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THE CARTESIAN CLOSED HULL OF THE CATEGORY OF APPROACH SPACES

by E. LOWEN-COLEBUNDERS, R. LOWEN and M. NAUWELAERTS

RESUME. Cet article décrit le plus petit élargissement cartésien fermé de la catégorie des espaces d'approximation ('approach space') **AP**, c'est-à-dire l'enveloppe cartésienne fermée de **AP**; il est construit comme une sous-catégorie des espaces de pseudo-approximation que deux des auteurs avait montré être le quasi-topos topologique enveloppe de **AP.**

1. INTRODUCTION.

It needs no argumentation that cartesian closedness is an important and useful property for a category. Unfortunately many categories are not cartesian closed and therefore it is interesting to look for a cartesian closed modification, especially a so-called cartesian closed hull, if it exists. Thus in [3], [11] and [4], Antoine, Machado and Bourdaud constructed the CCT hull of **TOP**, the category of topological spaces (and continuous maps) and in [2], Adámek and Reiterman constructed the CCT hull of **MET**, the category of metric spaces (and nonexpansive maps).

It is the purpose of this paper to construct the cartesian closed topological hull of **AP**, the category of approach spaces (and contractions). We do this by identifying it with a subcategory of **PsAP**, the category of pseudo-approach spaces, which, in [9], was shown to be the quasitopos hull, QTH(**AP**), of **AP**.

Remarkably, whereas the objects of **PsAP** can be described by axioms quite similar to those characterizing the objects of QTH(**TOP**), the situation for CCTH(**AP**) is somewhat different. The supplementary axiom required to characterize those objects of **PsAP** which are in CCTH(**AP**) is inspired not only by the approach of [11] and [4] for the case of **TOP**, but also by [2] for the case of **MET**.

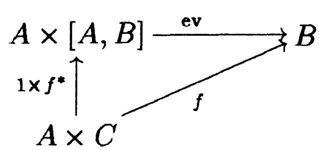
2. PRELIMINARIES.

As we will be talking about CCT categories (or constructs), we first note that a *topological* construct will stand for a concrete category over **Set** which is

a *well-fibred topological c-construct* in the sense of [1], i.e. each structured source has an initial lift, every set carries only a set of structures and each constant map (or empty map) between two objects is a morphism. We will always assume a functor to be concrete (unless this is clearly not the case from its definition) and subcategories (i.e. subconstructs) to be full.
We also recall that a construct $\mathbf{A}$ is *CCT (cartesian closed topological)* if $\mathbf{A}$ is a topological construct which has *canonical function spaces*, i.e. for every pair (A, B) of $\mathbf{A}$-objects the set $\hom(A, B)$ can be supplied with the structure of an $\mathbf{A}$-object, denoted by $[A, B]$, such that

(a) the evaluation map $\mathrm{ev} : A \times [A, B] \longrightarrow B$ is an $\mathbf{A}$-morphism,

(b) for each $\mathbf{A}$-object C and $\mathbf{A}$-morphism $f : A \times C \longrightarrow B$, the map $f^* : C \longrightarrow [A, B]$ defined by $f^*(c)(a) = f(a, c)$ is an $\mathbf{A}$-morphism (f^* is called the *transpose* of f). Note that given $f : A \times C \longrightarrow B$, the transpose $f^* : C \longrightarrow [A, B]$ is the map which makes the following diagram commute:



The CCT hull of a construct $\mathbf{A}$ (shortly denoted by $\mathrm{CCTH}(\mathbf{A})$) (if it exists) is defined as the smallest CCT construct $\mathbf{B}$ in which $\mathbf{A}$ is closed under finite products (see [7]). Also from [7], we recall that given a CCT construct $\mathbf{C}$ in which $\mathbf{A}$ is finally dense (i.e. each $\mathbf{C}$-object is a final lift of some structured sink in $\mathbf{A}$), the CCT hull of $\mathbf{A}$ is the full subconstruct of $\mathbf{C}$ determined by

$$\mathrm{CCTH}(\mathbf{A}) := \{C \in \mathbf{C} \mid \text{there exists an initial source } (f_i : C \longrightarrow [A_i, B_i])_{i \in I} \text{ where } \forall i \in I : A_i, B_i \in \mathbf{A}\}.$$

In short, the CCT hull of $\mathbf{A}$ is the initial hull in $\mathbf{C}$ of the power-objects of $\mathbf{A}$-objects.
A more recent survey of such properties and hull concepts can be found in [6] and [12].

Let us first introduce some notations in order to consider $\mathrm{CCTH}(\mathbf{TOP})$ as an example (which we shall be using later on). Given a set X, $\mathbf{F}(X)$ stands for the set of all filters on X; if $\mathcal{F} \in \mathbf{F}(X)$, then $\mathbf{U}(\mathcal{F})$ stands for the set of all ultrafilters on X finer than $\mathcal{F}$. In particular, $\mathbf{U}(X) := \mathbf{U}(\{X\})$ stands for the set of all ultrafilters on X. Given $A \subset X$, we recall that

$$\operatorname{stack} A := \{B \subset X \mid A \subset B\}$$

and if A consists of a single point a, we also denote $\operatorname{stack} a := \operatorname{stack} A$. If $\mathcal{F}$ is a filter on X, then the sec of $\mathcal{F}$ is defined as

$$\sec(\mathcal{F}) := \{A \subset X \mid \forall F \in \mathcal{F} : F \cap A \neq \emptyset\} = \{A \in \mathcal{U} \mid \mathcal{U} \in \mathbf{U}(\mathcal{F})\}.$$

Now recall that a *pseudotopology (on X)* is a function $q : X \longrightarrow \mathcal{P}(\mathbf{F}(X))$ assigning to each $x \in X$ a set of filters on X, the filters "converging to x", satisfying the properties:

(CT) $\forall x \in X : \operatorname{stack} x \in q(x)$.

(PsT) $\forall x \in X, \forall \mathcal{F} \in \mathbf{F}(X) : \mathbf{U}(\mathcal{F}) \subset q(x) \Rightarrow \mathcal{F} \in q(x)$.

The pair (X, q) is called a *pseudotopological space* and we also write $\mathcal{F} \longrightarrow x$ (or even $\mathcal{F} \xrightarrow{q} x$ or $\mathcal{F} \xrightarrow{X} x$) instead of $\mathcal{F} \in q(x)$. A map $f : (X, q_X) \longrightarrow (Y, q_Y)$ between pseudotopological spaces is called *continuous* whenever

$$\forall x \in X, \forall \mathcal{F} \in \mathbf{F}(X) : \mathcal{F} \xrightarrow{q_X} x \Rightarrow f(\mathcal{F}) \xrightarrow{q_Y} f(x).$$

The construct of pseudotopological spaces and continuous maps is denoted by **PsTOP**, which is a topological quasitopos in which **TOP** can be finally dense embedded (hence also bireflectively) by associating to each topological space $(X, \mathcal{T})$ the pseudotopological space $(X, q_{\mathcal{T}})$ such that $\mathcal{F} \xrightarrow{q_{\mathcal{T}}} x$ if and only if $\mathcal{F} \longrightarrow x$ (in $(X, \mathcal{T})$) (i.e. $\mathcal{V}(x) \subset \mathcal{F}$). It therefore follows from the general theory previously recalled that the CCT hull of **TOP** can be found within **PsTOP**, the concrete description of which requires the following concepts.

Let (X, q) be a pseudotopological space. We denote its **TOP**-bireflection by $(X, \bar{q})$ and define the *point-operator* (with respect to (X, q)) as

$$^{\bullet} : 2^X \longrightarrow 2^X : A \mapsto A^{\bullet} := \{x \in X \mid \operatorname{cl}_{\bar{q}}(\{x\}) \cap A \neq \emptyset\}.$$

A pseudotopological space X is called an *Antoine space* or *epi-topological space* if and only if it satisfies the following conditions (where $\mathcal{F}^{\bullet}$ is the filter generated by $\{F^{\bullet} \mid F \in \mathcal{F}\}$ and $\lim \mathcal{F} := \{x \in X \mid \mathcal{F} \longrightarrow x\}$):

(1) $\forall \mathcal{F} \in \mathbf{F}(X) : \lim \mathcal{F}$ is closed in $(X, \bar{q})$ (closed-domainedness),

(2) $\forall \mathcal{F} \in \mathbf{F}(X) : \lim \mathcal{F} = \lim \mathcal{F}^{\bullet}$ (point-regularity).

The full subconstruct of **PsTOP** consisting of Antoine spaces is denoted by **EpiTOP** and it was shown by work of A. Machado ([11]) and G. Bourdaud ([4]) that **EpiTOP** = CCTH(**TOP**).

Next, we turn to recalling some necessities regarding approach spaces.
An approach space is a pair (X, δ) where X is a set and where $\delta : X \times 2^X \longrightarrow [0, \infty]$ is a map, called a *distance*, which has the properties:

(D1) $\forall x \in X : \delta(x, \{x\}) = 0$.

(D2) $\forall x \in X : \delta(x, \emptyset) = \infty$.

(D3) $\forall x \in X, \forall A, B \subset X : \delta(x, A \cup B) = \delta(x, A) \wedge \delta(x, B)$.

(D4) $\forall x \in X, \forall A \subset X, \forall \epsilon \in [0, \infty] : \delta(x, A) \leq \delta(x, A^{(\epsilon)}) + \epsilon$ where

$$A^{(\epsilon)} := \{y \in X \mid \delta(y, A) \leq \epsilon\}.$$

Given approach spaces (X, δ_X) and (Y, δ_Y), a map $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is called a *contraction* if it fulfills the property that

$$\forall x \in X, \forall A \subset X : \delta_Y(f(x), f(A)) \leq \delta_X(x, A).$$

The category **AP** of approach spaces and contractions is a topological construct and is extensively studied and described in [10].

In [8]-[9], it is shown that **AP** is a subconstruct of **PsAP**, the category of pseudo-approach spaces, the objects of which are described by means of a concept of limits.

A map $\lambda : \mathbf{F}(X) \longrightarrow [0, \infty]^X$ is called a *pseudo-approach limit (on X)* if it fulfills the properties:

(CAL1) $\forall x \in X : \lambda(\text{stack}\, x)(x) = 0$.

(PsAL) $\forall \mathcal{F} \in \mathbf{F}(X) : \lambda(\mathcal{F}) = \sup_{\mathcal{U} \in \mathbf{U}(X)} \lambda(\mathcal{U})$.

Note that by (PsAL) a pseudo-approach limit is completely determined by its restriction to ultrafilters.

The pair (X, λ) is called a *pseudo-approach space*. Given pseudo-approach spaces (X, λ_X) and (Y, λ_Y), a map $f : (X, \lambda_X) \longrightarrow (Y, \lambda_Y)$ is called a *contraction* if it fulfills the property that

$$\forall \mathcal{F} \in \mathbf{F}(X) : \lambda_Y(f(\mathcal{F})) \circ f \leq \lambda_X(\mathcal{F}).$$

The category of pseudo-approach spaces and contractions is a topological construct, denoted **PsAP**. For more information on this category we refer the reader to [9], we only recall those properties which are required in the sequel.

A pseudo-approach limit $\lambda : \mathbf{F}(X) \longrightarrow [0,\infty]^X$ is called an *approach limit* if it satisfies some additional properties, such as a so-called diagonal property (see e.g. [10]), which however we shall not require to go into here, and which ensures it to be equivalent to a distance in the sense of the following result which allows us to conclude that $\mathbf{AP} \hookrightarrow \mathbf{PsAP}$.

2.1. Theorem ([8]).

(1) Given a set X and a distance δ on X, λ_δ defined by

$$\lambda_\delta(\mathcal{F}) := \sup_{U\in\sec(\mathcal{U})} \delta(-,U)$$

is an approach limit on X, and vice versa if λ is an approach limit on X then δ_λ defined by

$$\delta_\lambda(x,A) := \inf_{\mathcal{U}\in\mathbf{U}(\mathrm{stack}\,A)} \lambda(\mathcal{U})(x)$$

is a distance on X, such that $\lambda_{\delta_\lambda} = \lambda$ and $\delta_{\lambda_\delta} = \delta$.

(2) If (X,δ_X) and (Y,δ_Y) are $\mathbf{AP}$-objects and if $f : X \longrightarrow Y$, then f is a contraction if and only if it fulfills either of the following equivalent conditions:

(i) $\forall \mathcal{F} \in \mathbf{F}(X) : \lambda_Y(f(\mathcal{F})) \circ f \leq \lambda_X(\mathcal{F})$.

(ii) $\forall \mathcal{U} \in \mathbf{U}(X) : \lambda_Y(f(\mathcal{U})) \circ f \leq \lambda_X(\mathcal{U})$. ■

2.2. Proposition.

(1) ([9]) *Let (X,λ_X) and (Y,λ_Y) be pseudo-approach spaces, then the following are equivalent:*

- *$f : (X,\lambda_X) \longrightarrow (Y,\lambda_Y)$ is a contraction.*
- $\forall \mathcal{U} \in \mathbf{U}(X) : \lambda(f(\mathcal{U})) \circ f \leq \lambda(\mathcal{U})$.

(2) ([8]-[9, proposition 5.7]) *Let $(X_i,\lambda_i)_{i\in I}$ be a class of $\mathbf{PsAP}$ objects. If $(f_i : X \longrightarrow (X_i,\lambda_i))_{i\in I}$ is a source, then the initial lift λ on X is given by*

$$\lambda(\mathcal{U}) := \sup_{i\in I} \lambda_i(f_i(\mathcal{U})) \circ f_i \quad (\forall \mathcal{U} \in \mathbf{U}(X)),$$

hence (for $\mathcal{F} \in \mathbf{F}(X)$)

$$\begin{aligned}\lambda(\mathcal{F}) &:= \sup_{\mathcal{U}\in\mathbf{U}(\mathcal{F})} \lambda(\mathcal{U}) = \sup_{\mathcal{U}\in\mathbf{U}(\mathcal{F})} \sup_{i\in I} \lambda_i(f_i(\mathcal{U})) \circ f_i \\ &= \sup_{i\in I} \lambda_i(f_i(\mathcal{F})) \circ f_i\end{aligned}$$

(3) ([8]-[9, proposition 5.7]) *Let* $(X_i, \lambda_i)_{i\in I}$ *again be a class of* **PsAP** *objects. If* $(f_i : (X_i, \lambda_i) \longrightarrow X)_{i\in I}$ *now is a sink, then the final lift* λ *on* X *is obtained as follows. Given* $\mathcal{U} \in \mathbf{U}(X)$ *and* $x \in X$, *we then let*

$$\lambda(\mathcal{U})(x) := \begin{cases} 0 & \text{if } \mathcal{U} = \text{stack}\, x \\ \inf\limits_{i\in I} \inf\limits_{z\in f_i^{-1}(x)} \inf\limits_{\substack{\mathcal{G}\in\mathbf{F}(X_i) \\ f(\mathcal{G})\subset\mathcal{U}}} \lambda_i(\mathcal{G})(z) & \text{otherwise} \end{cases}$$

and

$$\lambda(\mathcal{F}) := \sup_{\mathcal{U}\in\mathbf{U}(X)} \lambda(\mathcal{U}) \ \ (\forall \mathcal{F} \in \mathbf{F}(X)). \quad \blacksquare$$

As we mentioned earlier, if we want to describe the CCT hull of **AP**, we require some CCT construct in which **AP** is fully embedded. The following result, shown in [9], indicates that **PsAP** is a good candidate.

2.3. **Proposition.** **PsAP** *is a cartesian closed topological construct. More precisely, given* **PsAP**-*objects* (X, λ_X) *and* (Y, λ_Y) *the pseudo-approach limit* λ *on* $\hom((X, \lambda_X), (Y, \lambda_Y))$ *is determined by*

$$\begin{aligned} \lambda(\Psi)(f) &:= \inf\{\alpha \in [0,\infty] \mid \forall \mathcal{F} \in \mathbf{F}(X) : \lambda_Y(\Psi(\mathcal{F})) \circ f \le \lambda_X(\mathcal{F}) \vee \alpha\} \\ &= \sup\{\lambda_Y(\Psi(\mathcal{F}))(f(x)) \mid \lambda_Y(\Psi(\mathcal{F}))(f(x)) > \lambda_X(\mathcal{F})(x)\}. \quad \blacksquare \end{aligned}$$

It is also shown in [9] that **AP** is bireflective in **PsAP** and we denote the bireflection of a **PsAP**-object (X, λ) by $(X, \bar{\lambda})$. Furthermore, **PsTOP** can be bireflectively bicoreflectively embedded in **PsAP** by associating (X, λ_q) with a **PsTOP**-object (X, q), where $\lambda_q(\mathcal{F})(x) = 0$ if $\mathcal{F} \xrightarrow{q} x$ and $\lambda_q(\mathcal{F})(x) = \infty$ otherwise. Conversely, the **PsTOP**-bicoreflection (X, q_λ) of (X, λ) is obtained by letting $\mathcal{F} \xrightarrow{q_\lambda} x$ if and only if $\lambda(\mathcal{F})(x) = 0$.
Also **TOP** is bireflectively bicoreflectively embedded in **AP** ([10]), where both the embedding and the bicoreflection are restrictions of the previously described functors. Furthermore, given $(X, \delta) \in$ **AP**, one also finds that the closure operator cl_δ of its **TOP**-bicoreflection $\mathcal{T}_\delta$ satisfies $\text{cl}_\delta(A) = \{x \in X \mid \delta(x, A) = 0\}$, hence, one finds its neighbourhood filters $(\mathcal{V}_\delta(x))_{x\in X}$ by $\mathcal{V}_\delta(x) = \{V \subset X \mid \delta(x, X \setminus V) > 0\}$.

Before we turn to the actual results and constructions, we recall some other concepts regarding approach spaces which we will be using to obtain our results (elegantly). We will for instance be making use of a characterization of approach spaces by means of approach systems.

First some terminology. A subset $\mathcal{A}$ of functions from a given set X to $[0,\infty]$ is called an *ideal* in $[0,\infty]^X$ if it is closed under the operation of taking finite suprema, and if it is closed under the operation of taking smaller functions. Given $\mathcal{A} \subset [0,\infty]^X$ and a function $\phi \in [0,\infty]^X$, then $\mathcal{A}$ is said to *dominate* ϕ if

$$\forall \epsilon > 0, \forall \omega < \infty : \exists \psi \in \mathcal{A} : \phi \wedge \omega \leq \psi + \epsilon$$

If $\mathcal{A}$ contains all $\phi \in [0,\infty]^X$ that are dominated by $\mathcal{A}$, then we say that $\mathcal{A}$ is *saturated.*

A subset $\mathcal{B}$ of $[0,\infty]^X$ is called an *ideal basis* in $[0,\infty]^X$ if for any $\alpha, \beta \in \mathcal{B}$, there exists $\gamma \in \mathcal{B}$ such that $\alpha \vee \beta \leq \gamma$.

A collection of ideal bases $(\mathcal{B}(x))_{x\in X}$ in $[0,\infty]^X$ is called an *approach basis* if for all $x \in X$ the following properties hold:

(B1) $\forall \phi \in \mathcal{B}(x) : \phi(x) = 0.$

(B2) $\forall \phi \in \mathcal{B}(x), \forall \epsilon > 0, \forall \omega < \infty : \exists (\phi_z)_{z\in X} \in \prod_{z\in X} \mathcal{B}(z)$ such that

$$\forall z, y \in X : \phi(y) \wedge \omega \leq \phi_x(z) + \phi_z(y) + \epsilon.$$

If, additionally:

$$\forall x \in X : \mathcal{B}(x) \text{ is a saturated ideal,}$$

then $(\mathcal{B}(x))_{x\in X}$ is called an *approach system.*

Given a subset $\mathcal{B} \subset [0,\infty]^X$, we define

$$\hat{\mathcal{B}} := \{\phi \in [0,\infty]^X \mid \mathcal{B} \text{ dominates } \phi\}.$$

We call $\hat{\mathcal{B}}$ the *saturation* of $\mathcal{B}$.

A collection of ideal bases $(\mathcal{B}(x))_{x\in X}$ is called an *approach basis for an approach system* $(\mathcal{A}(x))_{x\in X}$, if for all $x \in X$, $\mathcal{A}(x)$ equals the saturation of $\mathcal{B}(x)$. In this case, we also say that $(\mathcal{B}(x))_{x\in X}$ generates $(\mathcal{A}(x))_{x\in X}$ or that $(\mathcal{A}(x))_{x\in X}$ is generated by $(\mathcal{B}(x))_{x\in X}$.

For ease in notation we may also, whenever convenient, denote an approach system $(\mathcal{A}(x))_{x\in X}$ simply by $\mathcal{A}$.

We then have the following results from [10].

2.4. Proposition.

(1) Given a set X and a distance δ on X, $(\mathcal{A}_\delta(x))_{x\in X}$ defined by

$$\mathcal{A}_\delta(x) := \left\{\phi \in [0,\infty]^X \mid \forall A \subset X : \inf_{a\in A} \phi(a) \leq \delta(x, A)\right\}$$

is an approach system on X, and vice versa if $(\mathcal{A}(x))_{x\in X}$ is an approach system on X, then $\delta_{\mathcal{A}}$ defined by

$$\delta_{\mathcal{A}}(x,A) := \sup_{\phi\in\mathcal{A}(x)} \inf_{a\in A} \phi(a)$$

is a distance on X, such that $\mathcal{A}_{\delta_{\mathcal{A}}} = \mathcal{A}$ and $\delta_{\mathcal{A}_\delta} = \delta$.

(2) *If $(\mathcal{B}(x))_{x\in X}$ is an approach basis, then $(\widehat{\mathcal{B}(x)})_{x\in X}$ is an approach system with $(\mathcal{B}(x))_{x\in X}$ as basis and*

$$\delta_{\hat{\mathcal{B}}}(x,A) := \sup_{\phi\in\mathcal{B}(x)} \inf_{a\in A} \phi(a).$$

(3) *If (X,δ_X) and (Y,δ_Y) are* **AP***-objects and if $f : X \longrightarrow Y$, then the following are equivalent:*

(i) *f is a contraction.*

(ii) $\forall x \in X, \forall \phi \in \mathcal{A}_Y(f(x)) : \phi \circ f \in \mathcal{A}_X(x)$.

In (ii), if $(\mathcal{A}_Y(y))_{y\in Y}$ is replaced by an approach basis generating it, the equivalence remains valid. ■

In the following, we will also be needing a particular **AP**-object which we now introduce. Let $\mathbb{P} := ([0,\infty], \delta_{\mathbb{P}})$, where

$$\delta_{\mathbb{P}}(x,A) := \begin{cases} (x - \sup A) \vee 0 & \text{if } A \neq \emptyset, \\ \infty & \text{if } A = \emptyset. \end{cases}$$

Using the foregoing, one easily finds that $\mathcal{T}_{\mathbb{P}}$ is the *right order topology* on $[0,\infty]$ corresponding with the closure operator $\mathrm{cl}_r := \mathrm{cl}_{\delta_{\mathbb{P}}}$, i.e.

$$\mathcal{T}_r := \mathcal{T}_{\mathbb{P}} = \{(\alpha,\infty] \mid 0 \leq \alpha < \infty\} \cup \{[0,\infty]\}.$$

Further, that $\mathcal{F}$ converges to β in this topology is denoted by $\mathcal{F} \xrightarrow{r} \beta$. Since for every $\mathcal{F}$ on $[0,\infty]$, $\lim_r \mathcal{F}$ is closed in $\mathcal{T}_r$, we find that $\lim_r \mathcal{F} = [0, \mathcal{L}(\mathcal{F})]$, where $\mathcal{L}(\mathcal{F}) = \sup \lim_r \mathcal{F}$.
Recall the following useful expression of $\mathcal{L}(\mathcal{F})$ given in [9] (and an easy consequence).

2.5. **Lemma.** *Let $\mathcal{F}$ be a filter on $[0,\infty]$, then it holds that:*

(1) $\mathcal{L}(\mathcal{F}) = \inf_{U\in\sec\mathcal{F}} \sup U$ *and* $\lambda_{\mathbb{P}}(\mathcal{F})(\beta) = (\beta - \mathcal{L}(\mathcal{F})) \vee 0$.

(2) *Let $\epsilon \geq 0$ and $0 \leq y < \infty$, then:*

$$\lambda_{\mathbb{P}}(\mathcal{F})(y) \leq \epsilon \Longleftrightarrow \forall \beta > \epsilon :]y-\beta,\infty] \in \mathcal{F}. \quad ■$$

Some useful properties:

2.6. **Proposition** ([9]).

(1) Let (X, δ) be an approach space, $A \subset X$ and $\mathcal{F} \in \mathbf{F}(X)$, then $\delta(-, A) : (X, \delta) \longrightarrow \mathbb{P}$ and $\lambda_\delta(\mathcal{F})(-) : (X, \delta) \longrightarrow \mathbb{P}$ are contractions.

(2) $\mathbb{P}$ is initially dense in **AP**. ■

3. Construction of the hull.

3.1. **Definition.** *Given $(X, \lambda) \in \mathbf{PsAP}$, we define*

$$\dot{F}^\rho := \{y \in X \mid \exists x \in F : \delta_{\overline{\lambda}}(x, \{y\}) \leq \rho\}.$$

3.2. **Definition.** *Given $(X, \lambda) \in \mathbf{PsAP}$, we define*

$$\mathbf{d}_X : \mathbf{F}(X) \times \mathbf{F}(X) \longrightarrow [0, \infty] :$$
$$(\mathcal{F}, \mathcal{G}) \mapsto \mathbf{d}_X(\mathcal{F}, \mathcal{G}) := \inf\{\rho \geq 0 \mid \dot{\mathcal{G}}^\rho \subset \mathcal{F}\},$$

where $\dot{\mathcal{G}}^\rho :=$ stack$\{\dot{G}^\rho \mid G \in \mathcal{G}\}$.

Recall that an *extended pseudo-quasi-metric (on X)* is a function $d : X \times X \longrightarrow [0, \infty]$ such that (i) $d(x, x) = 0$ $(\forall x \in X)$ and (ii) $d(x, z) \leq d(x, y) + d(y, z)$ $(\forall x, y, z \in X)$ and that the pair (X, d) is then called an *extended pseudo-quasi-metric space.*

3.3. **Proposition.** *Given $(X, \lambda) \in \mathbf{PsAP}$, $(\mathbf{F}(X), \mathbf{d}_X)$ is an extended pseudo-quasi metric space.*

Proof. Let $\mathbf{d}_X(\mathcal{F}, \mathcal{G}) < \alpha$ and $\mathbf{d}_X(\mathcal{G}, \mathcal{H}) < \beta$. This implies that $\dot{\mathcal{G}}^\alpha \subset \mathcal{F}$ and $\dot{\mathcal{H}}^\beta \subset \mathcal{G}$. Consequently, $\dot{\mathcal{H}}^{\alpha+\beta} \subset \dot{\dot{\mathcal{H}}}^{\beta\,\alpha} \subset \dot{\mathcal{G}}^\alpha \subset \mathcal{F}$. ■

Note that $\mathbf{d}_X$ can attain the value ∞, by definition it is clearly not symmetric and for instance $\mathbf{d}_X(\dot{\mathcal{F}}^0, \mathcal{F}) = 0$ (hence, it is not possible to drop any prefix in the previous proposition).

We already know that in an approach space, for any $\mathcal{F} \in \mathbf{F}(X)$, the function $\lambda(\mathcal{F})(-)$ is a contraction. However we can also consider the function λ with two variables, filters on X and points of X. The foregoing definition of $(\mathbf{F}(X), \mathbf{d}_X)$ now makes it possible to consider contraction and/or continuity properties of this function of 2 variables.

3.4. **Definition.** *A pseudo-approach space (X, λ) is called an epi-approach space if it additionally satisfies the following condition:*

$(C):\quad \lambda : (\mathbf{U}(X), \mathcal{T}_{\mathbf{d}_X}) \times (X, \mathcal{T}_{\overline{\lambda}}) \longrightarrow ([0, \infty], \mathcal{T}_r)$ *is a continuous map.*

The full subconstruct of **PsAP** *consisting of epi-approach spaces is denoted by* **EpiAP**.

The following illustrates why we could restrict ourselves to ultrafilters (as usual) without causing any difficulties.

3.5. **Proposition.** *Given* $(X, \lambda) \in \mathbf{PsAP}$, *the following are equivalent:*

(1) $(X, \lambda) \in \mathbf{EpiAP}$.

(2) (X, λ) *satisfies the following condition:*

$(C)'$: $\lambda : (\mathbf{F}(X), \mathcal{T}_{\mathbf{d}_X}) \times (X, \mathcal{T}_{\bar{\lambda}}) \longrightarrow ([0, \infty], \mathcal{T}_r)$ *is a continuous map.*

Proof. The implication $\boxed{2 \Rightarrow 1}$ is clear.
To prove the implication $\boxed{1 \Rightarrow 2}$, assume that

$$\lambda : (\mathbf{U}(X), \mathcal{T}_{\mathbf{d}_X}) \times (X, \mathcal{T}_{\bar{\lambda}}) \longrightarrow ([0, \infty], \mathcal{T}_r) \text{ is continuous } (*).$$

Now let $\mathcal{F} \in \mathbf{F}(X)$ and $x \in X$ such that $K < \lambda(\mathcal{F})(x) = \sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \lambda(\mathcal{U})(x)$, hence, there exists some ultrafilter $\mathcal{U} \supset \mathcal{F}$ such that $\lambda(\mathcal{U})(x) > K$. By $(*)$, we find $V \in \mathcal{V}_{\bar{\lambda}}(x)$ and $\delta > 0$ such that $\mathbf{d}_X(\mathcal{U}, \mathcal{W}) \leq \delta$ (where $\mathcal{W} \in \mathbf{U}(X)$) and $y \in V$ implies that $\lambda(\mathcal{W})(y) > K$ $(**)$.
We now have to consider some $\mathcal{G} \in \mathbf{F}(X)$ and $y \in X$ such that $\mathbf{d}_X(\mathcal{F}, \mathcal{G}) < \delta$ and $y \in V$. Since $\dot{\mathcal{G}}^\delta \subset \mathcal{F} \subset \mathcal{U}$, we find some $\mathcal{W} \in \mathbf{U}(\mathcal{G}) : \dot{\mathcal{W}}^\delta \subset \mathcal{U}$. Indeed, assume otherwise that $\forall \mathcal{W} \in \mathbf{U}(\mathcal{G}), \exists W \in \mathcal{W} : \dot{W}^\delta \notin \mathcal{U}$. Hence ([10, proposition 1.2.2]), we can find $W_1, \ldots, W_n$ such that $\dot{W_i}^\delta \notin \mathcal{U}$ $(1 \leq i \leq n)$ and $W_1 \cup \ldots \cup W_n \in \mathcal{G}$. However, since $\mathcal{U} \supset \mathcal{F} \supset \dot{\mathcal{G}}^\delta \ni (W_1 \cup \ldots \cup W_n)^{\cdot\delta} = \dot{W_1}^\delta \cup \ldots \cup \dot{W_n}^\delta$, we find that $\dot{W_i}^\delta \in \mathcal{U}$ for some $1 \leq i \leq n$. Consequently, we obtained a contradiction and therefore we have some $\mathcal{W} \in \mathbf{U}(\mathcal{G}) : \dot{\mathcal{W}}^\delta \subset \mathcal{U}$, meaning $\mathbf{d}_X(\mathcal{U}, \mathcal{W}) \leq \delta$. By $(**)$, it follows that $\lambda(\mathcal{W})(y) > K$, hence also $\lambda(\mathcal{G})(y) \geq \lambda(\mathcal{W})(y) > K$. ∎

Looking for $\mathrm{CCTH}(\mathbf{AP})$ caused the following nice property to surface.

3.6. **Proposition.** *Let* $(X, \lambda) \in \mathbf{EpiAP}$ *and* $\mathbf{d}_X(\mathcal{F}, \mathcal{G}) = 0$, *then* $\lambda(\mathcal{F}) \leq \lambda(\mathcal{G})$. *If in particular,* $\mathcal{F} = \dot{\mathcal{G}}^0$, *then even* $\lambda(\mathcal{F}) = \lambda(\mathcal{G})$.

Proof. Let $K < \lambda(\mathcal{F})(x)$, then we find $\alpha > 0$ such that $\mathbf{d}_X(\mathcal{F}, \mathcal{G}') < \alpha$ implies that $\lambda(\mathcal{G}')(x) > K$, hence, in particular, $\lambda(\mathcal{G})(x) > K$. The arbitrariness of K now implies that $\lambda(\mathcal{F}) \leq \lambda(\mathcal{G})$.
As for the latter claim, since $\mathcal{F} = \dot{\mathcal{G}}^0 \subset \mathcal{G}$, we find that $\lambda(\mathcal{G}) \leq \lambda(\mathcal{F})$. Consequently, combining this with the previous inequality, we obtain $\lambda(\mathcal{F}) = \lambda(\mathcal{G})$. ∎

We are now in a position to state the main result.

3.7. **Theorem.** **EpiAP** *is the cartesian closed topological hull of* **AP**.

We shall prove this in several steps.

STEP 1: We first show that **AP** $\subset$ **EpiAP**.

3.8. **Lemma.** *Let* (X, λ) *be an approach space and let* $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$ *and* $\rho \geq 0$, *then:*

(1) $\lambda(\dot{\mathcal{F}}^{\rho}) \leq \lambda(\mathcal{F}) + \rho$.

(2) $\lambda(\mathcal{F}) \leq \lambda(\mathcal{G}) + \mathbf{d}_X(\mathcal{F}, \mathcal{G})$.

Proof. (1) Let $U \in \sec \dot{\mathcal{F}}^{\rho}$, we then claim that $U^{(\rho)} \in \sec \mathcal{F}$. Indeed, let $F \in \mathcal{F}$, then we find $z \in U$ such that also $z \in \dot{F}^{\rho}$, meaning $\delta(y, \{z\}) \leq \rho$ for some $y \in F$. Hence, also $\delta(y, U) \leq \rho$, i.e. $y \in F \cap U^{(\rho)}$. If we now recall from theorem 2.1 that

$$\lambda(\dot{\mathcal{F}}^{\rho})(x) = \sup_{U \in \sec \dot{\mathcal{F}}^{\rho}} \delta(x, U) \quad \text{and} \quad \lambda(\mathcal{F})(x) = \sup_{U \in \sec \mathcal{F}} \delta(x, U),$$

then the foregoing clearly demonstrates what was required.
(2) Let $\mathbf{d}_X(\mathcal{F}, \mathcal{G}) < \alpha$, hence $\dot{\mathcal{G}}^{\alpha} \subset \mathcal{F}$. Consequently, by (1), $\lambda(\mathcal{F}) \leq \lambda(\dot{\mathcal{G}}^{\alpha}) \leq \lambda(\mathcal{G}) + \alpha$. By the arbitrariness of α, we conclude that $\lambda(\mathcal{F}) \leq \lambda(\mathcal{G}) + \mathbf{d}_X(\mathcal{F}, \mathcal{G})$. ∎

The foregoing lemma shows that in an approach space, for any $x \in X$, also the function $\lambda(-)(x)$ is a contraction. However we can show more.

3.9. **Proposition.** *Let* (X, λ) *be an approach space and let* $(\mathcal{A}(x))_{x \in X}$ *be the approach system associated with* λ. *If we put*

$$\mathcal{B}_{\oplus}(\mathcal{F}, x) := \{\mathbf{d}_X(\mathcal{F}, \cdot) + \phi \mid \phi \in \mathcal{A}(x)\},$$

then $(\mathcal{B}_{\oplus})_{(\mathcal{F}, x) \in \mathbf{F}(X) \times X}$ *is an approach basis on* $\mathbf{F}(X) \times X$ *and*

$$\lambda : (\mathbf{F}(X) \times X, \widehat{\mathcal{B}_{\oplus}}) \longrightarrow \mathbb{P} \textit{ is a contraction.}$$

Proof. Let $(\mathcal{B}_{\mathbb{P}}(x)) := \{\phi \in [0, \infty]^{[0,\infty]} \mid \phi \leq d_{\mathbb{P}}(x, \cdot)\}$ if $x \in [0, \infty[$ and let $(\mathcal{B}_{\mathbb{P}}(\infty)) := \{\theta_{]a,\infty]} \mid 0 \leq a < \infty\}$ (where $\theta_A : [0, \infty] \longrightarrow [0, \infty]$ is such that $\theta_A(A) = \{0\}$ and $\theta_A([0, \infty] \setminus A) = \{\infty\}$ for any subset A of $[0, \infty]$). It then follows from proposition 2.4.(2) that this approach basis generates the approach system associated with $\delta_{\mathbb{P}}$.
Our claim will be shown by making use of proposition 2.4.(3). Therefore, let $(\mathcal{F}, x) \in \mathbf{F}(X) \times X$ and first assume that $\lambda(\mathcal{F})(x) < \infty$. Also let

$0 < \omega < \infty$ and $0 < \epsilon$ be fixed. As $\lambda(\mathcal{F}) : (X, \lambda) \longrightarrow \mathbb{P}$ is a contraction (by proposition 2.6), proposition 2.4.(3) allows us to find $\phi \in \mathcal{A}(x)$ such that

$$\forall y \in X : (\lambda(\mathcal{F})(x) - \lambda(\mathcal{F})(y)) \wedge \omega \leq \phi(y) + \epsilon.$$

We then find that

$$\begin{aligned} &(\lambda(\mathcal{F})(x) - \lambda(\mathcal{G})(y)) \wedge \omega \\ &\quad \leq \; (\lambda(\mathcal{F})(x) - \lambda(\mathcal{F})(y)) \wedge \omega + (\lambda(\mathcal{F})(y) - \lambda(\mathcal{G})(y)) \wedge \omega \\ &\quad \leq \; \phi(y) + \epsilon + (\lambda(\mathcal{G})(y) + \mathbf{d}_X(\mathcal{F}, \mathcal{G}) - \lambda(\mathcal{G})(y)) \wedge \omega \\ &\quad \leq \; \phi(y) + \epsilon + \mathbf{d}_X(\mathcal{F}, \mathcal{G}) \end{aligned}$$

Hence, by the arbitrariness of ω and ϵ, this demonstrates what was required (in case of the assumption $\lambda(\mathcal{F})(x) < \infty$).
Next we assume that $\lambda(\mathcal{F})(x) = \infty$, hence, it now needs to be shown that for arbitrary $\epsilon > 0$ and $0 \leq K, \omega < \infty$, there exists $\phi \in \mathcal{A}(x)$ such that

$$\forall \mathcal{G} \in \mathbf{F}(X), \forall y \in X : \theta_{]K,\infty]}(\lambda(\mathcal{G})(y)) \wedge \omega \leq \mathbf{d}_X(\mathcal{F}, \mathcal{G}) + \phi(y) + \epsilon.$$

As $\lambda(\mathcal{F}) : (X, \lambda) \longrightarrow \mathbb{P}$ is a contraction, we find $\phi \in \mathcal{A}(x)$ such that

$$\forall y \in X : \theta_{]K+\omega,\infty]}(\lambda(\mathcal{F})(y)) \wedge \omega \leq \phi(y) + \epsilon.$$

We now claim that

$$\theta_{]K,\infty]}(\lambda(\mathcal{G})(y)) \wedge \omega \leq \mathbf{d}_X(\mathcal{F}, \mathcal{G}) + \phi(y) + \epsilon.$$

If $\mathbf{d}_X(\mathcal{F}, \mathcal{G}) > \omega$ or $\lambda(\mathcal{G})(y) > K$, this is clearly satisfied, so let us assume that $\mathbf{d}_X(\mathcal{F}, \mathcal{G}) \leq \omega$ and $\lambda(\mathcal{G})(y) \leq K$. By the previous lemma, we then find that $\lambda(\mathcal{F})(y) \leq \lambda(\mathcal{G})(y) + \mathbf{d}_X(\mathcal{F}, \mathcal{G}) \leq K + \omega$, hence

$$\begin{aligned} \theta_{]K,\infty]}(\lambda(\mathcal{G})(y)) \wedge \omega &= \omega \\ &= \theta_{]K+\omega,\infty]}(\lambda(\mathcal{F})(y)) \wedge \omega \\ &\leq \phi(y) + \epsilon \leq \mathbf{d}_X(\mathcal{F}, \mathcal{G}) + \phi(y) + \epsilon. \quad \blacksquare \end{aligned}$$

3.10. **Corollary. AP $\subset$ EpiAP.**

Proof. Using notations as before, it is easily seen that the **TOP**-bicoreflection of $(\mathbf{F}(X) \times X, \widehat{\mathcal{B}_\oplus})$ is $(\mathbf{F}(X) \times X, \mathcal{T}_{\mathbf{d}_X} \times \mathcal{T}_{\overline{\lambda}})$ and, by definition, $\mathcal{T}_{\mathbb{P}} = \mathcal{T}_r$. ■

STEP 2: Our next goal is to show that **EpiAP** is a cartesian closed topological construct.

3.11. **Proposition.** *Let* $f : (X, \lambda_X) \longrightarrow (Y, \lambda_Y)$ *be a contraction between* **PsAP***-objects, then* $\bar{f} : (\mathbf{F}(X), \mathbf{d}_X) \longrightarrow (\mathbf{F}(Y), \mathbf{d}_Y) : \mathcal{F} \mapsto f(\mathcal{F})$ *is also a contraction.*

Proof. As $f : (X, \lambda_X) \longrightarrow (Y, \lambda_Y)$ is a contraction, we find that

$$\begin{aligned} f(\dot{F}^{\rho}) &= f(\{y \in X \mid \exists x \in F : \delta_{\bar{\lambda}_X}(x, \{y\}) \leq \rho\}) \\ &\subset \{y \in Y \mid \exists x \in f(F) : \delta_{\bar{\lambda}_Y}(x, \{y\}) \leq \rho\} = f(F)^{\cdot\rho} \end{aligned}$$

and, hence, $f(\mathcal{F})^{\cdot\rho} \subset f(\dot{\mathcal{F}}^{\rho})$ (for all $\mathcal{F} \in \mathbf{F}(X)$ and $\rho \geq 0$). Thus, $\dot{\mathcal{G}}^{\rho} \subset \mathcal{F}$ implies that $f(\mathcal{G})^{\cdot\rho} \subset f(\dot{\mathcal{G}}^{\rho}) \subset f(\mathcal{F})$, which means that $\mathbf{d}_Y(f(\mathcal{F}), f(\mathcal{G})) \leq \mathbf{d}_X(\mathcal{F}, \mathcal{G})$. ∎

3.12. **Proposition.** **EpiAP** *is bireflective in* **PsAP**, *in particular,* **EpiAP** *is a topological construct.*

Proof. Let $(f_i : (X, \lambda) \longrightarrow (X_i, \lambda_i))_{i \in I}$ be initial in **PsAP**, where all $(X_i, \lambda_i) \in$ **EpiAP**. To show that (C) is satisfied, assume that $A \in \mathcal{T}_r$, then it follows from 2.2(2) that

$$\lambda^{-1}(A) := \bigcup_{i \in I} \left(\lambda_i \circ (\bar{f}_i \times f_i)\right)^{-1}(A).$$

From the contractivity of all f_i, $i \in I$, the foregoing proposition and the fact that all λ_i, $i \in I$, satisfy (C) it follows that $\lambda^{-1}(A)$ is open. Hence λ is continuous and $(X, \lambda) \in$ **EpiAP**. ∎

3.13. **Proposition.** *Let* (X, λ_X) *and* (Y, λ_Y) *be* **PsAP***-objects, and let* $\mathcal{G}$ *be a filter on* X, *then the map* $\bar{\mathcal{G}} : \mathbf{F}(\mathrm{hom}((X, \lambda_X), (Y, \lambda_Y))) \longrightarrow \mathbf{F}(Y) : \Psi \mapsto \Psi(\mathcal{G})$ *is a contraction, i.e. for any pair* Φ *and* Ψ *of filters on* $\mathrm{hom}(X, Y)$*:* $\mathbf{d}_Y(\Phi(\mathcal{G}), \Psi(\mathcal{G})) \leq \mathbf{d}_{\mathrm{hom}(X,Y)}(\Phi, \Psi)$.

Proof. As $\mathrm{ev}_x : [X, Y] \longrightarrow Y$ $(x \in X)$ are contractions, we find that

$$\begin{aligned} \mathrm{ev}(G \times \dot{\phi}^{\rho}) &= \bigcup_{x \in G} \mathrm{ev}_x(\dot{\phi}^{\rho}) \subset \bigcup_{x \in G} (\mathrm{ev}_x(\phi))^{\cdot\rho} \\ &\subset \left(\bigcup_{x \in G} \mathrm{ev}_x(\phi)\right)^{\cdot\rho} = (\mathrm{ev}(G \times \phi))^{\cdot\rho}, \end{aligned}$$

and, hence, $\Phi(\mathcal{F})^{\cdot\rho} \subset \dot{\Phi}^{\rho}(\mathcal{F})$ (for all $\Phi \in \mathbf{F}(\mathrm{hom}(X, Y))$ and $\mathcal{F} \in \mathbf{F}(X)$). Consequently, $\dot{\Psi}^{\rho} \subset \Phi$ implies that $\Psi(\mathcal{G})^{\cdot\rho} \subset \dot{\Psi}^{\rho}(\mathcal{G}) \subset \Phi(\mathcal{G})$, which means that $\mathbf{d}_Y(\Phi(\mathcal{G}), \Psi(\mathcal{G})) \leq \mathbf{d}_{\mathrm{hom}(X,Y)}(\Phi, \Psi)$. ∎

3.14. **Proposition.** **EpiAP** *is closed under formation of power-objects in* **PsAP**. *Moreover, if* $(X, \lambda_X) \in$ **PsAP** *and* $(Y, \lambda_Y) \in$ **EpiAP**, *then* $[(X, \lambda_X), (Y, \lambda_Y)] \in$ **EpiAP**.
In particular, **EpiAP** *is a cartesian closed category.*

Proof. Let $(Z, \lambda) := [(X, \lambda_X), (Y, \lambda_Y)]$, where $(X, \lambda_X) \in$ **PsAP** and $(Y, \lambda_Y) \in$ **EpiAP**. To show that (Z, λ) satisfies (C), assume that $A \in \mathcal{T}_r$, then it follows from the formula of λ (see proposition 2.3) that

$$\lambda^{-1}(A) := \bigcup_{(\mathcal{F},x)\in \mathbf{F}(X)\times X} \Big(\big(\lambda_Y \circ (\bar{\mathcal{F}} \times \mathrm{ev}_x)\big)^{-1} \big(A \cap]\lambda(\mathcal{F})(x), \infty]\big) \Big).$$

Hence it follows from the fact that all ev_x, $x \in X$, are contractions, the foregoing proposition and the fact that λ_Y satisfies (C) that $\lambda^{-1}(A)$ is open. Hence $(Z, \lambda) \in$ **EpiAP**. ■

STEP 3: We now turn to showing that proper "density" conditions are satisfied.

Some of this, **AP** being finally dense in **EpiAP**, has already been considered in a more general way in [9, section 3] and we will recall the results needed here briefly.

3.15. **Proposition.** **AP** *is finally dense in* **EpiAP**.

Proof. As in [9], we first define the following approach spaces. Given a set X, $\mathcal{F} \in \mathbf{F}(X)$ and $f : X \longrightarrow [0, \infty]$, we define $\lambda_{(\mathcal{F},f)} : \mathbf{F}(X) \longrightarrow X^{[0,\infty]}$ by

$$\lambda_{(\mathcal{F},f)}(\mathcal{G})(x) := \begin{cases} f(x) & \mathcal{F} \cap \operatorname{stack} x \subset \mathcal{G}, \mathcal{G} \neq \operatorname{stack} x \\ 0 & \mathcal{G} = \operatorname{stack} x \\ \infty & \mathcal{F} \cap \operatorname{stack} x \not\subset \mathcal{G}. \end{cases}$$

It is then shown in [9, proposition 3.1] that $(X, \lambda_{(\mathcal{F},f)})$ is an approach space. If we now consider $(Z, \lambda) \in$ **EpiAP**, then one easily finds (as in [9, proposition 3.2]) that

$$(1_Z : (Z, \lambda_{(\mathcal{F},\lambda(\mathcal{F}))}) \longrightarrow (Z, \lambda))_{\mathcal{F}\in \mathbf{F}(X)}$$

is a final sink in **PsAP**, hence also a final sink in **EpiAP**. ■

Assume without restriction in the following that $X \neq \emptyset$.

3.16. **Proposition.** *Let* $(X, \lambda) \in$ **EpiAP**, *then* $j : X \longrightarrow [[(X, \lambda), \mathbb{P}], \mathbb{P}]$ *defined by* $j(x)(f) = f(x)$ *is an initial contraction.*

Proof. We first give the following diagram for clarity and mention that $j := \mathrm{ev}^*_{(X,\lambda),\mathbb{P}}$ is the map which makes the following diagram commute:

$$\begin{array}{ccc} [[(X,\lambda),\mathbb{P}],\mathbb{P}] \times [(X,\lambda),\mathbb{P}] & \xrightarrow{\mathrm{ev}} & \mathbb{P} \\ {\scriptstyle j\times 1}\uparrow & \nearrow_{\mathrm{ev}} & \\ (X,\lambda) \times [(X,\lambda),\mathbb{P}] & & \end{array}$$

Hence, by properties of power-objects, j is a contraction.
In the following we also let

$$(\mathrm{hom}((X,\lambda),\mathbb{P}),\lambda_H) := [(X,\lambda),\mathbb{P}],$$
$$(\mathrm{hom}([(X,\lambda),\mathbb{P}],\mathbb{P}),\lambda_{HH}) := [[(X,\lambda),\mathbb{P}],\mathbb{P}]$$

and $\hat{A}^\delta := \{f \in \mathrm{hom}((X,\lambda),\mathbb{P}) \mid f(A) \subset]\delta,\infty]\}$ $(A \subset X,\ 0 \le \delta < \infty)$.

To prove that j is initial, we will show for every ultrafilter $\mathcal{U}$ on X, $a \in X$ and $0 < K < \infty$ that $\lambda(\mathcal{U})(a) > K$ implies that $\lambda_{HH}(j(\mathcal{U}))(j(a)) \ge K$. By proposition 2.2, we then find that j is initial.
It will be shown that $\lambda_{HH}(j(\mathcal{U}))(j(a)) \ge K$ by defining an appropriate $g \in \mathrm{hom}((X,\lambda),\mathbb{P})$ and $\Psi \in \mathbf{F}(\mathrm{hom}((X,\lambda_X),\mathbb{P}))$ such that $\lambda_{\mathbb{P}}(\Psi(\mathcal{U}))(g(a)) = \lambda(j(\mathcal{U})(\Psi))(j(a)(g)) \ge K > \lambda_H(\Psi)(g)$, hence, by the description of function spaces in $\mathbf{PsAP}$, (see proposition 2.3), $\lambda_{HH}(j(\mathcal{U}))(j(a)) \ge K$.

Defining of $g \in \mathrm{hom}((X,\lambda),\mathbb{P})$ and $\Psi \in \mathbf{F}(\mathrm{hom}((X,\lambda_X),\mathbb{P}))$:
By proposition 3.5 and the fact that $(X,\lambda) \in \mathbf{EpiAP}$, we find $V \in \mathcal{V}_{\bar{\lambda}}(a)$ and $\delta' > 0$ such that for all $\mathcal{F}$ on X with $\mathbf{d}_X(\mathcal{U},\mathcal{F}) < \delta'$ and $x \in V$, we have $\lambda(\mathcal{F})(x) > K$ $(*)$.
Let $g_0 : (X,\lambda) \longrightarrow \mathbb{P} : \alpha \mapsto \delta_{\bar{\lambda}}(\alpha, X - V)$ (which is a contraction by proposition 2.6). As $V \in \mathcal{V}_{\bar{\lambda}}(a)$, we find that $\delta_1 := g_0(a) > 0$.
First assume that $g_0(a) < K$, then define $g_1 := g_0 + (K - \delta_1)$ and finally $g := g_1 \wedge K$ and $\delta'' := \delta_1$. It follows that $g \in \mathrm{hom}(X,\mathbb{P}), g \le K, g(a) = K$ and $\{g > K - \delta''\} \subset V$.
If however $g_0(a) \ge K$, then define $g := g_0 \wedge K$ and choose (any) $0 < \delta'' < K$, then g and δ'' also fulfill the foregoing properties.
Now choose $0 < \delta < \delta' \wedge \delta''$ and let Ψ'

$$\Psi' := \{\hat{F}^\delta \mid F \ne \emptyset, \dot{F}^\delta \notin \mathcal{U}\}.$$

Note that Ψ' is a filterbasis. Indeed, $\Psi' \ne \emptyset$, since for any $x \in V$, $\dot{\{x\}}^\delta \notin \mathcal{U}$, otherwise $\mathbf{d}_X(\mathcal{U}, \mathrm{stack}\, x) \le \delta < \delta'$, hence, by $(*)$, $\lambda(\mathrm{stack}\, x)(x) > K$, a contradiction. Furthermore, such $\hat{F}^\delta$ is never a void set, as it always contains the constant ∞-function. Also, it holds that $\hat{F_1}^\delta \cap \hat{F_2}^\delta = (F_1 \cup F_2)^{\wedge\delta}$, $(F_1 \cup F_2)^{\cdot\delta} = \dot{F_1}^\delta \cup \dot{F_2}^\delta$ and $\mathcal{U}$ is an ultrafilter, hence Ψ' is a filterbasis. Now let Ψ be the filter on $\hom(X, \mathbb{P})$ generated by Ψ'.

Proving that $\lambda_H(\Psi)(g) \leq K - \delta < K$:
By the description of function space pseudo-approach limits (see proposition 2.3), it needs to be shown for alll $x \in X$ and $\mathcal{F} \in \mathbf{F}(X)$ that

$$\lambda_{\mathbb{P}}(\Psi(\mathcal{F}))(g(x)) > K - \delta \text{ (I)} \Rightarrow \lambda_{\mathbb{P}}(\Psi(\mathcal{F}))(g(x)) \leq \lambda(\mathcal{F})(x).$$

To this end, let $x \in X$ and $\mathcal{F} \in \mathbf{F}(X)$ be such that (I) holds. Hence $K - \delta < \lambda_{\mathbb{P}}(\Psi(\mathcal{F}))(g(x)) \leq g(x)$, consequently $g(x) > K - \delta$ and so $x \in V$ (II). We also find that $\dot{\mathcal{F}}^\delta \subset \mathcal{U}$ (meaning $\mathbf{d}_X(\mathcal{U}, \mathcal{F}) \leq \delta < \delta'$ (III)).
Indeed, if this were not the case, then we could find $F \in \mathcal{F}$ such that $\dot{F}^\delta \notin \mathcal{U}$, implying $\hat{F}^\delta \in \Psi$, hence $]\delta, \infty] \in \Psi(\mathcal{F})$ and therefore in particular, $\forall \beta > K - \delta :]g(x) - \beta, \infty] \in \Psi(\mathcal{F})$ (as $g(x) \leq K$). It then follows from lemma 2.5.(2) that $\lambda_{\mathbb{P}}(\Psi(\mathcal{F}))(g(x)) \leq K - \delta$, which contradicts (I).
Consequently, it follows from $(*)$ (and (II) and (III)) that $\lambda(\mathcal{F})(x) > K$. Also, $\lambda_{\mathbb{P}}(\Psi(\mathcal{F}))(g(x)) \leq g(x) \leq K$, hence $\lambda_{\mathbb{P}}(\Psi(\mathcal{F}))(g(x)) \leq \lambda(\mathcal{F})(x)$.

Proving that $\lambda_{\mathbb{P}}(\Psi(\mathcal{U}))(g(a)) \geq K$:
Let us assume the contrary, i.e. $\lambda_{\mathbb{P}}(\Psi(\mathcal{U}))(g(a)) \leq K - \epsilon$, where $0 < \epsilon < K$. This implies that $\forall \beta > K - \epsilon : (g(a) - \beta, \infty] \in \Psi(\mathcal{U})$, hence $\forall \beta > K - \epsilon, \exists U \in \mathcal{U} : \hat{U}^{K-\beta} \in \Psi$. In particular, we have some $\gamma \geq 0$ and $U \in \mathcal{U}$ such that $\hat{U}^\gamma \in \Psi$. Consequently, $\hat{F}^\delta \subset \hat{U}^\gamma$ for some $F \neq \emptyset$, $\dot{F}^\delta \notin \mathcal{U}$. However, $\hat{F}^\delta \subset \hat{U}^\gamma$ implies that $U \subset \dot{F}^\delta$ (hence a contradiction). Indeed, let $z \notin \dot{F}^\delta$, implying $\delta_{\overline{\lambda}}(-, \{z\}) \in \hat{F}^\delta$, hence $\delta_{\overline{\lambda}}(-, \{z\}) \in \hat{U}^\gamma$, therefore $z \notin U$. ∎

STEP 4: Now we are in a position to combine all previous results and to prove the final step.

3.17. **Theorem.** **EpiAP** *is the cartesian closed topological hull of* **AP**.

Proof. For this to be the case, we need (as noted before) that **EpiAP** is a cartesian closed topological construct (which has been verified in step 2) and that **AP** is finally dense in **EpiAP** (which was verified in steps 1 and 3). We also need that the class

$$H := \{ [(X, \lambda_X), (Y, \lambda_Y)] \mid (X, \lambda_X), (Y, \lambda_Y) \in \mathbf{AP} \}$$

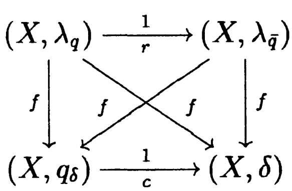
is initially dense in **EpiAP**. However, by the foregoing proposition, for any $(X,\lambda) \in$ **EpiAP** we have an initial map $j : X \longrightarrow [[(X,\lambda),\mathbb{P}],\mathbb{P}]$ and since the functor $[-,\mathbb{P}]$: **EpiAP** $\longrightarrow$ **EpiAP** transforms final epi-sinks into initial sources (see [7, lemma 6]) (and by proposition 3.15, we can obtain $[(X,\lambda),\mathbb{P}]$ as a final lift of an epi-sink involving **AP**-objects), we find that H is indeed initially dense in **EpiAP**. ■

4. *Epi*LOGUE

We now show that **EpiTOP** = CCTH(**TOP**) has a nice relation to **EpiAP**.

4.1. **Proposition.** *Let* $(X,q) \in$ **PsTOP**, *then* $(X,\lambda_{\bar{q}}) = (X,\bar{\lambda_q})$.

Proof. We will prove this by showing that $(X,\lambda_{\bar{q}})$ is also the **AP**-bireflection of (X,λ_q). To this end, let $f : (X,\lambda_q) \longrightarrow (X,\delta)$ be a contraction, where $(X,\delta) \in$ **AP**. But then also $f : (X,\lambda_q) \longrightarrow (X,q_\delta)$ is a contraction, where we recall that the latter space is the **PsTOP**-bicoreflection of (X,δ). Since we observed earlier that the **TOP**-bicoreflection in **AP** is just the restriction of the **PsTOP**-bicoreflection, it follows that (X,q_δ) is a topological space, hence $f : (X,\lambda_{\bar{q}}) \longrightarrow (X,q_\delta)$ is a contraction. Consequently, $f : (X,\lambda_{\bar{q}}) \longrightarrow (X,\delta)$ is a contraction. The following diagram illustrates this argumentation:



■

4.2. **Proposition. EpiAP ∩ PsTOP = EpiTOP.**

Proof. If $(X,\lambda) \in$ **PsTOP**, one easily finds that condition (C) is equivalent to

$$(C)^t : \ \forall \mathcal{F} \not\longrightarrow a, \exists V \in \mathcal{V}_{(X,\mathcal{T}_\lambda)}(a) : \dot{\mathcal{G}}^0 \subset \mathcal{F} \text{ and } x \in V \Rightarrow \mathcal{G} \not\longrightarrow x.$$

Also observe that in this case $\dot{\mathcal{G}}^0 = \mathcal{G}^\bullet$.

Now assume that $(C)^t$ holds (i.e. $(X,\lambda) \in$ **EpiAP ∩ PsTOP**). Since $\dot{\mathcal{F}}^0 \subset \mathcal{F}$ (for all $\mathcal{F} \in \mathbf{F}(X)$), we find that $\forall \mathcal{F} \not\longrightarrow a, \exists V \in \mathcal{V}_{(X,\mathcal{T}_\lambda)}(a)$: $V \subset (X - \lim \mathcal{F})$, meaning $\lim \mathcal{F}$ is closed in $(X,\mathcal{T}_\lambda)$. Also, by letting $\mathcal{F} = \dot{\mathcal{H}}^0$, we find that $\dot{\mathcal{H}}^0 \not\longrightarrow a$ implies $\mathcal{H} \not\longrightarrow a$, hence $\lim \mathcal{H} = \lim \dot{\mathcal{H}}^0$

(for all $\mathcal{H} \in \mathbf{F}(X)$). Consequently, (X,λ) is a closed-domained, point-regular pseudotopological space (i.e. an Antoine space).
Conversely, assume $(X,\lambda) \in \mathbf{EpiTOP}$ and $\mathcal{F} \not\longrightarrow a$. Let $V := X - \lim \mathcal{F} \in \mathcal{V}_{(X,\mathcal{T}_{\bar{\lambda}})}(a)$ (as (X,λ) is closed-domained) and suppose $\dot{\mathcal{G}}^0 \subset \mathcal{F}$ and $x \in V$. We then find that $\mathcal{G} \not\longrightarrow x$, for it this were not the case, then also $\dot{\mathcal{G}}^0 \longrightarrow x$ (as (X,λ) is point-regular), implying $\mathcal{F} \longrightarrow x$, which is a contradiction. ■

4.3. **Proposition.** **EpiTOP** *is bireflective and bicoreflective in* **EpiAP**.

Proof. The first claim is clear. As for the second claim, let $(X,\lambda) \in$ **EpiAP**, then we show that (X,λ'), the **PsTOP**-bicoreflection of (X,λ) belongs to **EpiTOP**.

To this end, assume that $\mathcal{F} \overset{q_{\lambda'}}{\not\longrightarrow} x$, i.e. $\lambda(\mathcal{F})(x) > 0$. Since $(X,\lambda) \in$ **EpiAP**, we find $V \in \mathcal{V}_{(X,\mathcal{T}_{\bar{\lambda}})}$ and $\delta > 0$ such that for $\mathbf{d}_{(X,\lambda)}(\mathcal{F},\mathcal{G}) < \delta$ and $y \in V$, we have that $\lambda(\mathcal{G})(y) > 0$.
As $1_X : (X,\lambda') \longrightarrow (X,\lambda)$ is a contraction, we find that $V \in \mathcal{V}_{(X,\mathcal{T}_{\bar{\lambda}'})}$ and that $\mathbf{d}_{(X,\lambda')}(\mathcal{F},\mathcal{G}) < \delta$ implies that $\mathbf{d}_{(X,\lambda)}(\mathcal{F},\mathcal{G}) < \delta$ (by proposition 3.11).
Putting together what we have found so far, we obtain:

$$\forall \mathcal{F} \overset{q_{\lambda'}}{\not\longrightarrow} x, \exists V \in \mathcal{V}_{(X,\mathcal{T}_{\bar{\lambda}'})} : (y \in V \wedge \mathcal{G}^{\bullet} \subset \mathcal{F}) \Rightarrow \lambda(\mathcal{G})(y) > 0 \Rightarrow \mathcal{G} \overset{q'_{\lambda}}{\not\longrightarrow} y.$$

Therefore, (X,λ') satisfies $(C)^t$ and belongs to **PsTOP**, hence $(X,\lambda') \in$ **EpiTOP**. ■

The foregoing results are combined in the following diagram:

$$\begin{array}{ccc}
\mathbf{PsTOP} & \xrightarrow[c]{r} & \mathbf{PsAP} \\
{\scriptstyle r}\uparrow & & {\scriptstyle r}\uparrow \\
\mathbf{EpiTOP} & \xrightarrow[c]{r} & \mathbf{EpiAP} \\
{\scriptstyle r}\uparrow & & {\scriptstyle r}\uparrow \\
\mathbf{TOP} & \xrightarrow[c]{r} & \mathbf{AP}
\end{array}$$

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