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ON *-AUTONOMOUS CATEGORIES OF TOPOLOGICAL VECTOR SPACES

by Michael BARR

Abstract

We show that there are two (isomorphic) full subcategories of the category of locally convex topological vector spaces—the weakly topologized spaces and those with the Mackey topology—that form $*$-autonomous categories.

Résumé

On montre qu'il y a deux sous-catégories (isomorphes) pleines de la catégorie des espaces vectoriels topologiques localement convexes—les espaces munis de la topologie faible et ceux munis de la topologie de Mackey—qui forment des catégories $*$-autonomes.

1 Introduction

From the earliest days of category theory, the concept of duality has been important. For almost as long, it has been recognized that certain categories had a very interesting and useful property: that the set of morphisms between two objects could be viewed in a natural way as an object of the category. Such a category is called closed. If there is also a tensor product, it is called a closed monoidal, or autonomous category. The two notions come together in the concept of a $*$-autonomous category in which the set of morphisms from any object to a fixed dualizing object gives a perfect duality. See [Barr, 1979] for details and some examples.

Let K denote either the real or complex number field, which will be fixed throughout. Let **TVS** denote the category of locally convex topological vector spaces over K. A space E will be said to be weakly topologized if it has the weakest possible topology for its set of continuous linear functionals. We denote by $\mathbf{T}_w$ the full subcategory of **TVS** of weakly topologized spaces. A space E will be said to have a Mackey topology if it has the strongest possible topology for its set of continuous linear functionals. We denote by $\mathbf{T}_m$ the full subcategory of spaces with a Mackey topology.

Let $\mathbf{T}$ denote one of the two categories $\mathbf{T}_w$ or $\mathbf{T}_m$. We will show that if E and F are two spaces in $\mathbf{T}$ the set of continuous linear maps $E \longrightarrow F$ has in a natural way the structure of an object of $\mathbf{T}$. We denote this object $E \multimap F$. It gives $\mathbf{T}$ the structure of a *closed* category. We will also show that if we define $E^* = E \multimap K$ the natural map $E \longrightarrow E^{**}$ is an isomorphism for every object E of $\mathbf{T}$. Thus $\mathbf{T}$ has the structure of a $*$-*autonomous* category [Barr, 1979]. See the conclusions of Theorem 2.1 for the full definition of $*$-autonomous.

It is well known and easy that given a locally convex space E there is a weakest topology on the underlying vector space of E that has the same functionals as E. We call this space σE. The identity function $E \longrightarrow \sigma E$ is continuous, but not generally an isomorphism. It is also well known, but not so easy, that there is a strongest such topology. We give what we believe to be a new proof of this fact in the appendix, 4.3. We denote by τE the resultant space. This time it is the direction $\tau E \longrightarrow E$ that is continuous and an isomorphism if and only if E is a Mackey space.

The main tool used in proving this is the category of *pairs* described in Section 2. The pairs, although not the category, were introduced by Mackey in his PhD thesis (published in [Mackey, 1945]).

A very early and more complicated version of the theory exposed here appeared in [Barr, 1976a,b]. A general theory appeared in [Barr, 1979]. A substantial simplification of the general theory can be found in [Barr, 1999] and the present paper is an exposition of the special case as it applies to locally convex vector spaces. A similar result in the category of balls has appeared in [Barr & Kleisli, 1999].

2 The category of pairs

In this section, no topology is assumed on the vector spaces. If V and W are two vector spaces (possibly infinite dimensional), we denote by $\hom(V, W)$ the vector space of linear transformations between them and by $V \otimes W$ the tensor product over K.

A *pair* is a pair $\mathbf{V} = (V, V')$ of vector spaces together with a pairing, that is a bilinear map $\langle -, - \rangle : V \times V' \longrightarrow K$ or, equivalently, a linear map $\langle -, - \rangle : V \otimes V' \longrightarrow K$. We make no assumption of non-singularity at this point. If $\mathbf{V}$ and $\mathbf{W} = (W, W')$ are pairs, a morphism $\mathbf{f} : \mathbf{V} \longrightarrow \mathbf{W}$ is a pair (f, f') where $f : V \longrightarrow W$ and $f' : W' \longrightarrow V'$ are linear transformations such that $\langle fv, w' \rangle = \langle v, f'w' \rangle$ for all $v \in V$ and $w' \in W'$.

There are two other equivalent definitions for morphisms. If we denote by $V^{\perp}$ the vector space dual $\hom(V, K)$, then a pairing $\langle -, - \rangle : V \times V' \longrightarrow K$ induces a homomorphism $V \longrightarrow V'^{\perp}$ and another one $V' \longrightarrow V^{\perp}$. We will use these arrows freely without explanation. If (V, V') and (W, W') are pairs, then a pair of arrows $f : V \longrightarrow V'$ and $f' : W' \longrightarrow V'$ gives an arrow $(f, f') : (V, V') \longrightarrow (W, W')$ if and only if either, and hence both, of the following squares commute:

$$
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\downarrow & & \downarrow \\
V'^{\perp} & \xrightarrow[f'^{\perp}]{} & W'^{\perp}
\end{array}
\qquad\qquad
\begin{array}{ccc}
W' & \xrightarrow{f'} & V' \\
\downarrow & & \downarrow \\
W^{\perp} & \xrightarrow[f^{\perp}]{} & V^{\perp}
\end{array}
\qquad\qquad (**)
$$

If $\mathbf{V}$ and $\mathbf{W}$ are pairs, it is clear that any linear combination of morphisms is a morphism and so we can let $\hom(\mathbf{V}, \mathbf{W})$ denote the vector space of morphisms of $\mathbf{V}$ to $\mathbf{W}$.

It is clear that if $\mathbf{V} = (V, V')$ is a pair, so is $\mathbf{V}^* = (V', V)$ with pairing given by $\langle v', v \rangle = \langle v, v' \rangle$. This is called the *dual* of $\mathbf{V}$. The definition of morphism makes it clear that $\mathbf{f} = (f, f')$ is a morphism $\mathbf{V} \longrightarrow \mathbf{W}$ if and only if $\mathbf{f}^* = (f', f) : \mathbf{W}^* \longrightarrow \mathbf{V}^*$ is one. Since the duality preserves linear combinations, it follows that $\hom(\mathbf{V}, \mathbf{W}) \cong \hom(\mathbf{W}^*, \mathbf{V}^*)$.

Because the categorical structure of the category of pairs was first investigated by P.-H. Chu [1979], this category of pairs is usually denoted

$\mathbf{Chu} = \mathbf{Chu}(\mathbf{Vect}, K)$.

Now we are in a position to give one of the main definitions of this note. If $\mathbf{V}$ and $\mathbf{W}$ are pairs, $\mathbf{V} \multimap \mathbf{W}$ denotes the pair $(\hom(\mathbf{V}, \mathbf{W}), V \otimes W')$ with pairing $\langle (f, f'), v \otimes w' \rangle = \langle fv, w' \rangle = \langle v, f'w' \rangle$. We denote by $\mathbf{K}$ the pair (K, K) with multiplication as pairing.

2.1 Theorem *For any pairs* $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ *we have*

1. $\hom(\mathbf{K}, \mathbf{V} \multimap \mathbf{W}) = \hom(\mathbf{V}, \mathbf{W})$;
2. $\mathbf{U} \multimap (\mathbf{V} \multimap \mathbf{W}) \cong \mathbf{V} \multimap (\mathbf{U} \multimap \mathbf{W})$;
3. $\mathbf{V} \multimap \mathbf{W} \cong \mathbf{W}^* \multimap \mathbf{V}^*$;
4. $\mathbf{V}^* = \mathbf{V} \multimap \mathbf{K}$.

See 4.1 for the proof.

The conclusions of this theorem constitute the formal definition of $*$-autonomous category.

2.2 Separated and extensional pairs A pair (V, V') is said to be *separated* if for each $v \neq 0$ in V, there is a $v' \in V'$ such that $\langle v, v' \rangle \neq 0$. We say that the pair is *extensional* if (V', V) is separated. The reason for the name "separated" is clear. The word "extensional" refers to the characteristic property of functions that two are equal if their values at every argument are the same.

2.3 Theorem *Let* $\mathbf{V} = (V, V')$ *and* $\mathbf{W} = (W, W')$ *be separated and extensional. Then so are* V^* *and* $\mathbf{V} \multimap \mathbf{W}$.

This is obvious for V^*; see 4.2 for the proof for $\mathbf{V} \multimap \mathbf{W}$.

The full subcategory of separated, extensional pairs has been widely denoted $\mathbf{chu} = \mathbf{chu}(\mathbf{Vect}, K)$.

2.4 Corollary *The category* $\mathbf{chu}$ *is* $*$*-autonomous.*

By refering to Diagram $(**)$ above, we see that if $\mathbf{W}$ is separated, the right hand arrow of the left hand diagram is injective and then for a given $f' : W' \longrightarrow V'$ there is at most one $f : V \longrightarrow W$ making the left hand square commute. Thus in that case f' determines f uniquely, if it exists. Dually, we see that if $\mathbf{V}$ is extensional, then f determines a unique f', if there is one. If **V** and **W** are each separated and extensional, then either of f or f' determines the other uniquely, if it exists.

2.5 The tensor product Most autonomous categories also have a tensor product. In fact, all $*$-autonomous categories do.

2.6 Theorem *A $*$-autonomous category has a tensor product, adjoint to the internal hom, given by $E \otimes F = (E \multimap F^*)^*$.*

Proof. We have, for objects E, F, and G,

$$\begin{aligned}(E \otimes F) \multimap G = (E \multimap F^*)^* \multimap G &\cong G^* \multimap (E \multimap F^*)\\ &\cong E \multimap (G^* \multimap F^*) \cong E \multimap (F \multimap G)\end{aligned}$$

which is the internal version of the characteristic property of the tensor product. By applying $\hom(-, K)$, we see that $\hom(E \otimes F, G) \cong \hom(E, F \multimap G)$, which is the external version. □

3 Weak spaces and Mackey spaces

If E is a topological vector space, let $|E|$ denote the underlying vector space and $E^\perp = \hom(E, K)$ denote the discrete space of continuous linear functionals on E. Then $TE = (|E|, E^\perp)$, with pairing given by evaluation, is an object of **chu**. It is extensional by definition. Assuming E is locally convex, there are, by the Hahn-Banach Theorem, enough continuous linear functionals to separate points, and so TE is also separated. Thus T is the object function of a functor T : **TVS** $\longrightarrow$ **chu**.

3.1 Theorem *The functor T has a right adjoint R and a left adjoint L, each of which is full and faithful. The image of R is the category of weak spaces and the image of L is the category of Mackey spaces, each of which is thereby equivalent to* **chu**. *Thus the categories of weak spaces and Mackey spaces are equivalent—in fact isomorphic—and each inherits a $*$-autonomous structure from* **chu**.

See 4.5 for the proof.

3.2 Explicit description of the internal hom The proof actually gives a description of the internal hom functors. In the weak case, $E \multimap F$ can be described as follows. For each element $e \in E$ and continuous linear functional $\varphi : F \longrightarrow K$, define a linear functional $(\varphi, e) : \hom(E, F) \longrightarrow K$ by $(\varphi, e)(f) = \varphi(f(e))$. Then $E \multimap F$ is the vector space $\hom(E, F)$ equipped with the weak topology for all the (φ, e).

If E and F are Mackey spaces, then the internal hom in the Mackey category, is given by $E \underset{\tau}{\multimap} F = \tau(E \multimap F)$, denoted $E \underset{\tau}{\multimap} F$.

4 Appendix: the gory details

4.1 Proof of Theorem 2.1 We begin with

1. An arrow $\mathbf{K} \longrightarrow \mathbf{V} \multimap \mathbf{W}$ consists of an arrow $f : K \longrightarrow \hom(\mathbf{V}, \mathbf{W})$ together with an arrow $f' : V \otimes W' \longrightarrow K$ such that for $\lambda \in K$, $v \in V$, and $w' \in W'$,

$$\langle f(\lambda), v \otimes w' \rangle = \langle \lambda, f'(v \otimes w') \rangle = \lambda f'(v \otimes w')$$

Since everything is K-linear, it is sufficient that this hold when $\lambda = 1$, which reduces to

$$\langle f(1), v \otimes w \rangle = f'(v \otimes w)) \qquad (*)$$

If we write $f(1) = \mathbf{g} = (g, g') : \mathbf{V} \longrightarrow \mathbf{W}$, then $(*)$ becomes $\langle \mathbf{g}, v \otimes w' \rangle = f'(v \otimes w')$. But $\langle \mathbf{g}, v \otimes w' \rangle$ is defined to be $\langle g(v), w' \rangle = \langle v, g'(w') \rangle$. Thus any such $\mathbf{f}$ is determined by a unique $\mathbf{g} : \mathbf{V} \longrightarrow \mathbf{W}$ by the formulas $f(1) = \mathbf{g}$ and $f'(v \otimes w') = \langle g(v), w' \rangle = \langle v, g'(w') \rangle$.

2. The definition gives that

$$\begin{aligned}\mathbf{U} \multimap (\mathbf{V} \multimap \mathbf{W}) &= \mathbf{U} \multimap (\hom(\mathbf{V}, \mathbf{W}), V \otimes W') \\ &= (\hom(\mathbf{U}, \mathbf{V} \multimap \mathbf{W}), U \otimes V \otimes W')\end{aligned}$$

and similarly,

$$\mathbf{V} \multimap (\mathbf{U} \multimap \mathbf{W}) = (\hom(\mathbf{V}, \mathbf{U} \multimap \mathbf{W}), V \otimes U \otimes W')$$

Thus it suffices to show that $\hom(\mathbf{U}, \mathbf{V} \multimap \mathbf{W}) \cong \hom(\mathbf{V}, \mathbf{U} \multimap \mathbf{W})$. What we will do is analyze the first of these and see that is symmetric between $\mathbf{U}$ and $\mathbf{V}$. A homomorphism $\mathbf{f} = (f, f') : \mathbf{U} \longrightarrow \mathbf{V} \multimap \mathbf{W}$ is determined by an arrow $f : U \longrightarrow \hom(\mathbf{V}, \mathbf{W})$ and an arrow $f' : V \otimes W' \longrightarrow U'$ subject to certain conditions that we will deal with later. For $u \in U$, let $f(u) = \mathbf{g}(u) = (g(u), g'(u))$ where $g(u) : V \longrightarrow W$ and $g'(u) : W' \longrightarrow V'$ such that for all $v \in V$ and $w' \in W'$, $\langle g(u)(v), w' \rangle = \langle v, g'(u)(w')$. Moreover, the compatibility condition on $\mathbf{f}$ is that

$$\langle u, f'(v \otimes w') \rangle = \langle f(u), v \otimes w' \rangle = \langle g(u)(v), w' \rangle = \langle v, g'(u)(w') \rangle$$

If we now identify the map $g : U \longrightarrow \hom(V, W')$ with a map we will still call $g : U \otimes V \longrightarrow W'$ and similarly for the map $g' : U \otimes W' \longrightarrow V'$, we see that a map $\mathbf{U} \longrightarrow \mathbf{V} \multimap \mathbf{W}$ is determined by three maps $g : U \otimes V \longrightarrow W'$, $g' : U \otimes W' \longrightarrow V$, and $f' : V \otimes W' \longrightarrow U$ subject to the condition that for all $u \in U$, $v \in V$, and $w' \in W'$

$$\langle u, f'(v \otimes w') \rangle = \langle g(u \otimes v), w' \rangle = \langle v, g'(u \otimes w) \rangle$$

which is symmetric between $\mathbf{U}$ and $\mathbf{V}$.

3. As above, it is sufficient to show that $\hom(\mathbf{V}, \mathbf{W}) \cong \hom(\mathbf{W}^*, \mathbf{V}^*)$. But if $\mathbf{f} = (f, f') : \mathbf{V} \longrightarrow \mathbf{W}$ is a morphism, it is purely formal to see that $\mathbf{f}^* = (f', f) : \mathbf{W}^* \longrightarrow \mathbf{V}^*$ is also a morphism.

4. We have that

$$\mathbf{V} \multimap \mathbf{K} = (\hom(\mathbf{V}, \mathbf{K}), V \otimes K) \cong (\hom(\mathbf{V}, \mathbf{K}), V)$$

so it is sufficient to show that $\hom(\mathbf{V}, \mathbf{K}) \cong V'$. A morphism $\mathbf{V} \longrightarrow \mathbf{K}$ is given by a pair (φ, v') where $\varphi : V \longrightarrow K$ and $\varphi' : K \longrightarrow V'$ such that for all $v \in V$ and $\lambda \in K$, $\langle \varphi(v), \lambda \rangle = \langle v, \varphi'(\lambda) \rangle$. If we write $v' = \varphi'(1)$, this equation becomes $\lambda\varphi(v) = \lambda\langle v, v' \rangle$ or $\varphi = \langle -, v' \rangle$. Thus a morphism is completely determined by the element $v' \in V'$. Conversely, such an element determines a unique morphism $\mathbf{V} \longrightarrow K$. □

4.2 Proof of Theorem 2.3 Let us write $\mathbf{U} = (U, U') = \mathbf{V} \multimap \mathbf{W}$. Then $U = \hom(\mathbf{V}, \mathbf{W})$ and $U' = V \otimes W'$. We begin by proving it is separated. Let $(f, f') : \mathbf{V} \longrightarrow \mathbf{W}$. Assuming $(f, f') \neq 0$, there is an element $v \in V$ with $f(v) \neq 0$ and then, since $\mathbf{W}$ is separated, there is an element $w' \in W'$ with $\langle f(v), w' \rangle \neq 0$. But $\langle (f, f'), v \otimes w' \rangle = \langle f(v), w' \rangle$.

For proving extensionality, it will simplify the notation to show that $\mathbf{V} \multimap \mathbf{W}^*$ is extensional. We need to show that for any element $\sum_{i=1}^{n} v_i \otimes w_i \in V \otimes W$, there is a morphism $(f, f') : (V, V') \longrightarrow (W', W)$ such that $\sum \langle w_i, f v_i \rangle = \sum \langle v_i, f' w_i \rangle \neq 0$. Let V_0 and W_0 be the (finite dimensional) subspaces of V and W generated by $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$, respectively. The inclusion $i : V_0 \longrightarrow V$ induces $V^\perp \longrightarrow {V_0}^\perp$. Composed with $V' \longrightarrow V^\perp$ we get a linear transformation $i' : V' \longrightarrow V^\perp$ for which

$$\begin{array}{ccc} V' & \xrightarrow{p} & V^\perp \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i^\perp} \\ {V_0}^\perp & \xrightarrow[\mathrm{id}]{} & {V_0}^\perp \end{array}$$

commutes. Here $p : V' \longrightarrow V^\perp$ is the transpose of the structure map. This commutation means that $(i, i') : \mathbf{V}_0 = (V_0, {V_0}^\perp) \longrightarrow \mathbf{V}$ is a morphism. There is a similar morphism $\mathbf{W}_0 \longrightarrow \mathbf{W}$ and hence $\mathbf{W}^* \longrightarrow {\mathbf{W}_0}^*$. Together they induce a morphism $\hom(\mathbf{V}, \mathbf{W}^*) \longrightarrow \hom(\mathbf{V}_0, \mathbf{W}_0^*)$. We will then complete the argument by showing that the latter map is surjective and that there is an $(f_0, f_0') \in \hom(\mathbf{V}_0, \mathbf{W}_0^*)$ such that $\sum \langle w_i, f_0 v_i \rangle \neq 0$.

Since $\sum v_i \otimes w_i \neq 0$ in $V \otimes W$, it is certainly non zero in $V_0 \otimes W_0$ and so there is a map $g : V_0 \otimes W_0$ such that $\sum g(v_i \otimes w_i) \neq 0$. This transposes to a map $f_0 : V_0 \longrightarrow {W_0}^\perp$ for which $\sum \langle w_i, f_0 v_i \rangle \neq 0$. Then $(f_0, {f_0}^\perp)$ is the required map.

In order to show that $\hom(\mathbf{V}, \mathbf{W}^*) \longrightarrow \hom(\mathbf{V}_0, \mathbf{W}_0^*)$ is surjective, it is sufficient to show that $\mathbf{V}_0 \longrightarrow \mathbf{V}$ and $\mathbf{W}_0 \longrightarrow \mathbf{W}$ are split monics. We do this for $\mathbf{V}_0 \longrightarrow \mathbf{V}$. First I claim that the composite

$$V' \xrightarrow{p} V^\perp \xrightarrow{i^\perp} {V_0}^\perp$$

is surjective. If not, it factors through a proper subobject of ${V_0}^\perp$ which has the form ${V_1}^{\perp}$, where $V_0 \longrightarrow V_1$ is a proper quotient mapping. But then the injection $V_0 \rightarrowtail V \rightarrowtail V'^{\perp}$ factors through the proper surjection $V_0 \twoheadrightarrow V_1$, which is impossible. Now let $j : {V_0}^{\perp} \longrightarrow V'$ be a right inverse to $i^{\perp} \circ p$. Then the square

$$\begin{tikzcd} {V_0}^{\perp} \arrow[r, "j"] \arrow[d, "\mathrm{id}"'] & V' \arrow[d, "p"] \\ {V_0}^{\perp} \arrow[r, "p \circ j"'] & V^{\perp} \end{tikzcd}$$

obviously commutes. This means that if $j' : V \longrightarrow {V_0}^{\perp\perp} \cong V_0$ is the double transpose of $p \circ j$, then $(j, j') : \mathbf{V} \longrightarrow \mathbf{V}_0$ is a morphism, one that evidently splits (i, i'). $\square$

4.3 The existence of the Mackey topology Although it is a standard fact of the theory of locally convex vector spaces that the Mackey topology exists, it is normally proved by defining it as the topology of uniform convergence on compact subsets of the dual with the weak topology. Here we give a proof that does not involve looking inside the space at all.

Let $\{E_i \mid i \in I\}$ range over the set of all topological vector spaces for which $TE_i = TE$, that is that $|E_i| = E$ and $\hom(E_i, K) = \mathrm{Hom}(E, K)$. Thus $\sigma E = \sigma E_i$, for each $i \in I$. Among the E_i is E itself. Now form the pullback

$$\begin{tikzcd} \tau E \arrow[r] \arrow[d] & \prod_{i \in I} E_i \arrow[d] \\ \sigma E \arrow[r] & \prod_{i \in I} \sigma E_i \end{tikzcd}$$

Of course, $\prod_{i \in I} \sigma E_i = (\sigma E)^I$, but we prefer to leave it in this form since it makes the right hand map evident. Since E is among the E_i, the bottom map, and hence the top map, is, up to isomorphism, a subspace inclusion. The space τE has, obviously, the supremum of the topologies on the E_i, but representing it by this pullback allows us to use arrow-theoretic reasoning. At this point, we require the following.

4.4 Proposition *Let $\{F_i \mid i \in I\}$ be a family of locally convex spaces. Then the natural map*

$$\sum_{i\in I} \hom(F_i, K) \longrightarrow \hom\left(\prod_{i\in I} F_i, K\right)$$

is an isomorphism.

Proof. Let U denote the open unit disk of K. That is, either the open unit disk of the complex plane or the open interval $(-1, 1)$ of the real numbers. If F is a topological vector space, it is easy to see that a linear functional $\varphi : F \longrightarrow K$ is continuous if and only if $\varphi^{-1}(U)$ is open. If $\varphi : F = \prod F_i \longrightarrow K$ is a continuous linear functional, $\varphi^{-1}(U)$ is open and hence there is a finite subset $J \subseteq U$ and an open 0-neighborhood $V_j \in F_j$ for $j \in J$ such that

$$\varphi^{-1}(U) \supseteq \prod_{j\in J} V_j \times \prod_{i\in I-J} F_i$$

In particular, $\varphi\left(\prod_{i\in I-J} F_i\right) \subseteq U$. Since U contains no non-zero subspace, it follows that $\varphi\left(\prod_{i\in I-J} F_i\right) = 0$. Thus φ is defined modulo $F_0 = \prod_{i\in I-J} F_i$. That means there is a linear functional $\psi : \prod_{j\in J} F_j \longrightarrow K$ that composed with the projection gives φ. Moreover, $\psi^{-1}(U) \supseteq \prod_{j\in J} V_j$ which implies that ψ is continuous. The category of topological vector spaces is additive, so that finite sums and products coincide. Thus, as J ranges over the finite subsets of I,

$$\begin{aligned}
\hom\left(\prod_{i\in I} F_i, K\right) &\cong \operatorname{colim}_J \hom\left(\prod_{i\in J} F_i, K\right) \cong \operatorname{colim}_J \hom\left(\sum_{i\in J} F_i, K\right)\\
&\cong \operatorname{colim}_J \prod_{i\in J} \hom\left(F_i, K\right) \cong \operatorname{colim}_J \sum_{i\in J} \hom\left(F_i, K\right)\\
&\cong \sum_{i\in I} \hom\left(F_i, K\right)
\end{aligned}$$

Now we return to the proof of the existence of the Mackey topology. A linear functional $\varphi : \tau E \longrightarrow K$ extends by the Hahn-Banach theorem, to an element

$$\psi \in \hom\left(\prod_{i\in I} E_i, K\right) \cong \sum_{i\in I} \hom(E_i, K)$$

$$\cong \sum_{i\in I} \hom(\sigma E_i, K) \cong \hom\left(\prod_{i\in I} \sigma E_i, K\right)$$

which restricts in turn to σE, which has the same continuous linear functionals as E. Thus every continuous linear functional on τE is also continuous on E. Since E is one of the factors in $\prod F_i$, it follows that the identity function is continuous from $\tau E \longrightarrow E$. Finally, suppose that $E' \longrightarrow \tau E$ is a bijection with a strictly finer topology. If there is no continuous functional on E' that is discontinuous on E, then E' would be among the E_i and hence has a coarser topology than τE. □

4.5 The proof of Theorem 3.1 Recall that $\mathbf{T}_w$ and $\mathbf{T}_m$ denote the full subcategories of the category of locally convex vector spaces consisting of the weak and the Mackey spaces, respectively. We begin by defining R : **chu** $\longrightarrow \mathbf{T}_m$ by letting $R(V, V')$ be the vector space V with the weak topology given by V'. That is, V is topologized as a subset of $K^{V'}$. It is clear that evaluation at every element of V' gives a continuous linear functional on $R(V, V')$. On the other hand, it follows from the Hahn-Banach theorem and Lemma 4.4 that for every continuous linear functional $\varphi : V \longrightarrow K$, there is a finite set of elements $v_1', v_2', \ldots, v_n' \in V'$ such that for all $v \in V$,

$$\varphi(v) = \langle v, v_1'\rangle + \langle v, v_2'\rangle + \cdots + \langle v, v_n'\rangle$$

but then for $v' = v_1' + v_2' + \cdots + v_n'$, $\varphi(v) = \langle v, v'\rangle$. Thus $TR(V, V') = (V, V')$. If we define $L(V, V') = \tau R(V, V')$, it also follows that $TL = \mathrm{Id}$. If E is a weakly topologized topological vector space, then E has the weak topology for its set of continuous linear functionals, so it is evident that $E = RF(E)$. If E is a Mackey space, then it is the finest topology compatible with its set of functionals, which is the condition that defines $LF(E) = \tau RF(E) = \tau\sigma(E) = E$. □

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