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The syntax of coherence

THE SYNTAX OF COHERENCE
by Noson S. YANOFSKY

RESUME. Cet article étudie la cohérence catégorique dans le cadre d'une généralisation 2-dimensionnelle de la sémantique fonctorielle de Lawvere. On y présente les 2-théories, manière syntactique de décrire des catégories avec structure. De la perspective ainsi obtenue, de nombreux résultats sur la cohérence deviennent de simples énoncés sur le lemme quasi-Yoneda et les morphismes de 2-théories. Etant donné deux 2-théories et un morphisme entre elles, on analyse la relation induite entre les 2-catégories correspondantes d'algèbres. La force des quasi-adjoints induits est classifiée par la force du morphisme de 2-théories. Ces quasi-adjoints reflètent à quel point l'une des structures peut être remplacée par l'autre. Un analogue 2-dimensionnel du produit de Kronecker est défini et construit. Cette opération permet d'engendrer de nouvelles lois de cohérence à partir d'anciennes.

1   Introduction

There has been much talk lately about higher-dimensional algebra. One-dimensional algebra is thought to be about sets with structure. Many branches of mathematics (low-dimensional topology, stable homotopy theory, etc), physics (quantum groups, quantum gravity, quantum field theory etc) and computer science (linear logic, programming semantics, etc) have made the move from sets with structure to categories with structure. This is thought of as two-dimensional algebra. One imagines

that n-categories with structure would be called n-dimensional algebra. This paper is an approach to two-dimensional universal algebra.

Ever since Mac Lane's classic paper [19], coherence questions have played a major role when studying categories with additional structure. Coherence deals with the relationship between two operations on a category. Whereas when dealing with sets, two operations can either be equal or not equal, when dealing with categories, many more options exist. Between any two operations on a category, there can be no relation, there can be a morphism, there can be an isomorphism, or there can be a unique isomorphism. Much effort has been exerted to characterize when one structure can be replaced by another. These theorems have been proved in an *ad hoc* fashion. We shall show that many of these theorems can be proven in a universal and organic manner.

The formalism that we chose to follow is Lawvere's functorial semantics [15], [16]. For each algebraic structure, one constructs a theory T whose objects are the natural numbers and whose morphisms $f : n \longrightarrow m$ correspond to operations. Composition of morphisms correspond to composition and substitution of operations. Models or algebras of a theory are product preserving functors F, from the theory to a category C with finite products. So $F(1)$ is an object $c \in C$. $F(n) \sim c^n$ and $F(f : n \longrightarrow 1)$ is an n-ary operation. Natural transformations between these functors are homomorphisms of the structures. Algebras and homomorphisms form a category $Alg(T, C)$. Between theories there are theory-morphisms $G : T \longrightarrow T'$. Precomposition with such a morphism induces $G^* : Alg(T', C) \longrightarrow Alg(T, C)$. The central theory of functorial semantics says that G^* has a left adjoint. Many functors throughout algebra turn out to be examples of such left adjoints. Other highlights of functorial semantics include the reconstruction of the theory T from its category of algebras in sets, $Alg(T, Sets)$. We also learn how to combine two algebraic structures using the Kronecker product construction [7].

This paper deals with the two-dimensional analog of functorial semantics. We start with the definition of an algebraic 2-theory, **T**, [10] which is a 2-category whose objects are the natural numbers, whose morphisms correspond to operations (functors) and whose 2-cells correspond to natural transformations between functors. We then go on to

define a 2-theory-morphisms and other morphisms in **2Theories** (following [9], we place a tilde over all 3-categories.) Connections between **Theories** and **2Theories** are enumerated.

Algebras for **T** are product preserving functors from $\mathbf{T}^{op}$ to a 2-category, **C**, with a product structure. If **C** is **Cat** then algebras are categories with extra structure. Not all structures that are put on a category can be represented by a 2-theory. We are restricted to structures with only covariant functors and hence can not deal with a closure or a duality structure. Methods of generalizing this work in order to handle such structures are discussed in section 5.

A search through the literature reveals that morphisms between algebras generally do not preserve the operations "on the nose." Rather, They are preserved up to a natural (iso)morphism. This translates to the notion of a quasi-natural transformation [5, 6, 9] between the product preserving functors from $\mathbf{T}^{op}$ to **C** i.e. a naturality square that commutes up to a 2-cell. It is important to realize that the quasi-natural transformations places our subject outside of enriched functorial semantics [3]. Between quasi-natural transformations there are modifications/2-cells. And so we have the 2-category $\mathbf{2Alg}(\mathbf{T}, \mathbf{C})$ of algebras, quasi-natural transformations and modifications.

Section 3 discusses the left quasi-adjoint Lan_G of $G^* : \mathbf{2Alg}(\mathbf{T}', \mathbf{C}) \longrightarrow \mathbf{2Alg}(\mathbf{T}, \mathbf{C})$ where $G : \mathbf{T} \longrightarrow \mathbf{T}'$ is a 2-theory-morphism. In order to construct this quasi-left-Kan extension we must first talk of the quasi-Yoneda lemma, quasi-comma categories, quasi-cocones, quasi-colimits etc. Our aim is not to repeat all the superb work of [5, 6, 9, 22, 4, 20] on quasi-(co)limits and quasi-adjoints, rather it is to state only what we need for functorial semantics. We have aimed at making this as readable as possible and we do not assume knowledge of any of the above papers. This paper is self-contained. The main idea behind section 3 is given two 2-theories and a 2-theory-morphism between them, one should explore the induced relationship between the corresponding 2-categories of algebras. The strength of the induced quasi-adjoints are classified by the strength of the 2-theory-morphism. These quasi-adjoints reflect the extent to which one structure can be replaced by another. Different types of 2-theory-morphisms induce quasi-adjoints of varying strength and these different adjoints express the coherence results. Whereas in

the 1-dimensional case, if the left adjoint is an equivalence of categories, the theories are isomorphic, in the 2-dimensional case, there are many intermediate possibilities. The aim is to simply look at the combinatorics of the 2-theory-morphism in order to understand the coherence result that is implied. Many examples are given. We also show how to reconstruct the 2-theory from the 2-category of algebras.

Section 4 is a discussion of a two-dimensional generalization of the Kronecker product. We show how one can combine one structure with another. This leads to many examples that are the standard fare of coherence theory. We go on to see how this Kronecker product respects the left quasi-adjoints of section 3. This helps us combine coherence results.

We end the paper with a look at the different directions that this project can proceed. Several conjectures are made. Some questions that seem interesting and important for future work are asked. Applications to representation theory and physics are discussed.

This paper was written to be self-contained. We assume only the basic definitions of 2-category theory. However, this work does not stand alone. This paper — as all papers in higher category theory — owes much to John Gray's important ground-breaking book [9]. We try to follow his names and notation when possible. Many of our examples come from Joyal and Street's wonderful paper [13] on the many structures and coherence theorems that are important for modern mathematics. This project would not exist without either of these important works.

A note on notation. In order to alleviate the pain of all the different types of morphisms, we will call objects and morphisms "0-cells" and "1-cells" respectively. However, an i-cell in one category can be an i'-cell in another category. All 2-categories will be in bold typeface. In contrast, (1-)categories will not.

This work would not have been possible without many enlightening and encouraging conversations with Alex Heller. Some of the ideas for this paper were formed while enjoying a postdoctoral position at McGill University. I would like to thank Mike Barr, Marta Bunge, Michael Makkai, Robert Seely and the entire Montreal categories group for that wonderful experience. Jim Stasheff and Mirco Mannucci have looked at earlier drafts of this paper and made many helpful comments. I am

indebted to them.

2   2-theories and 2-algebras

Consider the skeletal category of finite sets Fin_{sk}. Place a coproduct structure on this category. The coproduct structure allows one to write $n \cong \coprod_n 1$. For all positive integers m, n and p, one has the following induced isomorphisms

$$\sigma_m^n : \coprod_m n \cong \coprod_m \coprod_n 1 \longrightarrow \coprod_n \coprod_m 1 \cong \coprod_n m$$

$$\mu_{m,n}^p : \coprod_m p + \coprod_n p \longrightarrow \coprod_{m+n} p$$

$$\nu_p^{m,n} : \coprod_p m + \coprod_i n \longrightarrow \coprod_p (m+n).$$

These isomorphisms satisfy the following coherence condition:

$$\begin{array}{ccc}
\coprod_m p + \coprod_n p & \xrightarrow{\ \mu_{m,n}^p\ } & \coprod_{m+n} p \\
{\scriptstyle \sigma+\sigma}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\
\coprod_p m + \coprod_p n & \xrightarrow{\ \nu_p^{m,n}\ } & \coprod_p m + n.
\end{array} \tag{1}$$

Let $\overline{\mathbf{Fin}}_{sk}$ denote the 2-category with the same 0-cells and 1-cells as Fin_{sk} but with only identity 2-cells. $\overline{\mathbf{Fin}}_{sk}$ also has a coproduct structure. A coproduct structure for a 2-category is similar to a coproduct structure for a 1-category. However, there is an added requirement that for every finite family of 1-cells with common source and target, there is a 1-cell with injection 2-cells that satisfy the obvious universal property. When we talk of preserving coproduct structures, we mean preserving the coproduct strictly (equality).

Definition 1 *A (single sorted algebraic) 2-theory is a 2-category* $\mathbf{T}$ *with a given coproduct structure and a 2-functor* $G_{\mathbf{T}} : \overline{\mathbf{Fin}}_{sk} \longrightarrow \mathbf{T}$ *such that* $G_{\mathbf{T}}$ *is bijective on 0-cells and preserves the coproduct structure.*

The following examples are well known.

Example.2.1: $\overline{\text{Fin}}_{sk}$ is the initial 2-theory. Just as Fin_{sk} is the theory of sets, so too, $\overline{\text{Fin}}_{sk}$ is the theory of categories. $\square$

Example.2.2: Let $\overline{\text{Bin}}$ be $\overline{\text{Fin}}_{sk}$ with a nontrivial generating 1-cell $\otimes : 1 \longrightarrow 2$ thought of as a binary operation (bifunctor). $\square$

Example.2.3: $\mathbf{T}_{Mon}$ is the 2-theory of monoidal (tensor) categories. It is a 2-theory "over" $\overline{\text{Bin}}$ with a 1-cell $e : 1 \longrightarrow 0$. The isomorphic 2-cells are generated by
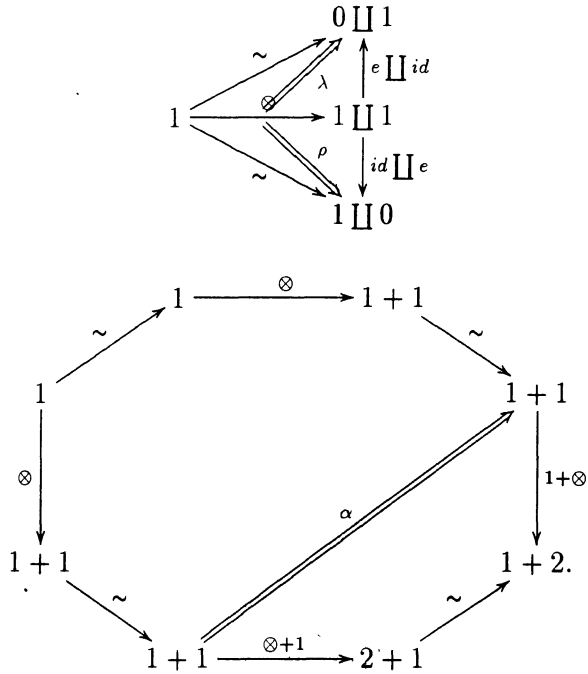
where the corner isomorphisms $n + m \longrightarrow m + n$ is an instance of σ_m^n in $\overline{\text{Fin}}_{sk}$. These 2-cells are subject to a unital equation (left for the reader)

and the now-famous pentagon condition:
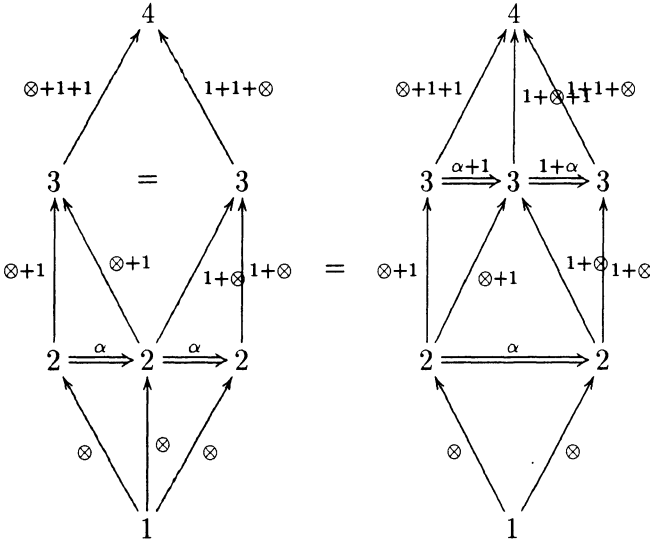
(We leave out the corner isomorphisms in order to make the diagram easier to read. However they are important and must be placed in the definition). □

Example.2.4: The theory of braided tensor categories $\mathbf{T}_{Braid}$ and balanced tensor categories [13],$\mathbf{T}_{Bal}$, are easily described in a similar manner. □
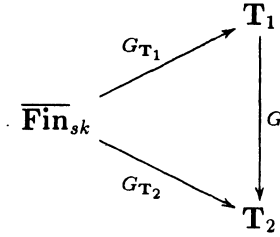
Example.2.5: Associative categories [27] which are monoidal categories in which the pentagon coherence does not necessarily hold are described by $\mathbf{T}_{Assoc}$. Similarly, commutative categories [28] which are braided tensor categories that do not necessarily satisfy the hexagon coherence condition are described by $\mathbf{T}_{Comm}$. □

Example.2.6: Whenever we have a theory with strict associativity, we denote it with a small "s" followed by the usual name e.g. $\mathbf{T}_{sMon}$, $\mathbf{T}_{sBraid}$, $\mathbf{T}_{sBal}$ etc. □

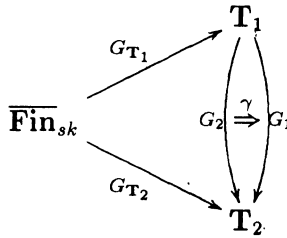**Definition 2** *A 2-theory-morphism from $\mathbf{T}_1$ to $\mathbf{T}_2$ is a 2-functor G:*

$\mathbf{T}_1 \longrightarrow \mathbf{T}_2$ *such that*

$$\begin{array}{ccc}
 & & \mathbf{T}_1 \\
 & \nearrow^{G_{\mathbf{T}_1}} & \\
\overline{\mathbf{Fin}}_{sk} & & \Big\downarrow G \\
 & \searrow_{G_{\mathbf{T}_2}} & \\
 & & \mathbf{T}_2
\end{array}$$

commutes.

Definition 3 *A 2-theory-natural transformation* $\gamma : G_1 \Longrightarrow G_2$ *between two 2-theory-morphisms is a natural transformation such that*

$$\begin{array}{ccc}
 & & \mathbf{T}_1 \\
 & \nearrow^{G_{\mathbf{T}_1}} & \\
\overline{\mathbf{Fin}}_{sk} & & G_2 \overset{\gamma}{\Rightarrow} G_1 \\
 & \searrow_{G_{\mathbf{T}_2}} & \\
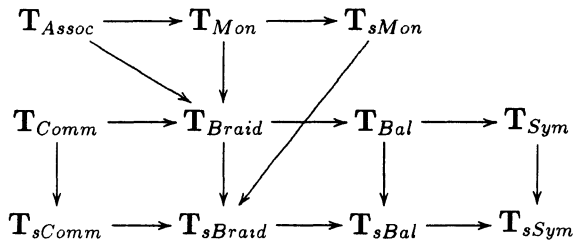 & & \mathbf{T}_2
\end{array}$$

commutes.

One goes on to define a 2-theory-modification $\Gamma : \gamma_1 \rightsquigarrow \gamma_2$ in the obvious way.

We shall denote the 3-category of 2-theories, 2-theory-morphisms, natural transformations and modifications as **2Theories**.

Here is a diagram of some of the 2-theories and 2-theory morphisms that we will work with.

$$\begin{array}{ccccc}
\mathbf{T}_{Assoc} & \longrightarrow & \mathbf{T}_{Mon} & \longrightarrow & \mathbf{T}_{sMon} \\
 & \searrow & \downarrow & & \\
\mathbf{T}_{Comm} & \longrightarrow & \mathbf{T}_{Braid} & \longrightarrow & \mathbf{T}_{Bal} & \longrightarrow & \mathbf{T}_{Sym} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{T}_{sComm} & \longrightarrow & \mathbf{T}_{sBraid} & \longrightarrow & \mathbf{T}_{sBal} & \longrightarrow & \mathbf{T}_{sSym}
\end{array}$$

Many examples of 2-theories and their morphisms come from one dimensional theories in the following manner. Let **Theories** denote the usual [15] 2-category of theories, theory-morphisms and theory-natural transformations. One can think of **Theories** as a 3-category $\widetilde{\textbf{Theories}}$ with only trivial 3-cells. Analogous to the relationship between sets and topological spaces, we have the following adjunctions:

$$
\widetilde{\textbf{Theories}} \quad
\begin{array}{c}
\xleftarrow{\quad\pi_0\quad} \\
\xrightarrow[\dashv]{\;\;d\;\;} \\
\xleftarrow[\perp]{\;\;U\;\;} \\
\xrightarrow[\perp]{\quad c\quad}
\end{array}
\quad \widetilde{\textbf{2Theories}}.
$$

$c(T)$ is the 2-theory with the same 1-cells as T and a unique 2-cell between nontrivial 1-cells. $d(T)$ has the same 1-cells as T and only trivial 2-cells. $U(\textbf{T})$ forgets the 2-cells of **T**. $\pi_0(\textbf{T})$ is a quotient theory of **T** where two 1-cells are set equal if there is a 2-cell between them. These functors extend in an obvious way to 3-functors. By adjunction we mean a strict 3-adjunction; that is the universal property is satisfied by a strict 2-category isomorphism. For example the following 2-categories are isomorphic

$$
Hom_{\widetilde{\textbf{Theories}}}(T, U(\textbf{T})) \cong Hom_{\widetilde{\textbf{2Theories}}}(d(T), \textbf{T})
$$

Example.2.7: $\overline{Fin}_{sk} = d(Fin_{sk})$, that is, the theory of categories is the discrete theory of sets. □

Example.2.8: $\overline{\textbf{Bin}} = d(T_{Magmas})$. □

Example.2.9: $d(T_{Monoids})$ is the theory of strict monoidal categories, $\textbf{T}_{sMon}$. □

Example.2.10: Let $T_{Magmas\bullet}$ be the theory of pointed magmas i.e. the theory of magmas with a distinguished element. $c(T_{Magmas\bullet})$ is the 2-theory of symmetric (monoidal) tensor categories. Warning: not all operations are made to be isomorphic. In particular, the projections (inclusions) live in $\overline{Fin}_{sk}$ and are not isomorphic. □

Example.2.11: Let $\textbf{T}_{Braid}$ denote the 2-theory of braided tensor categories. $\pi_0(\textbf{T}_{Braid})$ is the theory of commutative monoids. □

The units and counits of these adjunctions are of interest. ε : $\pi_0 dT \longrightarrow T$, $\mu : T \longrightarrow UdT$ and $\varepsilon : UcT \longrightarrow T$ are all identity

theory-morphisms. More importantly, $\mu : \mathbf{T} \longrightarrow d\pi_0\mathbf{T}$ is the 2-theory-morphism corresponding to "strictification". Every 2-cell becomes the identity. "Strictification" is often used in coherence theory and in section 3 we shall take (quasi-) Kan extensions along such 2-theory-morphisms. Similarly, $\mu : \mathbf{T} \longrightarrow cU\mathbf{T}$ might be called "coherification": a 2-theory is forced to be coherent. $\varepsilon : dU\mathbf{T} \longrightarrow \mathbf{T}$ is the injection of the 1-theory into the 2-theory.

Given a 2-theory $\mathbf{T}$ and a 2-category $\mathbf{C}$ with a product structure, an **algebra** of $\mathbf{T}$ in $\mathbf{C}$ is a product preserving 2-functor $F : \mathbf{T}^{op} \longrightarrow \mathbf{C}$.

A **quasi-natural transformation** (cf. pg. 26 of [9], [5, 6]) σ from an algebra F to an to an algebra F' is

- A family of 1-cells in $\mathbf{C}$, $\sigma_n : F(n) \longrightarrow F'(n)$ indexed by 0-cells of $\mathbf{T}$. This family must preserve products i.e. $\sigma_n = (\sigma_1)^n : F(1)^n \longrightarrow F'(1)^n$.

- A family of 2-cells in $\mathbf{C}$, σ_f, indexed by 1-cells $f : m \longrightarrow n$ of $\mathbf{T}$. σ_f makes the following diagram commute.

$$
\begin{array}{ccc}
 & F(1)^n \xrightarrow{\ \sigma^n\ } F'(1)^n & \qquad (2) \\
 \nearrow^{\sim} & & \searrow^{\sim} \\
 F(n) & & F'(n) \\
 \Big\downarrow{\scriptstyle Ff} & \qquad \sigma_f & \Big\downarrow{\scriptstyle F'f} \\
 F(m) & & F'(m) \\
 \searrow_{\sim} & & \nearrow_{\sim} \\
 & F(1)^m \xrightarrow{\ \sigma^m\ } F'(1)^m &
\end{array}
$$

These morphisms must satisfy the following conditions:

1. If f is in the image of $G_{\mathbf{T}} : \overline{\mathbf{Fin}}_{sk} \longrightarrow \mathbf{T}$, then $\sigma_f = id.$ That is, diagram (2) commutes strictly. This condition includes $\sigma_{id_n} = id_{\sigma_n}$.

2. σ preserves the coproduct structure: $\sigma_{f+f'} = \sigma_f \times \sigma_{f'}$. To be more exact, $\sigma_{f+f'}$ is the entire diagram in Figure I. The quadrilaterals in Figure I commute from the coproduct structure of $\mathbf{T}$ and the product structure of $\mathbf{C}$; see diagram (1).

3. $\sigma_{g \circ f} = \sigma_f \circ_v \sigma_g$ where $\circ_v$ is the vertical composition of 2-cells.

4. σ behaves well with respect to 2-cells of $\mathbf{T}$. That is, if we have

$$m \underset{f'}{\overset{f}{\rightleftarrows}} \Downarrow\alpha \; n$$

in $\mathbf{T}$, then the two diagrams of Figure II must be equal.

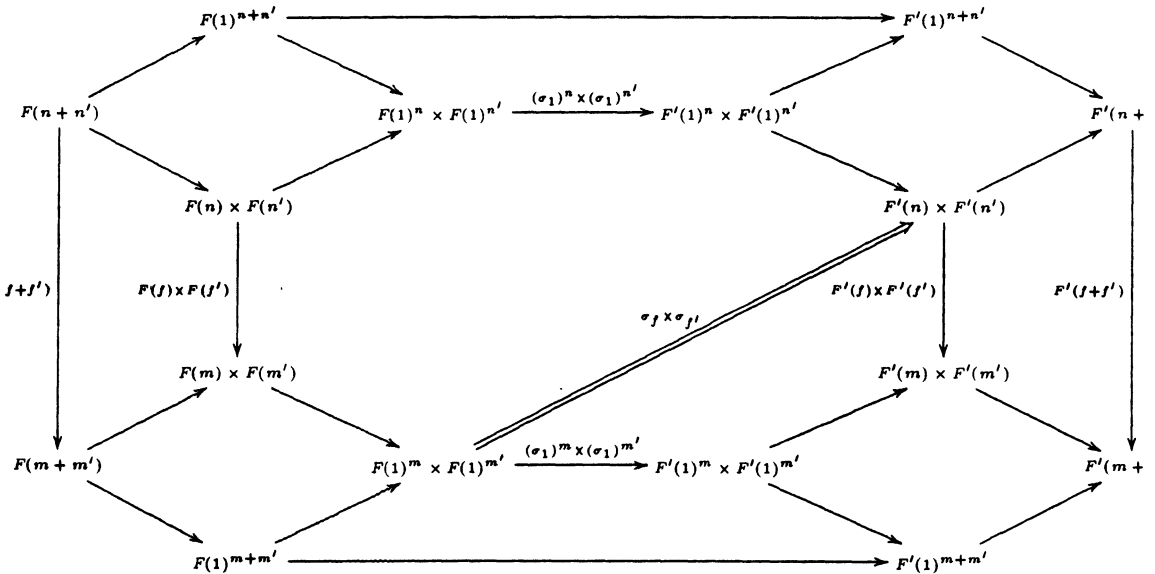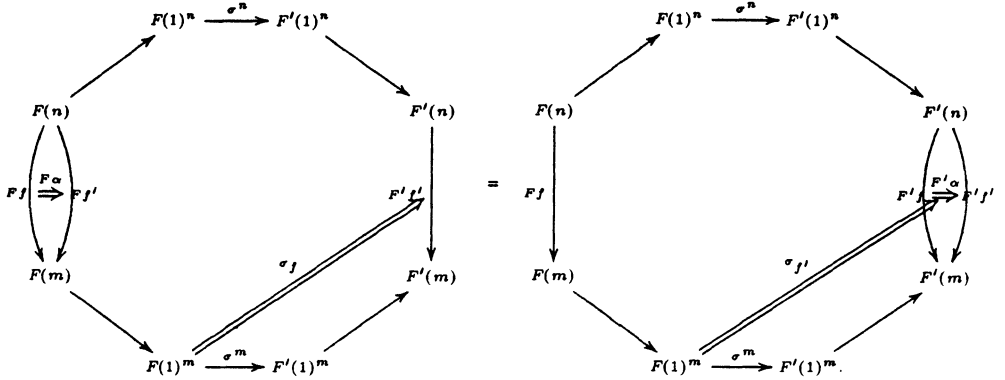**Figure I.**

Figure II.

Remark.2.1: We not only require σ to preserve the coproduct in **T** but also to preserve *all* the coherence properties of the coproduct. $\square$

Composition of quasi-natural transformations are given as

$$(\sigma'\sigma)_n = \sigma'_n\sigma_n \qquad (\sigma'\sigma)_f = \sigma'_f \circ_h \sigma_f$$

Given two quasi-natural transformations $\sigma, \sigma' : F \longrightarrow F'$, a **modification** $\Sigma : \sigma \rightsquigarrow \sigma'$ from σ to σ' is a family of 2-cells $\Sigma_n : \sigma_n \Longrightarrow \sigma'_n$ indexed by the 0-cells of **T**. These 2-cells must satisfy the following conditions:

1. Σ preserves products i.e. $\Sigma_n = (\Sigma_1)^n : (\sigma_1)^n \Longrightarrow (\sigma'_1)^n$.

2. Σ behaves well with respect to the 2-cells of **T**. That is, if we have

$$m \underset{f'}{\overset{f}{\rightrightarrows}} \Downarrow\alpha \; n$$

then we have the following "cube relation":

The left diagram shows $F(n) \xrightarrow{\sigma'_n} F'(n)$ with $F'(f')$, id, Σ_n, $F'(m)$, id, $F(n) \xrightarrow{\sigma_n} F'(n) \xRightarrow{F'(\alpha)} id$, $F'(f)$, σ_f, $F(f)$, $F(m) \xrightarrow{\sigma_m} F'(m)$.

The right diagram ($=$) shows $F(n) \xrightarrow{\sigma'_n} F'(n)$ with $F(f')$, $F'(f')$, $\sigma'_{f'}$, id, $F(m) \xrightarrow{\sigma'_m} F'($, $F(\alpha)$, $F(n) \xRightarrow{F(\alpha)} id$, id, $F(f)$, Σ_m, $F(m) \xrightarrow{\sigma_m} F'($.

$$(3)$$

Compositions of modifications are given as

$$(\Sigma' \circ_h \Sigma)_n = \Sigma'_n \circ_h \Sigma_n \qquad (\Sigma' \circ_v \Sigma)_n = \Sigma'_n \circ_v \Sigma_n$$

There is a need to generalize this definition. Let $G : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ be a 2-theory-morphism. Then $\mathbf{2Alg}_G(\mathbf{T}_2, \mathbf{C})$ will have the same 0-cells as $\mathbf{2Alg}(\mathbf{T}_2, \mathbf{C})$, however, $\mathbf{2Alg}_\mathbf{G}(\mathbf{T}_2, \mathbf{C})(F, F')$ will be the full sub-category of $\mathbf{2Alg}(\mathbf{T}_2, \mathbf{C})(F, F')$ consisting of those quasi-natural transformations that are actual natural transformations when precomposed with G i.e. those σ such that $\sigma_{G(f)} = id$ or in other words those σ such that

$$\mathbf{T}_1^{op} \xrightarrow{G^{op}} \mathbf{T}_2^{op} \xRightarrow[F']{\overset{F}{\Downarrow \sigma}} \mathbf{C}$$

$\sigma \circ G^{op}$ is a natural transformation (not quasi) from $F \circ G^{op}$ to $F' \circ G^{op}$.

$\mathbf{2Alg}_G^i(\mathbf{T}_1, \mathbf{C})$ is defined to be the locally full sub-2-category of $\mathbf{2Alg}_G(\mathbf{T}_1, \mathbf{C})$ consisting of those quasi-natural transformations where the σ_f's are isomorphisms.

It is obvious that $\mathbf{2Alg}_{\mathbf{G_T}}(\mathbf{T}, \mathbf{C}) = \mathbf{2Alg}(\mathbf{T}, \mathbf{C})$ and that $\mathbf{2Alg}_{id_\mathbf{T}}(\mathbf{T}, \mathbf{C})$ has only strict natural transformations. For every $\mathbf{2Alg}_\mathbf{G}(\mathbf{T}, \mathbf{C})$, there is a forgetful 2-functor $U : \mathbf{2Alg}_\mathbf{G}(\mathbf{T}, \mathbf{C}) \longrightarrow \mathbf{C}$ defined as $U(F) = F(1)$; $U(\sigma) = \sigma_1$ and $U(\Sigma) = \Sigma_1$.

Consider the 3-category $(\mathbf{2\widetilde{Theories}})^{\frown}$ which has as 0-cells 2-theory-morphisms. The i-cells for $i = 1, 2, 3$ are pairs of i-cells in $\mathbf{2\widetilde{Theories}}$ making the usual square commute. Thus we have the following 3-functor

$$\mathbf{2Alg}_{(?)}(t(?), \mathbf{C}) : ((\mathbf{2\widetilde{Theories}})^{\frown})^{op} \longrightarrow \mathbf{2Cat}/\mathbf{C}$$

where $t(?)$ is the target (codomain) of $(?)$.

3   Universal properties of coherence

Many coherence theorems are a result of the quasi-Yoneda lemma.

Lemma 1 (Quasi-Yoneda) *Let* $\mathbf{D}$ *be a 2-category. Let* $K : \mathbf{D} \longrightarrow$ $\mathbf{Cat}$ *be a 2-functor.* $\mathbf{qNat}(\mathbf{D}(r, -), K(-))$ *shall denote the category of quasi-natural transformations and modifications (not necessarily product preserving) between* $\mathbf{D}(r, -)$ *and* $K(-)$. *Then there are (quasi-)adjoint functors:*

$$\mathbf{qNat}(\mathbf{D}(r, -), K(-)) \underset{\hat{\Psi}_r}{\overset{\Psi_r}{\underset{\perp}{\rightleftarrows}}} K(r).$$

The unit of this adjunction, $id \longrightarrow \Psi_r \circ \hat{\Psi}_r$, *is quasi-natural. The counit of this adjunction,* $\hat{\Psi}_r \circ \Psi_r \longrightarrow id$, *is the identity.*

Proof. Definition of Ψ_r. Let $\sigma : \mathbf{D}(r, -) \longrightarrow K(-)$ be a quasi-natural transformation then $\Psi_r(\sigma) = \sigma_{r, id_r} \in K(r)$. For a modification $\Sigma :$ $\sigma \rightsquigarrow \sigma'$, we set $\Psi_r(\Sigma) = \Sigma_{r, id_r}$ where Σ_r is a 2-cell in $\mathbf{Cat}$ (a natural transformation):

$$\mathbf{D}(r, r) \underset{\sigma'_r}{\overset{\sigma_r}{\rightrightarrows}} {\Downarrow \Sigma_r} \quad K(r).$$

Definition of $\hat{\Psi}_r$. Let $U \in K(r)$. $\hat{\Psi}_r(U) = \sigma_U$ where $\sigma_{U,d} : \mathbf{D}(r, d) \longrightarrow$ $K(d)$ is defined as follows. For $f \in \mathbf{D}(r, d)$, $\sigma_{U,d}(f) = K(f)U \in K(d)$. For $\alpha : f \Longrightarrow f'$, $\sigma_{U,d}(\alpha) = K(\alpha)U$. One should have the following

picture in mind:

$$\underline{\mathbf{D}} \qquad\qquad\qquad \underline{\mathbf{Cat}}$$

$$U \in K(r)$$

Let $t : U \longrightarrow U'$ be a 1-cell in $K(r)$. Then $\hat{\Psi}_r(t) = \Sigma_t$ where $\Sigma_{t,d}$ fits in

$$\mathbf{D}(r,d) \underset{\sigma_{U'}}{\overset{\sigma_U}{\rightrightarrows}} \Downarrow \Sigma_{td} \quad K(d).$$

and is defined as follows:

$$[\Sigma_{t,d}(f : r \to d) = K(f)(t)] \quad : \quad [\sigma_{U,d}(f) = K(f)U] \quad \longrightarrow \quad [\sigma_{U',d}(f) = K(f)U'].$$

And finally

$$\Sigma_{t,d}(\alpha : f \Rightarrow f') \quad = \quad K(f')t \circ K(\alpha)_U \quad = \quad K(\alpha)_{U'} \circ K(f)t$$

i.e. the morphism described by the natural transformation

$$
\begin{array}{ccc}
K(f)U & \xrightarrow{K(\alpha)_U} & K(f')U \\
{\scriptstyle K(f)t}\downarrow & & \downarrow{\scriptstyle K(f')t} \\
K(f)U' & \xrightarrow{K(\alpha)_{U'}} & K(f')U'.
\end{array}
$$

The unit of the adjunction. Let σ be a quasi-natural transformation. $\Psi_r(\sigma) = \sigma_{r,id_r}$. Then $\hat{\Psi}_r\Psi_r(\sigma) = \hat{\Psi}_r(\sigma_{r,id_r})$.
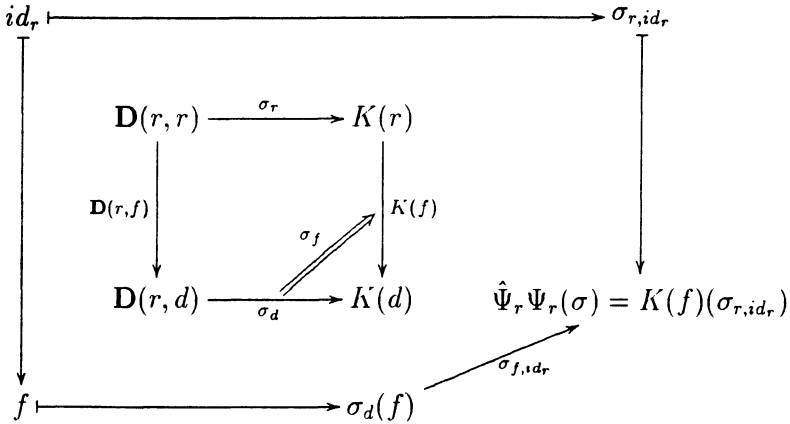
$$\hat{\Psi}_r(\sigma_{r,id_r})_d : \mathbf{D}(r,d) \longrightarrow K(d)$$

is defined as

$$\hat{\Psi}_r(\sigma_{r,id_r})_d(f) = K(f)(\sigma_{r,id_r})$$

The unit of the adjunction at $d \in \mathbf{D}$, $\sigma_d \longrightarrow \hat{\Psi}_r\Psi_r(\sigma)_d$, is defined at $f \in \mathbf{D}(r,d)$ as σ_{f,id_r}. The following picture is helpful:

Note that if we insist that σ_f is an isomorphism, then σ_{f,id_r} is also an isomorphism and hence the unit would be an isomorphism. The unit is a quasi-natural transformation.

The counit of the adjunction.

$$
\begin{aligned}
(\Psi_r \circ \hat{\Psi}_r)(U) \quad &= \quad \Psi_r(\sigma_U) \quad = \quad \sigma_{U,r,id_r} \quad = \quad K(id_r)U \\
&= \quad id_{K(r)}(U) \quad = \quad U \quad \square
\end{aligned}
$$

This theorem says that every 0-cell in $K(r)$ corresponds to a natural transformation. The unit of the adjunction is, in a sense, a reflection of the category of natural transformations inside the category of quasi-natural transformations.

The following facts about the quasi-Yoneda lemma are important. The proofs are trivial or tedious and we leave them for the readers leisure time.

Proposition 1 (On the quasi-Yoneda lemma) *Let $l : r \longrightarrow r'$ be a 1-cell in $\mathbf{D}$ and let $\kappa : K \longrightarrow K'$ be a quasi-natural transformation.*

(a) Ψ_r is quasi-natural with respect to r i.e. given $l : r \longrightarrow r'$, the obvious square commutes up to a natural transformation.

(b) Ψ_r is natural with respect to a K i.e. given a quasi-natural transformation $\kappa : K \longrightarrow K'$, the obvious square commutes strictly.

(c) $\hat{\Psi}_r$ is natural with respect to r.

(d) $\hat{\Psi}_r$ is quasi-natural with respect to K. If however, κ is natural (not quasi), then $\hat{\Psi}_r$ is also natural (not quasi).

(e) If we insist that the quasi-natural transformations σ have the usual square commuting up to a natural iso-2-cell, then the $\Psi_r, \hat{\Psi}_r$ adjunction becomes an equivalence of categories:

$$\mathbf{qNat}^{\mathbf{i}}(\mathbf{D}(r, -), K(-)) \cong K(r).$$

Warning: This is not natural in r.

(f) If we insist that the quasi-natural transformations σ be $\mathbf{Cat}$-natural transformations, then the $\Psi_r, \hat{\Psi}_r$ adjunction becomes an isomorphism of categories:

$$\mathbf{CatNat}(\mathbf{D}(r, -), K(-)) \cong K(r).$$

(g) If we insist that all the 2-cells are identities, then the $\Psi_r, \hat{\Psi}_r$ adjunction become the usual Yoneda lemma:

$$Nat(D(r, -), K(-)) \cong K(r).$$

(h) If $\mathbf{D}$ has a product structure and K, σ, Σ are assumed to preserve the product structure, then we still have the adjunction. Furthermore, $\Psi_{r \times r'} \cong \Psi_r \times \Psi_{r'}$ (similarly for $\hat{\Psi}_r$).

Let $[n]$ denote the discrete category whose objects are $\{0, \ldots, n-1\}$.

Proposition 2 Let $\mathbf{T}$ be a 2-theory. $\mathbf{T}^{\mathbf{op}}(n, -) : \mathbf{T}^{\mathbf{op}} \longrightarrow \mathbf{Cat}$ is a product preserving 2-functor and is the free $\mathbf{T}$-algebra on n generators in the sense that

$$\mathbf{2Alg}^{\mathbf{i}}(\mathbf{T}, \mathbf{Cat})(\mathbf{T}^{\mathbf{op}}(n, -), F(-)) = \mathbf{qNat}^{\mathbf{i}}((\mathbf{T}^{\mathbf{op}}(n, -), F(-)) \cong F(n)$$

$$\cong F(1)^n \cong \mathbf{Cat}([n], F(1))\square.$$

Notice the importance of insisting on *iso*-quasi-natural transformations since by (g) above, we have an equivalence of categories. From the universality and (quasi-)naturality of the quasi-Yoneda lemma, any other **T**-algebra that satisfies this universal property is equivalent to $\mathbf{T^{oP}}(n, -)$ in **Cat** and is equivalent to $\mathbf{T^{oP}}(n, -)$ in $\mathbf{2Alg^i(T, Cat)}$.

Example.3.1: Let B (see page 10 of [12]) be the category whose objects are the natural numbers and whose only morphisms are $Hom_B(n, n) = B_n$, the Artin braid group on n strings. B has a strict braided structure and is the free strict braided tensor category on one generator. Let $\mathbf{T}_{sBraid}$ be the 2-theory of strict braided tensor categories. $\mathbf{T}^{op}_{sBraid}(n, -)$ is the free braided tensor category generated by n objects Hence $B \cong \mathbf{T}^{op}_{sBraid}(1, -)$ as categories and as braided tensor categories. □

Example.3.2: Let S be the category similar to B but whose morphisms are the symmetric groups. S has a strict symmetric structure and is the free strict symmetric tensor category on one generator. Let $\mathbf{T}_{sSym}$ be the 2-theory of strict symmetric tensor categories. Hence $S \cong \mathbf{T}^{op}_{sSym}(1, -)$ as categories and as symmetric tensor categories. □

Example.3.3: Let $\tilde{B}$ be the free strict balanced tensor category on one generator (see pages 11, 41 of [12]). Let $\mathbf{T}_{Bal}$ be the 2-theory of strict balanced tensor categories. Hence $\tilde{B} \cong \mathbf{T}^{op}_{Bal}(1, -)$. □

On to the notion of quasi-cocones. Let **I** be a small, locally small 2-category. $\mathbf{C^I}$ shall denote the 2-category of **I**-diagrams in **C**. The 1-cells in $\mathbf{C^I}$ are quasi-natural transformations. In order to keep track of the morass of different types of morphisms in this discussion, we shall attempt to abide by the following table.

|	**I**	**C**	$\mathbf{C^I}$
0-cells	i	c	d
1-cells	$I : i \longrightarrow i'$	$\sigma : c \longrightarrow c'$	$\xi : d \longrightarrow d'$
2-cells	$\iota : I \Longrightarrow I'$	$\Sigma : \sigma \Longrightarrow \sigma'$	$\Xi : \xi \Longrightarrow \xi'$

To every **I**, there is a constant-diagram 2-functor

$$\Delta : \mathbf{C} \longrightarrow \mathbf{C^I}$$

which is defined on 0-cells as follows

$$\Delta(c)(i) = c \qquad \Delta(c)(I) = id_c \qquad \Delta(c)(\iota) = id_{id_c}.$$

$\Delta(\sigma)$ and $\Delta(\Sigma)$ are defined to be the usual morphisms between constant 2-diagrams.

The category $\mathbf{C}^{\mathbf{I}}(d, \Delta(c))$ is the category of cocones over d with vertex c and morphisms between such cocones. In detail, a cocone ξ over d with vertex c is a quasi-natural transformation in the 2-category $\mathbf{C}$. For every $I \in \mathbf{I}$, there is a $\xi_I : \xi_{i'} \circ d(I) \Longrightarrow \xi_i$ and for every $\iota : I \Longrightarrow I'$, we demand $\xi_{I'} \circ_h d\iota = \xi_I$
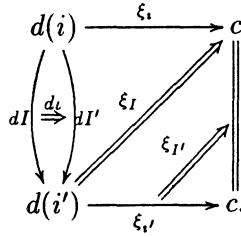
Let ξ' be another cocone over d with vertex c, then a morphism of cocones $\Xi : \xi \longrightarrow \xi'$ is a family of 2-cells $\Xi_i : \xi_i \Longrightarrow \xi_i'$ indexed by the 0-cells of $\mathbf{I}$. These 2-cells must satisfy

$$\Xi_i \circ_h \xi_I \quad = \quad \xi_{I'}' \circ_v (\Xi_{i'} \circ_h d(\iota))$$
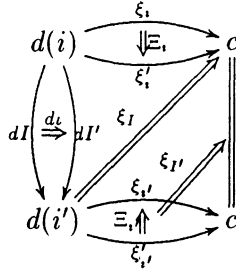
(This identity is nothing more than the cube relation (3) with $F(\alpha) = d(\iota)$, $F'(\alpha) = Id_c$, $\sigma_c = \xi_I$, $\sigma_{f'}' = \xi_{I'}'$, $\Sigma_n = \Xi_i$ and $\Sigma_m = \Xi_{i'}$).

There is a need to generalize this definition. Let $\gamma : \mathbf{I}' \longrightarrow \mathbf{I}$ be a 2-functor. $\mathbf{C}_\gamma^{\mathbf{I}}$ has the same 0-cells as $\mathbf{C}^{\mathbf{I}}$, however, $\mathbf{C}_\gamma^{\mathbf{I}}(d, d')$ is the full subcategory of $\mathbf{C}^{\mathbf{I}}(d, d')$ consisting of quasi-natural transformations $\xi : d \longrightarrow d'$ such that $\gamma \circ \xi$ is a strict natural transformation. There is also a generalization of Δ to $\Delta_\gamma : \mathbf{C} \longrightarrow \mathbf{C}_\gamma^{\mathbf{I}}$. In detail $\mathbf{C}_\gamma^{\mathbf{I}}(d, \Delta_\gamma(c))$ consists of cocones where $\xi_{\gamma(I')}$ is the identity for all $I' \in \mathbf{I}'$.

Let us move on to quasi-colimits [5, 6, 9] of 2-diagrams. $qcolim :$ $\mathbf{C^I} \longrightarrow \mathbf{C}$ is a 2-functor that is left $\mathbf{Cat}$-adjoint to Δ. That is, there is an isomorphism of categories

$$\mathbf{C}(qcolim(d); c') \cong \mathbf{C^I}(d, \Delta c').$$

In detail, a quasi-colimit of a diagram d is a pair $(qcolim(d), \xi)$ where $qcolim(d)$ is a 0-cell of $\mathbf{C}$ and ξ is a cocone over d with vertex $qcolim(d)$ that satisfies the following universal property: For any cocone over d with vertex $\Delta c'$, $\xi' : d \longrightarrow c'$ there is a unique $\tilde{\xi}' : qcolim(d) \longrightarrow c'$ such that $\tilde{\xi}' \circ \xi_i = \xi_i'$. For any $\Xi : \xi' \Longrightarrow \xi''$ there is a unique $\tilde{\Xi} : \tilde{\xi}' \Longrightarrow \tilde{\xi}''$ such that $\tilde{\Xi} \circ_h \xi_i = \Xi_i$.
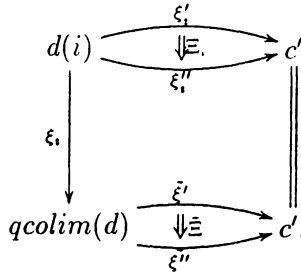
Let $\gamma : \mathbf{I'} \longrightarrow \mathbf{I}$ then $qcolim_\gamma$ is the left $\mathbf{Cat}$-adjoint to Δ_γ. In detail, we insist that ξ_I in
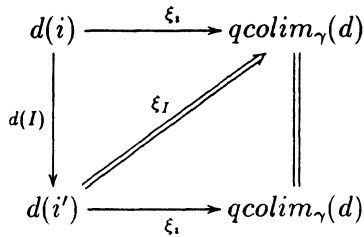
be the identity if $\gamma(I') = I$ for some $I' \in \mathbf{I'}$.

A 2-category is *quasi-cocomplete* if it has all quasi-colimits. For example, $\mathbf{Cat}$ is quasi-cocomplete. A model for $qcolim(d)$ is $\pi_0(1 \downarrow d)$ where π_0 is the functor from $\mathbf{2Cat}$ to $\mathbf{Cat}$ that forces all 2-cells to become identities and $(1 \downarrow d)$ is a 2-comma category (see page 29

of [9] or see below for a definition of a 2-comma category in another context.) The proof that this is a model for $qcolim(d)$ is similar to the one-dimensional colimit case. (Gray [9] (page 210) proves a marvelous theorem that says that **Cat** is the quasi-cocompletion of *Set*!) If **C** is quasi-cocomplete, then for all $\mathbf{T_2}$, and for all $G_2 : \mathbf{T_1} \longrightarrow \mathbf{T_2}$, $\mathbf{2Alg}_{G_2}(\mathbf{T_2}, \mathbf{C})$ is also quasi-cocomplete since one can put a $\mathbf{T_2}$ structure on the quasi-colimit.

A *weak-terminal object* of a 2-category **I** is a 0-cell $t \in \mathbf{I}$ with the following property: for every 0-cell $i \in \mathbf{I}$ there is a 1-cell $l : i \longrightarrow t$ and for any two 1-cells $l, l' : i \longrightarrow t$ there is a unique iso-2-cell $\iota : l \Longleftrightarrow l'$. Let $\gamma : \mathbf{I'} \longrightarrow \mathbf{I}$ be a 2-functor. A γ-relative terminal object is a weak terminal object in **I** with the added requirement that if $i = \gamma(i')$ for some $i' \in \mathbf{I'}$ then ι is the identity i.e. l is a *unique* 1-cell. If $\gamma = id_{\mathbf{I}}$ then a γ-relative terminal object is, in fact, a terminal object. If γ is the unique 2-functor from the empty 2-category to **I** then a γ-relative terminal object is a weak-terminal object. Whereas a terminal object is unique up to a unique isomorphism, a weak-terminal object is unique up to an equivalence. To see this, let t_1 and t_2 be weak-terminal objects. We then have

$$t_1 \xrightarrow{\ f\ } t_2 \xrightarrow{\ g\ } t_1 \xrightarrow{\ f\ } t_2.$$
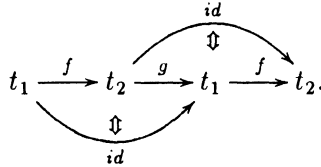
The reader inclined to think topologically should think of the terminal object as the one-point topological space and a weak-terminal object as a contractible pointed space.

Proposition 3 *Let $t \in \mathbf{I}$ be a γ-relative terminal object and let $d : \mathbf{I} \longrightarrow \mathbf{C}$ be a 2-diagram, then $qcolim_\gamma(d)$ is equivalent to $d(t)$. If $\gamma = id_{\mathbf{I}}$ then $qcolim_\gamma(d)$ is isomorphic to $d(t)$*

Given $G : \mathbf{T_1^{op}} \longrightarrow \mathbf{T_2^{op}}$ and a 0-cell $n \in \mathbf{T_2}$, we define the 2-comma category $(G \downarrow n)$. 0-cells are pairs $(Gm, g : Gm \longrightarrow n)$; 1-cells are pairs $(Gh : Gm \longrightarrow Gm', \tau_h : g' \circ Gh \Longrightarrow g)$ where τ_h is a 2-cell in $\mathbf{T_2^{op}}$ that

makes

$$
\begin{array}{ccc}
G(m) & \xrightarrow{\;\;g\;\;} & n \\
\Big\downarrow{\scriptstyle Gh} & {\scriptstyle \tau_h}\nearrow & \Big\| \\
G(m') & \xrightarrow[\;\;g'\;\;]{} & n
\end{array}
$$

commute; 2-cells are $G(\beta) : Gh \Longrightarrow Gh'$ that satisfy $\tau_h = \tau_{h'} \circ G(\beta)$. If $G_2 : (\mathbf{T}_0^{op})' \longrightarrow \mathbf{T}_2^{op}$ then $(G \downarrow n)_{G_2}$ is the locally full subcategory where $\tau_h = id$ if $h = G_2(h')$ for some h' in $(\mathbf{T}_0^{op})'$. $f : n \longrightarrow n'$ in $\mathbf{T}_2^{op}$ induces a 2-functor $(G \downarrow f) : (G \downarrow n) \longrightarrow (G \downarrow n')$. $\alpha : f \Longrightarrow f'$ in $\mathbf{T}_2^{op}$ induces a 2-natural transformation $(G \downarrow \alpha) : (G \downarrow f) \Longrightarrow (G \downarrow f')$. There is also an obvious forgetful 2-functor $P : (G \downarrow n) \longrightarrow \mathbf{T}_1^{op}$ that commutes with $(G \downarrow f)$ and $(G \downarrow \alpha)$. There are similar properties for $(G \downarrow n)_{G_2}$.

The final preliminary needed is

Definition 4 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be 2-categories. Let* $L : \mathbf{A} \longrightarrow \mathbf{B}$ *and* $G : \mathbf{B} \longrightarrow \mathbf{A}$ *be 2-functors.* L *is a (strict c.f. pg 168 of [9]) left quasi-adjoint of* G *if there exists two quasi-natural transformations* $\eta : id_{\mathbf{A}} \longrightarrow GL$ *and* $\varepsilon : FG \longrightarrow id_{\mathbf{B}}$ *strictly satisfying the usual two triangle identities.*

Every $G : \mathbf{T}_1^{op} \longrightarrow \mathbf{T}_2^{op}$ within a commutative square

$$
\begin{array}{ccc}
\mathbf{T}_0^{op} & \xrightarrow{\;\;G_1\;\;} & \mathbf{T}_1^{op} \\
\Big\downarrow{\scriptstyle G_0} & & \Big\downarrow{\scriptstyle G} \\
(\mathbf{T}_0')^{op} & \xrightarrow[\;\;G_2\;\;]{} & \mathbf{T}_2^{op}
\end{array}
\tag{4}
$$

induces a 2-functor $G^* : \mathbf{Hom}(\mathbf{T}_2^{op}, \mathbf{C}) \longrightarrow \mathbf{Hom}(\mathbf{T}_1^{op}, \mathbf{C})$ via precomposition. From the fact that G preserves products and the square (4) commutes, G^* restricts to an algebraic 2-functor $G^* : \mathbf{2Alg}_{G_2}(\mathbf{T}_2, \mathbf{C}) \longrightarrow \mathbf{2Alg}_{G_1}(\mathbf{T}_1, \mathbf{C})$.

Theorem 1 *Let* $\mathbf{C}$ *be a Cartesian closed quasi-cocomplete 2-category. Every* $G^* : \mathbf{2Alg}_{G_2}(\mathbf{T}_2, \mathbf{C}) \longrightarrow \mathbf{2Alg}_{G_1}(\mathbf{T}_1, \mathbf{C})$. *has a (strict) left quasi-adjoint* $Lan_G(F) : \mathbf{2Alg}_{G_1}(\mathbf{T}_1, \mathbf{C}) \longrightarrow \mathbf{2Alg}_{G_2}(\mathbf{T}_2, \mathbf{C})$ *which can be*

computed "pointwise" for F in $\mathbf{2Alg}_{G_1}(\mathbf{T}_1, \mathbf{C})$ as

$$Lan_G(F)(?) = qcolim_{G_1}(F \circ P : (G \downarrow (?))_{G_2} \longrightarrow \mathbf{T}_1^{op} \longrightarrow \mathbf{C}).$$

Furthermore: i) Lan_G takes quasi-natural transformations to natural transformations. ii)η_F has a left inverse and iii) ε_K has a right inverse.

Proof. By cocompleteness of $Lan_G(F)(n)$ is an object of $\mathbf{C}$. We prove the many steps in small bites:

$\underline{Lan_G(F)\text{ is a 2-functor.}}$ $f : n \longrightarrow n'$ induces $(G \downarrow f)_{G_2} : (G \downarrow n)_{G_2} \longrightarrow (G \downarrow n')_{G_2}$. which induces $\tilde{f} : Lan(F)(n) \longrightarrow Lan(F)(n')$. We must stress that for all $g : Gm \longrightarrow n$ in $(G \downarrow n)_{G_2}$

$$\begin{array}{ccc}
& & Lan(F)(n) \qquad\qquad (5) \\
& \overset{\xi_g}{\nearrow} & \Big| \\
Fm & & \Big| \tilde{f} \\
& \underset{\xi_{f \circ g}}{\searrow} & \downarrow \\
& & Lan(F)(n')
\end{array}$$

commutes strictly. There is a similar picture for $\alpha : f \Longrightarrow f'$ and the induced $\tilde{\alpha} : Lan(F)(f) \Longrightarrow Lan(F)(f')$.

$\underline{Lan_G(F)\text{ preserves products.}}$ This is very similar to the one dimensional case and we leave it for the reader. Preservation of products is not true for all 2-categories but it is true for the usual 2-categories that one takes algebras in, like $\mathbf{Cat}$ and any other Cartesian closed 2-category.

$\underline{Lan_G\text{ takes quasi-natural transformations to natural transformations.}}$ Let $\sigma : F \longrightarrow F'$ be a quasi-natural transformation in $\mathbf{2Alg}_{G_1}(\mathbf{T}_1, \mathbf{C})$. $\sigma_m : Fm \longrightarrow F'm$ makes $Lan(F')(n)$ satisfy the universal property of $Lan(F)(n)$ and so we have the commuting square

$$\begin{array}{ccc}
Fm & \overset{\sigma_m}{\longrightarrow} & F'm \qquad\qquad (6) \\
\Big\downarrow {\scriptstyle \xi} & & \Big\downarrow {\scriptstyle \xi'} \\
Lan(F)(n) & \overset{\sigma_n}{-\ -\ -\ \dashrightarrow} & Lan(F')(n).
\end{array}$$

For $f : n \longrightarrow n'$, there is

$$
\begin{array}{ccc}
Fm & \xrightarrow{\quad\sigma_m\quad} & F'm \\
\| & & \| \\
& Lan(F)(n) \dashrightarrow{\tilde{\sigma}_n} Lan(F')(n) & \\
& \big\downarrow{\scriptstyle LanFf} \qquad \big\downarrow{\scriptstyle LanF'f} & \\
& Lan(F)(n') \dashrightarrow{\tilde{\sigma}_{n'}} Lan(F')(n') & \\
\| & & \| \\
Fm & \xrightarrow{\quad\sigma_m\quad} & F'm
\end{array}
$$

where the left and right quadrilaterals commute from diagram (5). The top and bottom quadrilaterals commute from diagram (6). Since $Lan(F')(n')$ satisfy the universal properties of $Lan(F)(n)$ there is a unique $Lan(F)(n) \longrightarrow Lan(F')(n')$ which coheres with the surrounding commutative quadrilaterals. Hence the inner square commutes making $\tilde{\sigma}$ a natural transformation (*not quasi*) in $\mathbf{2Alg}_{G_2}(\mathbf{T}_2, \mathbf{C})$.

The unit of the $G^* \vdash Lan_G$ quasi-adjunction: $\eta_F : F \longrightarrow (G^* \circ Lan_G)(F)$

$$(G^* \circ Lan_G)(F) = Lan_G(F) \circ G = qcolim_{G_1}(F \circ P : (G \downarrow G(?)_{G_2}) \longrightarrow \mathbf{T}_1^{op} \longrightarrow \mathbf{C})$$

Within $(G \downarrow G(n))_{G_2}$ there is $id : G(n) \longrightarrow G(n)$. $(F \circ P)(id : G(n) \longrightarrow G(n)) = Fn$. And so we set

$$[\eta_{F,n} = \xi_{id}] : Fn \longrightarrow [qcolim_{G_1} = (G^* \circ Lan_G)(F)(n)].$$

Given $f : n \longrightarrow n'$ in $\mathbf{T}_1^{op}$, we have $G(f) : Gn \longrightarrow Gn'$ in $\mathbf{T}_2^{op}$ which induces $(G \downarrow G(f))_{G_2} : (G \downarrow Gn)_{G_2} \longrightarrow (G \downarrow Gn')_{G_2}$ and hence $\tilde{G}f : G^* Lan(F)(n) \longrightarrow G^* Lan(F)(n')$. Which makes η_F a quasi-natural

transformation:

$$
\begin{array}{ccc}
Fn & \xrightarrow{\ \eta_{F,n}\ } & G^*Lan(F)(n) \\
\end{array}
$$

where the upper right triangle commutes.

Remark.3.1: If $f = G_1(\bar{f})$ for some $\bar{f}$ in $\mathbf{T}_0^{op}$ then ξ_f is the identity making the square commute. If this is true for all $f \in \mathbf{T}_1^{op}$ then η_F is, in fact, a natural transformation. $\square$

One gets the left inverse of η_F from the commutativity of the bottom quadrilateral of
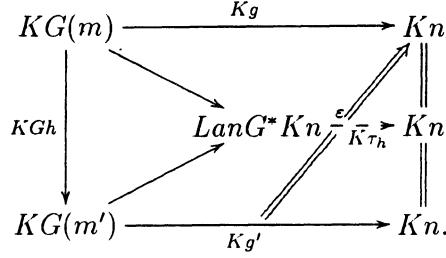
The counit of the $G^* \vdash Lan_G$ quasi-adjunction: $\varepsilon_K : (Lan_G \circ G^*)(K) \longrightarrow K$

$(Lan_G \circ G^*)(K) = Lan_G(KG) = qcolim_{G_1}(K \circ G \circ P : (G \downarrow (?))_{G_2} \longrightarrow \mathbf{T}_1^{op} \longrightarrow \mathbf{T}_2^{op} \longrightarrow \mathbf{C}).$

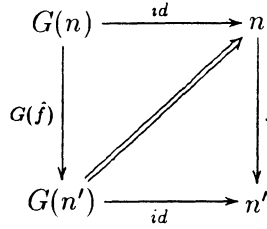Consider the following typical diagram in $(G \downarrow n)$

Applying $qcolim_{G_1}(K \circ G \circ P)$ and K to this diagram gives us:

$$
\begin{array}{ccc}
KG(m) & \xrightarrow{\ \ Kg\ \ } & Kn \\
\Big\downarrow{\scriptstyle KGh} & LanG^*Kn \xrightarrow[\ \overline{K\tau_h}\ ]{\varepsilon} Kn & \Big\| \\
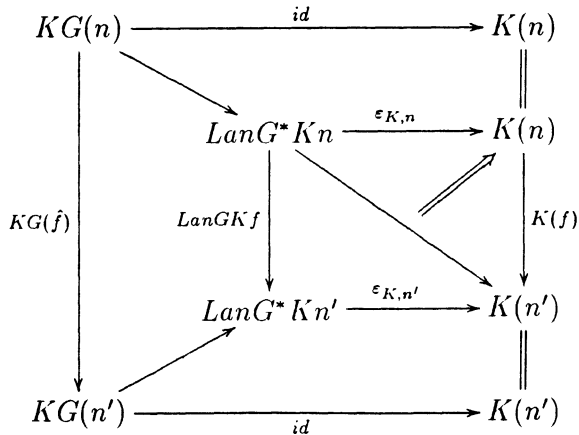KG(m') & \xrightarrow[\ \ Kg'\ \ ]{} & Kn.
\end{array}
$$

And so there is the induced $\varepsilon_{K,n} : (Lan_G \circ G^*)(K)(n) \longrightarrow Kn$.

ε_K is also quasi-natural. Given $f : n \longrightarrow n'$ in $\mathbf{T}_2^{op}$ we have the diagram in $(G \downarrow n')$

$$
\begin{array}{ccc}
G(n) & \xrightarrow{\ id\ } & n \\
\Big\downarrow{\scriptstyle G(\hat{f})} & & \Big\downarrow{\scriptstyle f} \\
G(n') & \xrightarrow[\ id\ ]{} & n'
\end{array}
$$

where $\hat{f}$ is in $\mathbf{T}_1^{op}$. This square commutes if $\hat{f} = G_1(\bar{f})$ for some $\bar{f} \in \mathbf{T}_0^{op}$. Applying K and taking appropriate quasi-colimits to this commutative or noncommutative diagram gives us:

$$
\begin{array}{ccc}
KG(n) & \xrightarrow{\qquad id \qquad} & K(n) \\
 & LanG^*Kn \xrightarrow{\ \varepsilon_{K,n}\ } K(n) & \Big\| \\
\Big\downarrow{\scriptstyle KG(\hat{f})}\ \ \Big\downarrow{\scriptstyle LanGKf} & & \Big\downarrow{\scriptstyle K(f)} \\
 & LanG^*Kn' \xrightarrow{\ \varepsilon_{K,n'}\ } K(n') & \\
KG(n') & \xrightarrow[\qquad id \qquad]{} & K(n')
\end{array}
$$

where the surrounding quadrilateral and the lower left triangle commute. If the outer square commutes then the inner square must also commute.

Remark.3.2: If $\hat{f} = G_1(\bar{f})$ for some $\bar{f} \in \mathbf{T}_0^{op}$, then the outer square commutes. If this is true for all $f \in \mathbf{T}_2^{op}$ then ε_K is, in fact, a natural transformation. □

One gets the right inverse of ε_K from the commutativity of the bottom quadrilateral of the diagram that gives ε_K when you set $m' = n$ and $g' = id_n$.

We leave the following usual two triangle identities for the reader's pleasure:

Q.E.D.

Now that we have these tools, we can go on and prove coherence theorems. Following Remark.3.1 (respectively Remark.3.2) we have

Theorem 2 *If there exists a 2-theory-morphism $H : \mathbf{T}_2^{op} \longrightarrow \mathbf{T}_0^{op}$ (resp. $H' : \mathbf{T}_1^{op} \longrightarrow (\mathbf{T}_0^{op})'$ such that*

the two triangles commute, then the unit (resp. counit) of the $G^* \vdash Lan_G$ adjunction is a natural transformation.□

Setting $\mathbf{T}_0^{op} = \mathbf{T}_1^{op} = \overline{\mathbf{Fin}}_{sk}^{op}$ gives us the naturality of the counit and so we have

Corollary 1 *For any $G_2 : \mathbf{T}_0^{op} \longrightarrow \mathbf{T}_2^{op}$ we have the following isomorphism of 2-categories:*

$$\mathbf{2Alg_{G_2}}(\mathbf{T}_2, \mathbf{Cat})(Lan_G F, K) \quad \cong \quad \mathbf{Cat}(F, G^* K)$$

where F is a category (functor from the trivial theory) and K is a $\mathbf{T}_2$-algebra. i.e. $Lan_G F$ is the free $\mathbf{T}_2$ category over F.

Proof. The only non-obvious part is the universality of the counit. This is similar to the one dimensional case. We leave the following diagram to help:

Setting $\mathbf{T}_0^{op}$ to also be $\overline{\mathbf{Fin}}_{sk}^{op}$ gives us an unrestricted $Lan F$. This is used in the reconstruction of a theory from its category of algebras (For technical reasons from the quasi-Yoneda lemma we insist that the category of algebras have quasi-natural transformations where the squares commute up to a iso-2-cell.)

Theorem 3 *Every theory $\mathbf{T}$ is quasi-equivalent to its 2-category of algebras, $\mathbf{2Alg^i}(\mathbf{T}, \mathbf{Cat})$.*

Proof Let $F_{[n]} : \overline{\mathbf{Fin}}_{sk}^{op} \longrightarrow \mathbf{Cat}$ be the "constant" functor on $[n]$ i.e. $F_{[n]}(m) \cong [n]^m$. $Lan F_{[n]}$ is the free $\mathbf{T}$-algebra on $[n]$ elements. By Proposition 2, $Lan F_{[n]} \cong \mathbf{T}^{op}(n, -)$. Using this and the quasi-Yoneda lemma (e), we get the following quasi-equivalence of categories

$$\begin{aligned} \mathbf{2Alg^i}(\mathbf{T}, \mathbf{Cat})(F_{[n]}, F_{[m]}) \quad &\cong \quad \mathbf{qNat^i}(F_{[n]}, F_{[m]}) \\ &\cong \quad \mathbf{qNat^i}(\mathbf{T}^{op}(n, -), \mathbf{T}^{op}(m, -)) \\ &\cong \quad \mathbf{T}^{op}(n, m) \quad \square . \end{aligned}$$

This quasi-equivalence is the counit of the adjunction:

$$\mathbf{2\widetilde{Theor}ies} \underset{\text{Free}}{\overset{2Alg(-,Cat)}{\rightleftarrows}} \bot \quad \mathbf{2\widetilde{Cat}/Cat}.$$

Where $\mathbf{2\widetilde{Cat}/Cat}$ denotes the tractable 2-functors $U : \mathbf{C} \longrightarrow \mathbf{Cat}$. Tractable means $\mathbf{C}$ must be a local groupoid and the category $\mathbf{2Cat}(U^n, U^m)$ must be small and locally small. There will be more about this adjunction at the end of section 4.

Definition 5 *A weakly-unique quasi-section of* $G : \mathbf{T}_1^{op} \longrightarrow \mathbf{T}_2^{op}$ *is a 2-theory-morphism* $H : \mathbf{T}_2^{op} \longrightarrow \mathbf{T}_1^{op}$ *satisfying:*

1. the diagram

$$\begin{array}{ccc} \mathbf{T}_0^{op} & \xrightarrow{\ \ G_1\ \ } & \mathbf{T}_1^{op} \\ \downarrow & & \uparrow{\scriptstyle H} \\ (\mathbf{T}_0^{op})' & \xrightarrow[\ \ G_2\ \ ]{} & \mathbf{T}_2^{op} \end{array}$$

commutes

2. for every 1-cell $f \in \mathbf{T}_2^{op}$ *there is a 2-cell* $\alpha : (G \circ H)(f) \Longrightarrow f$

3. H is unique up to a **unique** *2-cell.*

Theorem 4 *If* G *has a weakly-unique quasi-section, then* η_F *is an equivalence for every* $F \in \mathbf{2Alg}_{G_1}(\mathbf{T}_1^{op}, \mathbf{Cat})$.

Proof. The definition of a weakly-unique quasi-section insures that $id : G(n) \longrightarrow G(n)$ is a G_1-relative terminal object of $(G \downarrow G(n))$. Hence, using Proposition 3, we have that η_F is an equivalence. $\square$
Example.3.4: Setting $\mathbf{T}_0^{op} = (\mathbf{T}_0^{op})' = \overline{\mathbf{Fin}}_{sk}^{op}$ and $G : \mathbf{T}_{Mon} \longrightarrow \mathbf{T}_{sMon}$ (the obvious "strictification" functor), we have that every monoidal category is tensor equivalent to a strict monoidal category. $\square$

Example.3.5: Following the above, we have a more general theorem. Let $\mathbf{T}_X^{op}$ be any theory that "contains" the monoidal theory. Let $\mathbf{T}_{sX}^{op}$ be the strict version of that theory with G being the "strictification" 2-theory-morphism. Then η_F is an equivalence. $\square$

As with all conditions, the case where a condition fails is far more interesting. For example $G : \mathbf{T}_{Assoc} \longrightarrow \mathbf{T}_{sMon}$ has many quasi-sections but they are not unique up to a **unique** isomorphism. Similarly for $G : \mathbf{T}_{Braid} \longrightarrow \mathbf{T}_{Sym}$. Notice that in all these cases, $Lan_G F$ always exist and there are many things that one can say about η_F. But it is not an equivalence. There is much structure to explore.

Many other coherence theorems can be stated and proved on the syntactical level. For example, Corollary 2.4 (pg. 43) of [13] says that given

$\eta_F : F \longrightarrow G^* Lan_G F$ determines an equivalence of categories

$$\mathbf{2Alg}(\mathbf{T}_{Braid}, \mathbf{Cat})(G_1^*(Lan_G F), G_1^*(V)) \cong \mathbf{Cat}(F, G^*(V))$$

This is proven using the properties of G, G_1 and the universal properties of Lan_G.

One can go on to formalize many coherence statements like "If G is locally faithful etc ... and G_1 is faithful etc ... , then η_F is ... and ε_K is ...." . We leave this noble task for future explorers.

4 Kronecker product

It is common to look at the algebras of one theory in the category of algebras of another theory. The theory of such algebras is given as the Kronecker product of the two theories.

The Kronecker product [7] of (1-)theories is a well understood coherent symmetric monoidal 2-bifunctor $\otimes_K$: **Theories** $\times$ **Theories** $\longrightarrow$ **Theories**. Let T_1 and T_2 be two theories. $T_1 \otimes_K T_2$ is a theory that satisfies the universal property

$$Alg(T_1 \otimes_K T_2, C) \cong Alg(T_1, Alg(T_2, C)).$$

$T_1 \otimes_K T_2$ is constructed as follows. Construct the the coproduct in the category of theories (pushout in **Cat**)

Place a congruence on $T_1 \coprod T_2$ such that for all $f : m \longrightarrow m'$ in T_1 and $g : n \longrightarrow n'$ in T_2 the diagram

commutes. We have a full theory-morphism $T_1 \coprod T_2 \longrightarrow T_1 \otimes_K T_2$.

There is an analogous Kronecker product on the semantic level. Although we have not been able to find this construction in the literature, surely it is well known to the cognoscenti. Denote the tractable 2-functors from **Cat** to Set as **Cat**$/Set$. The semantic Kronecker product is a coherently monoidal symmetric 2-bifunctor $\oplus_K$: **Cat**$/Set$ $\times$

$\mathbf{Cat}/Set \longrightarrow \mathbf{Cat}/Set$. Given two such tractable functors $U_1 : C_1 \longrightarrow Set$ and $U_2 : C_2 \longrightarrow Set$, $C_1 \oplus_K C_2 \longrightarrow Set$ is constructed as follows. Construct the product in $\mathbf{Cat}/Set$ (pullback in $\mathbf{Cat}$)

$$
\begin{array}{ccc}
C_1 \times_{Set} C_2 & \longrightarrow & C_1 \\
\downarrow & & \downarrow \\
C_2 & \longrightarrow & Set.
\end{array}
$$

$C_1 \times_{Set} C_2$ is to be thought of as sets with both a C_1 structure and a C_2 structure. $C_1 \oplus_K C_2$ is the full subcategory of $C_1 \times_{Set} C_2$ consisting of those objects c that satisfy the following condition: for all $f : c^m \longrightarrow c^{m'}$ in C_1 and $g : c^n \longrightarrow c^{n'}$ in C_2,

$$
\begin{array}{ccc}
& U_2(c^n)^m \xrightarrow{U_2(g)^m} U_2(c^{n'})^m & \\
U_1(c^m)^n & & U_1(c^m)^{n'} \\
{\scriptstyle U_1(f)^n}\downarrow & & \downarrow{\scriptstyle U_1(f)^{n'}} \\
U_1(c^{m'})^n & & U_1(c^{m'})^{n'} \\
& U_2(c^n)^{m'} \xrightarrow{U_2(g)^{m'}} U_2(c^{n'})^{m'} &
\end{array}
$$

commutes. It is not hard to show that the structure - semantics adjunction (equivalence) takes the Kronecker product theories to the Kronecker product semantics and vice versa. See the end of section 4 for a large diagram showing what commutes.

There is a two-dimensional analogue to the Kronecker product. Rather than look at two 2-theories $\mathbf{T}_1$ and $\mathbf{T}_2$ that are disconnected, we shall assume that both of these theories have an underlying $\mathbf{T}_0$, i.e. there is

a diagram in $\widetilde{\mathbf{2Theories}}$

$$\mathbf{T}_1 \xleftarrow{\quad G_1 \quad} \mathbf{T}_0 \xrightarrow{\quad G_2 \quad} \mathbf{T}_2.$$

We can, however, give a similar bifunctor which assumes no underlying $\mathbf{T}_0$ (i.e. $\mathbf{T}_0 = \widetilde{\mathbf{Fin}}_{sk}$) or assuming a commutative square of 2-theories:

$$
\begin{array}{ccc}
\mathbf{T}_0 & \xrightarrow{\quad G_1 \quad} & \mathbf{T}_1 \\
\downarrow & & \downarrow \\
\mathbf{T}_0' & \xrightarrow{\quad G_2 \quad} & \mathbf{T}_2.
\end{array}
$$

However most examples are found with one underlying 2-theory.

Definition 6 *A (2-)Kronecker product of 2-theories is a 3-bifunctor*

$$\otimes_0^K : (\mathbf{T}_0 \downarrow \widetilde{\mathbf{2Theories}}) \times (\mathbf{T}_0 \downarrow \widetilde{\mathbf{2Theories}}) \longrightarrow (\mathbf{T}_0 \downarrow \widetilde{\mathbf{2Theories}})$$

that satisfies the following universal property: for all

$$\mathbf{T}_1 \xleftarrow{\quad G_1 \quad} \mathbf{T}_0 \xrightarrow{\quad G_2 \quad} \mathbf{T}_2$$

there is an induced

$$
\begin{array}{ccc}
\mathbf{T}_0 & \xrightarrow{\quad G_1 \quad} & \mathbf{T}_1 \\
G_2 \downarrow & \searrow^{G_1 \otimes_0^K G_2} & \downarrow \\
\mathbf{T}_2 & \xrightarrow{\quad\quad} & \mathbf{T}_1 \otimes_0^K \mathbf{T}_2.
\end{array}
$$

and for all 2-categories with finite products $\mathbf{C}$, *an isomorphism of 2-categories*

$$\mathbf{2Alg}_{G_1 \otimes_0^K G_2}(\mathbf{T}_1 \otimes_0^K \mathbf{T}_2, \mathbf{C}) \cong \mathbf{2Alg}_{G_1}(\mathbf{T}_1, \mathbf{2Alg}_{G_2}(\mathbf{T}_2, \mathbf{C})) \quad (7)$$

which is natural for all cells in $(\mathbf{T}_0 \downarrow \widetilde{\mathbf{2Theories}})$ *and for all cells in* $\mathbf{C}$.

When $\mathbf{C}$ is "nice" and the 2-theory is reconstructible from its 2-category of algebras we have

$$
\begin{aligned}
\mathbf{2Alg}(\mathbf{T}_1 \otimes_0^K (\mathbf{T}_2 \otimes_0^K \mathbf{T}_3), \mathbf{C}) \;&\cong\; \mathbf{2Alg}(\mathbf{T}_1, \mathbf{2Alg}(\mathbf{T}_2 \otimes_0^K \mathbf{T}_3, \mathbf{C})) \\
&\cong\; \mathbf{2Alg}(\mathbf{T}_1, \mathbf{2Alg}(\mathbf{T}_2, \mathbf{2Alg}(\mathbf{T}_3, \mathbf{C}))) \\
&\cong\; \mathbf{2Alg}((\mathbf{T}_1 \otimes_0^K \mathbf{T}_2), \mathbf{2Alg}(\mathbf{T}_3, \mathbf{C})) \\
&\cong\; \mathbf{2Alg}((\mathbf{T}_1 \otimes_0^K \mathbf{T}_2) \otimes_0^K \mathbf{T}_3, \mathbf{C}).
\end{aligned}
$$

and hence $\mathbf{T}_1 \otimes_0^K (\mathbf{T}_2 \otimes_0^K \mathbf{T}_3) \cong (\mathbf{T}_1 \otimes_0^K \mathbf{T}_2) \otimes_0^K \mathbf{T}_3$. It is conjectured that this bifunctor is actually coherently associative (c.f. [11]) but we leave this question for now. If we insist that the Kronecker product satisfy

$$
\mathbf{2Alg}^i_{G_1 \otimes_0^K G_2}(\mathbf{T}_1 \otimes_0^K \mathbf{T}_2, \mathbf{C}) \cong \mathbf{2Alg}^i_{G_1}(\mathbf{T}_1, \mathbf{2Alg}^i_{G_2}(\mathbf{T}_2, \mathbf{C}))
$$

then $\mathbf{T}_1 \otimes_0^K \mathbf{T}_2$ will be (coherently) isomorphic to $\mathbf{T}_2 \otimes_0^K \mathbf{T}_1$.

In order to construct $\mathbf{T}_1 \otimes_0^K \mathbf{T}_2$, we take the coproduct $\mathbf{T}_1 \coprod_{\mathbf{T}_0} \mathbf{T}_2$ in $(\mathbf{T}_0 \downarrow \mathbf{2Theories})$ and we freely add in the following 2-cells: For every $f : m \longrightarrow m'$ in $\mathbf{T}_1$ and $g : n \longrightarrow n'$ in $\mathbf{T}_2$ we add the 2-cell $\delta(f,g)$ that makes the following diagram commute:

$$(8)$$

[If $\otimes_0^K$ is to be symmetric, then we must insist that $\delta(f,g)$ be an isomorphism.] The δ's must satisfy the following coherence conditions that are compatible to the four coherence conditions in the definition of a quasi-natural transformation.

1. If f is in the image of G_1 [or if g is in the image of G_2], then $\delta(f,g)$ must be set to the identity.

2. δ must preserve products in f [and g] as in Figure I.

3. $\delta(f \circ f', g) = \delta(f,g) \circ_v \delta(f',g)$ [and $\delta(f, g \circ g') = \delta(f,g) \circ_h \delta(f,g')$.]

4. δ must preserve 2-cells. i.e. If there is a 2-cell in $\mathbf{T}_1$

$$\bullet \xRightarrow{\ \ \alpha\ \ } \bullet$$
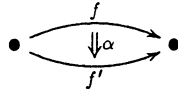
then we have the following equality of diagrams (we leave out the corner isomorphisms and the exponents)
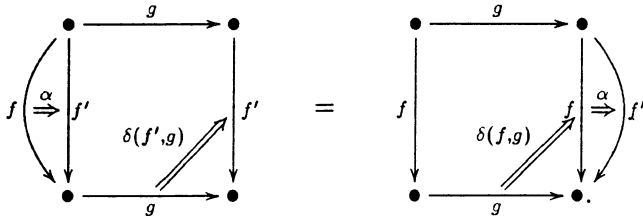
[For symmetry, a

$$\bullet \xRightarrow{\ \ \beta\ \ } \bullet$$
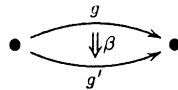
in $\mathbf{T}_2$ implies

Remark.4.1: We demand not only that the δ's preserve the Cartesian product, but that the δ's inherit *all* the coherence properties of the Cartesian product. $\square$

The fact that there is choice in the construction of $\mathbf{T}_1 \otimes_0^K \mathbf{T}_2$, should not disturb the reader too much since we never claimed that $\mathbf{T}_1 \otimes_0^K \mathbf{T}_2$ should be unique. Rather, it should be unique up to a (2-)isomorphism. In order to show that our construction of $\mathbf{T}_1 \otimes_0^K \mathbf{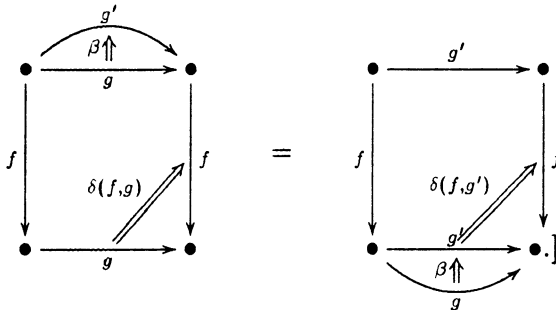T}_2$ satisfies the universal properties stated in 7, let us examine an algebra in $\mathbf{2Alg}_{G_1}(\mathbf{T}_1, \mathbf{2Alg}_{G_2}(\mathbf{T}_2, \mathbf{C}))$. An algebra is a finite product preserving functor $F : \mathbf{T}_1 \longrightarrow \mathbf{C}$. Assume $F(1) = G : \mathbf{T}_2 \longrightarrow \mathbf{C}$. Then $F(m) = F(1)^m = G^m : \mathbf{T}_2 \longrightarrow \mathbf{C}$ For every $f : m \longrightarrow m'$ in $\mathbf{T}_1$, $F(f)$ is a quasi-natural transformation from G^m to $G^{m'}$. In order for $F(f)$ to be such a quasi-natural transformation, we must have that every $g : n \longrightarrow n'$ in $\mathbf{T}_2$, makes a the following diagram:

$$\tag{9}$$

This is what is described by 8. $F(f)_g$ is what corresponds to $\delta(f,g)$ in our theory $\mathbf{T}_1 \otimes_0^K \mathbf{T}_2$. The rest of the tedious details melt away when one realizes that our construction was made to mimic the definition of a quasi-natural transformation in our 2-categories of algebras.

There is a similar construction for the Kronecker product on the semantic level:

$$\oplus_K : \mathbf{2\widetilde{Cat}/Cat} \times \mathbf{2\widetilde{Cat}/Cat} \longrightarrow \mathbf{2\widetilde{Cat}/Cat}.$$

We leave the details for the reader.

Our examples have all been proven in Joyal and Street's paper [13]. We are simply restating them in the language of Kronecker products.

All our examples have the δ's as isomorphisms. We, however, must stress that this is an historical accident rather then something intrinsicly important to coherence. Even though most coherence results are about natural *iso*morphisms, one should study the general case where the natural transformations are in not necessarily isomorphisms. The only example in the literature that we know of where coherence questions arise for natural transformations that are not isomorphisms is Yetter's notion of a pre-braiding [30].

In order to make the diagrams in the examples a little more readable, we shall write our morphisms of the theories the opposite way. In other words, we shall write them as if $\mathbf{T}$ was $\mathbf{T}^{op}$.

Example.4.1: Let $\mathbf{T}_{sMon}$ be the theory of strict (associativity) monoidal categories. Let $\mathbf{T}_0$ be the theory of pointed categories, that is the theory of categories with a distinguished element to be thought of as a unit of the tensor product(s). Then we have

$$\mathbf{T}_{sMon} \otimes_0^K \mathbf{T}_{sMon} \cong \mathbf{T}_{sBraid}$$

where $\mathbf{T}_{sBraid}$ is the theory of braided monoidal categories. This result is a two dimensional version of the fact that the Kronecker product of the theory of monoids with itself is the theory of commutative monoids.

In order to distinguish the two (isomorphic) multiplications, we shall denote one by $\otimes : 2 \longrightarrow 1$ and one by $\Phi : 2 \longrightarrow 1$. By the construction of the Kronecker product we have (abandoning corner isomorphisms)
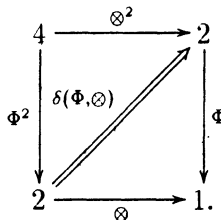
On the semantic level, $\delta(\Phi, \otimes)$ induces an isomorphism

$$\delta(\Phi, \otimes)_{A,A',B,B'} : (A\Phi B) \otimes (A'\Phi B') \longrightarrow (A \otimes A')\Phi(B \otimes B').$$

Setting $A' = B = I$, we get an isomorphism $A \otimes B' \longrightarrow A\Phi B'$. Setting $A = B' = I$ we get an isomorphism $B \otimes A' \longrightarrow A'\Phi B$. And so setting

$$\gamma_{A,B} = \delta_{I,A,B,I}^{-1} \circ \delta_{A,I,I,B} : A \otimes B \longrightarrow A\Phi B \longrightarrow B \otimes A.$$

Only the braiding relation is left to be shown. By creatively pasting the coherence conditions for δ (i.e. $\delta \circ (\delta \times 1) = \delta \circ (1 \times \delta)$), we have
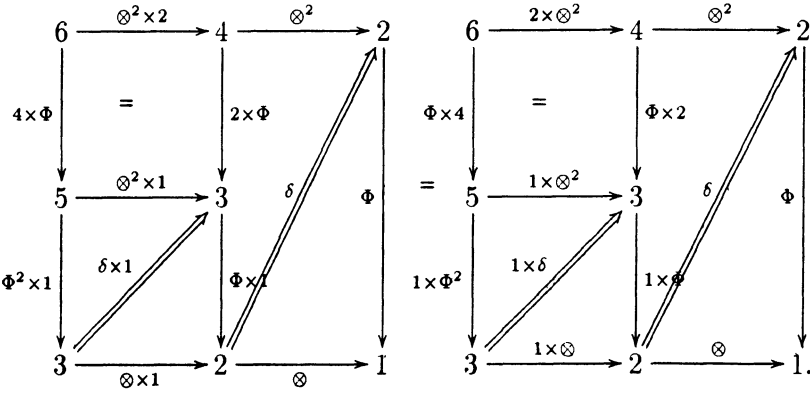
On the semantic level this means

$$(A\Phi A') \otimes (B\Phi B') \otimes (C\Phi C') \xrightarrow{\delta \otimes Id_{C\Phi C'}} (A \otimes B)\Phi(A' \otimes B') \otimes (C\Phi C')$$

$$Id_{A\Phi A'} \otimes \delta \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \delta_{A\otimes B, A'\otimes B', C, C'}$$

$$(A\Phi A') \otimes (B \otimes C)\Phi(B' \otimes C') \xrightarrow{\delta_{A,A',B\otimes C,B'\otimes C'}} (A \otimes B \otimes C)\Phi(A'' \otimes B' \otimes C')$$

commutes. Setting the appropriate letters to I (see page 58 of [13]) gives us the braiding relations: e.g.

$$\gamma_{A\otimes B, C} = (\gamma_{A,C} \otimes Id_B) \circ (Id_A \otimes \gamma_{B,C}).$$

□

Example.4.2: If we abandon the strictness (associativity) we get

$$\mathbf{T}_{Mon} \otimes_0^K \mathbf{T}_{Mon} \cong \mathbf{T}_{Braid}.$$

The multiplications are $\otimes : 2 \longrightarrow 1$ and $\Phi : 2 \longrightarrow 1$. Their respective reassociations are $\alpha : \oti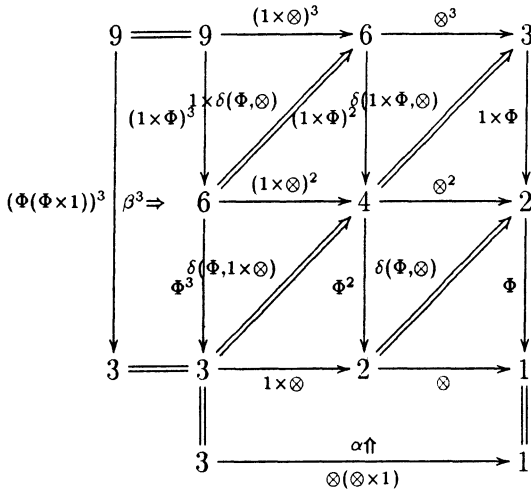mes(\otimes \times 1) \Longrightarrow \otimes(1 \times \otimes)$ and $\beta : \Phi(\Phi \times 1) \Longrightarrow \Phi(1 \times \Phi)$. Using similar results from last example, we can show that $A \otimes B \cong A\Phi B \cong B\Phi A \cong B \otimes A$ as well as $\alpha \cong \beta$. The braiding relation is the only difference. Creatively pasting coherence conditions of δ we get several diagrams of the form:

$$
\begin{array}{c}
9 = 9 \xrightarrow{(1\times\otimes)^3} 6 \xrightarrow{\otimes^3} 3 \\
(\Phi(\Phi\times 1))^3 \quad \beta^3\Rightarrow \\
1\times\delta(\Phi,\otimes) \quad \delta(1\times\Phi,\otimes) \\
(1\times\Phi)^3 \quad (1\times\Phi)^2 \quad 1\times\Phi \\
6 \xrightarrow{(1\times\otimes)^2} 4 \xrightarrow{\otimes^2} 2 \\
\delta(\Phi,1\times\otimes) \quad \delta(\Phi,\otimes) \\
\Phi^3 \quad \Phi^2 \quad \Phi \\
3 = 3 \xrightarrow{1\times\otimes} 2 \xrightarrow{\otimes} 1 \\
\alpha\Uparrow \\
3 \xrightarrow{\otimes(\otimes\times 1)} 1
\end{array}
$$

This diagram should be set equal to:

$$
\begin{array}{c}
9 \xrightarrow{(1\times\otimes)^3} 6 \xrightarrow{\otimes^3} 3 = 3 \\
\delta(\Phi\times 1,1\times\otimes) \quad \delta(\Phi\times 1,\otimes) \\
(\Phi\times 1)^3 \quad (\Phi\times 1)^2 \quad \Phi\times 1 \\
6 \xrightarrow{(1\times\otimes)^2} 4 \xrightarrow{\otimes} 2 \quad \beta\Rightarrow \quad \Phi(\Phi\times 1) \\
\delta(\Phi,1\times\otimes) \quad \delta(\Phi,\otimes) \\
\Phi^3 \quad \Phi^2 \quad \Phi \\
3 \xrightarrow{1\times\otimes} 2 \xrightarrow{\otimes} 1 = 1 \\
\alpha\Uparrow \\
3 \xrightarrow{\otimes(\otimes\times 1)} 1
\end{array}
$$
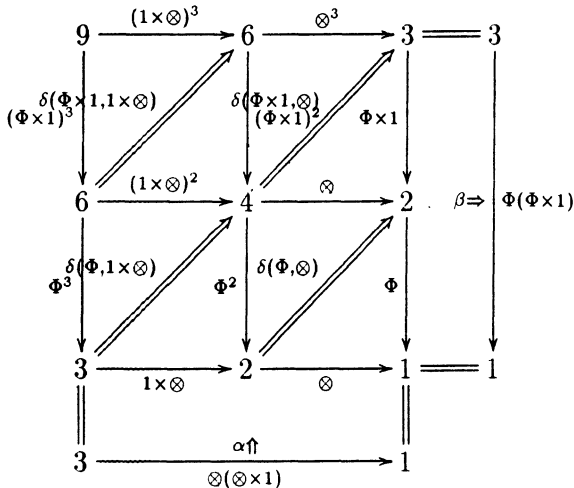
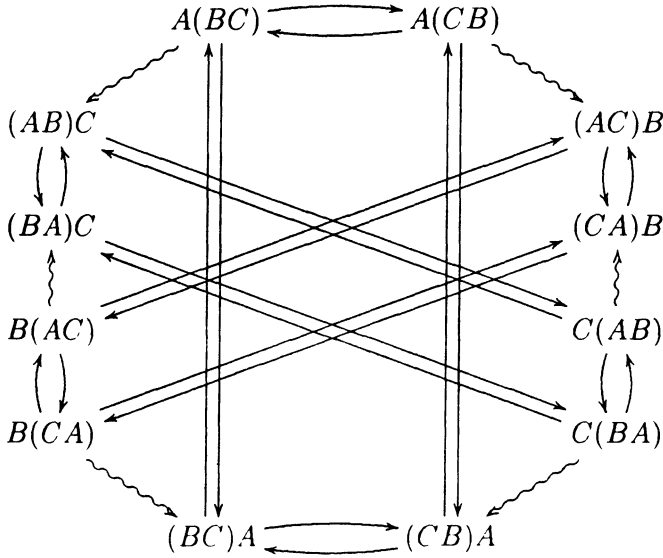Setting 6 of the 9 variables to be I (which will make some of the 2-cells into the identity) and combining different forms of these diagrams will give us the famous dodecahedron:

In order to distinguish the associativity isomorphisms from the commutativity isomorphisms we draw associativity as $\bullet \rightsquigarrow \bullet$. The commuting dodecahedron is worth a few minutes of meditation. The diagram actually indicates many equations. Notice also that the rectangles commute from that naturality of γ. This naturality is, however, a semantical notion! γ is merely a 2-cell in $\mathbf{T}_{Braid}$. The naturality comes from taking algebras in **Cat** where 2-cells are *natural transformations*. This leads us to ask what would happen if we took algebras in other 2-categories? Would the dodecahedron commute?

We would like to stress that the dodecahedron is not a "new" coherence structure. It is rather, a 2-dimensional statement that one reassociativity is a "homomorphism" of the other reassociativity. $\square$

Example.4.3: Let $\mathbf{T}_{sMon}$ be the theory of strict monoidal categories with multiplication $\Phi : 2 \longrightarrow 1$. Let $\mathbf{T}_{sBraid}$ be the theory of strict braided monoidal categories with multiplication $\otimes : 2 \longrightarrow 1$ and braiding $\gamma : \otimes \Longrightarrow \otimes \circ tw$. Let $\mathbf{T}_0$ be the theory of pointed categories, that is, the theory of categories with a distinguished element to be thought of as a unit element. Then we have

$$\mathbf{T}_{sMon} \otimes_0^K \mathbf{T}_{sBraid} \cong \mathbf{T}_{sBraid} \otimes_0^K \mathbf{T}_{sMon} \cong \mathbf{T}_{Sym}.$$

where $\mathbf{T}_{Sym}$ is the theory of symmetric monoidal categories.

From the last two example, we know that $\Phi \cong \otimes$ and that δ can be made into a braiding. This braiding will be isomorphic to γ.

The condition of the construction of $\otimes_0^K$ shows us that
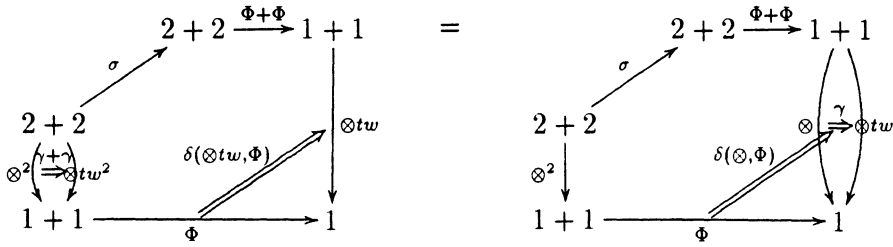
Which translates into

$$
\begin{array}{ccc}
A \otimes A' \otimes B \otimes B' & \xrightarrow{\gamma_{A,A'} \otimes \gamma_{B,B'}} & A' \otimes A \otimes B' \otimes B \\
\Big\downarrow{\scriptstyle Id \otimes \gamma_{A',B} \otimes Id} & & \Big\downarrow{\scriptstyle Id \otimes \gamma_{A,B'} \otimes Id} \\
A \otimes B \otimes A' \otimes B' & \xrightarrow{\gamma_{A \otimes B, A' \otimes B'}} & A' \otimes B' \otimes A \otimes B.
\end{array}
$$

commutes.

Setting $A = B' = I$ the unit (of both multiplications) makes the top horizontal map and the right vertical map the identity. That leaves us with

$$\gamma_{B,A'} \circ \gamma_{A',B} = Id$$

i.e. symmetry. $\square$

Example.4.4: This is actually an example of something that does not work. Let $\mathbf{T}_{sBraid}$ be the theory of strict (associativity) braided monoidal categories with multiplication $\otimes : 2 \longrightarrow 1$ and braiding $\gamma : \otimes \implies \otimes \circ tw$. Let $\mathbf{T}_{Twist}$ be the theory with only one nontrivial generating 2-cell $\theta : Id_1 \implies Id_1$ to be thought of as a twist of a ribbon. Let $\mathbf{T}_0 = \overline{\mathbf{Fin}}_{sk}$. One would expect that $\mathbf{T}_{sBraid} \otimes_0^K \mathbf{T}_{Twist}$ should be the theory of balanced categories (see pg 65 of [13] or page 349 of [14] where they are called ribbon categories. Truth be told, they assume a duality structure for the definition but it is not necessary for our needs.) The Kronecker product of these two theories forces the following equation
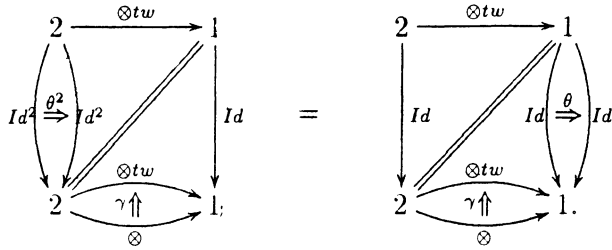
On the semantic level, this becomes

$$\gamma_{A,B} \circ (\theta_A \otimes \theta_B) \quad = \quad (\theta_{A \otimes B}) \circ \gamma_{A,B} \qquad (10)$$

This is very similar to the equation that is needed for a balanced category:

$$\gamma_{A,B} \circ (\theta_A \otimes \theta_B) \circ \gamma_{B,A} \quad = \quad \theta_{A \otimes B}. \qquad (11)$$

However these two equations are not the same! They would be the same if and only if the braiding were symmetric. Balanced tensor categories are "part-way between braiding and symmetry" and it seems that the Kronecker product is too strong because it makes the braiding symmetric. (See [28] for other structures that are between braiding and symmetry.) This (non)example is strikingly similar to the 1-dimensional case where the Kronecker product of the theory of monoids with the theory containing one endomorphism of 1 (to be thought of as the inverse) contains the theory of *commutative* (symmetric) monoids. □

To what extent does the Kronecker product preserve the left quasi-adjoint to G^*? Consider $G_1 : \mathbf{T}_1 \longrightarrow \mathbf{T}_1'$ and $G_2 : \mathbf{T}_2 \longrightarrow \mathbf{T}_2'$. They induce the following diagram

$$
\begin{array}{ccc}
\mathbf{2Alg}(\mathbf{T}_1 \otimes_0^K \mathbf{T}_2, \mathbf{Cat}) & \underset{(G_1 \otimes G_2)^*}{\overset{Lan_{G_1 \otimes G_2}}{\rightleftarrows}} & \mathbf{2Alg}(\mathbf{T}_1' \otimes_0^K \mathbf{T}_2', \mathbf{Cat}) \\
\cong \downarrow & & \downarrow \cong \\
\mathbf{2Alg}(\mathbf{T}_1, \mathbf{2Alg}(\mathbf{T}_2, \mathbf{Cat})) & \underset{(G_1)^*}{\overset{Lan_{G_1}}{\rightleftarrows}} & \mathbf{2Alg}(\mathbf{T}_1', \mathbf{2Alg}(\mathbf{T}_2', \mathbf{Cat})) \\
\cong \downarrow & & \downarrow \cong \\
\mathbf{2Alg}(\mathbf{T}_2, \mathbf{Cat}) & \underset{(G_2)^*}{\overset{Lan_{G_2}}{\rightleftarrows}} & \mathbf{2Alg}(\mathbf{T}_2', \mathbf{Cat}).
\end{array}
$$

From the uniqueness of the quasi-adjoint of $(G_1 \otimes G_2)^*$ we may write

$$Lan_{G_1 \otimes G_2} \cong Lan_{G_1} \otimes Lan_{G_2}$$

Using this diagram, we can write new coherence results about Kronecker product theories from old coherence results.

Operads and theories are intimately related. They are two ways of describing algebraic structures on an object in a category. Certain types of operads are in one-to-one correspondence with 2-theories (see [26, 27] for a worked out example and [28] for a general theory.) Markl [18] has worked on a construct called a topological relative operad. It is conjectured that this notion is nothing more then the operadic version of the Kronecker product.

We would like to finish this paper by putting some of the facts that we have worked with in one commutative diagram. This diagram takes place in the 4 category of $3\widetilde{\mathbf{Cat}}$. We shorten the triple adjunction $c \vdash U \vdash d \vdash \pi_0$ to

$$\bullet \underset{cUd\pi}{\Longleftrightarrow} \bullet$$

$$
\begin{array}{ccc}
((\widetilde{\textbf{Theories}})^{op})^2 & \overset{cU d\pi}{\Longleftrightarrow} & ((2\widetilde{\textbf{Theories}})^{op})^2 \\
 & \searrow^{\otimes\kappa} & \searrow^{\otimes\kappa} \\
(\widetilde{\textbf{Theories}})^{op} & \overset{cU d\pi}{\Longleftrightarrow} & (2\widetilde{\textbf{Theories}})^{op} \\
(\widetilde{\textbf{Cat}}/Set)^2 & \overset{cU d\pi}{\Longleftrightarrow} & (2\widetilde{\textbf{Cat}}/\textbf{Cat})^2 \\
 & \searrow^{\oplus\kappa} & \searrow^{\oplus\kappa} \\
\widetilde{\textbf{Cat}}/Set & \overset{cU d\pi}{\Longleftrightarrow} & 2\widetilde{\textbf{Cat}}/\textbf{Cat}
\end{array}
$$
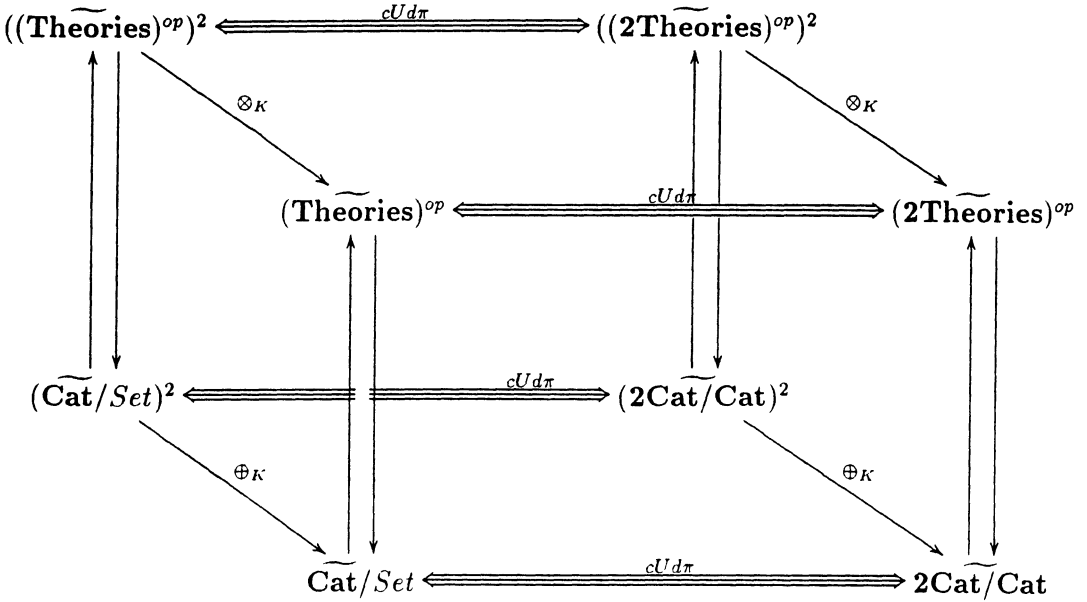
- Top is syntax.

- Bottom is semantics.

- Left is one-dimensional universal algebra.

- Right is two-dimensional universal algebra.

- All diagonal maps are Kronecker products.

The fact that each of the the squares commute was either done in the paper or is left for the reader.

5   Future directions

There are many different directions in which this work can be extended. An obvious generalization is multi-sorted 2-theories. More to the point, however, would be 2-theories whose 0-cells are the free

monoid on two generators λ and ρ corresponding to covariance and contravarience. We may call such 2-theories "bi-sorted 2-theories". Models/algebras of such theories will be in a 2-category $\mathbf{C}$ that has both a product structure and an involution $(?)^{op}$. The prototypical example of such a category is $\mathbf{Cat}$. Algebras of these theories would be functors that take λ to c and ρ to $(c)^{op}$. Using such a formalism, would help us understand the many structures that demand contravarient functors. The list of structures that we could represent with such theories abound: monoidal closed categories, ribbon categories, traced monoidal categories, spherical categories etc. Algebraic functors and their left adjoints connecting all these structures would enlighten us about the relationship between them.

A further generalization of this paper would be monoidal 2-theories. One can think of the our 2-theories as Cartesian 2-theories. A monoidal 2-theory is similar to a Cartesian 2-theory but with a monoidal product rather then a Cartesian product. Algebras will be (strict?) monoidal preserving functors. This generalization would be of use to those who study k-linear categories with extra structure, relative coherence theory (see [29]) and quantum field theory (see next paragraph).

With the above two generalizations of this paper (and a healthy love of science fiction) we can apply bi-sorted monoidal 2-theories to the study of quantum field theory. Following Graeme Segal's conception of conformal field theory, mathematical physicists have (see e.g. [23]) defined categories that look remarkably like 2-sorted monoidal 2-theories. The 0-cells are finite families of circles oriented in one of two ways (ρ or λ). The 1-cells are to be thought of as "space-time segments" from families of open circles to families of open circles. The 2-cells are isotopy classes of diffeomorphisms that fix the boundary. The 1-cells and 2-cells can have different structures depending on what type of physical structure is of interest. The "space-time segments" can be topological cobordisms (topological quantum field theory), or Riemann manifolds (conformal field theory), or symplectic manifolds (symplectic field theory). The tensor product in all of these theories is the disjoint union. There are many different functors between these 2-theories. For example there are forgetful functors $U : \mathbf{T_{cft}} \longrightarrow \mathbf{T_{tqft}}$ and $U' : \mathbf{T_{sft}} \longrightarrow \mathbf{T_{tqft}}$. What type of coherence results fall out of such 2-theory-morphisms?

What does the "free" tqft for a given cft look like? What can we say about the (quasi?) adjoint functors induced from the inclusion of the d-dimensional tqft into the $d + 1$-dimensional tqft? Is Tannaka duality [30] nothing more then the reconstruction of the monoidal 2-theory from its category of algebras? Is quantum field theory merely advanced universal algebra?

There are interesting questions arising from representation theory Besides the triple adjunction $c \vdash U \vdash d \vdash \pi_0$ between **Theories** and **2Theories** there is yet another relationship between these two levels of structure that is less clear and needs to be studied. For every suitable algebraic (1-)theory T and every $A \in Alg(T, Set)$ there is the category of modules (suitably defined) for A. One of the main ideas in quantum groups is that the structure of the algebra A is reflected in the structure of the category of modules of A. Hence there is a functor from **Theories** to **2Theories** that takes T to the 2-theory of the structure of its category of modules. For example, if A is an old-fashioned algebra, then the category of modules is simply a category. If we add a coassociative comultiplication to A, then the category of modules inherits a strict monoidal structure. If the algebra has an involution (R-matrix, Drinfeld weak comultiplicaton structure, etc) then the category of modules will have duality (braiding, monoidal structure, etc). Can this functor from **Theories** to **2Theories** be formalized? Is there some type of inverse of this functor? Do we really gain anything by going from the set with structure to the category with extra structure? Or can every theorem about categories with extra structure be understood on the set with structure level? These constructions and questions are the syntactical aspects of Tannaka duality.

Much work has been done lately to find the "right" definition of a weak n-category. Allow me to give a definition of a weak n-tuple category. A double category is a category object in **Cat**. Weakening this gives us a weak double category. Iterating the construction of a double category gives us n-tuple categories. We are left asking what is a weak n-tuple category. Let **T** be the 2-sketch of the theory of weak categories thought of as a set with endomorphisms and a partial operation. The partial operations make it a finite-limit sketch rather then a 2-theory. We must extend the work done in section 4 to construct the Kronecker

product of two finite-limit 2-sketches. $\mathbf{2Alg(T, Set)}$ is the category of weak categories. $\mathbf{2Alg(T \otimes T, Set)} \cong \mathbf{2Alg(T, 2Alg(T, Set))}$ is the 2-category of weak double categories. $\mathbf{2Alg(T^{\otimes n}, Set)}$ is the category of weak n-tuple categories. Coherence results will be induced by finite limit 2-sketch morphisms of the form $\mathbf{T^{\otimes m}} \longrightarrow \mathbf{T^{\otimes n}}$.

This paper does not close the door on functorial semantics. There are many other aspects of functorial semantics that we have not touched. For example, can we characterize when a 2-category is a category of algebras for some 2-theory? Can we characterize 2-functors as algebraic functors? When does a 2-theory morphism G induce a *right* (quasi-) adjoint to G^*? etc.

Further study needs to be done on the intimate relationship between 2-theories and 2-monads (see e.g. Blackwell et al [2] and Lawvere [17]). The study of the connection between theories and monads spawned much insight into both structures and we are sure that the same study of their two-dimensional analogs would be just as fruitful.

Computer science has long since coopted algebraic theories for its own use. Wagner [25] is a survey article of all these types of theories (e.g. ordered theories, iteration theories, rational theories, iterative theories etc.) Such generalizations have been used in diverse fields of computer science such as context-free grammars, flowchart semantics, recursion schemata and recursively defined domains. There is surely room to do similar generalizations for 2-theories. There are other areas of computer science that would benefit from a study of 2-theories. Seely [21] has an approach to lambda-calculus and computation using 2-categories. The entire area of linear logic uses categories with structure that could and should be put into a 2-theoretic context.

There is, obviously, a deep connection between higher dimensional category theory and homotopy theory. However, it is not too obvious what the connection actually is. Perhaps we would be able to better understand this connection by looking at the algebra case. Over the past few years there has been a tremendous amount of work "homotopy algebras" or "deformation algebras" (or " $(-)_\infty$ algebras"). These might all be formulated using theories. They all have some connections to homotopy. We conjecture that with all these algebras, their category of modules have added structure that can be formalized with a 2-theory.

So, in a sense, the homotopy aspects of these algebras can be seen in their 2-theoretic formulation. Can more be said on this topic?

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