **PRAP** which will permit an essential reduction of the problem of determining all SMC-structures, we will need some technical lemmas which we will mention without proof.

For the first statement of the next lemma, we extend the definition of the level $c(\mathfrak{G})$ to arbitrary $\mathfrak{G} \subset [0,\infty]^X$. The second statement below generalizes the concept of the product on two filters to the realm of ideals.

Lemma 3.8 *1. If X is a set and $\mathfrak{F} \subset [0,\infty]^X$, then $c(\langle\mathfrak{F}\rangle) = c(\mathfrak{F})$,*

2. If X, Y are sets, $\mathfrak{F} \in \mathbf{I}(X)$ and $\mathfrak{G} \in \mathbf{I}(Y)$, then with

$$\mathfrak{F} \rtimes \mathfrak{G} \doteqdot \{ (\varphi \circ \mathrm{pr}_X) \vee (\psi \circ \mathrm{pr}_Y) \mid \varphi \in \mathfrak{F}, \psi \in \mathfrak{G} \},$$

it follows that $\langle \mathfrak{F} \rtimes \mathfrak{G} \rangle \in \mathbf{I}(X \times Y)$. ■

Next two lemmas are required for showing case 2 of proposition 3.16.

Lemma 3.9 *If X, Y are sets, $f : X \longrightarrow Y$ is a function and $\mathfrak{F} \in \mathbf{I}(X)$, then $\langle f(\mathfrak{F})\rangle \in \mathbf{I}(Y)$ and $c(\langle f(\mathfrak{F})\rangle) \leq c(\mathfrak{F})$.* ■

Lemma 3.10 *Take X a set, $x \in X$, $\alpha \in [0,\infty[$ and $\mathfrak{F} \in \mathbf{I}(X)$ with $\mathfrak{F} \supset \mathfrak{F}_x^\alpha$. Then $\mathfrak{F} = \mathfrak{F}_x^\beta$ for some $\beta \in [\alpha,\infty[$.* ■

The technical construction below indeed shows us the desired final density of **NPPI** in **PRAP**. In fact, it all comes down for a given **PRAP** object, to forming a large enough coproduct of nppi-spaces, describing which non-principal prime ideals 'converge' to which points and then taking a suitable quotient of this coproduct, to end up with an isomorphic copy of the space we started with. We omit the long, technical and tedious proof of this result.

Theorem 3.11 *Let $X \in |\mathbf{PRAP}|$.*

1. *We write*

$$\mathcal{I}_1(X) \doteqdot \{ (x, \mathfrak{S}) \mid x \in X, \mathfrak{S} \in \mathbf{P}_{\mathrm{np}}(X) \cap I_X(x) \}.$$

Then for all $(x, \mathfrak{S}) \in \mathcal{I}_1(X)$, *it follows that*

$$\mathfrak{S}|_{X\setminus\{x\}} \in \mathbf{P}_{\mathrm{np}}(X \setminus \{x\})$$

and we define $X_1^{(x,\mathfrak{S})}$ *to be the nppi-space determined by* $\mathfrak{S}|_{X\setminus\{x\}}$ *and with new point* x.

2. *Put*

$$\mathcal{I}_2(X) \doteqdot \{ (x, y, \alpha) \mid x, y \in X, x \neq y, \alpha \in \mathbb{R}^+, \mathfrak{F}_y^\alpha \in I_X(x) \}.$$

We write

$$\mathfrak{F}_f \doteqdot \omega(\mathcal{F}_f) = \langle \{ \theta_{]k,\omega[} \mid k \in \mathbb{N} \} \rangle,$$

with $\mathcal{F}_f$ *the Frechet filter on* $\mathbb{N}$ *and for every* $k \in \mathbb{N}$, $]k, \omega[\doteqdot \{n \in \mathbb{N} \mid k < n\}$.

For all $(x, y, \alpha) \in \mathcal{I}_2(X)$, *we can pick*

$$\mathfrak{S}_{(x,y,\alpha)} \in \mathbf{P}_{\mathrm{m}}(\langle \mathfrak{F}_y^\alpha|_{X\setminus\{x\}} \rtimes \mathfrak{F}_f \rangle) \cap \mathbf{P}_{\mathrm{np}}((X \setminus \{x\}) \times \mathbb{N})$$

with $c(\mathfrak{S}_{(x,y,\alpha)}) = \alpha$. *We define* $X_2^{(x,y,\alpha)}$ *to be the nppi-space determined by* $\mathfrak{S}_{(x,y,\alpha)}$ *and with new point* (x, ω).

3. *With notations as in 1. and 2.*

$$\{ \mathrm{id}_X : X_1^{(x,\mathfrak{S})} \longrightarrow X \mid (x, \mathfrak{S}) \in \mathcal{I}_1(X) \} \cup \\ \{ \mathrm{pr}_X : X_2^{(x,y,\alpha)} \longrightarrow X \mid (x, y, \alpha) \in \mathcal{I}_2(X) \}$$

is a final episink in **PRAP**. ■

Corollary 3.12 **PRAP** *is the monocoreflective hull of* **NPPI** *in* **PRAP**.

Proof : This is clear since part 3. of the previous theorem yields that every pre-approach space is a quotient of a coproduct (both formed in **PRAP**) of a set-indexed family of nppi-spaces. ■

The reduction announced higher up now follows by exactly the same categorical arguments used in (J. Činčura [5]) for the pre-topological counterpart.

Corollary 3.13 *If $\Box$ and $\Box'$ are two tensorproducts on* **PRAP**, *then the following assertions are equivalent:*

(1) $\Box = \Box'$,

(2) $\forall\, X, Y \in |\mathbf{NPPI}| : X \Box Y = X \Box' Y$.

Proof : The implication (1) $\Rightarrow$ (2) is clear. If $\Box$ is a tensorproduct on **PRAP**, then $-\Box Z$ has a right adjoint for all $Z \in |\mathbf{PRAP}|$, whence $-\Box Z$ preserves quotients and coproducts in **PRAP** (or equivalently, final epi-sinks in **PRAP**) for each $Z \in |\mathbf{PRAP}|$. The same argument applies for the funtors $-\Box' Z$ with $Z \in |\mathbf{PRAP}|$. The implication (2) $\Rightarrow$ (1) now follows from 3.17 (or 3.16(3)), because, according to our convention, for every pair (f, g) of contractions, $f \Box g$ and $f \Box' g$ have the same underlying function. ■

Proposition 3.14 *If $\Box$ is a tensorproduct on* **PRAP** *it follows that for all $X, Y \in |\mathbf{PRAP}|$ we have*

$$X \times Y \leq X \Box Y \leq X \otimes Y,$$

i.e. that $\mathrm{id}_{X \times Y} : X \Box Y \longrightarrow X \times Y$ *and* $\mathrm{id}_{X \times Y} : X \otimes Y \longrightarrow X \Box Y$ *are contractions.*

Proof : This is proved in exactly the same way as 2.2 in (J. Činčura [5]). ■

Further on, the next criterion (the straightforward proof of which we omit) will be used to decide that certain functions are **PRAP**-quotients, i.e. **PRAP**-final surjections.

Lemma 3.15 *If $X, Y \in |\mathbf{PRAP}|$ and $f : X \longrightarrow Y$ is a function, the following assertions are equivalent:*

(1) *f is a quotient in* **PRAP**,

(2) *f is onto and* $\forall \psi \in [0, \infty]^Y : h_Y(\psi) = f(h_X(\psi \circ f))$. ■

We now prove some further parallels between the behaviour of the non-principal ultraspaces in the pre-topological case and the nppi-spaces in our case, showing that for any two nppi-spaces, the pre-hulls corresponding to their cartesian product, resp. inductive tensorproduct can only differ in one point, namely the pair of the two added points.

Proposition 3.16 *If X, Y are infinite sets, $\infty_X \notin X, \infty_Y \notin Y$, $\mathfrak{F} \in \mathbf{P}_{\mathrm{np}}(X)$, $\mathfrak{G} \in \mathbf{P}_{\mathrm{np}}(Y)$, $\gamma \in [0,\infty]^{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}$, then*

$$\forall (x,y) \in (X^{\mathfrak{F}} \times Y^{\mathfrak{G}}) \setminus \{(\infty_X, \infty_Y)\} : h_{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}(\gamma)(x,y) = h_{X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}}}(\gamma)(x,y).$$

Proof : Fix $(x,y) \in (X^{\mathfrak{F}} \times Y^{\mathfrak{G}}) \setminus \{(\infty_X, \infty_Y)\}$. Then according to 3.14 we have

$$h_{X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}}}(\gamma)(x,y) \geq h_{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}(\gamma)(x,y),$$

so we only need to verify the converse inequality. If $h_{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}(\gamma)(x,y) = \infty$, we are done, so assume without loss of generality that $h_{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}(\gamma)(x,y) < \alpha$ with $\alpha \in [0,\infty[$ (1). If $x \neq \infty_X$ and $y \neq \infty_Y$, then $(\theta_{\{x\}} \circ \mathrm{pr}_X) \vee (\theta_{\{y\}} \circ \mathrm{pr}_Y) \in \mathcal{A}_{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}((x,y))$, and therefore (1) implies that

$$h_{X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}}}(\gamma)(x,y) \leq \gamma(x,y) = \inf_{(s,t) \in X^{\mathfrak{F}} \times Y^{\mathfrak{G}}} (\gamma(s,t) + \theta_{\{x\}}(s) \vee \theta_{\{y\}}(t)) < \alpha,$$

so by arbitrariness of α, we are done in this case. Finally, we consider the case where $x \neq \infty_X$ and $y = \infty_Y$ (because the case that $x = \infty_X$ and $y \neq \infty_Y$ is treated in the same way.) Now take $\varphi \in [0,\infty]^{X^{\mathfrak{F}}}$ with $\varphi(x) = 0$ and $\psi \in \mathfrak{G}$. Then $(\theta_{\{x\}} \circ \mathrm{pr}_X) \vee (\psi^* \circ \mathrm{pr}_Y) \in \mathcal{A}_{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}((x,y))$, whence it follows from (1) that

$$\inf_{(s,t) \in X^{\mathfrak{F}} \times Y^{\mathfrak{G}}} (\gamma + \Gamma_{xy}(\varphi, \psi^*))(s,t) \leq \inf_{(s,t) \in X^{\mathfrak{F}} \times Y^{\mathfrak{G}}} (\gamma(s,t) + \theta_{\{x\}}(s) \vee \psi^*(t)) < \alpha.$$

This proves that $h_{X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}}}(\gamma)(x,y) \leq \alpha$ and again, arbitrariness of α concludes the proof. ■

Corollary 3.17 *Let $\Box$ be a tensorproduct on* **PRAP**, *X, Y infinite sets, $\infty_X \notin X$, $\infty_Y \notin Y$, $\mathfrak{F} \in \mathbf{P}_{\mathrm{np}}(X)$ and $\mathfrak{G} \in \mathbf{P}_{\mathrm{np}}(Y)$. Then the following assertions are equivalent:*

(1) $X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}} \neq X^{\mathfrak{F}} \Box Y^{\mathfrak{G}}$,
(2) $\exists\, \gamma \in [0,\infty]^{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}$:

$$h_{X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}}}(\gamma)(\infty_X, \infty_Y) > h_{X^{\mathfrak{F}} \Box Y^{\mathfrak{G}}}(\gamma)(\infty_X, \infty_Y)$$

Proof : It follows from 3.14 that (1) is equivalent to the statement

$$\exists\, \gamma \in [0,\infty]^{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}} : \exists\, (s,t) \in X^{\mathfrak{F}} \times Y^{\mathfrak{G}} :$$
$$h_{X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}}}(\gamma)(s,t) > h_{X^{\mathfrak{F}} \Box Y^{\mathfrak{G}}}(\gamma)(s,t) \geq h_{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}(\gamma)(s,t),$$

which according to the previous proposition, is equivalent to (2). ■

Let $\mathbf{2} \doteqdot (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$ be the Sierpinski topological space. We use the same notation for the (pre-)topological pre-approach space corresponding to it. The next major reduction step for the pre-topological problem proved in (J. Činčura [5]) consisted in showing that the equality between an arbitrary tensorproduct and the inductive one on **PRTOP** is completely decided by the equality of the two SMC structures on the pair of objects $(\mathbf{2},\mathbf{2})$. Our next goal is to prove that the equality of an arbitrary tensorproduct $\Box$ on **PRAP** and the inductive tensorproduct $\otimes$ only depends on the equality of $\mathbb{P}\Box\mathbb{P}$ and $\mathbb{P}\otimes\mathbb{P}$. Here $\mathbb{P}$ stands for the approach space with underlying set $[0,\infty]$ and

$$\delta_{\mathbb{P}} : [0,\infty] \times 2^{[0,\infty]} \longrightarrow [0,\infty] : (x,A) \mapsto (x - \sup A) \vee 0,$$

which is to be seen as the approach counterpart of the Sierpinski space, e.g. $\mathbb{P}$ is initially dense in **AP**. More surprising is the fact that in its turn, the equality of $\Box$ and $\otimes$ is again equivalent to the equality of $\mathbf{2}\Box\mathbf{2}$ and $\mathbf{2}\otimes\mathbf{2}$. Note that where, as will follow from some calculations below, $\mathbf{2}\otimes\mathbf{2}$ is topological, this is a priori not known for $\mathbf{2}\Box\mathbf{2}$ at this stage, making the situation more subtle.

Theorem 3.18 *For any tensorproduct $\Box$ on* **PRAP**, *the following assertions are equivalent:*
(1) $\Box = \otimes$
(2) $\mathbb{P}\Box\mathbb{P} = \mathbb{P}\otimes\mathbb{P}$
(3) $\mathbf{2}\Box\mathbf{2} = \mathbf{2}\otimes\mathbf{2}$

Proof : It is obvious that (1) implies (2) and (3) and because the implication from (2) to (1) is proved in exactly the same way as the one from (3) to (1), we only give a proof of the latter. According to 3.18 it is sufficient to prove that $\Box$ and $\otimes$ agree on pairs of nppi-spaces, so take two infinite sets X, Y, points $\infty_X \notin X, \infty_Y \notin Y$ and $\mathfrak{F} \in \mathbf{P}_{\mathrm{np}}(X)$, $\mathfrak{G} \in \mathbf{P}_{\mathrm{np}}(Y)$ arbitrary. Suppose on the contrary that

$$X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}} \neq X^{\mathfrak{F}} \Box Y^{\mathfrak{G}}.$$

According to 3.22, this yields the existence of $\gamma : X^{\mathfrak{F}} \times Y^{\mathfrak{G}} \longrightarrow [0, \infty]$ and $\alpha \in]0, \infty[$ such that

$$h_{X^{\mathfrak{F}} \Box Y^{\mathfrak{G}}}(\gamma)(\infty_X, \infty_Y) < \alpha < h_{X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}}}(\gamma)(\infty_X, \infty_Y).$$

Define the function

$$\tilde{\gamma} : X^{\mathfrak{F}} \times Y^{\mathfrak{G}} \longrightarrow [0, \infty] : (s,t) \mapsto \begin{cases} \infty & \text{if } s = \infty_X \text{ or } t = \infty_Y, \\ \gamma(s,t) & \text{if } (s,t) \in X \times Y. \end{cases}$$

First, we will show that

$$h_{X^{\mathfrak{F}} \Box Y^{\mathfrak{G}}}(\tilde{\gamma})(\infty_X, \infty_Y) < \alpha.$$

Because

$$h_{X^{\mathfrak{F}} \otimes Y^{\mathfrak{G}}}(\gamma)(\infty_X, \infty_Y) = h_{Y^{\mathfrak{G}}}(\gamma(\infty_X, \cdot))(\infty_Y) \wedge h_{X^{\mathfrak{F}}}(\gamma(\cdot, \infty_Y))(\infty_X),$$

it follows that $h_{X^{\mathfrak{F}}}(\gamma(\cdot, \infty_Y))(\infty_X) > \alpha$ and $h_{Y^{\mathfrak{G}}}(\gamma(\infty_X, \cdot))(\infty_Y) > \alpha$. Moreover, we obtain

$$\begin{aligned}
& h_{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}((\theta_{\{\infty_X\}} \circ \mathrm{pr}_{X^{\mathfrak{F}}}) \vee (\gamma(\infty_X, \cdot) \circ \mathrm{pr}_{Y^{\mathfrak{G}}}))(\infty_X, \infty_Y) \\
\geq & h_{X^{\mathfrak{F}} \times Y^{\mathfrak{G}}}(\gamma(\infty_X, \cdot) \circ \mathrm{pr}_{Y^{\mathfrak{G}}}))(\infty_X, \infty_Y) \\
= & \sup_{\varphi \in \mathfrak{F}} \sup_{\psi \in \mathfrak{G}} \inf_{(s,t) \in X^{\mathfrak{F}} \times Y^{\mathfrak{G}}} (\gamma(\infty_X, t) + \varphi^*(s) \vee \psi^*(t)) \\
= & \sup_{\varphi \in \mathfrak{F}} \sup_{\psi \in \mathfrak{G}} \inf_{t \in Y^{\mathfrak{G}}} (\gamma(\infty_X, t) + (\inf_{s \in X^{\mathfrak{F}}} \varphi^*(s)) \vee \psi^*(t)) \\
= & \sup_{\psi \in \mathfrak{G}} \inf_{t \in Y^{\mathfrak{G}}} (\gamma(\infty_X, t) + \psi^*(t)) \\
= & h_{Y^{\mathfrak{G}}}(\gamma(\infty_X, \cdot))(\infty_Y) > \alpha
\end{aligned}$$

and analogously we obtain that

$$h_{X^{\mathfrak{F}}\times Y^{\mathfrak{G}}}((\gamma(\cdot,\infty_Y)\circ \mathrm{pr}_{X^{\mathfrak{F}}})\vee(\theta_{\{\infty_Y\}}\circ \mathrm{pr}_{Y^{\mathfrak{G}}}))(\infty_X,\infty_Y)>\alpha.$$

Because

$$\gamma=\tilde{\gamma}\wedge((\theta_{\{\infty_X\}}\circ \mathrm{pr}_{X^{\mathfrak{F}}})\vee(\gamma(\infty_X,\cdot)\circ \mathrm{pr}_{Y^{\mathfrak{G}}})) \\ \wedge((\gamma(\cdot,\infty_Y)\circ \mathrm{pr}_{X^{\mathfrak{F}}})\vee(\theta_{\{\infty_Y\}}\circ \mathrm{pr}_{Y^{\mathfrak{G}}}))$$

we deduce that

$$h_{X^{\mathfrak{F}}\Box Y^{\mathfrak{G}}}(\tilde{\gamma})(\infty_X,\infty_Y)<\alpha.$$

since pre-hulls preserve finite infima. If $f: X^{\mathfrak{F}}\longrightarrow \mathbf{2}, g: Y^{\mathfrak{G}}\longrightarrow \mathbf{2}$ are given by $f(x)\doteqdot 1$ for $x\in X$, $f(\infty_X)\doteqdot 0$, $g(y)\doteqdot 1$ for $y\in Y$ and $g(\infty_Y)\doteqdot 0$, it is easily seen that f and g are contractions. Using the assumption (3) it then follows that

$$f\Box g: X^{\mathfrak{F}}\Box Y^{\mathfrak{G}}\longrightarrow \mathbf{2}\Box\mathbf{2}=\mathbf{2}\otimes\mathbf{2}:(s,t)\mapsto(f(s),g(t))$$

is a contraction too, so

$$h_{\mathbf{2}\otimes\mathbf{2}}((f\Box g)(\tilde{\gamma}))(0,0)\le(f\Box g)(h_{X^{\mathfrak{F}}\Box Y^{\mathfrak{G}}}(\tilde{\gamma}))(0,0) \\ =h_{X^{\mathfrak{F}}\Box Y^{\mathfrak{G}}}(\tilde{\gamma})(\infty_X,\infty_Y)<\alpha<\infty.$$

On the other hand, $((f\Box g)(\tilde{\gamma}))(0,\cdot)=((f\Box g)(\tilde{\gamma}))(\cdot,0)=\infty$ yielding

$$h_{\mathbf{2}\otimes\mathbf{2}}((f\Box g)(\tilde{\gamma}))(0,0)=h_{\mathbf{2}}(((f\Box g)(\tilde{\gamma}))(\cdot,0))(0) \\ \wedge h_{\mathbf{2}}(((f\Box g)(\tilde{\gamma}))(0,\cdot))(0)=\infty$$

which is a contradiction.

■

As a corollary, we now get that the equality of an arbitrary tensorproduct $\Box$ and the inductive tensorproduct on **PRAP** is in effect measured by the finiteness of one single parameter: the distance

$$\delta_{\mathbf{2}\Box\mathbf{2}}((0,0),\{(1,1)\})$$

which a priori can take values in the continuum $[0, \infty]$. Again note the parallel with the pre-topological case where the equality of a given tensorproduct with the inductive one is equivalent to the fact whether $(0, 0)$ belongs to the closure of $\{(1, 1)\}$ in the (considered) tensorproduct of $\mathbf{2}$ with itself. Where in the pre-topological case, an arbitrary tensorproduct of $\mathbf{2}$ with itself was either equal to the cartesian product or the inductive tensorproduct of these spaces, we now have to investigate whether there exist tensorproducts on **PRAP** corresponding to finite values of a parameter ranging through a continuum.

Corollary 3.19 *Let $\Box$ be a tensorproduct on* **PRAP**. *Then* $\mathbf{2}\times\mathbf{2}, \mathbf{2}\otimes\mathbf{2}$ *and* $\mathbf{2}\Box\mathbf{2}$ *are finite, whence ∞pqs-metric pre-approach spaces, whence determined by the point-point distances. We also have that*

$$\mathbf{2}\times\mathbf{2} \leq \mathbf{2}\Box\mathbf{2} \leq \mathbf{2}\otimes\mathbf{2}$$

and that

$$\forall((a,b),(c,d)) \neq ((0,0),(1,1)) :$$
$$\delta_{\mathbf{2}\times\mathbf{2}}((a,b),\{(c,d)\}) = \delta_{\mathbf{2}\Box\mathbf{2}}((a,b),\{(c,d)\}) = \delta_{\mathbf{2}\otimes\mathbf{2}}((a,b),\{(c,d)\}).$$

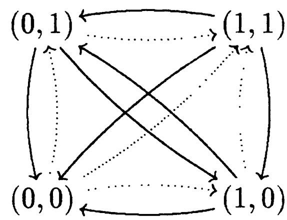
We also have that

$$\delta_{\mathbf{2}\times\mathbf{2}}((0,0),\{(1,1)\}) = 0 \leq \delta_{\mathbf{2}\Box\mathbf{2}}((0,0),\{(1,1)\}) \leq \infty = \delta_{\mathbf{2}\otimes\mathbf{2}}((0,0),\{(1,1)\})$$

So the following assertions are equivalent:
(1) $\Box \neq \otimes$
(2) $\delta_{\mathbf{2}\Box\mathbf{2}}((0,0),\{(1,1)\}) < \infty$.

Proof : $\mathbf{2}\times\mathbf{2}, \mathbf{2}\Box\mathbf{2}$ and $\mathbf{2}\otimes\mathbf{2}$ all have the same finite underlying set $\{0,1\}^2$ and therefore are ∞pqs-metric spaces. Therefore they are completely determined by specifying the point-point distances. Because **TOP** is a concretely bireflective subconstruct of **PRTOP** and **PRTOP** is a concretely bireflective subconstruct of **PRAP**, **TOP** is closed in **PRAP** with respect to forming products and the **PRAP**-product of a set-indexed family of topological spaces is exactly the approach space corresponding to their **TOP**-product. In particular this applies for $\mathbf{2}\times\mathbf{2}$.

The topological product of $\mathbf{2}$ and $\mathbf{2}$ is determined by the action of its closure operator $\bar{\cdot}$ on singletons, given by $\overline{\{(0,0)\}} = \{(0,0)\}, \overline{\{(0,1)\}} = \{(0,1),(0,0)\}, \overline{\{(1,0)\}} = \{(1,0),(0,0)\}, \overline{\{(1,1)\}} = \{0,1\}^2$. Because the distance of a point to itself equals 0, the following diagram exactly describes $\mathbf{2} \times \mathbf{2}$ where for all $(a,b),(c,d) \in \{0,1\}^2$ a dotted (resp. full) arrow from (a,b) to (c,d) means that $\delta_{\mathbf{2}\times\mathbf{2}}((a,b),\{(c,d)\}) = 0$ (resp. ∞):



We know that

$$\mathbf{2} \times \mathbf{2} \leq \mathbf{2}\Box\mathbf{2} \leq \mathbf{2} \otimes \mathbf{2},$$

or equivalently, that

$$\delta_{\mathbf{2}\times\mathbf{2}} \leq \delta_{\mathbf{2}\Box\mathbf{2}} \leq \delta_{\mathbf{2}\otimes\mathbf{2}},$$

whence the full arrows in the previous diagram at once contribute full arrows to the diagram for $\mathbf{2}\otimes\mathbf{2}$ and only the $\mathbf{2}\otimes\mathbf{2}$-distances corresponding to the dotted arrows in the diagram above have to be calculated. To start we e.g have that

$$\begin{aligned}
\delta_{\mathbf{2}\otimes\mathbf{2}}((0,0),\{(0,1)\}) &= h_{\mathbf{2}\otimes\mathbf{2}}(\theta_{\{(0,1)\}})(0,0) \\
&= h_{\mathbf{2}}(\theta_{\{(0,1)\}}(0,\cdot))(0) \wedge h_{\mathbf{2}}(\theta_{\{(0,1)\}}(\cdot,0))(0) \\
&= h_{\mathbf{2}}(\theta_{\{1\}})(0) \wedge h_{\mathbf{2}}(\infty)(0) \\
&= \delta_{\mathbf{2}}(0,\{1\}) \wedge \infty = 0
\end{aligned}$$

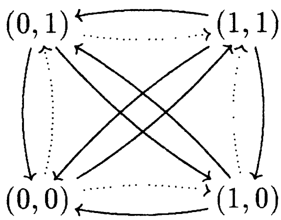
and in the same way we verify that

$$\delta_{\mathbf{2}\otimes\mathbf{2}}((0,0),\{(1,0)\}) = \delta_{\mathbf{2}\otimes\mathbf{2}}((0,1),\{(1,1)\}) = \delta_{\mathbf{2}\otimes\mathbf{2}}((1,0),\{(1,1)\}) = 0.$$

The only difference with the product case is

$$\begin{aligned}
\delta_{\mathbf{2}\otimes\mathbf{2}}((0,0),\{(1,1)\}) &= h_{\mathbf{2}\otimes\mathbf{2}}(\theta_{\{(1,1)\}})(0,0) \\
&= h_{\mathbf{2}}(\theta_{\{(1,1)\}}(0,\cdot))(0) \wedge h_{\mathbf{2}}(\theta_{\{(1,1)\}}(\cdot,0))(0) \\
&= h_{\mathbf{2}}(\infty)(0) \wedge h_{\mathbf{2}}(\infty)(0) \\
&= \infty \wedge \infty = \infty
\end{aligned}$$

All of this is summarized in the next diagram, with the same conventions concerning the arrows as above.



From this the conclusions stated in the corollary and the implication (2) $\Rightarrow$ (1) immediately follow. Now suppose that $\Box \neq \otimes$, which according to the previous theorem, is equivalent to saying that $\mathbf{2}\Box\mathbf{2}$ is strictly coarser than $\mathbf{2} \otimes \mathbf{2}$. Because, as indicated higher up, our considered pre-approach structures are completely determined by their respective point-point distances, this means that there exist $(a,b),(c,d) \in \{0,1\}^2$ with

$$\delta_{\mathbf{2}\times\mathbf{2}}((a,b),\{(c,d)\}) \leq \delta_{\mathbf{2}\Box\mathbf{2}}((a,b),\{(c,d)\}) < \delta_{\mathbf{2}\otimes\mathbf{2}}((a,b),\{(c,d)\})$$

and comparing the two diagrams, this is only possible if $(a,b) = (0,0)$ and $(c,d) = (1,1)$, yielding (2). ■

The next result provides an interpretation of the parameter

$$\delta_{\mathbf{2}\Box\mathbf{2}}((0,0),\{(1,1)\})$$

in relation to the action of the tensorproduct on pairs of nppi-spaces. Again note that this interpretation is a parallel to proposition 2.6 of [5], stating that if there existed a tensorproduct on **PRTOP**, different from the pretopological inductive one, then for all non-principal ultraspaces $X^{\mathcal{U}}, Y^{\mathcal{V}}$, the point (∞_X, ∞_Y) belongs to the closure of $X \times Y$ with respect to the particular tensorproduct of $X^{\mathcal{U}}$ and $Y^{\mathcal{V}}$.

Proposition 3.20 *If $\Box$ is a tensorproduct on* **PRAP** *and $\Box \neq \otimes$, then for all infinite sets X, Y, points $\infty_X \notin X, \infty_Y \notin Y$ and all $\mathfrak{F} \in$* $\mathbf{P}_{\mathrm{np}}(X), \mathfrak{G} \in \mathbf{P}_{\mathrm{np}}(Y)$ *with $c(\mathfrak{F}) = c(\mathfrak{G}) = 0$, we have that*

$$\delta_{X^{\mathfrak{F}}\Box Y^{\mathfrak{G}}}((\infty_X,\infty_Y), X \times Y) = \delta_{\mathbf{2}\Box\mathbf{2}}((0,0),\{(1,1)\}) < \infty.$$

Proof : Assume that $\Box \neq \otimes$, so it follows from the previous corollary that

$$\delta_{\mathbf{2}\Box\mathbf{2}}((0,0),\{(1,1)\}) \in \mathbb{R}^+.$$

Fix $\mathfrak{F} \in \mathbf{P}_{\rm np}(X), \mathfrak{G} \in \mathbf{P}_{\rm np}(Y)$ such that $c(\mathfrak{F}) = c(\mathfrak{G}) = 0$. Now define $f : X^{\mathfrak{F}} \longrightarrow \mathbf{2}$ and $g : Y^{\mathfrak{G}} \longrightarrow \mathbf{2}$ by $f(x) \doteq 1$ for every $x \in X$, $f(\infty_X) \doteq 0, g(y) \doteq 1$ for all $y \in Y$ and $g(\infty_Y) \doteq 0$. Then f and g are **PRAP**-quotients. We verify this for f. To do so we fix $\gamma \in [0,\infty]^{\{0,1\}}$. According to 3.20 . it now suffices to check that

$$h_{\mathbf{2}}(\gamma) = f(h_{X^{\mathfrak{F}}}(\gamma \circ f))$$

because f is onto. We distinguish between three cases. If γ is constant, so is $\gamma \circ f$ and we are done since f is onto. Next, suppose that $\alpha \doteq \gamma(0) > \gamma(1) =: \beta$. On the one hand, clearly

$$h_{\mathbf{2}}(\gamma)(0) = \inf_{t\in\{0,1\}}(\gamma(t) + 0) = \beta$$

and

$$h_{\mathbf{2}}(\gamma)(1) = (\inf_{t\in\{0,1\}}(\gamma(t) + 0)) \vee (\inf_{t\in\{0,1\}}(\gamma(t) + \theta_{\{1\}}(t))) = \beta \vee \beta = \beta.$$

On the other hand we have

$$\begin{aligned} f(h_{X^{\mathfrak{F}}}(\gamma \circ f))(0) &= h_{X^{\mathfrak{F}}}(\gamma \circ f)(\infty_X) \\ &= \sup_{\psi\in\mathfrak{F}} \inf_{z\in X\cup\{\infty_X\}}((\gamma \circ f)(z) + \psi^*(z)) \\ &= (\sup_{\psi\in\mathfrak{F}} \inf_{z\in X}(\psi(z) + \beta)) \wedge \alpha \\ &= (c(\mathfrak{F}) + \beta) \wedge \alpha = \beta \end{aligned}$$

and

$$\begin{aligned} f(h_{X^{\mathfrak{F}}}(\gamma \circ f))(1) &= \inf_{x\in X} h_{X^{\mathfrak{F}}}(\gamma \circ f)(x) \\ &= \inf_{x\in X} \inf_{z\in X\cup\{\infty_X\}}((\gamma \circ f)(z) + \theta_{\{x\}}(z)) \\ &= \inf_{x\in X} \gamma(f(x)) = \beta, \end{aligned}$$

showing the desired equality in this case. Finally, we consider the case where $\alpha \doteqdot \gamma(0) < \gamma(1) =: \beta$. In this case,

$$h_2(\gamma)(0) = \inf_{t\in\{0,1\}}(\gamma(t) + 0) = \alpha$$

and

$$h_2(\gamma)(1) = (\inf_{t\in\{0,1\}}(\gamma(t) + 0)) \vee (\inf_{t\in\{0,1\}}(\gamma(t) + \theta_{\{1\}}(t))) = \alpha \vee \beta = \beta.$$

On the other hand we find , just as in the previous case, that

$$f(h_{X^{\mathfrak{F}}}(\gamma \circ f))(0) = \sup_{\psi\in\mathfrak{F}} \inf_{z\in X\cup\{\infty_X\}}((\gamma \circ f)(z) + \psi^*(z)) = \gamma(f(\infty_X)) = \alpha$$

and

$$f(h_{X^{\mathfrak{F}}}(\gamma \circ f))(1) = \inf_{x\in X} \gamma(f(x)) = \beta,$$

concluding the verification. We continue with the main proof. Because $\square$ is a tensorproduct, it follows that

$$f\square g : X^{\mathfrak{F}}\square Y^{\mathfrak{G}} \longrightarrow \mathbf{2}\square\mathbf{2} : (s,t) \mapsto (f(s), g(t))$$

is a **PRAP**-quotient as well, so thanks to 3.20 and the fact that

$$\theta_{\{(1,1)\}} \circ (f\square g) = \theta_{(f\square g)^{-1}(1,1)} = \theta_{X\times Y},$$

we find that

$$\begin{aligned}
\mathbb{R}^+ \ni \delta_{\mathbf{2}\square\mathbf{2}}((0,0),\{(1,1)\}) &= h_{\mathbf{2}\square\mathbf{2}}(\theta_{\{(1,1)\}})(0,0) \\
&= (f\square g)(h_{X^{\mathfrak{F}}\square Y^{\mathfrak{G}}}(\theta_{\{(1,1)\}} \circ (f\square g)))(0,0) \\
&= (f\square g)(\delta_{X^{\mathfrak{F}}\square Y^{\mathfrak{G}}}(\cdot, X\times Y))(0,0) \\
&= \delta_{X^{\mathfrak{F}}\square Y^{\mathfrak{G}}}((\infty_X, \infty_Y), X\times Y))
\end{aligned}$$

and the proof is complete. ■

As might be expected the non-principal ultraspaces playing a fundamental role in the reduction of the question for **PRTOP**, can be obtained as particular cases of nppi-spaces.

Proposition 3.21 *If X is an infinite set, $\infty_X \notin X$ and $\mathcal{U}$ is a non-principal ultrafilter on X, then $\omega(\mathcal{U}) \in \mathbf{P}_{\mathrm{np}}(X)$ and $X^{\omega(\mathcal{U})}$ is the topological approach space associated with the topological non-pricipal ultra-space $X^{\mathcal{U}}$*
$\doteqdot (X, \mathcal{U}, \infty_X)$.

Proof : It is routine to verify that $\omega(\mathcal{U}) \in \mathbf{P}(X)$. If $\omega(\mathcal{U})$ were principal, then we would have $x \in X$ and $\alpha \in \mathbb{R}^+$ with $\omega(\mathcal{U}) = \mathfrak{F}_x^\alpha$. Because $c(\omega(\mathcal{U})) = 0$ and $c(\mathfrak{F}_x^\alpha) = \alpha$, clearly $\alpha = 0$, so $\theta_{\{x\}} \in \omega(\mathcal{U})$. Therefore there would exist $U \in \mathcal{U}$ with $\theta_{\{x\}} \wedge 2 \leq \theta_U + 1$, whence $\{x\} \in \mathcal{U}$ or equivalently $\mathcal{U} = \dot{x}$ which is a contradiction. This proves $\omega(\mathcal{U})$ to be non-principal. The topological space $X^{\mathcal{U}}$ and the approach space $X^{\omega(\mathcal{U})}$ share the same underlying set $X \cup \{\infty_X\}$ by definition. For $x \in X$, the approach system for $X^{\omega(\mathcal{U})}$ at x is given by

$$\begin{aligned}\mathcal{A}_{X^{\omega(\mathcal{U})}}(x) &\doteqdot \{\gamma \in [0,\infty]^{X \cup \{\infty_X\}} \mid \gamma(x) = 0\} \\ &= \langle\{\theta_Y \mid Y \in 2^{X \cup \{\infty_X\}}, x \in Y\}\rangle = \langle\{\theta_V \mid V \in \mathcal{V}_{X^{\mathcal{U}}}(x)\}\rangle.\end{aligned}$$

In ∞_X, the approach system for $X^{\omega(\mathcal{U})}$ is given by

$$\begin{aligned}\mathcal{A}_{X^{\omega(\mathcal{U})}}(\infty_X) \doteqdot \{\varphi^* \mid \varphi \in \omega(\mathcal{U})\} &= \langle\{\theta_{U \cup \{\infty_X\}} \mid U \in \mathcal{U}\}\rangle \\ &= \langle\{\theta_V \mid V \in \mathcal{V}_{X^{\mathcal{U}}}(\infty_X)\}\rangle\end{aligned}$$

where the second equality is clear because for all $U \in \mathcal{U}$, $(\theta_U)^* = \theta_{U \cup \{\infty_X\}}$ and this proves the claim.

■

We now come to the main theorem which has the same general line as the proof of the pre-topological situation established in theorem 2.7 in [5]. Again note the remarkable fact that the spaces which satisfy to provoke a contradiction from the assumption that there would exist a tensorproduct $\Box$ on **PRAP** other than $\otimes$ are precisely the same topological (!) objects used to derive a contradiction in the proof of the pre-topological case, but that the situation here becomes more subtle because we do not know at this stage that the $\Box$-product of two topological spaces is necessarily pre-topological. We therefore need to go through the arguments of the proof of [5]2.7 again, explaining why we can take over certain parts and proving the additional 'approach' steps required.

Theorem 3.22 *If $\Box$ is a tensorproduct on* **PRAP**, *then* $\Box = \otimes$.

Proof : Suppose that $\Box \neq \otimes$. We use the same notations as in the proof of theorem 2.7 of (J. Činčura [5]). Let $(\mathbb{N}_k^*)_{k\in\mathbb{N}}$ be a sequence of pairwise disjoint copies of the topological **PRAP**-objects corresponding to the Alexandroff compactification of the natural numbers equipped with the discrete topology. Just as in [5], we write $\mathbb{N}_k^* \doteqdot \mathbb{N}_k \cup \{\omega_k\}$ and we label the elements of $\mathbb{N}_k$ by the index k. Note that for all $k \in \mathbb{N}$,

$$\mathcal{A}_{\mathbb{N}_k^*}(n_k) = \langle\{\theta_{\{n_k\}}\}\rangle$$

for every $n_k \in \mathbb{N}_k$ and

$$\mathcal{A}_{\mathbb{N}_k^*}(\omega_k) = \langle\{\theta_{\mathbb{N}_k^*\setminus F_k} \mid F_k \in 2^{(\mathbb{N}_k)}\}\rangle = \langle\{\theta_{]n_k,\omega_k]} \mid n_k \in \mathbb{N}_k)\}\rangle.$$

(Here $]n_k,\omega_k] = \{m_k \in \mathbb{N}_k \mid n_k \leq m_k\} \cup \{\omega_k\}$). Because **PRTOP** is a concretely bireflective (resp. concretely bicoreflective) subconstruct of **PRAP**, **PRTOP** is closed in **PRAP** with respect to the formation of initial (resp. final) structures in **PRAP**. Moreover, the initial (resp. final) lift for a structured **PRTOP**-source (resp. sink) in **PRAP** is precisely the pre-topological pre-approach space corresponding to the initial (resp. final) lift of this source (resp. sink) in **PRTOP**. Define

$$\mathcal{N} \doteqdot \coprod_{k\in\mathbb{N}} \mathbb{N}_k^*$$

in **PRAP** (which, as particular case of the previous remark, corresponds to the pre-topological coproduct). Define the equivalence relation $\sim$ on $\mathcal{N}$ by

$$x \sim y \Leftrightarrow (x, y \in \{\omega_k \mid k \in \mathbb{N}\} \vee x = y).$$

Define Ω to be the equivalence class of all $\omega_k, k \in \mathbb{N}$ and let

$$e : \mathcal{N} \longrightarrow \mathcal{N}' : x \mapsto \begin{cases} x & \text{if } x \in \mathcal{N} \setminus \{\omega_k \mid k \in \mathbb{N}\}, \\ \Omega & \text{if } x \in \{\omega_k \mid k \in \mathbb{N}\}. \end{cases}$$

be the **PRAP**-quotient defined by $\sim$, which again by the remark higher up, is the pre-approach object corresponding to the quotient in **PRTOP** which will also be denoted by the same symbol. It was noted in [5] that both $\mathcal{N}$ and $\mathcal{N}'$ are in effect topological, whence e amounts to a **TOP** quotient. For each $k \in \mathbb{N}$, define

$$A_k \doteqdot \bigcup_{j=k}^{\infty} \mathbb{N}_j \subset \mathcal{N}'$$

and let

$$M \doteqdot \bigcup_{k\in\mathbb{N}} (A_k \times \{k\}) \subset \mathcal{N}'\Box\mathbb{N}^*$$

where $\mathbb{N}^* \doteqdot \mathbb{N} \cup \{\omega\}$ is another copy of the topological **PRAP**-object corresponding to the Alexandroff compactification of $\mathbb{N}$ with the discrete topology. Now we claim that

$$\delta_{\mathcal{N}'\Box\mathbb{N}^*}((\Omega,\omega), M) < \infty.$$

In order to verify this, as in [5] put $\mathcal{F}_\Omega \doteqdot \{U \setminus \{\Omega\} \mid U \in \mathcal{V}_{\mathcal{N}'}(\Omega)\}$ and $\mathcal{S} \doteqdot \mathcal{F}_\Omega \cup \{A_k \mid k \in \mathbb{N}\}$. It was shown in [5] that with

$$X \doteqdot \mathcal{N}' \setminus \{\Omega\} = \bigcup_{k\in\mathbb{N}} \mathbb{N}_k,$$

$\mathcal{S} \subset 2^X$ has the finite intersection property and that $\bigcap \mathcal{S} = \emptyset$ so there exists a non-principal ultrafilter $\mathcal{U}$ on X with $\mathcal{U} \supset \mathcal{S}$. As a consequence of 3.26, $\omega(\mathcal{U}) \in \mathbf{P}_{\mathrm{np}}(X)$ and the nppi-space $X^{\omega(\mathcal{U})}$ with new point $\infty_X \doteqdot \Omega$ is the topological pre-approach space associated with the non-principal ultraspace $X^{\mathcal{U}} \doteqdot (X, \mathcal{U}, \Omega)$. In [5] it is proved that $j : X^{\mathcal{U}} \longrightarrow \mathcal{N}' : x \mapsto x$ is continuous, so $j : X^{\omega(\mathcal{U})} \longrightarrow \mathcal{N}' : x \mapsto x$ is a contraction. If $f : X \longrightarrow \mathbb{N}$ is defined by $f(n_k) \doteqdot k$ for $k \in \mathbb{N}, n_k \in \mathbb{N}_k$ it was proved in [5] that $\mathcal{V} \doteqdot \{f(U) \mid U \in \mathcal{U}\}$ is a non-principal ultrafilter on $\mathbb{N}$ and we denote $\mathbb{N}^{\mathcal{V}} \doteqdot (\mathbb{N}, \mathcal{V}, \omega)$. Proposition 3.26 yields that $\omega(\mathcal{V}) \in \mathbf{P}_{\mathrm{np}}(\mathbb{N})$ and that $\mathbb{N}^{\omega(\mathcal{V})}$ (with $\infty_{\mathbb{N}} \doteqdot \omega$) is the pre-approach space corresponding to $\mathbb{N}^{\mathcal{V}}$. Since it was shown in [5] that

$$g : X^{\mathcal{U}} \longrightarrow \mathbb{N}^{\mathcal{V}} : s \mapsto \begin{cases} f(s) & \text{if } s \in X \\ \omega & \text{if } s = \Omega \end{cases}$$

is a quotient in **PRTOP**, we obtain that $g : X^{\omega(\mathcal{U})} \longrightarrow \mathbb{N}^{\omega(\mathcal{V})}$ is a **PRAP**-quotient. Therefore

$$g\Box\mathrm{id}_{\mathbb{N}^{\omega(\mathcal{V})}} : X^{\omega(\mathcal{U})}\Box\mathbb{N}^{\omega(\mathcal{V})} \longrightarrow \mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})} : (s,t) \mapsto (g(s),t)$$

is a contraction and as in [5] we get that

$$L \doteqdot (g\Box\mathrm{id}_{\ \mathbb{N}^{\omega(\mathcal{V})}})(M) = \bigcup_{k\in\mathbb{N}}([k,\omega[\times\{k\})$$

and that $M = (g\Box\mathrm{id}_{\ \mathbb{N}^{\omega(\mathcal{V})}})^{-1}(L)$. As an intermediate claim we show that

$$\delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),L) < \infty.$$

(Suppose on the contrary that

$$\delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),L) = \infty.$$

Note that

$$\delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),\mathbb{N}^2) = \delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),L) \wedge \delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),\mathbb{N}^2 \setminus L).$$

Because $c(\omega(\mathcal{V})) = 0$, it follows from 3.25 that

$$\delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),\mathbb{N}^2) < \infty$$

so

$$\delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),\mathbb{N}^2 \setminus L) < \infty.$$

Because $\Box$ is symmetric,

$$c \doteqdot \nu_{\mathbb{N}^{\omega(\mathcal{V})},\mathbb{N}^{\omega(\mathcal{V})}} : \mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})} \longrightarrow \mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})} : (s,t) \mapsto (t,s)$$

is an isomorphism in **PRAP**. With

$$L' \doteqdot c(L) = \bigcup_{k\in\mathbb{N}}(\{k\} \times [k,\omega[)$$

as in [5], it follows that $\mathbb{N}^2 \setminus L \subset L'$ which yields that, because c is iso,

$$\begin{aligned}\delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),L) &= \delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}(c(\omega,\omega),c(L))\\ &= \delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),L')\\ &\leq \delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),\mathbb{N}^2\setminus L) < \infty,\end{aligned}$$

conflicting with our assumption). Since g is a quotient in and $\Box$ is a tensorproduct on **PRAP**, the map $g\Box\mathrm{id}_{\mathbb{N}^{\omega(\mathcal{V})}}$ is a quotient in **PRAP** as well, so by 3.20 and because

$$\theta_L \circ (g\Box\mathrm{id}_{\mathbb{N}^{\omega(\mathcal{V})}}) = \theta_{(g\Box\mathrm{id}_{\mathbb{N}^{\omega(\mathcal{V})}})^{-1}(L)} = \theta_M$$

it follows that

$$\begin{aligned}\infty &> \delta_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\omega),L)\\ &= h_{\mathbb{N}^{\omega(\mathcal{V})}\Box\mathbb{N}^{\omega(\mathcal{V})}}(\theta_L)(\omega,\omega)\\ &= ((g\Box\mathrm{id}_{\mathbb{N}^{\omega(\mathcal{V})}})(h_{X^{\omega(\mathcal{U})}\Box\mathbb{N}^{\omega(\mathcal{V})}}(\theta_L \circ (g\Box\ \mathrm{id}_{\mathbb{N}^{\omega(\mathcal{V})}}))))(\omega,\omega)\\ &= ((g\Box\mathrm{id}_{\mathbb{N}^{\omega(\mathcal{V})}})(h_{X^{\omega(\mathcal{U})}\Box\mathbb{N}^{\omega(\mathcal{V})}}(\theta_M)))(\omega,\omega)\\ &= ((g\Box\mathrm{id}_{\mathbb{N}^{\omega(\mathcal{V})}})(\delta_{X^{\omega(\mathcal{U})}\Box\mathbb{N}^{\omega(\mathcal{V})}}(\cdot,M))(\omega,\omega)\\ &= \delta_{X^{\omega(\mathcal{U})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\omega,\Omega),M).\end{aligned}$$

Since it is shown in [5] that $h : \mathbb{N}^{\mathcal{V}} \longrightarrow \mathbb{N}^* : t \mapsto t$ is continuous, $h : \mathbb{N}^{\omega(\mathcal{V})} \longrightarrow \mathbb{N}^* : t \mapsto t$ is a contraction, yielding that

$$j\Box h : X^{\omega(\mathcal{U})}\Box\mathbb{N}^{\omega(\mathcal{V})} \longrightarrow \mathcal{N}'\Box\mathbb{N}^* : (s,t) \mapsto (s,t)$$

is a contraction as well, which means that $X^{\omega(\mathcal{U})}\Box\mathbb{N}^{\omega(\mathcal{V})}$ is finer than $\mathcal{N}'\Box\mathbb{N}^*$. Therefore,

$$\delta_{\mathcal{N}'\Box\mathbb{N}^*}((\Omega,\omega),M) \leq \delta_{X^{\omega(\mathcal{U})}\Box\mathbb{N}^{\omega(\mathcal{V})}}((\Omega,\omega),M) < \infty$$

and this completes the proof of the claim. Since e is a **PRAP**-quotient and $\Box$ is a tensorproduct on **PRAP**,

$$e\Box\mathrm{id}_{\mathbb{N}^*} : \mathcal{N}\Box\mathbb{N}^* \longrightarrow \mathcal{N}'\Box\mathbb{N}^*$$

is also **PRAP**-quotient and as noted in [5], $(e\Box\mathrm{id}_{\mathbb{N}^*})^{-1}(M) = M$. Again because $\Box$ is a tensorproduct, $-\Box\mathbb{N}^*$ preserves coproducts, whence

$$\mathcal{N}\Box\mathbb{N}^* = \coprod_{k\in\mathbb{N}}(\mathbb{N}^*_k\Box\mathbb{N}^*)$$

so applying 3.12 yields that

$$\forall k \in \mathbb{N} : \delta_{\mathcal{N}\Box\mathbb{N}^*}((\omega_k, \omega), M) = \delta_{\mathbb{N}_k^*\Box\mathbb{N}^*}((\omega_k, \omega), M \cap (\mathbb{N}_k^* \times \mathbb{N}^*))$$

and using from [5] that for every $k \in \mathbb{N}$, $M \cap (\mathbb{N}_k^* \times \mathbb{N}^*) = \mathbb{N}_k \times \{0, \ldots, k\}$, we get

$$\forall k \in \mathbb{N} : \delta_{\mathcal{N}\Box\mathbb{N}^*}((\omega_k, \omega), M) = \delta_{\mathbb{N}_k^*\Box\mathbb{N}^*}((\omega_k, \omega), \mathbb{N}_k \times \{0, \ldots, k\}).$$

Because $\mathbb{N}^*$ and $\mathbb{N}_k^*$ are both topological pre-approach spaces, so is their **PRAP**-product and because it moreover corresponds to their topological product, the fact, proved in [5], that (ω_k, ω) does not belong to the closure of $\mathbb{N}_k \times \{0, \ldots, k\}$ in this topological product, implies that

$$\delta_{\mathbb{N}_k^*\times\mathbb{N}^*}((\omega_k, \omega), \mathbb{N}_k \times \{0, \ldots, k\}) = \infty$$

for all $k \in \mathbb{N}$ and since $\mathbb{N}_k^* \times \mathbb{N}^* \leq \mathbb{N}_k^*\Box\mathbb{N}^*$, we conclude that

$$\forall k \in \mathbb{N} : \delta_{\mathbb{N}_k^*\Box\mathbb{N}^*}((\omega_k, \omega), \mathbb{N}_k \times \{0, \ldots, k\}) = \infty$$

yielding that

$$\forall k \in \mathbb{N} : \delta_{\mathcal{N}\Box\mathbb{N}^*}((\omega_k, \omega), M) = \infty.$$

Because $e\Box\mathrm{id}_{\mathbb{N}^*}$ is a **PRAP**-quotient, applying 3.20 together with

$$\theta_M \circ (e\Box\mathrm{id}_{\mathbb{N}^*}) = \theta_{(e\Box\mathrm{id}_{\mathbb{N}^*})^{-1}(M)} = \theta_M$$

yields that

$$\begin{aligned}
\delta_{\mathcal{N}'\Box\mathbb{N}^*}((\Omega, \omega), M) &= h_{\mathcal{N}'\Box\mathbb{N}^*}(\theta_M)(\Omega, \omega) \\
&= (e\Box\mathrm{id}_{\mathbb{N}^*})(h_{\mathcal{N}\Box\mathbb{N}^*}(\theta_M \circ (e\Box\mathrm{id}_{\mathbb{N}^*})))(\Omega, \omega) \\
&= (e\Box\mathrm{id}_{\mathbb{N}^*})(\delta_{\mathcal{N}\Box\mathbb{N}^*}(\cdot, M))(\Omega, \omega) \\
&= \inf_{k\in\mathbb{N}} \delta_{\mathcal{N}\Box\mathbb{N}^*}((\omega_k, \omega), M) = \infty,
\end{aligned}$$

contradicting the claim.

■

Taking into account that we indeed could assume all the considered tensorproducts to be in standard form, without loss of generality, this main theorem in fact becomes, stated in a mathematically more complete way:

Theorem 3.23 *(Up to natural equivalence,)* $(-\otimes -, [-,-])$ *is the only SMC structure on* **PRAP**.

Proof : This is a direct consequence of theorem 3.27 which, according to theorem 2.9, should in fact be stated up to natural equivalence and the fact that pairs of adjoint functors are determined up to natural equivalence. ∎

To conclude, we want to note the peculiarity of this result, showing that the numerified supercategory **PRAP** behaves in the same way as its classical underlying construct **PRTOP** concerning the existence of symmetric monoidal closed structures. On the contrary it is known that this similar behaviour between 'classical' topological constructs and their numerical approach counterparts is not there for other categorical problems, like e.g. the existence of simultaneously bireflective and bicoreflective subconstructs: where **TOP** is known only to have trivial such subconstructs (see V. Kannan [8]), it was shown in (H. Herrlich and R. Lowen [7]) that **AP** has infinitely many.

References

[1] Adámek, J. Herrlich, H. and Strecker, G. *Abstract and Concrete Categories*, John Wiley, New York (1990)

[2] Banaschewski, B. and Nelson, E. *Tensorproducts and Bimorphisms*, Canad. Math. Bull., 19(4) (1976), pp. 385-402

[3] Borceux, F. *Handbook of Categorical Algebra I, II, III*, Encyclopedia of Mathematics Vol. 50, 51, 52, Cambridge University Press (1994)

[4] Brock, P and Kent, D. C. *Approach Spaces, Limit Tower Spaces and Probabilistic Metric Spaces*, Appl. Categ. Struct., 5 (1997), pp. 99-110

[5] Činčura, J. *On a Tensorproduct in Initially Structured Categories*, Math. Slovaca, 29(3) (1979), pp. 245-255

[6] Eilenberg, S. and Kelley, G. M. *Closed Categories*, Proc. Conf. on Categorical Algebra, La Jolla 1965, Springer Verlag, New-York (1966), pp. 421-562

[7] Herrlich, H. and Lowen, R. *On Simultaneously Reflective and Coreflective Subconstructs*, to appear in G.C.L. Brümmer Festschrift

[8] Kannan, V. *Reflexive cum Coreflexive Subcategories in Topology*, Math. Ann., 195 (1972), pp. 168-174

[9] Logar, A. and Rossi, F. *Monoidal Closed Structures on Categories with Constant Maps*, J. Australian Math. Soc. (Ser. A), 38 (1985), pp. 175-185

[10] Lowen, E. and Lowen, R. *Characterization of Convergence in Fuzzy Topological Spaces*, Internat. J. Math. Math. Sci., 8(3) (1985), pp. 497-511

[11] Lowen, E. and Lowen, R. *A Quasi-topos Containing* **CONV** *and* **MET** *as Full Subcategories*, Int. J. Math. Math. Sci., 11 (1988), pp. 417-438

[12] Lowen, E. and Lowen, R. *Topological Quasitopos Hulls of Categories Containing Topological and Metric Objects*, Cah. Top. Géom. Diff. Catég., 30 (1989), pp. 213-238

[13] Lowen, E., Lowen, R. and Verbeeck, C. *Exponential Objects in the Construct* **PRAP**, Cah. Top. Géom. Diff. Catég., XXXVIII(4) (1997), pp. 259-276

[14] Lowen-Colebunders, E. and Sonck, G. *Exponential Objects and Cartesian Closedness in the Construct* **PRTOP**, Appl. Categ. Struct., 1 (1993), pp. 345-360

[15] Lowen, R. *Convergence in fuzzy topological spaces*, Gen. Top. Appl., 10 (1979), pp. 147-166.

[16] Lowen, R. *Approach Spaces: a Common Supercategory of* **TOP** *and* **MET**, Math. Nachr. 141 (1989), pp. 183-226

[17] Lowen, R. *Approach Spaces: the Missing Link in the Topology-Uniformity-Metric Triad*, Oxford Mathematical Monographs, Oxford University Press (1997)

[18] Lowen, R., Vaughan, D. and Verbeeck, C. *Convergence Using Generalizations of Filters*, to appear

[19] MacLane, S. *Natural Associativity and Commutativity*, Rice Univ. Studies, 49 (1963), pp. 28-46

[20] MacLane, S. *Categories for the Working Mathematician* , Graduate Texts in Mathematics, Springer Verlag, New-York (1971)

[21] Nel, L. D. *Initially Structured Categories and Cartesian Closedness*, Canad. J. Math., 27 (1975), 1361-1377

[22] Pumplün, D. *Das Tensorprodukt als Universelles Problem*, Math. Ann., 171 (1967), pp. 247-262

[23] Wischnewsky, M. B. *Aspects of Categorical Algebra in Initialstructure Categories*, Cah. Top. Géom. Diff., XV (1974), pp. 419-444

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