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A PRIESTLEY VIEW OF SPATIALIZATION OF FRAMES

by A. PULTR and J. SICHLER

Résumé : La représentation des "frames" par la dualité de Priestley fournit un critère simple de spatialité (au sens d'être isomorphe à une topologie). De ce critère on peut en particulier deduire facilement la spatialité des "frames" G_δ-absolus (Isbell), ou celle des treillis continus distributifs (Hofmann et Lawson, Banaschewski).

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In an earlier paper [**10**], we have characterized the Priestley duals of frames and the maps of these spaces corresponding to frame homomorphisms. Extending the Stone duality, which represents complete Boolean algebras by extremally disconnected Stone spaces, the Priestley duals of frames are precisely the "extremally disconnected" Priestley spaces, that is, those in which the closure of every open down-set is an open down-set – see 2.1 below. The aim of the present article is to show how to apply the Priestley duality to obtain a straightforward characterization of spatial frames (Theorem 4.1 below). From this, in turn, one can very easily derive certain well-known spatialization results (Isbell [**4**], Hofmann, Lawson and Banaschewski [**3**], [**2**]). Using the Priestley approach, these results can be extended to cover completely the duality between distributive continuous lattices and locally compact spaces (Section 5).

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Needless to say, since Priestley duality is logically equivalent to the Boolean Ultrafilter Theorem, the results depend on this (weaker) form of the axiom of choice. It cannot be otherwise: although the frame version of Tychonoff product theorem is choice-free ([**6**]), the classical one is equivalent to the BUT ([**7**]), and hence any spatiality theorem involving all compact regular frames has to use BUT.

We wish to emphasize the simplicity of all the proofs involved. In order to assure the reader that this simplicty is an inherent feature of our approach – and is not due to any 'harder' results proved elsewhere and quoted here – we also include proofs of some known facts we use.

1. Preliminaries

1.1. Recall that a *frame* is a complete lattice L satisfying the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge b \mid b \in S\}$$

for every $a \in L$ and every $S \subseteq L$, and a *frame homomorphism* $h : L \to M$ is a mapping preserving all joins (including the least element 0) and finite meets (including the greatest element 1).

The two-element frame (Boolean algebra) $\{0, 1\}$ is denoted by $\mathbf{2}$.

For a topological space X we have the frame $\mathfrak{O}(X) = \{U \mid U \subseteq X,\ U \text{ open }\}$, and for a continuous $f : X \to Y$ we have the frame homomorphism $\mathfrak{O}(f) : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ defined by $\mathfrak{O}(f)(U) = f^{-1}(U)$.

A frame L is called *spatial* if it is isomorphic to $\mathfrak{O}(X)$ for a space X. It is well-known (see, e.g., [**5**]) that

> *L is spatial iff for any two $a, b \in L$ such that $a \not\leq b$ there is a frame homomorphism $h : L \to \mathbf{2}$ such that $h(a) > h(b)$.*

The reader interested in details may consult [**5**] or [**11**].

1.2. A triple $(X, \tau, \leq)$ is an ordered topological space if (X, τ) is a topological space and $(X, \leq)$ is a poset. Let $Y \subseteq X$. The set Y is decreasing (resp. increasing) if $x \leq y \in Y$ implies that $x \in Y$ (resp. $x \geq y \in Y$ implies that $x \in Y$). For an ordered topological space $(X, \tau, \leq)$, let $\downarrow\tau$ (resp. $\uparrow\tau$) denote the set of all decreasing (resp. increasing) open sets, and $\downarrow CO\tau$ (resp. $\uparrow CO\tau$) the set of all decreasing (resp. increasing) clopen sets.

A *Priestley space* is a compact ordered topological space such that for any $x \not\leq y$ there exists $U \in \uparrow CO\tau$ such that $x \in U$ and $y \notin U$. We recall that in a Priestley space

1.2.1. $\downarrow CO\tau$ (resp. $\uparrow CO\tau$) is a basis of $\downarrow\tau$ (resp. $\uparrow\tau$), and

1.2.2. $\downarrow CO\tau \cup \uparrow CO\tau$ is a subbasis of τ.

As usual, for a subset A of an ordered set write $\downarrow A$ for $\{x \mid x \leq a \text{ for some } a \in A\}$ and $\uparrow A$ for $\{x \mid x \geq a \text{ for some } a \in A\}$. By a standard compactness argument one proves that

1.2.3. in a Priestley space X, if $A \subseteq X$ is closed then $\downarrow A$ and $\uparrow A$ are closed,

and from this fact one easily obtains that

1.2.4. if Y is closed in a Priestley space $(X, \tau, \leq)$ then each $U \in \downarrow(\tau|Y)$ can be expressed as $V \cap Y$ with $V \in \downarrow\tau$.

We shall also use the following two well-known facts:

1.2.5. For each x in a Priestley space X there is a minimal y and a maximal z such that $y \leq x \leq z$.

(Indeed, let $C \subseteq X$ be linearly ordered by $\leq$. Let $C' \subseteq C$ be finite. Then the intersection $\bigcap\{\downarrow c \mid c \in C'\}$ is $\downarrow c_0$ where c_0 is smallest in C'. Using compactness we get $\bigcap\{\downarrow c \mid c \in C\} \neq \emptyset$. Hence there is a $b \in X$ such that $b \leq c$ for all $c \in C$. Use Zorn's lemma.)

1.2.6. For any closed decreasing $Y \subseteq X$ and any $x \in X \setminus Y$ there is $V \in \uparrow CO\tau$ such that $x \in V$ and $Y \cap V = \emptyset$.

Given a poset $(X, \leq)$, the set of all maximal elements of a subset $Y \subseteq X$ will be denoted by

$$\mathsf{max}(Y).$$

1.3. Recall that the famous Priestley duality between the category of Priestley spaces and the monotone continuous maps, and the category of distributive (0,1)-lattices and (0,1)-lattice homomorphisms associates with $(X, \tau, \leq)$ the lattice

$$\mathcal{D}(X) = (\downarrow CO\tau, \cap, \cup),$$

and with an $f : X \to Y$ the homomorphism $\mathcal{D}(f) = (U \mapsto f^{-1}(U)) : \mathcal{D}(Y) \to \mathcal{D}(X)$; the Priestley space $\mathcal{P}(D)$ associated with a distributive (0,1)-lattice D is carried by the set of all prime filters on D ordered by reversed inclusion, and $\mathcal{P}(h)(F) = h^{-1}(F)$ for any (0,1)-homomorphism $h : D \to D'$ (see [**8**],[**9**]).

Proposition. *The closed decreasing subsets of X and the filters in $\mathcal{D}(X)$ are in a one-one onto correspondence provided by*

$$\varphi = (Y \mapsto \{U \mid U \supseteq Y\}), \quad \psi = (\mathcal{F} \mapsto \bigcap \mathcal{F}).$$

PROOF: Trivially $Y \subseteq \bigcap\{U \mid U \supseteq Y\}$. If $x \notin Y$ then by 1.2.6 we have a clopen increasing $V \ni x$, $V \cap Y = \emptyset$, and hence $x \notin X \setminus V \supseteq Y$. Thus $\psi\gamma$ =id.

If $\mathcal{F} \not\subseteq \mathcal{G}$ then (by a standard argument) there is a prime filter $\mathcal{H}$ such that $\mathcal{G} \subseteq \mathcal{H}$ and $\mathcal{F} \not\subseteq \mathcal{H}$. In X, this $\mathcal{H}$ is represented by an element $x \in \bigcap \mathcal{F} \setminus \bigcap \mathcal{G}$. Thus, ψ is one-one and hence also $\varphi\psi$ =id. □

1.4. We shall repeatedly use the folowing trivial topological fact:

If U is open then $U \cap \overline{A} \subseteq \overline{U \cap A}$ for any A. Consequently, if U is clopen then $U \cap \overline{A} = \overline{U \cap A}$ for any A.

2. LP-spaces

2.1. By [**10**], in Priestley duality *frames* correspond exactly to the Priestley spaces in which

(LP-obj) $\qquad$ for each $U \in \downarrow\tau$, $\quad \overline{U} \in \downarrow CO\tau$

and *frame homomorphisms* to the continuous maps $f : X \to Y$ for which

(LP-morph) $\qquad$ for each $U \in \downarrow\tau$, $\; f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$.

We will speak of *LP-spaces* and *LP-maps*.

2.2. Proposition. *In an LP-space, if U is open then $\uparrow U$ is open. In particular, if U is clopen then $\uparrow U$ is clopen.*

PROOF: In view of 1.2.1 it suffices to prove that if U is clopen then $\uparrow U$ is open. Thus, let U be clopen. We have $U \cap (X \setminus \uparrow U) = \emptyset$ and hence, as U is is open, $U \cap \overline{(X \setminus \uparrow U)} = \emptyset$. By 1.2.3 and (LP-obj), $\overline{X \setminus \uparrow U}$ is decreasing and hence $\uparrow U \cap \overline{(X \setminus \uparrow U)} = \emptyset$ and we have $X \setminus \uparrow U \supseteq \overline{X \setminus \uparrow U}$. Thus, $X \setminus \uparrow U$ is clopen. □

2.3. For a given $U \in \mathcal{D}(X)(= \downarrow CO\tau)$ let U^* designate the pseudocomplement (in $\mathcal{D}(X)$).

Proposition. $U^* = X \setminus \uparrow U$. *Consequently, the "rather below" relation $U \prec V$ is expressed by $\uparrow U \subseteq V$.*

PROOF: By 2.2, $X \setminus \uparrow U \in \downarrow CO\tau$, and obviously if $V \in \downarrow CO\tau$ and $U \cap V = \emptyset$ then $X \setminus \uparrow U \supseteq V$. □

2.4. A subset Y of an LP-space X is called an *L-set* if

(1) Y is closed, and
(2) for each $U \in \downarrow\tau$, $\overline{U} \cap Y = \overline{U \cap Y}$.

Since onto frame homomorphisms ("sublocales") correspond in Priestley duality to embeddings of LP-subspaces into LP-spaces, we immediately see from the (LP-morph) in 2.1 that the following three statements about subsets Y of an LP-space X are equivalent:

- Y is an L-set,
- Y with the induced topology and order is an LP-space,
- the relation $U \sim V$ defined on $\mathcal{D}(X)$ by $U \cap Y = V \cap Y$ is a frame congruence.

2.4.1. Using 1.4 for the second statement below we easily obtain

Proposition. *1. Let Z be an L-set in Y and Y an L-set in X. Then Z is an L-set in X.*

2. Each clopen $Y \subseteq X$ is an L-set.

2.4.2. Using 1.4 and (LP-obj) we conclude

Lemma. *Let Y be a subset of an LP-space such that for each $U \in \downarrow\tau$, $\overline{U} \cap Y \subseteq \overline{U \cap Y}$. Then the closure $\overline{Y}$ of Y is an L-set.*

2.5. For any L-space X, the symbol

$$\mathsf{LS}(X)$$

will denote the system of all its L-sets.

Proposition. $\mathsf{LS}(X)$ *is a complete lattice with the suprema*

$$Y \vee Z = Y \cup Z, \quad \bigvee Y_i = \overline{\bigcup Y_i}.$$

PROOF: If $U \in \downarrow\tau$, $\overline{U}$ is clopen and hence

$$(\bigvee Y_i) \cap \overline{U} = \overline{\bigcup Y_i} \cap \overline{U} = \overline{(\bigcup Y_i) \cap \overline{U}} = \overline{\bigcup (Y_i \cap \overline{U})} =$$
$$= \overline{\bigcup \overline{Y_i \cap \overline{U}}} \subseteq \overline{\bigcup (Y_i \cap U)} = \overline{(\bigcup Y_i) \cap U} \subseteq \overline{(\bigvee Y_i) \cap U}$$

so that $\bigvee Y_i$ is an L-set. □

2.6. Obviously, the open sublocales of $\mathcal{D}(X)$ are represented by the members of $\downarrow CO\tau$. We will refer to them as *L-open*. Consequently, the closed sublocales are represented by the members of $\uparrow CO\tau$, which will be referred to as *L-closed*. From 2.5 and the second claim in 2.4.1 we easily infer the following formula for the meet of L-closed sets:

2.6.1. *Let A_i be L-closed. Then $\bigwedge A_i = X \setminus \overline{\bigcup (X \setminus A_i)}$.*

2.7. Proposition. *Let $Y \subseteq X$ be an L-set. Then $\uparrow Y$ is clopen.*

PROOF: As $\uparrow Y$ is closed, $U = X \setminus \uparrow Y$ is in $\downarrow\tau$ and $\overline{U} = \overline{X \setminus \uparrow Y}$ is in $\downarrow CO\tau$. We have $Y \cap \overline{(X \setminus \uparrow Y)} = \overline{Y \cap (X \setminus \uparrow Y)} = \emptyset$ and hence $X \setminus Y \supseteq \overline{X \setminus \uparrow Y}$. But then $X \setminus \uparrow Y \supseteq \overline{X \setminus \uparrow Y}$ because the latter set is decreasing. □

2.8. Proposition. *Let Y be an L-set in an LP-space X. Let $U \in \downarrow CO(\tau|Y)$. Then there is a $W \in \downarrow CO\tau$ such that $U = W \cap Y$.*

PROOF: By 1.2.4, $U = V \cap Y$ with $V \in \downarrow\tau$. Thus, $U = \overline{U} = \overline{V \cap Y} = \overline{V} \cap Y$, and $W = \overline{V} \in \downarrow CO\tau$. □

2.9. An element x of X is said to be an *L-point* if $\{x\}$ is an L-set. The system of all L-points of X will be denoted by

$$\mathsf{Pt}(X).$$

More generally, for any subset Y of X we set

$$\mathsf{Pt}(Y) = \mathsf{Pt}(X) \cap Y.$$

Proposition. *The following statements about a point x of an LP-space X are equivalent:*

(1) *x is an L-point,*
(2) *$U \in \downarrow\tau$ and $x \in \overline{U}$ imply $x \in U$,*
(3) *$\uparrow x$ is clopen.*

PROOF: (2) is just a reformulation of (1). Now let (2) hold. By 1.2.3, $U = X \setminus \uparrow x \in \downarrow\tau$ and as $x \notin U$, $x \notin \overline{U}$. As $\overline{U}$ is decreasing, $\uparrow x \cap \overline{U} = \emptyset$ and hence $X \setminus \uparrow x \supseteq \overline{X \setminus \uparrow x}$. On the other hand, if $\uparrow x$ is open and $x \in \overline{U}$ for a $U \in \downarrow\tau$, then $\uparrow x \cap U \neq \emptyset$ and hence $x \in U$. □

2.10. Obviously each subset of $\mathsf{Pt}(X)$ satisfies (2) from 2.4. Thus, by 2.4.2 we obtain

Corollary. *Let Y be any subset of an LP-space X. Then $\overline{\mathsf{Pt}(Y)}$ is an L-set.*

2.11. In the Priestley duality, the frame **2** corresponds to the one-point space. Consequently, an L-point x corresponds to the frame homomorphism $h : \mathcal{D}(X) \to \mathbf{2}$ defined by $h(U) = 1$ iff $x \in U$. Thus,

$\mathcal{D}(X)$ is spatial iff

(L-sp) *whenever $U, V \in \downarrow CO\tau$ are such that $U \nsubseteq V$, there is an L-point $x \in U \setminus V$.*

In view of this, an LP-space $(X, \leq, \tau)$ will be called *L-spatial* if (L-sp) holds.

2.12. Proposition. *Let X be an L-spatial LP-space. Then $U = \overline{\mathsf{Pt}(U)}$ for every $U \in {\downarrow}CO\tau$.*

PROOF: The inclusion $\supseteq$ is obvious. Now let $x \in U$ and let $x \in W \in \tau$. Then (recall 1.2.2) there are $W_1 \in {\downarrow}CO\tau$, $W_2 \in {\uparrow}CO\tau$ such that $x \in W_1 \cap W_2 \subseteq W$. Hence we have $x \in U \cap W_1 \in {\downarrow}CO\tau$ and $x \notin X \setminus W_2$. Thus $U \cap W_1 \not\subseteq X \setminus W_2$ and by L-spatiality there is an L-point y such that $y \in U \cap W_1$ and $y \notin X \setminus W_2$. Hence $W \cap \mathsf{Pt}(U) \neq \emptyset$ and we see that $x \in \overline{\mathsf{Pt}(U)}$. □

3. L-compact sets

3.1. The following statement has been proved in [**10**]:

Proposition. *Let X be an LP-space. Then the frame $\mathcal{D}(X)$ is compact iff*

(Comp) *if $U \in {\downarrow}\tau$ and $\overline{U} = X$ then $U = X$.*

PROOF: Let $\mathcal{D}(X)$ be compact and $\overline{U} = X$. We have $U = \bigcup_{i \in J} U_i$ with $U_i \in {\downarrow}CO\tau$. Thus, $X = \bigvee U_i$ and we have $X = \bigvee_{i \in K} U_i = \bigcup_{i \in K} U_i$ for a finite $K \subseteq J$. Thus, $X \subseteq U$. On the other hand, let (Comp) hold. If $X = \bigvee_{i \in J} U_i$ with $U_i \in {\downarrow}CO\tau$, we have $X = \overline{\bigcup_{i \in J} U_i}$ and hence $X = \bigcup_{i \in J} U_i$. As X is a compact space, we can find a finite subcover of $\{U_i \mid i \in J\}$. □

An LP-space satisfying (Comp) will be called *L-compact.* More generally, a closed subset $Y \subseteq X$ of an LP-space X is called *L-compact* if

for every $U \in {\downarrow}\tau$, $Y \subseteq \overline{U}$ implies $Y \subseteq U$.

3.2. Proposition. *Let Y be an L-compact subset of an LP-space X. Then*

$$\mathsf{max}(Y) \subseteq \mathsf{Pt}(X).$$

PROOF: Let $x \in \max(Y)$. If $x \in \overline{U}$ and $U \in \downarrow\tau$ then $Y \subseteq \overline{U} \cup (X \setminus \uparrow x) \subseteq \overline{U \cup (X \setminus \uparrow x)}$ and hence $U \subseteq U \cup (X \setminus \uparrow x)$. Thus, as $x \notin (X \setminus \uparrow x)$, we have $x \in U$. □

3.3. Proposition. *In an LP-space X, let a subset Y be L-compact and Z be L-closed. Then $Y \cap Z$ is L-compact. In particular, all L-closed subsets of an L-compact LP-space are L-compact.*

PROOF: Let $Y \cap Z \subseteq \overline{U}$, $U \in \downarrow\tau$. Then $Y \subseteq \overline{U} \cup (X \setminus Z) \subseteq \overline{U \cup (X \setminus Z)}$ as $X \setminus Z$ is open. Hence $Y \subseteq U \cup (X \setminus Z)$, and finally $Y \cap Z \subseteq U$. □

3.4. Proposition. *Let $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k \supseteq \cdots$ be L-closed subsets of an L-compact LP-space X. If $U \in \downarrow CO\tau$ is such that $A_k \nsubseteq U$ for all k, then $\bigwedge A_k \nsubseteq U$.*

PROOF: Suppose $\bigwedge A_k \subseteq U$. By 2.6.1, $X = \overline{\bigcap (X \setminus A_i)} \cup U$. By the L-compactness, $X = \bigcup (X \setminus A_i) \cup U$ and, by the compactness of X, there is a k such that $(X \setminus A_k) \cup U = X$. Hence $A_k \subseteq U$. □

4. Spatiality

4.1. Theorem. *An LP-space X is L-spatial iff for any two $U, V \in \downarrow CO\tau$ such that $U \nsubseteq V$ there is an L-compact $Y \subseteq X$ such that $Y \subseteq U$ and $Y \nsubseteq V$.*

PROOF: The trivial implication follows from the fact that an L-point constitutes an L-compact set. On the other hand, let Y satisfy the condition. As $Y \nsubseteq V$ and V is decreasing, $\max(Y) \nsubseteq V$. Take any $x \in \max(Y) \setminus V$. As $x \in U$, the statement follows from 3.2. □

4.2. By the formula in 2.3, the Priestley dual $\mathcal{D}(X)$ of an LP-space is regular iff $U = \bigvee\{V \in \downarrow CO\tau \mid \uparrow V \subseteq U\}$ for every $U \in \downarrow CO\tau$. We have (see **[10]**)

Proposition. *Let X be an LP-space. Then $\mathcal{D}(X)$ is a regular frame iff*

(Reg) *for each $U \in \downarrow CO\tau$ there are $U_1 \in \downarrow\tau$ and $U_2 \in \uparrow\tau$ such that $U_1 \subseteq U_2$ and $\overline{U_1} = \overline{U_2} = U$.*

PROOF: Let $\mathcal{D}(X)$ be regular, $U \in \downarrow CO\tau$. Set $U_1 = \bigcup\{V \mid \uparrow V \subseteq U\}$ and $U_2 = \bigcup\{\uparrow V \mid \uparrow V \subseteq U\}$. On the other hand, let (Reg) hold. By 1.2.1, $U_1 = \bigcup V_i$ for some $V_i \in \downarrow CO\tau$ and hence $U = \bigvee V_i$. Now if $V_i \in \downarrow CO\tau$ and $V_i \subseteq U_1$ we have $\uparrow V_i \subseteq U_2 \subseteq U$. □

4.3. From 4.1 we immediately see that, in particular,
L-open LP-subspaces of L-compact L-regular LP-spaces, corresponding to locally compact regular frames, and hence all L-regular L-compact LP-spaces are L-spatial.

(Indeed, let $U \not\subseteq V$ be in $\downarrow CO\tau$ of an L-compact L-regular LP-space X. By L-regularity there is a $W \in \downarrow CO\tau$ such that $\uparrow W \subseteq U$ and $W \not\subseteq V$. The set $\uparrow W$ is L-closed in X and hence L-compact.)

Using 4.1 we obtain the much stronger Isbell's spatialization theorem for absolutely G_δ frames ([**4**], see also Lemma 9 in [**2**]):

Theorem. *Let X be an L-compact L-regular LP-space. Let Y be a meet of a countable system of L-open subsets of X. Then Y is L-spatial.*

PROOF: Let $Y = \bigwedge_{n=1}^{\infty} U_n$ with $U_n \in \downarrow CO\tau$. We can assume that $U_1 \supseteq U_2 \supseteq \cdots$. Let $U \not\subseteq V$ be in $\downarrow CO(\tau|Y)$. By 2.8 there are $U', V' \in \downarrow CO\tau$ such that $U = U' \cap Y$ and $V = V' \cap Y$.

CLAIM : *If $W \in \downarrow CO\tau$ is such that $W \cap Y \not\subseteq V'$ then for any n there is a $W' \in \downarrow CO\tau$ such that $\uparrow W' \subseteq W \cap U_n$ and $W' \cap Y \not\subseteq V'$.*

(Indeed, we have $W \cap U_n = \bigvee\{W' \in \downarrow CO\tau \mid \uparrow W' \subseteq W \cap U_n\}$. If all the W' in the set were such that $W' \cap Y \subseteq V'$ we would have $W \cap Y = W \cap U_n \cap Y = \bigvee\{W' \cap Y \mid \cdots\} \subseteq V'$.)

Starting with $W = U'$, we can now inductively choose sets $W_n \in {\downarrow}CO\tau$ so that

$$\uparrow W_n \subseteq U' \cap U_n \quad \text{and} \quad W_n \cap Y \nsubseteq V'.$$

Take $Z = \bigwedge \uparrow W_n$. Then $Z \subseteq \bigwedge U_n = Y$. By 3.4, $Z \nsubseteq V'$ and hence $Z \nsubseteq V$. By 2.7 and 3.3, Z is L-compact. Use 4.1. □

4.4. For locally compact LP-spaces (see 5.1 below) we do not need the regularity.

The well-known *way-below* relation $\ll$ on $\mathcal{D}(X)$ can be expressed as follows (see [**10**]; but it is also easy to infer it from 1.2.1 and the formula in 3.2):

$$V \ll U \ \text{ iff for each } \ W \in {\downarrow}\tau, \ \ U \subseteq \overline{W} \ \textit{ implies } \ V \subseteq W.$$

We immediately see that

$$\text{if} \quad V_i \ll U \quad \text{for} \quad i = 1, 2, ..., n \quad \text{then} \quad V_1 \cup \cdots \cup V_n \ll U.$$

Following lattice terminology we say that an *LP-space is continuous* if

$$U = \bigvee \{V \mid V \in {\downarrow}CO\tau, \ V \ll U\}$$

for every set $U \in {\downarrow}CO\tau$.

4.5. Lemma. *For $n = 1, 2, \ldots$, let $\emptyset \neq U_n \in {\downarrow}CO\tau$ be such that*

$$U_1 \gg U_2 \gg \cdots \gg U_n \gg \cdots.$$

Then $Y = \bigcap_{n=1}^{\infty} U_n$ is L-compact.

PROOF: Since $\mathcal{F} = \{V \mid V \in {\downarrow}CO\tau, \text{ and } \exists n \ V \supseteq U_n\}$ is a filter and obviously $Y = \bigcap \mathcal{F}$, by Proposition in 1.3 we have, for any $V \in {\downarrow}CO\tau$,

$$V \supseteq Y \quad \text{iff} \quad \exists n \ V \supseteq U_n.$$

Let $Y \subseteq \overline{U}$. Then $\overline{U} \supseteq U_n$ for some n and hence $Y \subseteq U_n \subseteq U$. □

4.6. Proposition. *Every continuous LP-space is L-spatial.*

PROOF: Let $U, V \in {\downarrow}CO\tau$ be such that $U \not\subseteq V$. As X is continuous, we can pick, by induction, sets $U_n \in {\downarrow}CO\tau$, $U_n \subseteq U$, such that $U_{n+1} \ll U_n$ and $U_n \not\subseteq V$. Set $Y = \bigcap_{n=1}^{\infty} U_n$. By 4.5, Y is L-compact and obviously $Y \subseteq U$. By 1.3, $Y \not\subseteq V$ since else, as $Y = \bigcap\{V \mid V \supseteq U_n$ for some $n\}$, we would have $U_n \subseteq V$ for some n. □

5. Local compactness

5.1. Proposition 4.6 constitutes a part of the well-known duality between continuous frames and locally compact topological spaces (Hofmann and Lawson [**3**], Banaschewski [**1**]). In this section we will show that the Priestley approach covers also its remaining fact, namely that the continuous LP-spaces are exactly the locally compact L-spatial ones.

In Priestley duality, the classical notion of local compactness directly translates as follows. An *LP-space* $(X, \leq, \tau)$ is said to be *locally compact* if it is L-spatial and if for each L-point $x \in X$ and each $U \in {\downarrow}CO\tau$ such that $x \in U$ there is a $V \in {\downarrow}CO\tau$ and an L-compact L-set Y such that $x \in V \subseteq Y \subseteq U$.

5.2. The definition above had to be formulated with an L-compact L-set because such a set corresponds to a compact sublocale (resp. subspace); a set which is just L-compact is not necessarily an L-set. We have, however,

Proposition. *An L-spatial LP-space $(X, \leq, \tau)$ is locally compact iff for each L-point $x \in X$ and each $U \in {\downarrow}CO\tau$ such that $x \in U$ there is a $V \in {\downarrow}CO\tau$ and an L-compact Y such that $x \in V \subseteq Y \subseteq U$.*

PROOF: Let $V, U \in {\downarrow}CO\tau$, let Y be L-compact and let $V \subseteq Y \subseteq U$. Then $\overline{\mathsf{Pt}(Y)} \subseteq U$ because U is closed in τ. By 2.12, we have $V = \overline{\mathsf{Pt}(V)} \subseteq \overline{\mathsf{Pt}(Y)}$. Since $\overline{\mathsf{Pt}(Y)}$ is an L-set, by 2.10, it suffices to show that it is L-compact.

Thus let $\overline{\mathsf{Pt}(Y)} \subseteq \overline{W}$ for some $W \in {\downarrow}\tau$. As $\overline{W}$ is decreasing, by 3.2 we have $Y \subseteq {\downarrow}\mathrm{Pt}(Y) \subseteq \overline{W}$, and hence $Y \subseteq W$. Since Y is closed, $\overline{\mathsf{Pt}(Y)} \subseteq Y \subseteq U$. □

5.3. Lemma. *The relation $\ll$ interpolates in any continuous LP-space X.*

PROOF: Let $V \ll U$. By the continuity,

$$U = \bigvee\{W \mid W \ll W' \ll U \text{ for some } W'\} = \overline{\bigcup\{W \mid \cdots\}}$$

and hence $V \subseteq \bigcup\{W \mid \cdots\}$. The compactness of X (and, consequently, that of V) implies the existence of $W_i \ll W_i' \ll U$ with $i = 1, \ldots, n$ such that $V \subseteq W_1 \cup \cdots \cup W_n$. Set $W = W_1' \cup \cdots \cup W_n'$. Then $V \ll W \ll U$, by 4.4. □

5.4. Theorem. *An LP-space is continuous iff it is locally compact.*

PROOF: A locally compact space is continuous since from $V \subseteq Y \subseteq U$ for $V, U \in {\downarrow}\mathcal{CO}\tau$ and an L-compact Y it immediately follows that $V \ll U$.

For the converse, let X be continuous. Then X is L-spatial, by 4.6. Choose an L-point X and a $U \in {\downarrow}\mathcal{CO}\tau$ such that $x \in U$. Then $x \in \bigvee\{V \mid V \ll U\} = \overline{\bigcup\{V \mid V \ll U\}}$, and hence $x \in \bigcup\{V \mid V \ll U\}$. Thus there is a $V \ll U$ such that $x \in V$. By 5.3, we can inductively choose U_n so that $V \ll \cdots \ll U_n \ll U_{n-1} \ll \cdots \ll U_1 \ll U$. But then $Y = \bigcap_{n=1}^{\infty} U_n$ is L-compact, by 4.5. Use 5.2. □

REFERENCES

1. B. Banaschewski, *The duality of distributive continuous lattices*, Canadian J. of Math. **XXXII** (1980), 385-394.
2. B. Banaschewski and A. Pultr, *Samuel compactification and completion of uniform frames*, Math. Proc. Camb. Phil. Soc. **108** (1990), 63-78.
3. K.H. Hofmann and J.D. Lawson, *The spectral theory of distributive continuous lattices*, Trans. Amer. Math. Soc. **246** (1978), 285-310.

4. J.R. Isbell, *Function spaces and adjoints*, Math. Scand. **36** (1975), 317-339.
5. P.T. Johnstone, *"Stone Spaces"*, Cambridge University Press, Cambridge, 1982.
6. P.T. Johnstone, *Tychonoff's theorem without the axiom of choice*, Fund. Math. **113** (1981), 31-35.
7. J.L. Kelley, *The Tychonoff Product Theorem implies the Axiom of Choice*, Fund. Math. **37** (1950), 75-76.
8. H.A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. **2** (1970), 186-190.
9. H.A. Priestley, *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. **24** (1972), 507-530.
10. A. Pultr and J. Sichler, *Frames in Priestley's duality*, Cahiers de Top. et Géom. Diff. Cat. **XXIX-3** (1988), 193-202.
11. S. Vickers, *"Topology via Logic"*, Cambrige Tracts in Theor. Comp. Sci., Number 5, Cambridge University Press, Cambridge, 1985.

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