CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

EDUARDO J. DUBUC
JORGE C. ZILBER

# Inverse function theorems for Banach spaces in a topos

*Cahiers de topologie et géométrie différentielle catégoriques*, tome 41, nº 3 (2000), p. 207-224
<http://www.numdam.org/item?id=CTGDC_2000__41_3_207_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLI-3 (2000)

# INVERSE FUNCTION THEOREMS FOR BANACH SPACES IN A TOPOS

*by Eduardo J. DUBUC and Jorge C. ZILBER*

**Résumé.** En [4] nous avons construit une immersion de la catégorie d'ouverts d'espaces de Banach et des fonctions holomorphes dans un topos modèle analytique de la GDS [3]. Cette immersion préserve les produits finis et elle est consistante avec le calcul différentiel.

Nous étudions ici les flèches dans le topos entre des espaces de banach. Nous démontrons qu'elles peuvent être considérées comme des fonctions entre ces espaces-là, et qu'elles s'avèrent Goursat ou G-holomorphes. En plus, elles doivent être compatibles, dans un certain sens, avec les congruences définies par les idéaux dans les anneaux de germes qui définissent les objets du site. La continuité de la flèche dans le topos par rapport à la topologie banach correspond exactement à la condition d'être holomorphe dans la fonction. Cependant, ce n'est pas le cas en ce qui concerne les variables internes de type exponentiel. Nous avons introduit une condition plus forte qui détermine un subobjet de l'exponentielle strictement contenu dans celui déterminé par la condition de continuité et qui définit la notion interne correcte de fonction holomorphe.

En nous servant de ces résultats, nous avons dévéloppé l'infrastructure nécessaire afin de pouvoir quantifier sur des variables holomorphes internes dans le topos et nous avons démontré des théorèmes internes de la fonction inverse pour les espaces de Banach.

## Introduction.

In [4] we have constructed an embedding $j: \boldsymbol{B} \to \boldsymbol{T}$ from the category $\boldsymbol{B}$ of open sets of complex Banach spaces and holomorphic functions into the analytic (well adapted) model of SDG $\boldsymbol{T}$ introduced in [3]. This embedding preserves finite products and is consistent with the differential calculus. Then, in [5] we consider a topological structure in the topos (in the sense of [1]) on objects of the form $jB$ and the exponentials $jB^X$, with $X$ any object in the site of definition. The topology in $jB^X$ strictly generalizes the "canonical" topology in the set of complex valued morphisms of analytic spaces considered, for example, in [6].

Here in section 1 we investigate the nature of the arrows $\lambda: jB_1 \to jB_2$ in the topos between open sets of banach spaces. We prove that the global sections functor $\Gamma: \boldsymbol{T} \to \boldsymbol{Ens}$ is faithful, when restricted to the full subcategory of objects in the image of the embedding $j: \boldsymbol{B} \to \boldsymbol{T}$. This means that the arrows in the topos can be considered as functions between the sets of global sections $f = \Gamma(\lambda): B_1 \to B_2$ such that satisfy some condition. This condition turns out to be related to the notion of Goursat or G-holomorphic function. It follows that an arrow $\lambda: jB_1 \to jB_2$ is continuous for the inherited topology if and only if it is of the form $j$ f for a (unique) holomorphic function $f: B_1 \to B_2$. We can say then that continuity for the inherited opens in the topos is a condition that means that an arrow $jB_1 \to jB_2$ is holomorphic. It defines a subobject of the exponential $jB_2^{jB_1}$ allowing internal quantification (on inherited continuous) variables of type $jB_2^{jB_1}$. However, this condition is not enough to grasp the meaning "holomorphic" for the internal logic of the topos. This lead us to study several stronger conditions which define subobjects of the exponential, and which will allow the correct internal quantification on holomorphic maps.

In section 2 we consider infinitesimal and local inverse function theorems for objects of the form $jB$. It turns out that the infinitesimal inverse function theorem for an arrow of the form $jf$ is equivalent to the classical local inverse function theorem for the holomorphic function $f$. The inverse function theorems are internal statements in the logic of the topos, and can be stated and proved using the quantification on holomorphic variables developed in section 1.

For the convenience of the reader and to set the notation we start recalling some facts.

## 0. Recall of some definitions and notation.

The topos $\boldsymbol{T}$ is the topos of sheaves on the category $\boldsymbol{H}$ of (affine) analytic schemes.

**0.1.1** Recall briefly from [3] the construction of $\boldsymbol{T}$. We consider the category $\boldsymbol{H}$ of (affine) analytic schemes. An object $E$ in $\boldsymbol{H}$ is an $A$-ringed space $E = (E, \mathcal{O}_E)$ (by abuse we denote also by the letter $E$ the underlying topological space of the $A$-ringed space) which is given by an open subset $D$ of $\mathbb{C}^m$, and two coherent sheaves of ideals $R$, $S$ in $\mathcal{O}_D$, $R \subset S$. Then, $E = supp(S)$, $\mathcal{O}_E = (\mathcal{O}_D/R)$ (the support or set of zeroes of $S$ and the quotient by $R$ respectively). The arrows in $\boldsymbol{H}$ are the morphism of $A$-ringed spaces. We denote by $\boldsymbol{T}$ the topos of sheaves on $\boldsymbol{H}$ for the (sub canonical ) Grothendieck topology given by the open coverings. There is a full (Yoneda) embedding $\boldsymbol{H} \to \boldsymbol{T}$. Notice that for an infinite dimensional banach open $B$, the $A$-ringed space $(B, \mathcal{O}_B)$ is not in $\boldsymbol{H}$.

Let $\boldsymbol{B}$ be the category of open sets of complex Banach spaces and holomorphic functions. The embedding $j\colon \boldsymbol{B} \to \boldsymbol{T}$ is defined in [4] as follows:

**0.1.2** Let $E$ be an object in $\boldsymbol{H}$, $E = (E, \mathcal{O}_E)$ as above, let $B$ be an open subset of a complex Banach space, and let $t = (t, \tau)$ be a morphism of $A$-ringed spaces, $t\colon (E, \mathcal{O}_E) \to (B, \mathcal{O}_B)$ (we adopt the corresponding abuse of notation for morphisms). We say that $t$ has "local extensions", if for each $x \in E$, there is an open neighborhood $U$ of $x$ in $\mathbb{C}^m$ and an extension $(f, f^*)\colon (U, \mathcal{O}_U) \to (B, \mathcal{O}_B)$, $t(x) = f(x)$, and $\tau = \rho \circ f^*$ (where $\rho$ is the quotient map). The set

$$jB(E) = \{t\colon (E, \mathcal{O}_E) \to (B, \mathcal{O}_B) \quad \text{such that t has local extensions}\}$$

defines the sheaf $jB \in \boldsymbol{T}$. It follows in a natural way that $jB$ is a sheaf and that $j$ is functorial. It is clear that $\Gamma(jB) = B$ for all $B \in \boldsymbol{B}$, and $\Gamma(jf) = f$ for all arrows $f\colon B_1 \to B_2$ in $\boldsymbol{B}$.

**0.1.3** An arrow $t\colon E \to jB$ in $\boldsymbol{T}$ corresponds to an element $t \in jB(E)$, that is, $t\colon (E, \mathcal{O}_E) \to (B, \mathcal{O}_B)$ is a morphism of $A$-ringed spaces with local extensions. Then, $t = (t, \tau)$ where $t\colon E \to B$ is continuous, and it is immediate that $\Gamma(t) = t$ (notice here the abuse of language). If $E$ is of the form $(U, \mathcal{O}_U)$ for an open set $U \subset \mathbb{C}^m$, then $t$ is an holomorphic function and $\tau = t^*$, see [4, 1.2]. Thus, if $t$, $r\colon U \to jB$ are such that $\Gamma(t) = \Gamma(r)$, then $t = r$.

Concerning the inherited topological structures we recall from [5] the following:

**0.2.1** Consider a (fixed) object $F$ in $\boldsymbol{T}$ such that the set of global sections $\Gamma(F)$ is furnished with a (unnamed) arbitrary topology. Assume that the topology in $\Gamma(F)$ is Haussdorf and that for any object $E$ in the site and anyarrow $q: E \to F$ in the topos $\boldsymbol{T}$, the function $q = \Gamma(q) : E \to \Gamma(F)$ is continuos. Then, an inherited topological structure in the sense of [1] is determined in the object $F$. Essentially we can think that the inherited-open subobjects are those subobjects whose sets of global sections are open subobjects of $\Gamma(F)$ (all the technical requirements for this are developed in detail in [5], were we develop as well the basic properties of this construction). In particular, we have:

**0.2.2** Let $F_1$ and $F_2$ be objects in the topos such that their sets of global sections $\Gamma(F_1)$ and $\Gamma(F_2)$ are furnished with topologies as in 0.2.1 above. Then, an arrow in the topos $f: F_1 \to F_2$ is continuos for the inherited topological structures if and only if the function between the global sections $\Gamma(f): \Gamma(F_1) \to \Gamma(F_2)$ is continuos.

## 1. On the arrows between open sets of banach spaces in the topos.

We shall investigate now the nature of the arrows $\lambda: jB_1 \to jB_2$ in the topos between open sets of banach spaces. We prove that the functor $\Gamma: \boldsymbol{T} \to \boldsymbol{Ens}$, when restricted to the full subcategory of objects in the image of the embedding $j: \boldsymbol{B} \to \boldsymbol{T}$ is faithful. This means that the arrows in the topos can be considered as functions between the sets of global sections $f = \Gamma(\lambda): B_1 \to B_2$ such that satisfy some condition. This condition turns out to be related to the notion of Goursat or $G$-holomorphic function.

**1.1 Proposition.** *Let $B_1$ and $B_2$ be open subsets of complex Banach spaces and let $\lambda$ and $\mu$ be arrows in $\boldsymbol{T}$, $\lambda$ , $\mu: jB_1 \to jB_2$. If $\Gamma(\lambda) = \Gamma(\mu)$, then $\lambda = \mu$.*

**Proof.** We have to see that $\lambda \circ r = \mu \circ r$ for any given $E \in \boldsymbol{H}$, and $r: E \to jB_1$ in $\boldsymbol{T}$. But $\Gamma(\lambda \circ r) = \Gamma(\lambda) \circ \Gamma(r) = \Gamma(\mu) \circ \Gamma(r) = \Gamma(\mu \circ r)$. If $E$ is of the form $(U, \mathcal{O}_U)$ for an open set $U \subset \mathbb{C}^n$, then $\lambda \circ r$, $\mu \circ r: U \to jB_2$ are just holomorphic functions, thus $\lambda \circ r = \mu \circ r$, see 0.1.3 above. In the general case we have an (open) covering $E_\alpha \to E$, $E_\alpha \subset U_\alpha \subset \mathbb{C}^n$, and holomorphic extensions $f_\alpha: U_\alpha \to jB_1$, $f_\alpha|_{E_\alpha} = r|_{E_\alpha}$, see 0.1.2 above. Thus, as we have just seen, $\lambda \circ f_\alpha = \mu \circ f_\alpha$. It follows that $\lambda \circ r|_{E_\alpha} = \mu \circ r|_{E_\alpha}$, and in consequence also $\lambda \circ r = \mu \circ r$. □

In view of this proposition, the arrows $\lambda: jB_1 \to jB_2$ can be considered to be functions $f: B_1 \to B_2$ which satisfy a certain condition. In our next theorem we explicitly describe this condition, which turns out to be related to the notion of Goursat or $G$-holomorphic function. Before it is convenient to consider some definitions. Recall that a function $f: B_1 \to B_2$ between open sets of banach spaces is Goursat or $G$-holomorphic if given any $U \subset \mathbb{C}$ open, and any line segment defined on $U$, $a + xb: U \to B_1$ ($x$ a variable in $U$, $a \in B_1$, and $b$ any vector in the banach space containing $B_1$), the composite $f(a + xb): U \to B_2$ is holomorphic (see [7, II, 8]). In this paper we shall adopt a seemingly stronger condition as the defining property for $G$-holomorphic functions:

**1.2 Definition.** *We say that a function $f: B_1 \to B_2$ between open sets of banach spaces is Goursat or $G$-holomorphic if given any $U \subset \mathbb{C}$ open, and any holomorphic curve defined on $U$, $r: U \to B_1$, the composite $f \circ r: U \to B_2$ is holomorphic.*

Remark that with this definition it is immediate to see that the composite of two $G$-holomorphic functions is $G$-holomorphic. Conversely, if the weaker definition quoted above happens to be closed under composition, then given any holomorphic curve $h: U \to B_1$, the composite $f \circ h: U \to B_2$ would be $G$-holomorphic, and thus clearly also holomorphic (any $G$-holomorphic function defined on $U$ is holomorphic, just take the line $0 + x1$). It follows that definition 1.2 is the "correct" (closed under composition) notion.

We recall that the difference between $G$-holomorphic and holomorphic functions is in the continuity condition. In fact, a function is holomorphic if and only if it is continuous and $G$-holomorphic [7, II, 8.7].

**1.3 Proposition.** *All $G$-holomorphic functions $h: U \to B$, defined in an open set $U \subset \mathbb{C}^n$ with values in any open set $B$ of a Banach space are holomorphic.*

**Proof.** Clearly the function $h$ is separately holomorphic, that is, it is holomorphic in each variable when the others are held fixed. But then it is known that any such function is holomorphic, see [7, VIII, 36]. □

**1.4 Corollary.** *Let $f: B_1 \to B_2$ be a $G$-holomorphic function between open sets of banach spaces . Then, given any open set $U \subset \mathbb{C}^n$ and* *an holomorphic function* $r: U \to B_1$*, the composite* $h \circ r: U \to \mathbb{C}$ *is holomorphic.*

Given a point $x \in \mathbb{C}^n$ and an ideal $I_x \subset \mathcal{O}_x^n$ in the ring of germs of holomorphic functions, consider an open subset $B \subset C$ of a banach space $C$. Recall from [5, 3.1 b] that we say that two $B$-valued germs of holomorphic functions $[s]_x$ , $[t]_x$ are equivalent modulo $I_x$ if for all continuos linear forms $\alpha \in C'$, $[\alpha \circ (s-t)]_x \in I_x$. That is, $s = t \,\mathrm{mod}(I_x)$ if for all $\alpha \in C'$ $\alpha \circ s = \alpha \circ t \,\mathrm{mod}(I_x)$.

**1.5 Definition.** *Let* $x$ *be a point* $x \in \mathbb{C}^n$*, and* $I_x \subset \mathcal{O}_x^n$ *an ideal in the ring of germs of holomorphic functions. We say that a* $G$*-holomorphic function* $f: B_1 \to B_2$ *between open sets of banach spaces,* $B_1 \subset C_1$*,* $B_2 \subset C_2$ *has good reduction modulo* $I_x$ *if given any pair of holomorphic* $B_1$ *valued germs* $[s]_x$ *,* $[r]_x$ *, if* $r = s \,\mathrm{mod}(I_x)$*, then also* $f \circ r = f \circ s \,\mathrm{mod}(I_x)$*. That is, if the following implication holds:*

*If for all continuous linear forms* $a \in C_1'$*,* $[\alpha \circ r - \alpha \circ s]_x \in I_x$*, then also for all continuous linear forms* $\beta \in C_2'$*,* $[\beta \circ (f \circ r) - \beta \circ (f \circ s)]_x \in I_x$*.*

If we think the composition with the continuous linear forms as the "coordinates" of an holomorphic function, then this condition on $f$ says that if any pair of holomorphic functions $r$ and $s$ as above have the same coordinates modulo $I_x$ , then this is also the case for the composites $f \circ r$ and $f \circ s$.

An important consequence of a result we proved in [4] ([4, 2.6]) is the following:

**1.6 Proposition.** *With the notations in the previous definition, a* $G$*-holomorphic function* $f$ *has good reduction modulo* $I_x$ *if and only if given any pair of* $B_1$ *valued germs* $[s]_x$*,* $[r]_x$*,* $s(x) = r(x) = p$*, the following implication holds:*

*If for all germs* $[t]_p \in \mathcal{O}_{B_1,p}$*,* $[t \circ r - t \circ s]_x \in I_x$*, then also for all germs* $[u]_q \in \mathcal{O}_{B_2,q}$*,* $[u \circ (f \circ r) - u \circ (f \circ s)]_x \in I_x$*. Here* $q = f(p)$*.*

**Proof.** It is an immediate consequence of [5, 3.3]. □

**1.7 Remark.** *All holomorphic functions have good reduction.*

**Proof.** With the notations in 1.6, consider the germ $t = u \circ f$. □

Notice that $u \circ f$ is not necessarily holomorphic in general. $G$-holomorphic functions which do no have good reduction are necessarily non holomorphic, which amounts to say that they are not continuous. We do not know yet any explicit example of a $G$-holomorphic function which does not have good reduction.

**1.8 Theorem.** *A function $f: B_1 \to B_2 \subset C_2$ between open sets of banach spaces determines a (unique) arrow in the topos $\lambda: jB_1 \to jB_2$ such that $f = \Gamma(\lambda)$ if and only if it is $G$-holomorphic and it has good reduction modulo $I_x$ for any point $x \in \mathbb{C}^n$, and any ideal $I_x \subset \mathcal{O}_x^n$.*

**Proof.** By proposition 1.1 a function $f: B_1 \to B_2$ is of the form $f = \Gamma(\lambda)$ for at most one arrow $\lambda$.

Assume $f = \Gamma(\lambda)$ and let $r: U \to B_1$ be an holomorphic function. This defines a section of $jB_1$, $r: U \to jB_1$, and the composite $\lambda \circ r: U \to jB_2$ is a section of $jB_2$. Thus (see 0.1.3 above) $f \circ r = \Gamma(\lambda) \circ r = \Gamma(\lambda \circ r)$ is an holomorphic function. This shows that $f$ is $G$-holomorphic. Conversely, if $f$ is $G$-holomorphic, a section of $jB_1$ defined in $U$, $r: U \to jB_1$, determines an holomorphic function $U \to B_1$ (0.1.3 above), and then the composite $f \circ r$ in turn defines a section $U \to jB_2$. This is the data that determines the action $\lambda \circ r$ (on sections $r$ defined on open sets $U$) of an arrow $\lambda$ in the topos. We shall see now that the possibility to extend this data to sections defined in a general object $E$ in the site is equivalent to the good reduction of $f$.

Let $s: E \to jB_1$ be a section. Then (see 0.1.1 and 0.1.2 above) $s$ is a morphism of $A$-ringed spaces $s: (E, \mathcal{O}_E) \to (B_1, \mathcal{O}_{B_1})$, $s = (s, \sigma)$, $\sigma_x: \mathcal{O}_{B_1,p} \to E_x = \mathcal{O}_x^n / I_x$, $p = s(x)$, $E \subset U \subset \mathbb{C}^n$. Since $s$ has local extensions, there is an open covering $E_\alpha \subset E$, $E_\alpha = U_\alpha \cap E$, $U_\alpha \subset U$, and holomorphic functions $r_\alpha: U_\alpha \to B_1$, such that $\forall x \in E_\alpha$, $r_\alpha(x) = s(x)$ and $\rho_x \circ r_\alpha * = \sigma_x$, where $\rho_x: \mathcal{O}_x^n \to E_x$ is the quotient map. Assume $f$ has good reduction. We then define for each $\alpha$, $(\lambda \circ s)|_{E_\alpha} = ((f \circ s)|_{E_\alpha}, \theta_\alpha): (E_\alpha, O_{E_\alpha}) \to (B_2, \mathcal{O}_{B_2})$, where, for $x \in E_\alpha$, $\theta_{\alpha x} = \rho_x \circ (f \circ r_\alpha)^*$. We have to see that this is well defined, that is, it does not depend on the choice of the extension $r_\alpha$, and (to determine an arrow $\lambda \circ s: E \to jB_2$ in the topos), that it defines a compatible family for the covering. Both these statements are clear for the first coordinate of the morphism $(f \circ s)|_{E_\alpha}$. We now pass to see that they hold also for the second coordinate $\theta_\alpha$. Let $[t]_p$, $[u]_q$ be any two germs $[t]p \in \mathcal{O}_{B_1,p}$, $[u]q \in \mathcal{O}_{B_2,q}$, $p = s(x) = r_\alpha(x)$, $q = f(p)$. Then we have $\sigma_x([t]_p) = \rho_x \circ r_\alpha^*([t]_p) = \rho_x([t \circ r_\alpha]_x)$, and

$\theta_{\alpha x}([u]_q) = \rho_x \circ (f \circ r_\alpha)^*([u]_q) = \rho_x([u \circ (f \circ r_\alpha)]_x)$. Considering that $\sigma_x$ is defined as part of the data for the section $s: E \to jB_1$, the needed statements for $\theta_\alpha$ follow simultaneously and immediately from the good reduction of $f$ and proposition 1.6.

Conversely, assume $f = \Gamma(\lambda)$. Let $x$ be a point $x \in \mathbb{C}^n$, and $I_x \subset \mathcal{O}_x^n$ be an ideal in the ring of germs of holomorphic functions. It is known (see [4, 2.1], [3, pp. 191]) that there is an open set $U \subset \mathbb{C}^n$ and a finitely generated ideal $I(U) \subset \mathcal{O}(U)$ which generate $I_x$. Clearly $I(U)$ determines a coherent sheaf of ideals $I$ in $\mathcal{O}_U$ equal to $I_x$ at the point $x$. Let $J$ be the principal ideal $J = \{h \in \mathcal{O}(U) \mid h(x) = 0\}$. This pair of ideals define an object $(E, \mathcal{O}_E)$ in the site such that $E = \{x\}$, and $\mathcal{O}_{E,x} = \mathcal{O}_x^n / I_x$. Remark that $E \subset U$ as objects in the topos. Given any two $B_1$ valued germs $[s]_x$, $[r]_x$, as in definition 1.5, defined, say, in an open set $U$, $r$, $s: U \to B_1$. Assume that $r(x) = s(x) = p$, and that for all germs $[t]_p \in \mathcal{O}_{B_1,p}$, $[t \circ r - t \circ s]_x \in I_x$. Then the restrictions of $r$ and $s$ to $E$ in the topos are equal, $r|_E = s|_E$. Thus, also $(\lambda \circ r)|_E = (\lambda \circ s)|_E$. Since $f \circ r = \Gamma(\lambda \circ r)$ and $f \circ s = \Gamma(\lambda \circ s)$, it follows that for all germs $[u]_q \in \mathcal{O}_{B_2,q}$, $[u \circ (f \circ r) - u \circ (f \circ s)]_x \in I_x$. Thus, again by proposition 1.6, $f$ has good reduction modulo $I_x$. □

We shall determine now which are the arrows in the topos that correspond to holomorphic functions. The answer is given by the topological structure (in the sense of [1]) on $jB$ inherited from the topology of the banach space $B$ (see 0.2.1 above).

We denote this structure with the letter '$\kappa$', $\kappa[jB] \subset [jB] = \Omega[jB] = \Omega^B$ (see [5]). Essentially we can think that the $\kappa$-open subobjects are those subobjects whose sets of global sections are open subobjects of $B$.

We shall see that an arrow $\lambda: jB_1 \to jB_2$ is continuous for the inherited topology if and only if the function $f = \Gamma(\lambda): B_1 \to B_2$ is holomorphic. In this case, it follows from 1.1 that $\lambda$ is of the form $\lambda = jf$ for a (unique) holomorphic function $f: B_1 \to B_2$, $f = \Gamma(\lambda)$, thus $j\Gamma(\lambda) = \lambda$.

**1.9 Proposition.** *Let $B_1$ and $B_2$ be open subsets of complex Banach spaces and let $f$ be an holomorphic function, $f: B_1 \to B_2$. Then, $jf$ is continuous for the $\kappa$-structure.*

**Proof.** Since $f = \Gamma(jf)$, the proof follows immediately from 0.2.2 above. □

**1.10 Proposition.** *Let $B_1$ and $B_2$ be open subsets of complex Banach spaces and let $\lambda$ be an arrow in $\boldsymbol{T}$, $\lambda: jB_1 \to jB_2$. Then, $\lambda$ is continuous for the $\kappa$-structure if and only if $\Gamma(\lambda): B_1 \to B_2$ is holomorphic. In this case, $j\Gamma(\lambda) = \lambda$.*

**Proof.** By 0.2.2 above if follows that $\lambda$ is continuous for the $\kappa$-structure if and only if $\Gamma(\lambda): B_1 \to B_2$ is continuous. By theorem 1.8 we know that, in particular, $\Gamma(\lambda)$ is $G$-holomorphic. The proof follows recalling that a $G$-holomorphic function is holomorphic if and only if it is continuous (see [7, II, 8.7]). The final statement is clear by 1.1. □

We shall see now that continuity for the $\kappa$ topological structure actually defines a subobject of the exponential $jB_2^{jB_1}$ in the topos. This allows internal quantification on $\kappa$-continuous arrows.

**1.11 Definition-Proposition.** *Let $B_1$ and $B_2$ be open subsets of complex Banach spaces. We define a subobject $\kappa[jB_1, jB_2] \subset jB_2^{jB_1}$ by the following condition:*

*$\forall X \in \boldsymbol{H}$, $\forall q: X \to jB_2^{jB_1}$ in $\boldsymbol{T}$, $q$ factors through $\kappa[jB_1, jB_2]$ iff the corresponding arrow $jB_1 \times X \to jB_2$ is $\kappa$-continuous. By $\kappa$-continuous we mean continuous for the $\kappa$-structures and the product structure on $jB_1 \times X$ (for precision on the product structure see [5, 2.5])*

**Proof.** We have to show that this property defines a subsheaf of $jB_2^{jB_1}$:

a) Given $r: E \to X$ in $\boldsymbol{H}$, and $q: X \to jB_2^{jB_1}$,

if $jB_1 \times X \to jB_2$ is $\kappa$-continuous, then so is

$$jB_1 \times E \to jB_1 \times X \to jB_2.$$

b) Given a covering $r_i: U_i \to X$, and $q: X \to jB_2^{jB_1}$,

if $\forall i \; jB_1 \times U_i \to jB_1 \times X \to jB_2$ is $\kappa$-continuous, then so is $jB_1 \times X \to jB_2$.

Both statements follow easily using 0.2.2 above (see also [5, 1.7]). For the second notice that since $U_i \to X$ is an open covering, then so is

$$jB_1 \times U_i \to jB_1 \times X.$$

□

In proposition 1.10 we have shown that for arrows $jB_1 \to jB_2$ in the topos, the property of $\kappa$-continuity captures the meaning of being holomorphic. However, for variables of type $jB_2^{jB_1}$ the internal notion of $\kappa$-continuity defined by the subobject $\kappa[jB_1, jB_2] \subset jB_2^{jB_1}$ is not strong enough to capture correctly the internal meaning of being holomorphic.

**1.12 Definition-Proposition.** *Let $B_1$ and $B_2$ be open subsets of complex Banach spaces. We define a subobject $Hol[jB_1, jB_2] \subset jB_2^{jB_1}$ by the following condition:*

*$\forall X \in \boldsymbol{H}$, $X \subset \mathbb{C}^n$ and $\forall q: X \to jB_2^{jB_1}$ in $\boldsymbol{T}$, $q$ factors through $Hol[jB_1, jB_2]$ iff the corresponding arrow $q: jB_1 \times X \to jB_2$ has local extensions.*

*By this later condition we mean that given any point $(p, x) \in B_1 \times X$, there are open sets $W \subset \mathbb{C}^n$, $H \subset B_1$, $(p, x) \in H \times W$ and an holomorphic function $g: H \times W \to B_2$ such that the following diagram commutes in the topos:*

$$\begin{tikzcd} jH \times (X \cap W) \arrow[dr, "q"'] & \subset & jH \times W \arrow[dl, "jg"] \\ & jB_2 & \end{tikzcd}$$

Recall that $j(H \times W) = jH \times W$ (see [4, 3.2]), and remark that this condition is much stronger that just the commutativity on the global sections.

**Proof.** We have to show that this property defines a subsheaf of $jB_2^{jB_1}$:

a) Given $r: E \to X$ in $\boldsymbol{H}$, and $q: X \to jB_2^{jB_1}$, if $jB_1 \times X \to jB_2$ has local extensions, then so has $jB_1 \times E \to jB_1 \times X \to jB_2$.

b) Given a covering $r_i: U_i \to X$, and $q: X \to jB_2^{jB_1}$, if

$$\forall i \;\; jB_1 \times Ui \to jB_1 \times X \to jB_2$$

have local extensions, then so has $jB_1 \times X \to jB_2$.

The statement a) follows from the fact that the arrows $r: E \to X$ in $\boldsymbol{H}$, $E \subset \mathbb{C}^m$, $X \subset \mathbb{C}^n$ have local extensions in the sense that given any $x \in E$ and open $W \subset \mathbb{C}^n$, $r(x) \in W$, there is an open $V \subset \mathbb{C}^m$ and an holomorphic function $g: V \to W$ which is an extension of $r$ in the obvious sense. The statement b) is straightforward from the fact that $U_i \to X$ is an open covering and $X \subset \mathbb{C}^n$ has the subspace topology. $\square$

The inherited $CU$-topological structure in the exponential $jB_2^X$ (see [5, section 3]) can also be used in the obvious way to define a subobject of $jB_2^{jB_1}$, namely:

**1.13 Definition-Proposition.** *Let $B_1$ and $B_2$ be open subsets of complex Banach spaces. We define a subobject $CU[jB_1, jB_2] \subset jB_2^{jB_1}$ by the following condition:*

$\forall X \in \boldsymbol{H}$, $\forall q: X \to jB_2^{jB_1}$ in $\boldsymbol{T}$, $q$ factors through $CU[jB_1, jB_2]$ iff the corresponding arrow $jB_1 \to jB_2^X$ is continuous for the inherited $\kappa$ and $CU$ structures.

**Proof.** We have to show that this property defines a subsheaf of $jB_2^{jB_1}$.

a) Given $r: E \to X$ in $\boldsymbol{H}$, and $q: X \to jB_2^{jB_1}$,

if $jB_1 \to jB_2^X$ is continuous, then so is $jB_1 \to jB_2^X \to jB_2^E$.

b) Given a covering $r_i: U_i \to X$, and $q: X \to jB_2^{jB_1}$,

if $\forall i \; jB_1 \to jB_2^X \to jB_2^{U_i}$ is continuous, then so is $jB_1 \to jB_2^X$.

The statement a) follows immediately from the fact proved in [5, 3.11] that the arrow induced by $r$, $jB_2^X \to jB_2^E$ is continuos for the $CU$ topological structure.

The proof of b) is easy. By 0.2.2 above we can work with the sets of global sections. The part corresponding to the map $[X, B_2] \to C(X, B_2)$ follows using the fact that a compact set $K$ of $X$ is of the form $K = K_1 \cup K_2 \cup \ldots \cup K_m$, $K_i \subset U_i$, compact subsets of $U_i$. The part corresponding to the morphisms $[X, B_2] \to \mathcal{O}_x^n(B_2)/I_x$ is immediate since the fibers are the same. $\square$

The three conditions defined on the variables of type $jB_2^{jB_1}$ are obviously related, we have:

**1.14 Proposition.** *With the notations in 1.11, 1.12, and 1.13. there is an inclusion of subobjects $Hol[jB_1, jB_2] \subset CU[jB_1, jB_2] \subset \kappa[jB_1, jB_2]$.*

*These inclusions collapse for the global sections, that is, for actual arrows* $jB_1 \to jB_2$ *in* $\boldsymbol{T}$,

$$\Gamma(Hol[jB_1, jB_2]) = \Gamma(CU[jB_1, jB_2]) = \Gamma(\kappa[jB_1, jB_2]).$$

**Proof.** Let $q: X \to jB_2^{jB_1}$. Assume that $q \in Hol[jB_1, jB_2]$. We will show that the corresponding arrow $q: jB_1 \to jB_2^X$ is continuous in the sense of 1.13. By 0.2.2 we have to do this for the function $q: B_1 \to [X, B_2]$. Let $(p, x) \in B_1 . \times X$ and consider an holomorphic extension $g: H \times W \to B_2$. With these data at hand, a proof of the continuity of $q : B_1 \to [X, B_2]$ can be done following exactly the same lines that the proof of proposition 3.8 in [5]. Thus, we have proved that $q \in CU[jB_1, jB_2]$. Then in particular the composite $B_1 \to [X, B_2] \to C(X, B_2)$ is continuous. Since $X$ is locally compact, the topology on $C(X, B_2)$ is exponential, thus the corresponding function $B_1 \times X \to B_2$ is continuous. Thus by 0.2.2 above it follows that $q \in \kappa[jB_1, jB_2]$. This finishes the proof of the first statement. The second statement clearly follows from proposition 1.10. □

We see in this proof that the condition $q \in \kappa[jB_1, jB_2]$ is clearly weaker than the condition $q \in CU[jB_1, jB_2]$ since it corresponds to only the continuity of the composite $B_1 \to [X, B_2] \to C(X, B_2)$. When $q \in CU[jB_1, jB_2]$, the continuity of all the composites

$$B_1 \to [X, B_2] \to \mathcal{O}_x^n(B_2)/I_x$$

imposes an stronger condition. This condition is probably strong enough to guarantee the existence of local extensions. Recall that if a sequence of germs in $\mathcal{O}_x^n(B_2)$ is convergent then the whole sequence "lift" to some $Hol(W, B_2)$ with $W$ open, $x \in W$ (see [5, 3.4]). Using this fact, since $B_1$ is first countable, it can easily be seen that given a map $B_1 \to \mathcal{O}_x^n(B_2)$, there are open sets $H \times W$, $(p, x) \in H \times W$, and a lifting $H \to Hol(W, B_2) \to \mathcal{O}_x^n(B_2)$. If this lifting is continuous, then it follows that we have local extensions in the sense of 1.12. We conjecture that this is the case, so that we would have the interesting fact $Hol[jB_1, jB_2] = CU[jB_1, jB_2]$.

## 2. The inverse function theorems

In this section we shall prove an infinitesimal inverse function theorem in the topos. This theorem turns out to be equivalent to the classical local inverse function theorem for holomorphic maps between open sets of banach spaces. We shall also prove a local inverse function theorem in the topos.

Let $B$ be an open set of a banach space $C$ and consider the inherited topological structure on $jB$, $\kappa[jB] \subset \Omega[jB] = \Omega^B$ (see 0.2.1 above). Recall that given any $x \in jB$ in the topos, the infinitesimal neighborhood of $x$ is defined as the intersection of all $\kappa$-open neighborhoods, $\kappa_x(jB) = \bigcap\{\, U \in \kappa[jB] \mid x \in U \,\}$. On the other hand there exist the Penon or intrinsic infinitesimal neighborhood $\Delta_B(x) = \neg\neg\{x\}$, where '$\neg\neg$' indicates the double negation in the logic of the topos. In [5, 1.13, 2.5] it is proved that $\kappa_x(jB) = \Delta_B(x)$, and that $\Delta_B(x) \cong \Delta_C(0) = \neg\neg\{0\}$. Thus the objects $\Delta_B(x)$ are isomorphic for all the points $x$ in $B$, they define the object of infinitesimals of $B$, that we simply denote by $\Delta_B$. This object can be thought in a sense as to be represented by the ringed space consisting on a single point of $B$ structured with the whole ring of complex valued holomorphic germs (for more precision see [5, 2.2, 2.3]).

From now on, we assume, without any loss in generality, that $0 \in B$ and $\Delta_B = \Delta_B(0)$.

**2.1 Definition.** *We say that a function $f \in \Delta_B^{\Delta_B}$ is holomorphic if it has an holomorphic extension. This determines the subobject $Hol[\Delta_B, \Delta_B] \subset \Delta_B^{\Delta_B}$ of holomorphic functions. It is defined by means of the following formula:*

$$f \in Hol[\Delta_B, \Delta_B] \iff$$

$$\exists H \in \kappa[jB], 0 \in H \text{ and } g \in Hol[H, jB] \text{ such that } g(x) = f(x) \forall\, x \in \Delta_B$$

Recall that (internal) existential quantification means local existence. It can be checked by Kripki-Joyal semantics that given $X \subset \mathbb{C}^n$ in the site, and $f\colon X \to \Delta_B^{\Delta_B}$ (with corresponding $f\colon \Delta_B \times X \to \Delta_B$), then $f$ factors through $Hol[\Delta_B, \Delta_B]$ if and only if for each $x$ in $X$, there is an open subset $S \subset B \times \mathbb{C}^n$, $(0, x) \in S$, and an holomorphic function $g\colon S \to B$ such that the diagram below commutes in the topos. Notice that $S$ can be taken of the form $S = H \times W$, for open sets $H \subset B$, $W \subset \mathbb{C}^n$, and that $\Delta_B \subset jH$:

$$\begin{tikzcd} \Delta_B \times (X \cap W) \arrow[r, phantom, "\subset"] \arrow[d, "f"'] & jS = jH \times W \arrow[d, "jg"'] \\ DB \arrow[r, phantom, "\subset"] & jB \end{tikzcd}$$

In particular ( $X = 1$ ), any actual arrow in the topos $\Delta_B \to \Delta_B$ which is in $Hol[\Delta_B, \Delta_B]$ has an holomorphic extension into some $jH$, for an open subset $H$ of $B$.

Recall that $j(H \times W) = jH \times W$ [4, 3.2], and remark that this condition is stronger that just the commutativity on the global sections.

If $D \subset \mathbb{C}$ is the object of infinitesimals of square $0$, recall that the tangent bundle of any object $F$ in the topos is the exponential $T(F) = F^D$, and that the derivative of any map $g: F \to G$ in the topos is defined by $Dg = g^D$. Given any $p \in F$, we have the derivative at $p$ between the tangent spaces $Dg(p): T_p(F) \to T_q(G)$, $q = g(p)$. Since $D \subset \neg\neg\{0\}$, it follows that any tangent vector $\zeta \in F^D$ factors through $\neg\neg\{p\} = \Delta_F(p)$, the infinitesimal neighborhood of its base point $p = \zeta(0)$. Thus, given any $p \in F$, there is an identification of tangent spaces $T_p(\Delta_F(p)) = T_p(F)$. Also, given any $g: F \to G$, $g(\Delta_F(p)) \subset \Delta_G(q)$, $q = g(p)$, thus the restriction of $g$ defines a map $f = g|_\Delta: \Delta_F(p) \to \Delta_G(q)$. Via the identification of tangent spaces, we have $Df(p) = Dg(p)$. In particular we have:

**2.2 Observation.** *Let $B$ be an open subset of a banach space, $f \in Hol[\Delta_B, \Delta_B]$, and $g \in Hol[H, jB]$ be any extension of $f$ into an open subobject $H \in \kappa[jB]$. Then $Df(0) = Dg(0)$ (notice that it follows that $\Delta_H = \Delta_B$ since $H$ is $\kappa$-open).*

Recall that all this synthetic differential calculus is compatible with the classical calculus on banach spaces, in a precise sense described in [4, section 4]. From this it follows

**2.3 Definition-Proposition.** *Let $B$ be an open subset of a complex Banach space $C$. Let $f \in Hol[\Delta_B, \Delta_B]$. Then $Df(0) \in Hol[jC, jC]$. We say that $Df(0)$ is invertible if it has an inverse which it is also in $Hol(jC, jC)$.*

We will need the following observation on holomorphic functions.

**2.4 Observation.** *Let $C$, $L$ and $R$ be banach spaces, and let $B \subset C$, $A \subset R$, and $S \subset B \times L$ be open subsets. Let $g: S \to A$ be an holomorphic function, and consider the map $(g, \pi_2): S \to A \times L$. Let $(p, x) \in S$ be such that the derivative $D(g(-, x))(p): C \to R$ is invertible (with continuous inverse). Then, the derivative $D(g, \pi_2)(p, x): C \times L \to R \times L$ is also invertible (with continuous inverse).*

We are now in condition to state and prove the inverse function theorems.

**2.5 Theorem (infinitesimal inverse function theorem).** *Let $B$ be an open subset of a complex Banach space $C$. Let $f \in Hol[\Delta_B, \Delta_B]$ be such that $Df(0)$ is invertible. Then $f$ is invertible, that is, $(\exists h \in Hol[\Delta_B, \Delta_B] \mid f \circ h = h \circ f = id)$.*

**Proof .** In this proof we will use the letter 's' for the points of the space $X$, and we reserve the letter 'x' for a variable of type $X$ in the topos.

Let $f \in Hol[\Delta_B, \Delta_B]$ be given by an arrow

$$f: X \to Hol[\Delta_B, \Delta_B] \subset \Delta_B^{\Delta_B}$$

with corresponding $f: \Delta_B \times X \to \Delta_B$ as in definition 2.1. Consider the arrow $(f, \pi_2): \Delta_B \times X \to \Delta_B \times X$. To prove the statement we have to show that this arrow has an inverse, which will be necessarially of the form $(h, \pi_2): \Delta_B \times X \to \Delta_B \times X$, and that $h$ has local extensions in the sense of definition 2.1.

For each $s$ in $X$, there is an open subset $S_s \subset B \times \mathbb{C}^n$, $(0, s) \in S_s$, and an holomorphic extension $g_s: S_s \to B$. The assumptions mean that, for all $s \in X$, $f(0, s) = 0$ and $D(f(-, s))(0)$ is invertible. Thus $jg_s(0, s) = 0$, and by 2.2, $D(jg_s(-, s))(0)$ is invertible. It follows then from the compatibility of the synthetic calculus with the differential calculus on banach spaces, and more precisely [4, theorem 4.6], that the usual derivative $D(g_s(-, s))(0)$ is invertible (with continuous inverse by 2.3). Consider the holomorphic function $(g_s, \pi_2): S_s \to B \times \mathbb{C}^n$. By 2.4 we have that

$$D(g_s, \pi_2)(0, s): C \times \mathbb{C}^n \to C \times \mathbb{C}^n$$

is invertible (with continuous inverse). We can apply then the classical inverse function theorem for banach spaces to the function $(g_s, \pi_2)$. It follows that there are open subsets $S_s \subset B \times \mathbb{C}^n$ (that we can call with the same letter) and $T_s \subset B \times \mathbb{C}^n$ such that $(0,s) \in S_s, (0,s) \in T_s$ and $(g_s, \pi_2): S_s \to T_s$ has an holomorphic inverse which is necessarily of the form $(t_s, \pi_2): T_s \to S_s$. It follows we have in the topos a pair of inverse arrows $(jg_s, \pi_2): jS_s \to jT_s$ and $(jt_s, \pi_2): jT_s \to jS_s$. Let $W_s \subset X$, be the open set $W_s = \pi_2(V_s)$, $V_s = S_s \cap (\{0\} \times X) = T_s \cap (\{0\} \times X)$. This equality holds since $(g_s, \pi_2)|_{V_s} = (f, \pi_2)|_{V_s} = id$. As usual, we shall indicate also by $W_s$ the corresponding open subobject in the site or in the topos. By definition we have $\{0\} \times W_s = V_s \subset S_s$. Since $S_s$ is open, it follows that $\Delta_B \times W_s \subset jS_s$. In the same way, $\Delta_B \times W_s \subset jT_s$. The $W_s$ form an open covering, thus we have $X = \cup_s W_s$ in the topos.

We are going to prove now in the internal logic of the topos that the arrow $(f, \pi_2): \Delta_B \times X \to \Delta_B \times X$ is injective and surjective, and thus invertible. Let $\varepsilon$, $\varepsilon' \in \Delta_B$ and $x$, $x' \in X$ be such that $(f, \pi_2)(\varepsilon, x) = (f, \pi_2)(\varepsilon', x')$. Clearly $x = x'$. Let $s$ be such that $x \in W_s$. Then $(\varepsilon, x) \in \Delta_B \times W_s$ and $(\varepsilon', x) \in \Delta_B \times W_s$. Thus $(f, \pi_2)(\varepsilon, x) = (jg_s, \pi_2)(\varepsilon, x)$ and $(f, \pi_2)(\varepsilon', x) = (jg_s, \pi_2)(\varepsilon', x)$. It follows that $\varepsilon = \varepsilon'$, showing the injectivity. Given $(\varepsilon, x)$ as above, take $s$ such that $x \in W_s$. Then $(\varepsilon, x) = (jg_s, \pi_2)(\varepsilon', x)$, where $(\varepsilon', x) = (jt_s, \pi_2)(\varepsilon, x)$, showing the surjectivity. If $(h, \pi_2)$ is the inverse of $(f, \pi_2)$, clearly the functions $t_s$ are local extensions for $h$ in the sense of 2.1. This finishes the proof. Notice that we have proved more, namely, that the local extensions are also inverses of each other. $\square$

**2.6 Remark (infinitesimal invertibility = local invertibility).** *Theorem 2.5 holds in particular when $f$ is an actual arrow $f: \Delta_B \to \Delta_B$, in which case $h$ is given also by an arrow $h: \Delta_B \to \Delta_B$ inverse of $g$ in the topos. Given any holomorphic function $g: B \to B$ such that $g(0) = 0$, the restriction of $jg$ to infinitesimals determines a map $jg|_{\Delta_B}: \Delta_B \to \Delta_B$. Observe then that it follows (from [5, example 3 after remark 3.7]) that the infinitesimal invertibility of $jg|_{\Delta_B}$ in the topos is actually equivalent to the local invertibility of $g$ at $0$.*

Analyzing the proof of theorem 2.5 it readily follows a stronger theorem:

**2.7 Theorem (second version of 2.5).** *With the same hypothesis than in 2.5, let $h$ be the inverse of $f$. Then, given any extension $g$ of $f$, an extension $t$ of $h$ can be choosed so that it is an inverse of $g$. Concretely, $\exists\ H \in \kappa[jB]$,*

*$\exists\ G \in \kappa[jB]|$, $0 \in H$, $0 \in G$, and $\exists\ t \in Hol[G, H]$ such that $g \circ t = id_G$, $t \circ g = id_H$, and $f = g|_\Delta$, $h = t|_\Delta$ extensions of $f$ and $h$ respectively.*

**2.8 Corollary (local inverse function theorem).** *Let $B$ be an open subset of a complex Banach space. Let $g \in Hol[jB, jB]$ be such that $g(0) = 0$ and $Dg(0)$ is invertible. Then, $g$ is locally invertible at $0$. That is, $\exists H \in \kappa[jB]$, $\exists G \in \kappa[jB]$, $0 \in H$, $0 \in G$, and $\exists\ t \in Hol[G, H]$, $g|_H \circ t = id_G$, $t \circ g|_H = id_H$.*

**Proof of 2.7 and 2.8.** It is clear that in the proof above we have also shown the statement in 2.7. Finally, it is also clear that 2.8 follows immediately from 2.7. □

## References.

[1] Bunge M., Dubuc E. J. *Local Concepts in Synthetic Differential Geometry and Germ Representability*, Lectures Notes in Pure and Applied Mathematics, Marcel Dekker, New York, (1989).

[2] Cartan H. *Idéaux de Fonctions Analytiques de $n$ variables complexes*, Annales de L' Ecole Normale, 3e serie, **61**, (1944).

[3] Dubuc E. J., Zilber J. C., *On Analytic Models of Synthetic Differential Geometry*, Cahiers de Topologie et Geometrie Differential Categoriques, Vol **XXXV-1** (1994).

[4] Dubuc E. J., Zilber J. C., *Banach Spaces in an Analytic Model of Synthetic Differential Geometry*, Cahiers de Topologie et Geometrie Differentielle Categoriques, Vol **XXXIX-2** (1998).

[5] Dubuc E. J., Zilber J.C., *Infinitesimal, local structure for Banach Spaces and its exponentials in a topos*, Cahiers de Topologie et Geometrie Differentielle Categoriques, Vol

[6] Kaup L., Kaup B., *Holomorphic Functions of Several Variables*, Walter de Gruyter, Berlin, New York (1983).

[7] Mujica J., *Holomorphic Functions and Domains of Holomorphy in Finite and Infinte Dimensions*, North Holland Mathematics Studies **120** (1986).

**Eduardo J. Dubuc, Jorge C. Zilber**

Departamento de Matemática.
Facultad de Ciencias Exactas y Naturales
Ciudad Universitaria
1428 Buenos Aires, Argentina