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FIBRATIONS AND CLASSIFYING SPACES : OVERVIEW AND THE CLASSICAL EXAMPLES

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RESUME. Soit G un groupe topologique. La construction et les propriétés de l'espace fibré principal de Milnor rattaché à G sont reconnues comme étant le modèle principal dans le développement des théories des fibrations et de leurs classifications.

Dans cet article, l'auteur développe une telle théorie pour des fibrations structurées générales. Des cas particuliers incluent les résultats analogues pour les fibrations principales, celles de Hurewicz et les fibrations avec sections.

Certains articles antérieurs sur ce sujet ont obtenu des résultats aussi satisfaisants que la construction de Milnor en termes de simplicité et de généralité. D'autres ont fait de même en termes de généralité et de potentiel pour les applications; d'autres encore l'ont fait en termes de simplicité et de potentiel pour les applications. Les résultats présentés ici pour les fibrations structurées et pour le niveau de fibration considéré sont les premiers à réussir dans ces trois sens à la fois.

1 Introduction

Let G be a topological group. Milnor constructed a universal principal G-bundle [Mi] and Dold showed that its base space acts as a classifying space for principal G-bundles [D1, thms.7.5 and 8.1]. These results have provided a model for the development of analogous theories of fibrations and their associated classifying spaces.

This theory is *simple* in the senses that it utilizes a standard definition of principal G-bundle and classifies such bundles up to the most natural equivalence relation between such bundles. The theory is *general*, in that it is valid for arbitrary topological groups G. Classifying spaces are produced by a relatively simple procedure, i.e. as quotients of joins of copies of G. The details of this construction are directly useful in applications of this work, i.e. they *enhance the applicability* of the whole theory.

The analogous problems of classifying the three types of *classical fibrations,* i.e. *principal, Hurewicz* and *sectioned fibrations*, have been considered by numerous authors. Principal and sectioned fibrations have been defined in different ways in different publications. Each type of classical fibration has been classified by means of more than one equivalence relation. In some cases the distinguished fibre is arbitrary, in others it is required to satisfy a topological condition. Some of the classifying spaces are produced by *geometric bar construction* or *GBC* procedures, others by methods that may be less helpful when applications are involved. Also, when the *Brown Representability Theorem* or *BRT* is used, the proof often incorporates a set theoretical difficulty. *We will show that none of these results have all three of the above desirable features of the Milnor-Dold theory.*

We now consider theories that describe *structured fibrations* in general. It is obviously important for them to incorporate satisfactory versions of the classical theories as particular cases. To clarify this, we need a few ideas concerning categories of *enriched spaces.* More precise explanations of the concepts involved will be given in section 2.

Let $\mathcal{E}$ denote a category of enriched topological spaces and F be a given $\mathcal{E}$-space. We define the associated *category of fibres* $\mathcal{F}$ as consisting of all $\mathcal{E}$-spaces that are of the same $\mathcal{E}$-homotopy type as F and all $\mathcal{E}$-homotopy equivalences between such $\mathcal{E}$-spaces. Then *$\mathcal{E}$-fibrations* are fibrations whose fibres are prescribed to lie in $\mathcal{E}$. Different choices of $\mathcal{E}$ lead into discussions of different types of fibrations. Thus taking $\mathcal{E}$ to be a suitable category of G-spaces and G-maps, the category of spaces and maps and the category of pointed spaces and pointed maps, we obtain theories of principal G-fibrations, Hurewicz fibrations and sectioned fibrations, respectively.

Thus, for a result at the structured fibration level, there is an important *unity* condition. In general terms, the classifying result for structured fibrations should do a very good job of unifying the classifying results for the classical fibrations.

We do not claim that an optimal theory of fibrations would match that of Milnor-Dold in all respects. Thus [D1, thm.8.1] showed that the total space of Milnor's universal principal G-bundle is contractible. However, a result of this type for principal G-fibrations - such as [F, thm.p.334] - has both advantages and disadvantages. Let G be a topological monoid and $p_G : X_G \to B_G$ be a principal G-fibration, with X_G contractible. It follows, via [Va, thm.1.3] and [BHMP, lem.3.2], that the coboundary map $\Omega B_G \to G$ is a homotopy equivalence and an H-map. So G has a homotopy inverse. We prefer to avoid this approach since, as we explain in the discussion before proposition 5.10, the condition on G may fail to hold in some relevant examples.

A space X will be said to be *weakly contractible* if it is path connected and all of its homotopy groups are zero. To avoid the aforementioned limitation on G, we prefer to use weak contractibility in place of contractibility. We expect that a theory of principal fibrations, or a theory of structured fibrations that includes principal fibrations, should satisy a *weak contractibility* condition, i.e. that the total spaces of universal principal fibrations should be at least weakly contractible.

There are three classification results for structured fibrations in general, i.e. a result from [B3] (= theorem 2.3 of this paper) and two GBC style results of [M1]. Thus theorem 9.2(i) of that paper - parts (a)(i) and (b)(i) to be precise - classifies such fibrations up to a form of fibrewise weak homotopy equivalence. Theorem 9.2(ii) - actually (a)(ii) and (b)(ii) - classifies such fibrations up to a form of fibrewise homotopy equivalence. *We will show that none of these accounts have all of the previously listed desirable features.*

One of the main aims of this series of papers, i.e. [B1], [B2], [B3], [B4] and this paper, can be summarized as follows.

Objective A *We wish to develop a theory of structured fibrations and their classifying spaces that will meet the seven criteria that we list in section 3. These include the aforementioned simplicity, generality, enhanced applicability, unity and weak contractibility conditions.*

Results Obtained *This objective is achieved by further developing the theory of our theorem 2.3. The classifying spaces and universal structured fibrations of theorem 4.7 will be shown to satisfy all of our criteria.*

A general theory of fibrations and classifying spaces should do much more that upgrade existing theories; it should facilitate the development of new theories. This leads to a second major goal of this work.

Objective B *Our theory should provide the machinery for manufacturing classifying spaces for a great variety of types of structured fibrations. Further, the conditions that have to be verified, i.e. to show that it is applicable to specific types of structured fibration, should be uncomplicated and easily managed.*

Results Obtained *This objective is met by our theorems 4.7 and 6.6.*

We have already explained that suitable choices of $\mathcal{E}$ allow us to derive theories of classifying spaces for the classical theories of fibrations. There are many other examples of categories $\mathcal{E}$ for which the associated $\mathcal{F}$-fibrations have good theories of classifying spaces. These examples include the cases where $\mathcal{E}$ is (i) the category of pairs of spaces and maps of pairs [Sp, p.22-23], (ii) the category of spaces under a fixed space and maps under that space, (iii) the fibrewise category, i.e. the category of spaces over a fixed space and maps over that space, and (iv) the category whose objects are Dold fibrations that are also identifications and whose morphisms are pairwise maps between such fibrations.

These lead us into classification theories for (i) pairs of fibrations, (ii) fibrations that extend trivial fibrations, (iii) fibrations over product spaces and (iv) composites of Dold fibrations, respectively. These extensive and interesting theories are beyond the scope of this paper. However, some preliminary ideas on these topics are given in section 8 of [B1].

The techniques of this present group of papers make heavy use of concepts from [M1], even though our methods of proof are quite different from those used there. Thus our terms categories of fibres $\mathcal{F}$, $\mathcal{F}$-fibrations and $\mathcal{F}$-fibre homotopy equivalence derive from - although they do not coincide with - the corresponding concepts of that memoir. The use of the GBC construction of classifying spaces, however, transfers directly to this paper. The following account of the use of that procedure to define universal $\mathcal{F}$-quasifibrations is a condensed version of material in section 3 of [B4]. This in turn derives from section 7 and remark 9.3 of [M1].

Let G be a topological monoid with a strongly non-degenerate base point. If X is a right G-space and Y is a left G-space, then there is an associated *GBC-space* $B(X,G,Y)$. This construction is functorial so, given a left G-space F and the map $F \to *$, there is an associated map $p_{*,F}: B(*,G,F) \to B(*,G,*)$. If G is *grouplike*, i.e. such that $\pi_0(G)$ with the induced operation is a group, then $p_{*,F}$ is a quasifibration.

If X is a pointed space, then X' will denote X with a unit interval "whisker" grown at its distinguished point. The base point of X', i.e. the point at the end of that whisker, is strongly non-degenerate. Let G be a grouplike topological monoid, with the identity as distinguished point. Then G' is a grouplike topological monoid, with the operation that extends the operation on G and multiplication on the unit interval I, as well as satisfying $tg = gt = g$, where $t \in I$ and $g \in G$.

There is a canonical action of $\mathcal{F}(F)'$ on F, with $\mathcal{F}(F)$ acting by evaluation and the whisker acting trivially. This action determines a GBC-space $B(*,\mathcal{F}(F)',F)$. The map $F \to *$ then induces a quasifibration $p_{*,F}: B(*,\mathcal{F}(F)',F) \to B(*,\mathcal{F}(F)',*)$, with $\mathcal{F}$-space fibres. Further, this $\mathcal{F}$-*quasifibration* is *universal*, in the sense that May's universal $\mathcal{F}$-fibrations correspond to and derive from such quasifibrations.

A key difference between the approaches of May and the present author can now be clarified. Thus [M1, thm.9.2] uses Γ-*completeness* and

Γ'-*completeness* assumptions to transform universal $\mathcal{F}$-quasifibrations $p_{*,F}$ into universal $\mathcal{F}$-fibrations $\Gamma(p_{*,F})$ or $\Gamma'(p_{*,F})$, over the corresponding classifying spaces $B(*, \mathcal{F}(F)', *)$. We construct universal fibrations by the BRT approach [B2, thm.8.1]. We prove that these $\mathcal{F}$-fibrations are equivalent to universal $\mathcal{F}$-quasifibrations by means of a fibred mapping space argument (see [B4, thm.4.2]). Hence we are able to equate our BRT classifying spaces to the corresponding spaces $B(*, \mathcal{F}(F)', *)$. *This procedure is a key step in our argument. It enables us to blend the two theories together. Thus we are able to manufacture a theory with the advantages of both approaches and the disadvantages of neither.*

We discuss terminology and notation in section 2. The strengths and weaknesses of the existing general theories of structured fibrations are reviewed in section 3. This leads to our list of seven criteria - including the five discussed previously - that describe what we regard as the desirable characteristics of a classification theorem for structured fibrations. In section 4 we establish such a result for classes of $\mathcal{F}$-fibrations (theorem 4.7); this result is later shown to meet all of our criteria. This theorem is applied to principal, Hurewicz and sectioned fibrations in sections 5, 7 and 8, respectively. A review of what was known previously is included in each of these three last mentioned sections. We note that our results are, in each case, improvements on those results. Section 6 looks at relations between structured fibrations and their associated principal fibrations.

The author thanks the referee for making numerous helpful comments on papers in this series.

2 Terminology and Notation

Our terminology and notation grows out of, and in most cases coincides with, that developed in [B1], [B2], [B3], and [B4].

We will work in the context of the category $\mathcal{T}$ of *weak Hausdorff compactly generated spaces* and maps between such spaces [B1, p128-129]. The symbol $\mathcal{W}$ will be used to denote the class of such spaces that have the homotopy types of CW-complexes.

Let A be a given space. Then a *space* (X, i) *under* A consists of a space X and a homeomorphism i from A onto a closed subspace of X. *Maps under* A are defined in the usual and obvious fashion. We will use $\mathcal{A}$ to denote the *category of spaces under* A *and maps under* A.

We recall the concept of a *category of enriched spaces* $(\mathcal{E}, U)$, where $\mathcal{E}$ is a category and $U : \mathcal{E} \to \mathcal{T}$ is a faithful functor. The *underlying space functor* U is such that if Y is an $\mathcal{E}$-object, S is a space and $f : S \to UY$ is a homeomorphism onto the space UY, then there exists a unique $\mathcal{E}$-object X and a unique $\mathcal{E}$-isomorphism $g : X \to Y$ such that $UX = S$ and $Ug = f$ (see [B2, p.86]). The objects and morphisms of such $\mathcal{E}$ will henceforth be called *$\mathcal{E}$-spaces* and *$\mathcal{E}$-maps*, respectively. In practise we will frequently simplify our terminology by omitting reference to U. Thus we may state that $\mathcal{E}$ is a category of enriched spaces when we really mean that $(\mathcal{E}, U)$ is a category of enriched spaces.

If X and Y are $\mathcal{E}$-spaces and there is an $\mathcal{E}$-map $i : X \to Y$ such that $Ui : UX \to UY$ is a homeomorphism into, then the $\mathcal{E}$-space X will be said an *$\mathcal{E}$-subspace* of the $\mathcal{E}$-space Y.

A *category of well-enriched spaces under* A [B2, def.2.3] is a triple $(\mathcal{E}, U_{\mathcal{A}}, \{X \times_{\mathcal{E}} I\})$. In this situation, $\mathcal{E}$ is a category and $U_{\mathcal{A}} : \mathcal{E} \to \mathcal{A}$ is an *underlying space under* A *functor*. Further, for each $\mathcal{E}$-space X, there is an associated $\mathcal{E}$-space $X \times_{\mathcal{E}} I$, i.e. the *$\mathcal{E}$-cylinder* of X. These data are required to satisfy conditions that $\mathcal{E}$ and $U_{\mathcal{A}}$ are well behaved with regard to *$\mathcal{E}$-subspaces*, $\mathcal{E}$-cylinders, and *$\mathcal{E}$-mapping cylinders*.

It is known that every category of well-enriched spaces also carries the structure of a category of enriched spaces [B2, lem.2.4].

We now review a number of definitions that apply in the context of a category of enriched spaces, and therefore also in a category of well enriched spaces.

An *$\mathcal{E}$-overspace* is a map $q : Y \to C$ together with, for each $c \in C$, an associated structure of an $\mathcal{E}$-space on the *fibre* $Y|c = q^{-1}(c)$. If V is a subspace of C, then $Y|V = q^{-1}(V)$. The restriction $q|V : Y|V \to V$ is clearly also an $\mathcal{E}$-overspace.

Given an $\mathcal{E}$-overspace $q : Y \to C$ and a map $f : B \to C$, we can form the associated pullback space $Y \sqcap B$. Then q_f will denote the projection and *induced* $\mathcal{E}$*-overspace* $Y \sqcap B \to B$. The projection $Y \sqcap B \to Y$ will be denoted by f_q.

Let $p: X \to B$ and $q: Y \to C$ be $\mathcal{E}$-overspaces. An $\mathcal{E}$*-pairwise map* $<f, g>$ from p to q consists of maps $f: X \to Y$ and $g: B \to C$. These must satisfy the conditions that $qf = gp$ and that, for each $b \in B$, $f|(X|b): X|b \to Y|g(b)$ is an $\mathcal{E}$-map.

An $\mathcal{E}$-pairwise map with domain the $\mathcal{E}$-overspace $p \times 1_I$ will be called an $\mathcal{E}$*-pairwise homotopy.*

Taking $B = C$ and fixing g to be 1_C, the $\mathcal{E}$-pairwise map concept reduces to $\mathcal{E}$*-map over* C or $\mathcal{E}$*-fibrewise map.* Taking B to be $C \times I$ and fixing g to be the projection $C \times I \to C$, there is an obvious associated concept of $\mathcal{E}$*-homotopy over* C, and hence of $\mathcal{E}$*-fibre homotopy equivalence* or $\mathcal{E}FHE$ *over* C.

We recall the concepts of $\mathcal{E}$*-covering homotopy property* or $\mathcal{E}CHP$ and $\mathcal{E}$*-weak covering homotopy property* or $\mathcal{E}WCHP$. These ideas, the $\mathcal{E}$-versions of the CHP and WCHP of [D1], are described in sections 3 and 4 of [B1], respectively. An $\mathcal{E}$*-fibration* is an $\mathcal{E}$-overspace that satisfies the $\mathcal{E}$WCHP. We recall that if $q: Y \to C$ is an $\mathcal{E}$-overspace and $C \in \mathcal{W}$, then q is an $\mathcal{E}$-fibration if and only if q is $\mathcal{E}$*-locally homotopy trivial* or $\mathcal{E}LHT$ (see [B1, p.142 and thm.6.3], and also [B1, p.141] for *numerably contractible*).

Given an object F in $\mathcal{E}$, then there is an associated *category of fibres* $\mathcal{F}$ in $\mathcal{E}$, consisting of all $\mathcal{E}$-spaces that are $\mathcal{E}$-homotopy equivalent to F and all $\mathcal{E}$-homotopy equivalences between such spaces. The *monoid of self-*$\mathcal{F}$*-homotopy equivalences of* F, under composition, will be denoted by $\mathcal{F}(F)$. It carries the cg-ified compact-open topology, i.e. the compact-open topology modified in standard fashion to make it compactly generated [B1, p.129].

Let $q: Y \to C$ is an $\mathcal{F}$-overspace. We define $Prin_F(Y)$ to be the space with underlying set $\bigcup_{c \in C} \mathcal{F}(F, Y|c)$ and the cg-ified compact-open topology. There is a principal overspace $prin_F(q): Prin_F(Y) \to C$, defined by $prin_F(q)(f) = c$, where $f \in \mathcal{F}(F, Y|c)$ [B4, p.276].

If we take $\mathcal{E} = \mathcal{T}$ and U to be the identity functor on $\mathcal{T}$, then we have a category of enriched spaces $(\mathcal{T}, U)$ and $\mathcal{T}$-fibrations are Dold fibrations, i.e. fibrations satisfying the WCHP of [D1]. However, we are also interested in classifying Hurewicz fibrations, a proper subclass of the class of Dold fibrations. In a similar way our preferred definition of principal fibration involves a global action on the total space and our preferred definition of well sectioned fibration involves the section being a cofibration. So, in each of the classical cases, we have to deal with a subclass of the class of all $\mathcal{E}$-fibrations, for a suitable choice of $\mathcal{E}$. *Hence we will develop a theory that classifies such subclasses of $\mathcal{E}$-fibrations.*

Let $\mathcal{E}$ be a category of enriched spaces and $\mathbb{S}\mathcal{E}$*fibn* be a particular class of $\mathcal{E}$-fibrations. The members of $\mathbb{S}\mathcal{E}$*fibn* will be called $\mathbb{S}\mathcal{E}$*-fibrations.* The class of all $\mathcal{E}$-fibrations will be denoted by $\mathcal{E}$*fibn.*

Definition 2.1 *$\mathbb{S}\mathcal{E}$fibn will be said to be a closed class of $\mathcal{E}$-fibrations if, whenever $q: Y \to C$ is a $\mathbb{S}\mathcal{E}$-fibration and $f: B \to C$ is a map, the induced $\mathcal{E}$-fibration $q_f: Y \sqcap B \to B$ is also an $\mathbb{S}\mathcal{E}$-fibration.*

Let F be a given $\mathcal{E}$-space, $\mathcal{F}$ be the category of fibres determined by F and $\mathbb{S}\mathcal{E}$*fibn* be a closed class of $\mathcal{E}$-fibrations. We define $\mathbb{S}\mathcal{F}$*fibn* to be the intersection of $\mathbb{S}\mathcal{E}$*fibn* with the class of all $\mathcal{F}$-fibrations. Hence an $\mathbb{S}\mathcal{F}$*-fibration* is an $\mathbb{S}\mathcal{E}$-fibration that is also an $\mathcal{F}$-fibration. Clearly $\mathbb{S}\mathcal{F}$*fibn* inherits the closed class property from $\mathbb{S}\mathcal{E}$*fibn.*

Given a space B, then $\mathbb{S}\mathcal{F}FHE(B)$ will denote the set - assuming that it is a set - of all $\mathcal{F}$FHE-classes of $\mathbb{S}\mathcal{F}$-fibrations over B. If, for every $B \in \mathcal{W}$, $\mathbb{S}\mathcal{F}FHE(B)$ is a set, then $\mathbb{S}\mathcal{F}$*fibn* will be said to be $\mathbb{S}\mathcal{F}FHE$ *set-valued.* In particular, $\mathcal{F}FHE(B)$ will denote the set - again if it is a set - of $\mathcal{F}$FHE-classes of $\mathcal{F}$-fibrations over B. If, for every $B \in \mathcal{W}$, $\mathcal{F}FHE(B)$ is a set, then $\mathcal{F}$*fibn* will be said to be $\mathcal{F}FHE$ *set-valued.*

If $B \in \mathcal{W}$ and $\mathcal{F}$*fibn* is $\mathcal{F}$FHE set-valued, then $\mathcal{F}FHE(B)$ is a set and its subclass $\mathbb{S}\mathcal{F}FHE(B)$ is a set. Hence, if $\mathcal{F}$*fibn* is $\mathcal{F}$FHE set-valued, then $\mathbb{S}\mathcal{F}$*fibn* will be $\mathbb{S}\mathcal{F}FHE$ *set-valued.* In that case, $\mathbb{S}\mathcal{F}FHE(-)$ is a contravariant functor from the homotopy category of spaces in $\mathcal{W}$ to the category of sets and functions (see the paragraph before [B2, defs.2.2]).

We are now ready to commence a discussion of universality amongst $\mathbb{S}\mathcal{F}$-fibrations.

Definition 2.2 *Let $\mathcal{E}$ be a category of enriched spaces, F be a given $\mathcal{E}$-space, $\mathcal{F}$ be the category of fibres in $\mathcal{E}$ that is determined by F, $\mathbb{S}\mathcal{E}$fibn* *be a closed class of $\mathcal{E}$-fibrations and $r: Z \to D$ be a $\mathbb{S}\mathcal{F}$-fibration. We will assume that $\mathbb{S}\mathcal{F}fibn$ is $\mathbb{S}\mathcal{F}FHE$ set-valued.*

Then r will be said to be free universal amongst $\mathbb{S}\mathcal{F}$-fibrations, or to be a free universal $\mathbb{S}\mathcal{F}$-fibration, if the natural function

$$[B, D] \to \mathbb{S}\mathcal{F}FHE(B), \quad [f] \to [r_f],$$

is a bijection for all $B \in \mathcal{W}$. Further, D will then be said to be a classifying space for $\mathbb{S}\mathcal{F}$-fibrations and f a classifying map for the $\mathbb{S}\mathcal{F}$-fibrations in the $\mathcal{F}FHE$ class $[r_f]$.

Let $\mathcal{E}$ be a category of well enriched spaces under the space A. If, for every choice of a category of fibres $\mathcal{F}$ in $\mathcal{E}$ and of a space X under A, the class of all associated $\mathcal{F}$-space structures on X is a set, then $\mathcal{E}$ will be said to be *proper.*

The following classification result for $\mathcal{F}$-fibrations is theorem 5.3 of [B3] and theorem 3.2 of [B4]. It is the main classification result derived from the BRT approach of [B2] and [B3] and is the foundation of our main line of argument in section 4.

Theorem 2.3 *Let $\mathcal{E}$ carry the structure of a proper category of well-enriched spaces under a space A, F be an $\mathcal{E}$-space and $\mathcal{F}$ be the associated category of fibres. Then there exists an $\mathcal{F}$-fibrataion, $p_{\mathcal{F}}: X_{\mathcal{F}} \to B_{\mathcal{F}}$, that is free universal amongst all $\mathcal{F}$-fibrations, where $B_{\mathcal{F}}$ is a path connected CW-complex.*

We conclude this section by reviewing some terminology from [M1].

An *[M1]-sense category of fibres* $\mathcal{F}$ has a faithful underlying space functor $\mathcal{F} \to \mathcal{T}$ and a distinguished object F. It satisfies the condition that, for each object X in $\mathcal{F}$, the associated products with one-point spaces, i.e. spaces $X \times *$ and $* \times X$, are in $\mathcal{F}$. Also the evident homeomorphisms between these spaces and X are in $\mathcal{F}$. Further, every map in $\mathcal{F}$ is a weak homotopy equivalence, $\mathcal{F}(F, X)$ is non-empty for each space X in $\mathcal{F}$, and composition with ϕ

$$\mathcal{F}(1, \phi) : \mathcal{F}(F, F) \to \mathcal{F}(F, X)$$

is a weak homotopy equivalence, for each $\phi \in \mathcal{F}(F, X)$. In this case the objects of $\mathcal{F}$ will be called *[M1]-sense $\mathcal{F}$-spaces.*

A given map $q : Y \to C$ will be said to be an *[M1]-sense $\mathcal{F}$-overspace* (called an $\mathcal{F}$-space in [M1]) if, for each $c \in C$, there is an associated [M1]-sense $\mathcal{F}$-space structure on the fibre $Y|c$.

Let $p: X \to C$ and $q: Y \to C$ be $\mathcal{F}$-overspaces in the [M1]-sense. A map $X \to Y$ over C, whose restrictions to individual fibres are in $\mathcal{F}$, will be called an an *[M1]-sense $\mathcal{F}$-map over C.*

An *[M1]-sense $\mathcal{F}$-fibration* is an [M1]-sense $\mathcal{F}$-overspace that satisfies the corresponding version of the $\mathcal{F}$CHP.

If a pair of [M1]-sense $\mathcal{F}$-fibrations over C are equivalent under the equivalence relation generated by [M1]-sense $\mathcal{F}$-maps over C, then they will be said to be *$\mathcal{F}$-fibre weak homotopy equivalent* or *$\mathcal{F}FWHE$.*

A category of fibres $\mathcal{F}$, in our sense, is clearly an [M1]-sense category of fibres. For such $\mathcal{F}$, many [M1]-sense $\mathcal{F}$-concepts coincide with the corresponding concepts in our sense. The [M1]-sense $\mathcal{F}$-fibration idea then agrees with our concept of an $\mathcal{F}$-overspace satisfying the $\mathcal{F}$CHP.

3 Assessing theories of Structured Fibrations and their Classifying Spaces

In this section we will compare three "free universality style" classification results for $\mathcal{F}$-fibrations, i.e. our theorem 2.3 and parts (i) and (ii) of [M1, thm.9.2]. In later sections we will compare numerous published classification results for the classical fibrations. Our method will involve listing seven desirable criteria that can be used to assess and evaluate the basic characteristics of such theories. We will determine the extent to which the results meet - or fail to meet - the criteria. *This will help us decide what is required of a "smoother" account, i.e. one that combines advantageous aspects of each approach.*

In the case of the two general results of [M1], terms such as category of fibres $\mathcal{F}$ and $\mathcal{F}$-fibration should be understood in the sense of [M1].

Let $\mathcal{E}$ be a category of enriched spaces, $\mathcal{F}$ be the category of fibres determined by a given $\mathcal{E}$-space F and $\mathbb{S}\mathcal{E}$fibn be a class of $\mathcal{E}$-fibrations. Our criteria will refer to a classification theorem for $\mathbb{S}\mathcal{F}$-fibrations.

The three aforementioned results classify $\mathcal{F}$-fibrations. So, in reviewing these results, we will be asking whether or not they satisfy the critieria in the case where $\mathbb{S}\mathcal{E}fibn$ means $\mathcal{E}fibn$, $\mathbb{S}\mathcal{F}$-fibration means $\mathcal{F}$-fibration and $\mathbb{S}\mathcal{F}$FHE(B) means $\mathcal{F}$FHE(B).

Our first criterion is just a statement of the problem considered.

(1) Simplicity. *Our preferred result will classify $\mathbb{S}\mathcal{F}$-fibrations, or at least those that are over spaces in $\mathcal{W}$, up to $\mathcal{F}FHE$.*

This condition is satisfied by the approach of our theorem 2.3.

Theorem 9.2(i) of [M1] classifies May's $\mathcal{F}$-fibrations up to $\mathcal{F}$FWHE, which does not in general conform to condition (1).

Theorem 9.2(ii) of [M1] requires that categories of fibres be either Γ-complete in $\mathcal{W}$ [M1, def.5.1] or Γ'-complete in $\mathcal{W}$ [M1, def.5.4]. So the spaces of $\mathcal{F}$ must be in $\mathcal{W}$. Hence the morphisms of $\mathcal{F}$ are both $\mathcal{F}$-maps and homotopy equivalences (see [Sp, cor.7.6.24]), but may not be $\mathcal{F}$-homotopy equivalences. So the equivalence relation that classifies $\mathcal{F}$-fibrations may not be $\mathcal{F}$FHE and (1) has not been justified.

Our next two conditions are valid for, and strong points of, bar construction approaches to the classification of fibrations. In particular, this applies to the two approaches of [M1].

(2) Firm Foundations. *Any theory of $\mathbb{S}\mathcal{F}$-fibrations, and their classifying spaces, should include a verification that $\mathbb{S}\mathcal{F}FHE(B)$ is a set, for all choices of $B \in \mathcal{W}$.*

This is not a problem with the bar construction approach since, in that case, every $\mathbb{S}\mathcal{F}$-fibration over B is $\mathcal{F}$FHE to one of a constructible set of $\mathcal{F}$-fibrations. In fact this set theoretical difficulty arises only when the BRT approach is used. However, on the approach of our theorem 2.3, (2) follows from [B3, prop.3.7] and the existence (see the proof of [B3, thm.5.3]) of a weakly contractible universal $\mathcal{F}$-fibration.

(3) Enhanced Applicability. *Classifying spaces for $\mathbb{S}\mathcal{F}$-fibrations should be constructed by a geometric bar construction procedure.*

This is advantageous, because known properties of such constructions can sometimes be used in applications. We are not claiming that (3) is an absolute principle. If an alternative construction of classifying spaces were to be found, that was advantageous in tems of applications, then that construction would be preferable.

It is shown in [B4, thm.4.2] that (3) is within range of being valid for our theory, i.e. that our classifying spaces have the weak homotopy types of the corresponding bar construction spaces. In fact this deficiency of our theorem 2.3 will be eliminated in theorems 4.5 and 4.7.

(4) Generality of Fibres. *A classification theorem for $\mathbb{S}\mathcal{F}$-fibrations should not require that any unnecessary topological condition be imposed on any fibre or fibres.*

This is valid for our theorem 2.3, where no conditions are imposed on the spaces of $\mathcal{F}$.

The criterion holds when [M1, thm.9.2(i)] is applied to the three classical theories (see "reviews" (iv), (vi) and (ii) of chapters 5, 7 and 8, respectively). However, it is not clear that it holds for [M1, thm.9.2(i)] in general.

In the case of [M1, thm.9.2(ii)], it is required that $\mathcal{F}$ should be either Γ-complete in $\mathcal{W}$ or Γ'-complete in $\mathcal{W}$. This requires that, for all X in $\mathcal{F}$, the spaces X and $\mathcal{F}(F, X)$ should be in $\mathcal{W}$, so the criterion does not hold. The space $\mathcal{F}(F, X)$, of morphisms from F to X, carries the cg-ified compact-open topology.

(5) The Equivalence of Theories of Structured and Principal Fibrations. *Let $\mathcal{E}$ be a category of enriched spaces, $\mathcal{F}$ be the category of fibres determined by a particular $\mathcal{E}$-space F, $\mathbb{S}\mathcal{E}fibn$ be a given class of $\mathcal{E}$-fibrations and r be an $\mathbb{S}\mathcal{F}$-fibration.*

Then r should be free universal amongst $\mathbb{S}\mathcal{F}$-fibrations if and only if $prin_F(r)$ is free universal amongst principal $\mathcal{F}(F)$-fibrations.

This will be verified on our approach in theorem 6.4(i). It tells us that the theories of $\mathcal{F}$-fibrations and principal $\mathcal{F}(F)$-fibrations are, in a sense, equivalent. Thus (ii) and (iii) of theorem 6.4, which explain this equivalence, depend on (5).

Similar arguments apply in the two cases of [M1, thm.9.2]. However the limitations on F, referred to under (4), prevent us from achieving full generality.

(6) Unity of Theories. *For each specific type of classical fibration, a suitable choice of $\mathcal{E}$ and $\mathbb{S}\mathcal{E}$fibn should ensure that the $\mathbb{S}\mathcal{E}$-fibration concept coincides with a standard or optimal definition of that type of classical fibration. Our general classification theorem should then imply that the conditions (1), (2), ... (5) are valid in the context of each such theory of classical fibrations.*

We explained in our Introduction that [M1, thm.9.2(i)] is actually two disjoint results, i.e. parts (a)(i) and (b)(i) of that theorem. [M1, thm.9.2(ii)] is also two disjoint results, i.e. parts (a)(ii) and (b)(ii) of that theorem. In each case, (a) refers to the situation where the corresponding space A is empty and applies to both principal and Hurewicz fibrations, and (b) to the situation where A is a one point space and applies to sectioned fibrations. So these two general results, in the form given, do not meet (6) since neither is a single theorem that covers all of the classical theories.

We delay further discussion of (6) until sections 5, 7 and 8.

Our final criterion refers only to principal G-fibrations and is the fibration analogue of [D1, thm.7.5]. It is not verified in any of the three accounts of structured fibrations. However it is shown, in each case, that there are weakly contractible universal principal G-fibrations that are also free universal. The criterion will be verified, for our approach, in corollary 6.2. Proposition 5.10(i), in conjunction with (5), implies that an assumption that G is *grouplike* is required in our theory.

(7) Weak Contractibility. *Let G be a topological monoid. A principal G-fibration should be free universal if and only if its total space is weakly contractible.*

Our first objective can now be made more precise.

Clarified Statement of Objective A. *We wish to develop a theory of $\mathbb{S}\mathcal{E}$-fibrations and their classifying spaces that satisfies our criteria (1), (2), ... (7).*

Review of Progress made towards achieving Objective A. It should be understood that, in *some* of the cases where a given published theory is not shown to satisfy a given criterion, the criterion can be verified without major changes to the work in question. However, the critical difficulty with the BRT approach lies in ensuring that (2) and (3) are satisfied. The critical difficulty with the GBC approach lies in ensuring that (1), (4) and (6) are satisfied. It was explained, in the previous section, that earlier papers of this series have simplified the situation. Thus (2) has been resolved and (3) partly resolved for our version of the BRT approach. In the case of (3), this was done by equating the two types of classifying spaces.

The remaining problem consists of modifying theorem 2.3 to obtain a classification result for $\mathcal{F}$-fibrations that satisfies (3), extending that result to classify subclasses $\mathbb{S}\mathcal{F}fibn$ of $\mathcal{F}$-fibrations, noticing that (1) to (4) remain valid in the $\mathbb{S}\mathcal{F}fibn$ context, and verifying (5), (6) and (7) in that context.

4 On the Classification of Structured Fibrations

We now introduce three more universal $\mathbb{S}\mathcal{F}$-fibration concepts, the second of them being closely related to our criterion (3).

Definitions 4.1 *Let $\mathcal{E}$ be a category of enriched spaces, F be a given $\mathcal{E}$-space, $\mathcal{F}$ be the category of fibres in $\mathcal{E}$ that is determined by F, $\mathbb{S}\mathcal{E}fibn$ be a closed class of $\mathcal{E}$-fibrations and $r : Z \to D$ be an $\mathbb{S}\mathcal{F}$-fibration.*

(i) The $\mathbb{S}\mathcal{F}$-fibration r will be said to be weakly contractible universal if $Prin_F(Z)$ is a weakly contractible space.

(ii) We recall that there is a "univeral" $\mathcal{F}$-overspace and quasifibration $p_{*,F} : B(*, \mathcal{F}(F)', F) \to B(*, \mathcal{F}(F)', *)$ (see the introduction). Then r will be said to be bar construction universal if there are weak homotopy

equivalences, $h: Z \to B(*, \mathcal{F}(F)', F)$ and $k: D \to B(*, \mathcal{F}(F)', *)$, such that $< h, k >$ is an $\mathcal{F}$-pairwise map from r to $p_{*,F}$.

(iii) An $\mathbb{S}\mathcal{F}$-fibration will be said to be a triple universal $\mathbb{S}\mathcal{F}$-fibration, or to be triple universal amongst $\mathbb{S}\mathcal{F}$-fibrations, if it is universal in the free, weakly contractible and bar construction senses.

We now determine some relations between our various types of universality, a discussion that is completed in theorem 6.1.

Proposition 4.2 *Let $\mathcal{E}$ be a category of enriched spaces, F be a given $\mathcal{E}$-space, $\mathcal{F}$ be the category of fibres in $\mathcal{E}$ that is determined by F, $\mathbb{S}\mathcal{E}fibn$ be a closed class of $\mathcal{E}$-fibrations and $r: Z \to D$ be a $\mathbb{S}\mathcal{F}$-fibration. Then the conditions:*

(i) The class $\mathbb{S}\mathcal{F}fibn$ is $\mathbb{S}\mathcal{F}FHE$ set-valued and r is a free universal $\mathbb{S}\mathcal{F}$-fibration,

(ii) r is weakly contractible universal,

(iii) r is bar construction universal, and

(iv) r is a triple universal $\mathbb{S}\mathcal{F}$-fibration

are related according to the scheme:

$$(i) \Leftarrow (ii) \Leftarrow (iii) \Leftrightarrow (iv).$$

If $D \in \mathcal{W}$, then this relation becomes:

$$(i) \Leftarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv).$$

Proof. $(ii) \Rightarrow (i)$. This is similar to the proof of [B3, thm.3.4], except that we now forget about base points and fibrations being grounded. The closed condition ensures that the argument is valid in the context of $\mathbb{S}\mathcal{F}$-fibrations.

$(iii) \Rightarrow (ii)$. We recall that $prin_F(p_{*,F})$ is a quasifibration [M1, prop.7.10]. The $\mathcal{F}$-pairwise map of our data induces a pairwise map from $prin_F(r)$ and $prin_F(p_{*,F})$. This in turn induces a sequence of homomorphisms from the groups in the exact homotopy sequence of $prin_F(r)$ to the groups in the exact homotopy sequence of $prin_F(p_{*,F})$. The homomorphisms between the homotopy groups of the base spaces and the homomorphisms between the homotopy groups of the fibres are all isomorphisms. In the former case this follows from the data; in the latter case it is a consequence of the morphisms of $\mathcal{F}$ all being $\mathcal{F}$-homotopy equivalences. It then follows, using the 5-lemma and the path-connectedness of $B(*, \mathcal{F}(F)', *)$ [M1, prop.7.1], that the homomorphisms between the homotopy groups of the total spaces are also isomorphisms. Now the total space of $prin_F(p_{*,F})$ is contractible [B4, prop.3.1], hence the total space of $prin_F(r)$ is weakly contractible.

(ii) $\Rightarrow$ *(iii)*, where $D \in \mathcal{W}$. This is similar to the proof of [B4, thm.4.2(i)]. The reader will notice that this argument uses the path connectivity of the base space. This last fact follows from the free universality of the given fibration, since there is just a single $\mathcal{F}$FHE class of $\mathcal{F}$-fibrations over a point.

The conditions involving (iv) are immediate from the definition of that concept and the other parts of this result.

For a triple universal $\mathbb{S}\mathcal{F}$-fibration with BRT type classifying spaces, these spaces are CW-approximations to the corresponding GBC-spaces $B(*, (\mathcal{F}(F))', *)$ [B4, thm.4.2(i)]. Hence *such BRT spaces are determined up to homotopy type.* We will specify a CW-approximation to $B(*, (\mathcal{F}(F))', *)$ and utilize it as our preferred classifying space.

Given any space X, we will use $|S(X)|$ to denote Milnor's geometric realization of the singular complex $S(X)$. Then $|S(X)|$ is a CW-complex and there is an associated weak homotopy equivalence $j = j_X : |S(X)| \to X$ [FP, thm.4.5.30].

Definition 4.3 *For each topological monoid G, we define the associated geometric bar construction CW-complex $\mathbf{B}_G$ to be $|S(B(*, G', *))|$.*

Proposition 4.4 *If G is a topological monoid, then $\mathbf{B}_G$ is path connected. Further, taking $\mathcal{M}$ to be the category of topological monoids and continuous homomorphisms, $\mathbf{B} = |S(B(*, -', *))|$ is a covariant functor from $\mathcal{M}$ to the category of path connected CW-complexes and maps.*

Proof. We already know that $\mathbf{B}_G$ is a CW-complex. Now $B(*, G', *)$ is path connected [M1, prop.7.1] and there is a weak homotopy equivalence $j : \mathbf{B}_G \to B(*, G', *)$, so $\mathbf{B}_G$ is also path connected.

The functoriality of $\mathbf{B}$ follows from the functoriality of $|S(-)|$ and $B(*, -', *)$ (see [M1, p.31]).

We now recall a weakness of theorem 2.3, i.e. that Criterion (3) is not valid. The following classification result for $\mathcal{F}$-fibrations is a modification of theorem 2.3, the BRT classifying spaces of that theorem being replaced by GBC CW-complex classifying spaces. The advantageous features of theorem 2.3, i.e. the validity of the criteria (1), (2) and (4), still apply. Furthermore, Criterion (3) now holds.

Theorem 4.5 : The Classification of $\mathcal{F}$-fibrations. *Let A be a given space, $\mathcal{E}$ be a category that carries the structure of a proper category of well enriched spaces under A, F be a given $\mathcal{E}$-space and $\mathcal{F}$ be the category of fibres determined by F. Then there is an $\mathcal{F}$-fibration, $p_{\mathcal{F}\mathcal{F}} : X_{\mathcal{F}\mathcal{F}} \to \mathbf{B}_{\mathcal{F}(F)}$, that is triple universal amongst $\mathcal{F}$-fibrations.*

Proof. We know, via [B3, thm.5.3], that there is a weakly contractible universal $\mathcal{F}$-fibration $p_{\mathcal{F}} : X_{\mathcal{F}} \to B_{\mathcal{F}}$, over a path connected CW-complex $B_{\mathcal{F}}$. Also, there is an $\mathcal{F}$-pairwise map $< h', k'>$ from $p_{\mathcal{F}}$ to $p_{*,F}$ (see $(ii) \Rightarrow (iii)$ of proposition 4.2).

Now the weak homotopy equivalences, $k' : B_{\mathcal{F}} \to B(*, \mathcal{F}(F)', *)$ and $j : \mathbf{B}_{\mathcal{F}(F)} \to B(*, \mathcal{F}(F)', *)$, ensure that $B_{\mathcal{F}}$ and $\mathbf{B}_{\mathcal{F}(F)}$ are CW-approximations to $B(*, \mathcal{F}(F)', *)$. Hence there exists a homotopy equivalence $k'' : \mathbf{B}_{\mathcal{F}(F)} \to B_{\mathcal{F}}$ such that $k'k'' \simeq j$ [Sp, thm.7.8.1]. Let us define $p_{\mathcal{F}\mathcal{F}} : X_{\mathcal{F}\mathcal{F}} \to \mathbf{B}_{\mathcal{F}(F)}$ to be $(p_{\mathcal{F}})_{k''} : X_{\mathcal{F}} \sqcap \mathbf{B}_{\mathcal{F}(F)} \to \mathbf{B}_{\mathcal{F}(F)}$.

If $h'' : X_{\mathcal{F}\mathcal{F}} \to X_{\mathcal{F}}$ denotes the projection map, then $<h'', k''>$ is an $\mathcal{F}$-pairwise map from $p_{\mathcal{F}\mathcal{F}}$ to $p_{\mathcal{F}}$. Defining $k = k'k''$ and $h = h'h''$, we see that $<h, k> \; = \; <h', k'> \circ <h'', k''>$ is an $\mathcal{F}$-pairwise map from $p_{\mathcal{F}\mathcal{F}}$ to $p_{*,F}$. Now k', k'' and h' are known to be a weak homotopy equivalences. The same property for h'' follows from the five lemma and the path connectivity of $\mathbf{B}_{\mathcal{F}(F)}$ and $B(*, \mathcal{F}(F)', *)$.

Hence k and h are weak homotopy equivalences. So $p_{\mathcal{F}\mathcal{F}}$ is bar construction universal. It is triple universal amongst all $\mathcal{F}$-fibrations by proposition 4.2.

Definition 4.6 *Let $\mathcal{E}$ be a category of enriched spaces and $\mathbb{S}\mathcal{E}fibn$ be a closed class of $\mathcal{E}$-fibrations. Then $\mathbb{S}\mathcal{E}fibn$ will be said to be a diffuse* *class of $\mathcal{E}$-fibrations if, whenever $B \in \mathcal{W}$ and $p: X \to B$ is an $\mathcal{E}$-fibration, there is a $\mathbb{S}\mathcal{E}$-fibration $q: Y \to B$ that is $\mathcal{E}FHE$ to p.*

Our next classification result generalizes theorem 4.5, applying to $\mathbb{S}\mathcal{F}$-fibrations rather than just to $\mathcal{F}$-fibrations.

Theorem 4.7 : The Classification of $\mathbb{S}\mathcal{F}$-fibrations. *Let A be a given space, $\mathcal{E}$ a category that carries the structure of a proper category of well enriched spaces under A, F a given $\mathcal{E}$-space, $\mathcal{F}$ the category of fibres in $\mathcal{E}$ determined by F and $\mathbb{S}\mathcal{E}fibn$ a diffuse closed class of $\mathcal{E}$-fibrations. Then there exists a $\mathbb{S}\mathcal{F}$-fibration, $p_{\mathbb{S}\mathcal{F}}: X_{\mathbb{S}\mathcal{F}} \to \mathbf{B}_{\mathcal{F}(F)}$, that is triple universal amongst $\mathbb{S}\mathcal{F}$-fibrations.*

Proof. We know, from theorem 4.5, that there exists a triple universal $\mathcal{F}$-fibration, i.e. $p_{\mathcal{F}\mathcal{F}}: X_{\mathcal{F}\mathcal{F}} \to \mathbf{B}_{\mathcal{F}(F)}$. The diffuse property of $\mathbb{S}\mathcal{E}$ ensures that $\mathbb{S}\mathcal{F}$ is also diffuse. It follows that there is a $\mathbb{S}\mathcal{F}$-fibration $p_{\mathbb{S}\mathcal{F}}$ that is $\mathcal{F}$FHE to $p_{\mathcal{F}\mathcal{F}}$. Then $prin_F(p_{\mathbb{S}\mathcal{F}})$ and $prin_F(p_{\mathcal{F}\mathcal{F}})$ are $\mathcal{F}(F)$-FHE and therefore FHE. It follows that $p_{\mathbb{S}\mathcal{F}}$ is weakly contractible universal. We know that $\mathbf{B}_{\mathcal{F}(F)} \in \mathcal{W}$, so $p_{\mathbb{S}\mathcal{F}}$ is triple universal by proposition 4.2.

The validity of criteria (1), (2) and (3) carry over from theorem 4.5 to theorem 4.7. In general (4) remains valid. In the case of a specific theory, the verification of the closed and diffuse conditions may require additional assumptions concerning $\mathcal{E}$. When this occurs the issue of the validity of (4) is reopened. We will verify the remaining criteria later in this paper.

5 Principal G-Fibrations

Let G be a topological monoid. The category of (right-) G-spaces that are (right-) G-homotopy equivalent to the (right-) G-space G, together with the (right-) G-homotopy equivalences between such G-spaces, will be denoted by $\mathcal{G}$. The term *G-homotopy equivalence* should be understood, of course, in the sense that the homotopy inverse and homotopies involved are themselves G-maps. Taking $U: \mathcal{G} \to \mathcal{T}$ to denote the functor that forgets the action of G, we see that $\mathcal{G}$ carries the structure of a category of enriched spaces. Then $\mathcal{G}$-pairwise maps will be called *G-pairwise maps*. The equivalence relations $\mathcal{G}FHE$ and $\mathcal{G}FWHE$ will be written as G-FHE and G-$FWHE$, respectively.

Our theory in this section will include a classification result for generic principal fibrations, i.e. *$\mathcal{G}$-fibrations*, and another such result for our preferred version of the principal fibration concept.

We will first review the existing literature on this topic. In these publications classifying spaces B_G are produced by bar construction related procedures. These spaces are then used to classify some sort of principal G-bundles or some sort of principal G-fibrations up to some sort of equivalence relation.

Before getting into a review of the individual results, we will comment on the relevance of three of our criteria from section 3.

We explained, in that section, that (2) is always satisfied on bar construction approaches to the classification of fibrations. So (2) is satisfied for all of our principal fibration examples.

If G is a grouplike topological monoid and q is a $\mathcal{G}$-fibration then we will show, in lemma 5.6, that q is G-FHE to its associated principal G-fibration $prin_G(q)$. So, whilst (5) is not verified in any of these publications, it holds for such G and need not be an issue here.

Criterion (7) is not verified in any of the following publications. However it is shown, in each case, that there is a free universal principal fibration that is either weakly contactible universal or has a weakly contractible total space.

So our discussion of specific examples will refer only to criteria (1), (3) and (4).

Relevant published material includes [Mi], [D1], [DL], [F], [P], and [M1, cor.9.4]. Here [Mi] and [D1] are part of an earlier series of papers that investigates principal bundles and their classification; they will not be discussed further here.

(i) Dold and Lashof [DL, thm.6.2] Let G be a topological monoid. This result refers to principal G-bundles classified up G-FHE. So it does not belong to the family of results that we are primarily concerned with here.

(ii) Fuchs [F, thm.p.334] The principal G-fibrations of this paper have fibres of the G-homotopy type of G. The principal G-fibration concept, and the G-FHE relation up to which such fibrations are classified, are the same in [F] as in this paper. Criterion (1) holds. The classifying spaces are of the same homotopy type as the corresponding Dold-Lashof constructions [F, p.335] and so (3) is also satisfied. A further review of this result, including consideration of the extent to which it conforms to (4), will be given in proposition 5.10 and the associated discussion.

(iii) Porter [P, thms.10 and 11] An (almost) standard definition of principal fibration is used and fibres are required to be *multiplicatively equivalent* to the monoid of the principal fibration in question. These fibrations are classified by the equivalence relation generated by maps satisfying higher homotopy conditions, i.e. *strong homotopy homomorphisms.* In fact, two definitions of strong homotopy homomorphisms are used, leading to two equivalence relations and the two classification theorems. All spaces used are required to be in $\mathcal{W}$, so fibres are not arbitrary. Hence (1) and (4) have not been justified in this case. The classifying spaces used are of the same homotopy types as the corresponding Dold-Lashof constructions [P, p.48], and so (3) is satisfied.

(iv) May [M1, cor.9.4(i)] (= a special case of [M1, thm.9.2(i)]) This classifies a non-standard type of principal G-fibration, with fibres that are G-spaces of the G-weak homotopy type as G (see [M1, Examples 6.2(i)] for a precise definition of these fibres). The equivalence relation used is G-FWHE and GBC classifying spaces are used. It is only required that G be a grouplike topological monoid, a condition which, as we explained in section 3, seems essential for a satisfactory theory of principal G-fibrations. Hence (1) fails. However (3) and (4) hold.

(v) May [M1, cor.9.4(ii)] (= a special case of [M1, thm.9.2(ii)]) This classifies another non-standard type of principal G-fibration, in this case with G-space fibres that are homotopy equivalent to G via homotopy equivalences that are also G-maps (see [M1, Examples 6.2(ii)] for a precise definition of these fibres). The equivalence relation used is generated by fibre homotopy equivalences that are also G-maps. GBC classifying

spaces are used. It is required that $G \in \mathcal{W}$. Hence (1) fails, (3) holds and (4) has not been verified.

We now develop our own theory of principal fibrations.

Theorem 5.1 : The Classification of $\mathcal{G}$-fibrations. *Let G be a topological monoid. Then there is a $\mathcal{G}$-fibration $p_{\mathcal{G}\mathcal{G}} : X_{\mathcal{G}\mathcal{G}} \to \mathbf{B}_G$ that is triple universal amongst all $\mathcal{G}$-fibrations.*

Proof. It is shown in [B3, sec.6, ex.(ii)] that, taking $\mathcal{E} = \mathcal{G}$, the enriched stucture on $\mathcal{G}$ extends to a proper well enriched structure. We recall that the corresponding A, F and $\mathcal{F}$ are the empty space, G and $\mathcal{G}$, respectively. The result then follows from theorem 4.5.

The following definition of principal G-fibration is arguably the natural definition of this concept. It is the definition used on p.329 of [F] and is a direct extension of the usual definition of principal bundle.

Definition 5.2 *Let G be a topological monoid. A principal G-fibration will consist of a pair (q, α), where $q : Y \to C$ is a map and $\alpha : Y \times G \to Y$ is a right action of G on Y, such that:*

(i) $q\alpha = q\pi_Y$, where $\pi_Y : Y \times G \to Y$ denotes the projection, and

(ii) there is a numerable cover $\mathcal{V}$ of C such that, for each $V \in \mathcal{V}$, $q|V$ is G-FHE to the projection and G-overspace $G \times V \to V$.

In practise, we will omit α and simply state that q *is a principal G-fibration.*

We now consider an important example of principal fibrations.

Proposition 5.3 *(i) Let $\mathcal{E}$ be a category of enriched spaces, $\mathcal{F}$ the category of fibres in $\mathcal{E}$ that contains an $\mathcal{E}$-space F, C a space in $\mathcal{W}$, $q : Y \to C$ an $\mathcal{F}$-fibration and $prin_F(q) : Prin_F(Y) \to C$ the associated principal $\mathcal{F}(F)$-overspace. Then composition defines a right action α of $\mathcal{F}(F)$ on $Prin_F(Y)$. This action makes $prin_F(q)$ into a principal $\mathcal{F}(F)$-fibration, i.e. the principal fibration associated with q.*

(ii) Let $<h, k>$ be an $\mathcal{F}$-pairwise map, from an $\mathcal{F}$-fibration $p : X \to B$ to an $\mathcal{F}$-fibration $q : Y \to C$. Then there is an associated $\mathcal{F}(F)$-pairwise

map $<Prin_F(h), k>$, *from the principal* $\mathcal{F}(F)$*-fibration* $prin_F(p)$ *to the principal* $\mathcal{F}(F)$*-fibration* $prin_F(q)$, *where composition defines the obvious map* $Prin_F(h): Prin_F(X) \to Prin_F(Y)$.

(iii) the rule $<h, k> \to <Prin_F(h), k>$ *is covariantly functorial.*

Proof. (i) It is a standard result in $\mathcal{T}$ that, if P, Q and R are spaces, then the composition map $\mathcal{T}(Q, R) \times \mathcal{T}(P, Q) \to \mathcal{T}(P, R)$ is continuous. The continuity of α follows by restriction.

We know, via [B1, thm.6.3], that q carries a $\mathcal{F}LHT$ structure. Let us assume that V is a subspace of C and that there is an $\mathcal{F}$FHE from $q|V : Y|V \to V$ to the projection and trivial $\mathcal{F}$-fibration $\pi_V : F \times V \to V$. It is easily seen that the obvious induced $\mathcal{F}(F)$-FHE, from $Prin_F(q|V)$ to $Prin_F(\pi_V)$, is an $\mathcal{F}(F)$-FHE from $Prin_F(q)|V$ to the projection $\mathcal{F}(F) \times V \to V$. It follows that $prin_F(q)$ is a principal $\mathcal{F}(F)$-fibration.

(ii) and (iii). These proofs are straightforward.

Example 5.4 *Let* G *be a grouplike topological monoid and* $q : Y \to C$ *be a* $\mathcal{G}$*-fibration. Then there is a right action*

$$\alpha : Prin_G(Y) \times G \to Prin_G(Y), \quad (h, g) \to h \circ (g.-),$$

making $prin_G(q)$ *into a principal* G*-fibration.*

Proof. Let us view G as a right G-space. We will use $GM(G)$ to denote the space of all G-maps of G into itself, topologized using the cg-ified compact-open topology.

The map $GM(G) \to G$, that evaluates at the identity of G, is a homeomorphism. It has inverse the G-map $G \to GM(G)$, that takes any element of G to left-multiplication by that element. The grouplike condition ensures that $GM(G)$ is grouplike, and hence that $GM(G) = \mathcal{G}(G)$, the space of self-G-homotopy equivalences of G. Clearly $\mathcal{G}(G)$ is a monoid under composition. It is easily verified that the evaluation map is a homomorphism, so $\mathcal{G}(G)$ is isomorphic-homeomorphic to G.

The result follows from proposition 5.3, with $\mathcal{F}(F) = \mathcal{G}(G) \cong G$.

Let G be a grouplike topological monoid and Z be a right G-space that is G-homotopy equivalent to G. The space $\mathcal{G}(G, Z)$, of right G-maps from G to Z, will be given the (of course cg-ified) compact-open topology.

Lemma 5.5 *With the preceding assumptions, there is a right action*

$$\mathcal{G}(G,Z)\times G\to\mathcal{G}(G,Z),\ (h,g)\to h\circ(g.-),\ h\in\mathcal{G}(G,Z),\ g\in\mathcal{G}.$$

Then, using ι to denote the identity of G, the evaluation map

$$e:\mathcal{G}(G,Z)\to Z,\ e(f)=f(\iota),\ f\in\mathcal{G}(G,Z)$$

is a G-homeomorphism.

Proof. We saw above that $GM(G)=\mathcal{G}(G)$. Now Z has the same G-homotopy type as G, so all G-maps from G to Z are G-homotopy equivalences. Hence $\mathcal{G}(G,Z)$ is closed under the specified action of G. The continuity of the action is a standard convenient category, i.e. exponential law, type argument.

Further, the evaluation map is a G-homeomorphism, with inverse the G-map

$$Z\to\mathcal{G}(G,Z),\ z\to(g\to z.g),\ z\in Z,\ g\in G.$$

Lemma 5.6 *Let G be a grouplike topological monoid, $\mathcal{G}$ be the associated category of fibres and $q:Y\to C$ be a $\mathcal{G}$-fibration, with $C\in\mathcal{W}$.*

Then the map $e:Prin_G(Y)\to Y$, that evaluates at the identity of G, is a G-FHE from $prin_G(q)$ to q

Proof. We see, as in lemma 5.5, that e is a G-map. Now e is a map over C from the $\mathcal{G}$-fibration $prin_G(q)$ (see example 5.4) to the $\mathcal{G}$-fibration q, whose restrictions to individual fibres are G-homeomorphisms (lemma 5.5). Now q and $prin_G(q)$ satisfy the $\mathcal{G}$WCHP [B1, prop.6.2], so the result follows from [B1, thm.5.4].

Proposition 5.7 *If G is a grouplike topological monoid, then the class $\mathbb{S}\mathcal{G}fibn$ of all principal G-fibrations is a diffuse closed class of $\mathcal{G}$-fibrations.*

Proof. We recall that our definition of $\mathcal{G}$-fibration involves the $\mathcal{G}WCHP$, whereas our definition of principal G-fibration involves the $\mathcal{G}$-locally homotopy trivial condition. Hence every principal G-fibration is also a $\mathcal{G}$-fibration [B1, prop.6.2]. However a $\mathcal{G}$-fibration, even if it is over a space in $\mathcal{W}$, may not carry a continuous global action by G. Hence such a $\mathcal{G}$-fibration may not be a principal G-fibration.

If $q: Y \to C$ is a principal G-fibration and $f: B \to C$ is a map, then it is easily seen that q_f is also a principal G-fibration. So the closed condition is satisfied.

We know, by example 5.4, that $prin_G(p)$ is a principal G-fibration; so the diffuse condition follows from lemma 5.6.

Theorem 5.8 : The Classification of Principal G-Fibrations. *Let G be a grouplike topological monoid. Then there exists a triple universal principal G-fibration $p_G: X_G \to \mathbf{B}_G$.*

Proof. We noted, in the proof of theorem 5.1, that $\mathcal{G}$ carries the structure of a proper category of well enriched spaces under the empty space. Taking $\mathbb{S}\mathcal{G}fibn$ to be the class of principal G-fibrations, the result follows from theorem 4.7 and proposition 5.7. The universal fibration p_G is $prin_G(p_{\mathcal{G}\mathcal{G}})$.

The following result provides an alternative and easy characterization of triple universal principal G-fibrations. The proof follows from the equivalence of triple and weakly contractible universality (proposition 4.2) and from the G-homotopy equivalence e of lemma 5.6.

Proposition 5.9 *Let G is a grouplike topological monoid, $D \in \mathcal{W}$ and $r: Z \to D$ be a principal G-fibration. Then r is triple universal amongst principal G-fibrations if and only if Z is a weakly contractible space.*

We conclude this section with a review of the relation between our classification result for principal G-fibrations and that of [F].

Our work assumes that G is a grouplike topological monoid; that of [F] requires that G is a topological monoid with an (unbased) homotopy inverse. It is easily seen that the latter condition implies the former. The converse, however, may be false for some important examples.

If we are given a catgory of fibres $\mathcal{F}$ determined by a space F, and an $\mathcal{F}$-fibration, then the associated principal fibration has monoid $G =$

$\mathcal{F}(F)$. We know that $\mathcal{F}(F)$ is a grouplike topological monoid, but not that it has an (unbased) homotopy inverse.

For example, in the case of Hurewicz fibrations, we only know how to verify the homotopy inverse condition in cases where $\mathcal{H}(F)$, the monoid of self-homotopy equivalences of F, is in $\mathcal{W}$ [Sib, p.20-21]. This is true, of course, when F has the homotopy type of a finite CW-complex.

Proposition 5.10 *Let G be a topological monoid.*

(i) There exists a principal G-fibration whose total space is weakly contractible if and only if G is grouplike. Such principal G-fibrations are free universal.

(ii) There exists a principal G-fibration whose total space is contractible if and only if G has an (unbased) homotopy inverse. Such principal G-fibrations are free universal in a more general sense. Thus they classify all numerable principal G-fibrations, i.e. without requiring that the base spaces of these fibrations are in $\mathcal{W}$.

Proof. $(i)(\Leftarrow)$ This is immediate from theorem 5.8 and proposition 5.9.

$(i)(\Rightarrow)$ Let us assume that $r : Z \to D$ is a principal G-fibration such that Z is weakly contractible. This fibration has an associated sequence

$$\ldots \to \Omega Z \to \Omega D \xrightarrow{\delta} G \to Z \to D.$$

The weakly contractible condition ensures that δ is a weak homotopy equivalence. Now δ is also an H-map [BHMP, lem.3.2], so it induces an isomorphism $\pi_1(D) \to \pi_0(G)$. Hence G is grouplike.

(i) (free universality) This is the $(ii) \Rightarrow (i)$ part of proposition 4.2.

$(ii)(\Leftarrow)$ This is the main result of [F].

$(ii)(\Rightarrow)$ Our argument follows that of the corresponding part of (i). The difference is that, as we explained in our Introduction, δ is now a homotopy equivalence as well as being an H-map. The result follows.

(ii) (free universality) This runs parallel to the proof of the corresponding portion of (i) of this result. In both cases we have to follow the proof of [B3, thm.3.4] and produce sections to, and liftings over, a map $p\Box_1 q$. In (i) this is possible since $p\Box_1 q$ is a weak homotopy equivalence. In this case it follows because $p\Box_1 q$ satisfies the Section

Extension Property (see [D1, cor.2.8(α)]).

The free universal aspects of (i) and (ii) of proposition 5.10 are best viewed as complementary results.

(i) Our result is more general in the sense that our G is only required to be a grouplike topological monoid, rather than to be a topological monoid with a homotopy inverse.

(ii) The result of [F] is more general in the sense that it classifies numerable principal fibrations, without our requirement that base spaces be in $\mathcal{W}$.

However, in terms of our criteria, the latter approach does not satisfy (4). Further, it does not allow us to verify criterion (5) for certain categories of fibres $\mathcal{F}$, i.e. those for which we do not know that the monoid $\mathcal{F}(F)$ has a homotopy inverse.

6 Structured Fibrations and their Associated Principal Fibrations

In this section we will complete our argument that, in favourable circumstances, various definitions of universal $\mathbb{S}\mathcal{F}$-fibration are equivalent. We will verify criteria (5) and (7), and will give sufficient conditions for our full set of criteria to be valid for an arbitrary class of $\mathbb{S}\mathcal{F}$-fibrations.

Theorem 6.1 : Equivalence of Alternative Definitions of Universal Structured Fibration. *(see also [Mo, thm.3]). Let A be a given space, $\mathcal{E}$ a category that carries the structure of a proper category of well enriched spaces under A, F a given $\mathcal{E}$-space, $\mathcal{F}$ the category of fibres in $\mathcal{E}$ that contains F, $\mathbb{S}\mathcal{E}$fibn a diffuse, closed class of $\mathcal{E}$-fibrations and $p: X \to B$ an $\mathbb{S}\mathcal{F}$-fibration, where $B \in \mathcal{W}$. The conditions of p being (i) a free universal $\mathbb{S}\mathcal{F}$-fibration, (ii) weakly contractible universal, (iii) a bar construction universal $\mathbb{S}\mathcal{F}$-fibration and (iv) a triple universal $\mathbb{S}\mathcal{F}$-fibration are equivalent.*

Proof. We will show that, if p is free universal, then it is also weakly contractible universal. The result then follows from proposition 4.2.

We know, via theorem 4.7, that there is a triple universal, and hence free universal, $\mathbb{S}\mathcal{F}$-fibration $r: Z \to D$ over a GBC CW-complex D. Then there are maps $f: D \to B$ and $g: B \to D$ such that r is $\mathcal{F}$FHE to p_f and p is $\mathcal{F}$FHE to r_g. So p is $\mathcal{F}$FHE to $p_{(fg)}$ and there is a homotopy $K: B \times I \to B$ from 1_B to fg.

Now p satisfies the $\mathcal{F}$WCHP, i.e the $\mathcal{F}$-version of the CHP for homotopies into B that are stationary on $[0, \frac{1}{2}]$. We can, if necessary, modify K to make it satisfy this condition. Applying the $\mathcal{F}$WCHP, there is a homotopy $H: X \times I \to X$ such that $H|(X \times \{0\})$ is the "identity" map $X \times \{0\} \to X$ and $<H, K>$ is an $\mathcal{F}$-pairwise homotopy from $p \times 1_I$ to p. It follows, by the universal property of pullbacks, that $L = (H, p \times 1_I): X \times I \to X \sqcap (B \times I)$ is an $\mathcal{F}$-homotopy over $B \times I$, i.e. such that $p_K L = p \times 1_I$. Here $X \sqcap (B \times \{0\})$ can be identified with $X \times \{0\}$ and hence with X, i.e. via the homeomorphism $(x, b, 0) \to x$, where $(x, b, 0) \in X \sqcap (B \times \{0\})$. So $L|X \times \{0\}$ can be viewed as the identity on X. Further, $X \sqcap (B \times \{1\})$ can be identified with the space $X \sqcap B$, i.e. the space obtained by pulling p back over fg. This identification is via the homeomorphism $(x, b, 1) \to (x, b)$, where $(x, b, 1) \in X \sqcap (B \times \{1\})$. So we can regard $h = L|X \times \{1\}$ as an $\mathcal{F}$-map $X \to X \sqcap B$ over B.

We notice that $<K_p L, K>$ is an $\mathcal{F}$-pairwise homotopy from $\mathrm{p} \times 1_I$ to p. Thus it is an $\mathcal{F}$-pairwise homotopy between the $\mathcal{F}$-pairwise maps $<1_X, 1_B>$ and $<(fg)_p h, fg>$, each from p to p. Now $X \sqcap B$ can be identified with $(X \sqcap D) \sqcap B$, by $(x, b) \to (x, g(b), b)$, where $(x, b) \in X \sqcap B$. This allows us to take $(fg)_p$ as $(f_p)(g_{(p_f)})$, where $f_p: X \sqcap D \to X$ and $g_{(p_f)}: (X \sqcap D) \sqcap B \to X \sqcap D$ denote the projections. Hence $<1_X, 1_B>$ is $\mathcal{F}$-pairwise homotopic to $<f_p(g_{(p_f)})h, fg> = <f_p, f> \circ <g_{(p_f)}h, g>$.

Let 1 denote the identity map on $Prin_F(X)$. If we apply the functor $Prin_F(-)$, we then obtain an $\mathcal{F}(F)$-pairwise homotopy between the $\mathcal{F}(F)$-pairwise maps $<1, 1_B>$ and $<Prin_F(f_p)Prin_F(g_{(p_f)}h), fg>$ (proposition 5.3(iii)), where $Prin_F(g_{(p_f)}h): Prin_F(X) \to Prin_F(X \sqcap D)$ and $Prin_F(f_p) : Prin_F(X \sqcap D) \to Prin_F(X)$. Now $Prin_F(X \sqcap D)$ has the homotopy type of the weakly contractible space $Prin_F(Z)$, hence $Prin_F(X \sqcap D)$ is weakly contractible. So 1 induces zero homomorphisms on the homotopy groups of $Prin_F(X)$. Hence $Prin_F(X)$ is weakly contractible and p is weakly contractible universal.

Corollary 6.2 *If G is a grouplike topological monoid, then our theory of principal G-fibrations satisfies Criterion (7).*

Proof. It follows, from proposition 5.9 and theorem 6.1, that this property holds for principal G-fibrations whose base spaces are in $\mathcal{W}$.

Let $r : Z \to D$ be a principal G-fibration and K be a CW-approximation to D via a weak homotopy equivalence $j : K \to D$. Then the free universal property is valid for r if and only if it is valid for the induced principal G-fibration r_j (see [Sp, thm.7.8.12]). It follows from routine properties of fibrations that r has a weakly contractible total space if and only if r_j has the same property. The result follows.

Let $\mathcal{F}$ be a category of enriched fibres containing the $\mathcal{F}$-space F. Also, let $q : Y \to C$ be an $\mathcal{F}$-fibration and $f : B \to C$ be a map. If $b \in B$, then $\pi_b : Y|f(b) \to (Y|f(b)) \times \{b\}$ will be used to denote the canonical $\mathcal{F}$-homeomorphism. Given a function $k : F \to Y|f(b)$, we notice that $k \in \mathcal{F}(F, Y|f(b))$ if and only if $\pi_b \circ k \in \mathcal{F}(F, (Y|f(b)) \times \{b\})$.

Lemma 6.3 *The function $\xi : Prin_F(Y) \sqcap B \to Prin_F(Y \sqcap B)$ defined by $\xi(k, b) = \pi_b \circ k$ is an $\mathcal{F}(F)$-homeomorphism over B, i.e. it is an isomorphism of principal $\mathcal{F}(F)$-fibrations from $(prin_F(q))_f$ to $prin_F(q_f)$.*

Proof. The fibres of $(prin_F(q))_f$ and $prin_F(q_f)$ over b are the spaces $\mathcal{F}(F, Y|f(b)) \times \{b\}$ and $\mathcal{F}(F, (Y|f(b)) \times \{b\})$, respectively. Hence the function $\mathcal{F}(F, Y|f(b)) \times \{b\} \to \mathcal{F}(F, (Y|f(b)) \times \{b\})$, $(k, b) \to \pi_b \circ k$, is a bijection and ξ is a bijective function over B.

Let us consider a space W and a function $h : W \to Prin_F(Y) \sqcap B$. It can be seen, using the exponential law [B1, (0.1)], the definition of *prin* and properties of pullbacks, that h is continuous if and only if $\xi \circ h$ is continuous. It follows that ξ is a homeomorphism.

The actions of $\mathcal{F}(F)$ on $Prin_F(Y) \sqcap B$ and $Prin_F(Y \sqcap B)$ agree via the homeomorphim ξ, so the principal fibrations $(prin_F(q))_f$ and

$prin_F(q_f)$ are equivariantly isomorphic.

Criterion (5) will be verified in part (i) of the following result. In (ii) and (iii), we show that this criterion implies that the theories of $\mathbb{S}\mathcal{F}$-fibrations and principal $\mathcal{F}(F)$-fibrations are, in a sense, equivalent.

Theorem 6.4 : On the Equivalence of the Theories of Structured and Principal Fibrations. *Let A be a given space, $\mathcal{E}$ a category that carries the structure of a proper category of well enriched spaces under A, F a given $\mathcal{E}$-space, $\mathcal{F}$ the category of fibres in $\mathcal{E}$ that contains F, $\mathbb{S}\mathcal{E}fibn$ a diffuse closed class of $\mathcal{E}$-fibrations and $r: Z \to D$ an $\mathbb{S}\mathcal{F}$-fibration over a space $D \in \mathcal{W}$. Then:*

(i) r is a free universal $\mathbb{S}\mathcal{F}$-fibration if and only if $prin_F(r)$ is a free universal principal $\mathcal{F}(F)$-fibration.

(ii) If $q: Y \to C$ is an $\mathbb{S}\mathcal{F}$-fibration and $f: C \to D$ is a map, then f is a classifying map for q if and only if f is a classifying map for $prin_F(q)$.

(iii) If p and q are $\mathbb{S}\mathcal{F}$-fibrations, then p is $\mathcal{F}$FHE to q if and only if $prin_F(p)$ is $\mathcal{F}(F)$-FHE to $prin_F(q)$.

Proof. (i) The $\mathbb{S}\mathcal{F}$-fibration r is triple universal if and only if it is weakly contractible universal (proposition 4.2), i.e. if $prin_F(r)$ is triple universal (propositions 4.2 and 5.9).

(ii) Let us now assume that r is a triple universal $\mathbb{S}\mathcal{F}$-fibration. If $f: C \to D$ is a classifying map for q, then q is $\mathcal{F}$FHE to r_f. It is easily seen that $prin_F(q)$ is then $\mathcal{F}(F)$-FHE to $prin_F(r_f)$, i.e. to $(prin_F(r))_f$ (see lemma 6.3) Hence f is a classifying map for the associated principal $\mathcal{F}(F)$-fibration $prin_F(q)$.

Conversely, let $g : C \to D$ classify $prin_F(q)$. If f classifies q, then f classifies $prin_F(q)$. So $f \simeq g$. Hence g classifies q.

(iii) The result follows from (ii) and the free universality concept.

Theorem 6.5 : Validity of Criteria for Structured Fibrations. *Let A be a given space, $\mathcal{E}$ a category that carries the structure of a proper category of well enriched spaces under A, F a given $\mathcal{E}$-space and $\mathcal{F}$ the category of fibres in $\mathcal{E}$ determined by F.*

(i) The criteria (1), (2) ... (5) are valid in the context of theorem 4.5, our classification result for $\mathcal{F}$-fibrations.

Let $\mathbb{S}\mathcal{E}$fibn be a diffuse closed class of $\mathcal{E}$-fibrations.

(ii) The criteria (1), (2) ... (5) are valid in the context of theorem 4.7, our classification result for $\mathbb{S}\mathcal{F}$-fibrations.

Proof. The validity of (1), (2), (3) and (4) was noted before theorem 4.5 and after theorem 4.7; (5) is (i) of theorem 6.4.

We now reconsider the questions discussed in the last theorem, but in the context of a specific theory of $\mathbb{S}\mathcal{F}$-fibrations. In this case we may have introduced additional assumptions, concerning $\mathcal{E}$, to ensure that that the diffuse closed condition of theorem 4.7 is satisfied. *We then have to reconsider whether (4) is still valid.*

Theorem 6.6 : Validity of Criteria for Specific Theories of Structured Fibrations. *Let A be a given space, $\mathcal{E}$ a category that carries the structure of a proper category of well enriched spaces under A, F a given $\mathcal{E}$-space and $\mathcal{F}$ the category of fibres determined by F.*

(i) The criteria (1), (2) ... (5) are valid in the context of theorem 4.5, our classification result for $\mathcal{F}$-fibrations.

Let us also assume that $\mathbb{S}\mathcal{E}$fibn is a diffuse closed class of $\mathcal{E}$-fibrations and that any assumptions concerning $\mathcal{E}$, that are introduced to verify the diffuse closed condition in $\mathbb{S}\mathcal{E}$fibn, are minimal requirements for verifying that condition.

(ii) The criteria (1), (2) ... (5) are valid in the context of theorem 4.7, our classification result for $\mathbb{S}\mathcal{F}$-fibrations.

Proof. The only issue concerns whether or not (4) is still valid. In case (i) the diffuse closed condition is trivially true, without any extra assumption on $\mathcal{E}$, so there is nothing to verify. In (ii) we have added an assumption that ensures that the validity of (4) is preserved.

The second sentence of the statement of Objective B is imprecise, since the terms "uncomplicated " and "easily managed" are relative and informal concepts. Subject to this proviso, however, we make the following claim.

Conclusion concerning Objective B. *Theorems 4.7 and 6.6, taken together, mean that Objective B has been met.*

Theorem 6.7 : Validity of Criteria for Principal Fibrations.
(i) Let G be a topological monoid. The theory of $\mathcal{G}$-fibrations and their classifying spaces, as presented in theorem 5.1, meets our criteria (1), (2), ... (5).

(ii) Let G be a grouplike topological monoid. The theory of principal G-fibrations and their classifying spaces, as presented in theorem 5.8, meets our criteria (1), (2), ... (5) and (7). Hence (6) is valid for principal G-fibrations.

Proof. (i) This follows from (i) of theorem 6.6, since theorem 5.1 is a particular case of theorem 4.5.

(ii) The main part of the result follows from theorem 6.6(ii), since theorem 5.8 is a particular case of theorem 4.7. It follows from corollary 6.2 that criterion (7) is justified and from (i) of proposition 5.10 that the grouplike condition on G is essential. With regard to the condition concerning additional assumptions on $\mathcal{E} = \mathcal{G}$, as stated in theorem 6.6(ii), our verification of the diffuse condition (proposition 5.7) requires that G be grouplike. However, we have just seen that this condition is essential, so it is a minimal condition as required.

7 Dold and Hurewicz Fibrations

A *Dold fibration* is a map that satisfies the *weak covering homotopy property* or $WCHP$ of [D1, sec.5]. A *Hurewicz fibration* is, of course, a map that satisfies the *covering homotopy property* or CHP of [D1, sec.4]. We recall that $\mathcal{T}$ denotes the category of weak Hausdorff compactly generated spaces and their maps, i.e. the category of enriched spaces that carries no extra structure. The equivalence relations $\mathcal{T}FHE$ and $\mathcal{T}FWHE$ will be written in simplified form as FHE and $FWHE$, respectively.

We will review the previously published results concerning the classification of Dold and Hurewicz fibrations. The first listed result considers Dold fibrations and the second *grounded* Hurewicz fibrations [B2, defs.6.1]; the rest apply to Hurewicz fibrations. Result (vi) classifies fibrations, whose fibres have the weak homotopy type of a given space F, up to FWHE. The others classify fibrations, whose fibres have the homotopy type of a given space F, up to FHE. The arguments of the first four results follow the BRT approach; the last four are based on bar construction procedures. Criterion (5) is not discussed in any of these publications. However, it is shown in several cases that the free universal fibrations considered are actually weakly contractible universal, which goes some way towards establishing (5).

(i) Dold [D2, 16.9 korollar] This classifies Dold fibrations, whose fibres have a given homotopy type, up to FHE. Hence (1) is satisfied. The theory also satisfies (4); the issues of (2) and (3) are not considered.

(ii) Allaud [A, thm.2.1] Grounded Hurewicz fibrations are classified up to grounded FHE [B2, defs.6.1]. This implies a corresponding classification of Hurewicz fibrations, with fibres of a given homotopy type, up to FHE [B2, prop.7.4]. So (1) is (essentially) satisfied. The result is proved only for fibres in $\mathcal{W}$, so (4) is not verified here. Issues (2) and (3) are not discussed. This paper includes the first proof that a BRT grounded universal fibration is also weakly contractible universal.

(iii) Dold [D2, satz of 6.5 and 16.9 korollar] These two results give a classification of Hurewicz fibrations, whose fibres have a given homotopy type, up to FHE. Hence (1) is satisfied. It follows from (i) that Criterion (4) is also satisfied. The issues of (2) and (3) are not addressed in (i), so no conclusions can be drawn about those criteria in this case.

(iv) Schön [Sc, 2.thm.] Hurewicz fibrations, with fibres of a given homotopy type, are classified up to FHE in this paper. So (1) holds. This is the first BRT approach to successfully resolve the issue of (2). This theory also satisfies (4); the issue of (3) is not addressed.

(v) Stasheff [Sta, classification theorem] This is the original FHE classification theorem for Hurewicz fibrations whose fibres have a given homotopy type. So (1) is valid. Criteria (2) and (3) are also satisfied. The result is established for fibres that have the homotopy type of a finite CW-complex, so (4) is not verified.

(vi) May [M1, cor.9.5(i)] (= a special case of [M1, thm.9.2(i)]) This classifies Hurewicz fibrations, whose fibres have the weak homotopy type of a given space $F \in \mathcal{W}$, up to FWHE. So (1) is not satisfied. However, (4) holds since all spaces (= all possible fibres) have the weak homotopy type of some choice of a space $F \in \mathcal{W}$. Criteria (2) and (3) are also valid.

(vii) May [M1, cor.9.5(ii)] (= a special case of [M1, thm.9.2(ii)]) Hurewicz fibrations, whose fibres have a given homotopy type, are classified up to FHE. Hence (1) holds. Criteria (2) and (3) are also satisfied. It is assumed that F is compact and that all fibres are in $\mathcal{W}$, so (4) is not verified here.

(viii) May [M2, thm.1.2] Hurewicz fibrations, whose fibres have given homotopy type, are classified up to FHE. So (1) is valid. As in the preceding result, (2) and (3) are also satisfied. The condition on F that occurs in (vii) is now loosened up, in that it is only required that all fibres are in $\mathcal{W}$ [M2, lem.1.1]. Hence (4) is closer to being justified.

We first establish an improved classification result for Dold fibrations. Taking U to denote the identity functor on $\mathcal{T}$, we see that $\mathcal{T}$ carries the structure of a category of enriched spaces. The Dold fibration and *$\mathcal{T}$-fibration* concepts coincide.

Let $\mathcal{H}$ denote the category of spaces that are homotopy equivalent to a given space F and of homotopy equivalences between such spaces. Then the *Dold fibration with fibres that have the homotopy type of F* and *$\mathcal{H}$-fibration* concepts coincide. We use $\mathcal{H}(F)$, of course, to denote the monoid of self homotopy equivalences of F.

Theorem 7.1 : The Classification of Dold Fibrations. *Let F be a given space. There exists a Dold fibration $p_{\mathcal{H}\mathcal{H}} : X_{\mathcal{H}\mathcal{H}} \to \mathbf{B}_{\mathcal{H}(F)}$, with fibres of the homotopy type of F, that is triple universal amongst Dold fibrations with such fibres.*

Proof. We saw, in [B3, ch.6, ex.(i)], that $\mathcal{T}$ carries the structure of a proper category of well enriched spaces under the empty space. Taking $\mathcal{F} = \mathcal{H}$, the result follows from theorem 4.5.

We now develop our theory of Hurewicz fibrations. Such fibrations are always Dold fibrations and therefore $\mathcal{T}$-fibrations, but the converse statement is false [D1, p.238-239].

Proposition 7.2 *The class $\mathbb{S}\mathcal{T}$fibn of Hurewicz fibrations is a diffuse closed class of $\mathcal{T}$-fibrations.*

Proof. If $q: Y \to C$ is a Hurewicz fibration and $f: B \to C$ is a map, then it is standard (and easily seen) that q_f is also a Hurewicz fibration.

Any map p can be factored, in a standard fashion, as the composite qh of Hurewicz fibration q and a homotopy equivalence h [Mn, thm.6.5.10]. If p is a Dold fibration, then it follows by [D1, thm.6.1] that h is a fibre homotopy equivalence from p to q.

Theorem 7.3 : The Classification of Hurewicz Fibrations. *Let F be a given space. Then there is a Hurewicz fibration $p_\infty : X_\infty \to B_\infty$, with fibres that are of the homotopy type of F, that is triple universal amongst Hurewicz fibrations with such fibres.*

The Hurewicz fibration p_∞ is obtained when we factor the Dold fibration $p_{\mathcal{H}\mathcal{H}} : X_{\mathcal{H}\mathcal{H}} \to \mathbf{B}_{\mathcal{H}(F)}$, in standard fashion, as the composite of a homotopy equivalence and a Hurewicz fibration. In particular, the classifying space B_∞ is the bar construction CW-complex $\mathbf{B}_{\mathcal{H}(F)}$.

Proof. We saw, in the proof of theorem 7.1, that the enriched structure on $\mathcal{T}$ extends to a proper well enriched structure under the empty space. Taking $\mathbb{S}\mathcal{T}$*fibn* be the class of Hurewicz fibrations, the result follows via theorem 4.7 and proposition 7.2.

Theorem 7.4 : Validity of Criteria for Dold and Hurewicz Fibrations.

(i) The theory of Dold fibrations and their classifying spaces, as presented in theorem 7.1, meets our criteria (1), (2), ... (5).

(ii) The theory of Hurewicz fibrations and their classifying spaces, as presented in theorem 7.3, meets our criteria (1), (2), ... (5). In other words, (6) is valid for Hurewicz fibrations.

Proof. (i) This follows from (i) of theorem 6.6, since theorem 7.1 is a particular case of theorem 4.5.

(ii) Theorem 7.3 is a particular case of theorem 4.7. The result then follows from (ii) of theorem 6.6 , since no additional assumptions are required on $\mathcal{E}$ to validate the diffuse closed condition of proposition 7.2.

8 Sectioned Fibrations

The preferred definition of sectioned fibration is not settled with the same degree of certainty as, for example, the definition of Hurewicz fibration. It certainly involves (some sort of) fibration that is equipped with (some sort of) cross section. The section should probably satisfy some sort of cofibration property. We will prove a classification result for generic *sectioned fibrations*, and another such result for our preferred version of that concept, i.e. *well sectioned fibrations.*

We first review some classification results for sectioned fibrations that have been obtained elsewhere. Criterion (5) is not discussed in any of these cases.

(i) Siegel [Sie, thm.1.5a] The sectioned fibrations considered here are Hurewicz fibrations with associated sections. Grounded sectioned fibrations, with fibres of a given pointed homotopy type, are classified up to grounded sectioned FHE [B2, defs.6.1]. This implies a corresponding classification of sectioned fibrations, with fibres of a given pointed homotopy type, up to sectioned FHE [B2, prop.7.4]. Thus these fibrations would be classified up to the equivalence relation required by (1). All spaces are assumed to have the homotopy types of countable connected CW-complexes, so (4) does not hold. Two methods of proof are referred to. One is to mimic the techniques of [A] and is just mentioned. The other, which is based on the results of [A], is presented in a fairly condensed fashion. On this approach the universal sectioned fibration is the induced fibration, obtained by pulling p_∞ back over itself, together with the diagonal section. However, the present author has not been able to verify a key step in that argument. Thus [Sie, cor.1.9] seems to

require an additional assumption concerning the sections of sectioned fibrations. Siegel's methods are based on those of [A], so they have other characteristics of that account. In particular (2) and (3) are not verified.

(ii) May [M1, cor.9.8(i)] (= a special case of [M1, thm.9.2(i)]). The sectioned fibrations of this result satisfy what we (later) refer to as the sectioned CHP. The fibres are required to be non-degenerately pointed spaces of a given pointed weak homotopy type. The equivalence relation used is a section preserving version of FWHE. This is pointed version of (vi) of section 7, so (1) does not apply, but (2), (3) and (4) are valid.

(iii) May [M1, cor.9.8(ii)] (= a special case of [M1, thm.9.2(ii)]). Another classification theorem for sectioned CHP style fibrations is given here. The fibres are required to be non-degenerately pointed and to have a given pointed homotopy type. The equivalence relation used agrees with our sectioned FHE. The spaces of all fibres are in $\mathcal{W}$ and the space of the distinguished fibre F is compact. This is pointed version of (vii) of section 7; (1), (2) and (3) hold, but (4) is not verified.

(iv) May [M2, thm.1.2] This result concerns the same types of fibres, fibrations and equivalence relation as does the previous example. The difference is that the compactness condition on F has now been eliminated: it is only required that F be non-degenerately pointed and in $\mathcal{W}$. This is pointed version of (viii) of section 7; (1), (2) and (3) hold and (4) is closer to being verified than it is in (iii) above.

We now develop our own results concerning sectioned fibrations. Let us use $\mathcal{T}^0$ to denote the category of pointed spaces and pointed maps and take $U:\mathcal{T}^0 \to \mathcal{T}$ to be the functor that forgets basepoints. We see that $(\mathcal{T}^0, U)$ is an enriched category of spaces. In the discussion that follows we will make reference to various $\mathcal{T}^0$-concepts, as defined on pages 130, 135, 136 and 142 of [B1].

Let us define a *sectioned overspace* to be a pair (q, t), where $q: Y \to C$ is a map (= a continuous function), $t: C \to Y$ is a (not necessarily continuous) function with $qt = 1_C$. Equivalently, this is the concept of a $\mathcal{T}^0$*-overspace.*

For example, if C is a space and $(Z, *)$ is a pointed space, then there is a projection $\pi_{Z,C}: Z \times C \to C$, $(z,c) \to c$ and also an injection $\sigma_{Z,C}: C \to Z \times C$, with $c \to (*,c)$. Then $(\pi_{Z,C}, \sigma_{Z,C})$ is a *trivial sectioned overspace* or, equivalently, a *trivial* $\mathcal{T}^0$*-overspace.*

Further, if (q,t) is a sectioned overspace, there is an associated sectioned overspace $(q \times 1_I : Y \times I \to C \times I,\ t \times 1_I : C \times I \to Y \times I)$.

Let $f: B \to C$ be a map and (q,t) be a sectioned overspace. We define a function $t^f: B \to Y \sqcap B$ by $t^f(b) = (tf(b), b)$, where $b \in B$. Then (q_f, t^f) is the *sectioned overspace induced from* (q,t) *by* f. Thus we have the concept of an *induced* $\mathcal{T}^0$*-overspace.*

Let (p,s) and (q,t) be sectioned overspaces of B and $< h, k >$ be a pairwise map from p to q. If $tk = hs$, then $< h, k >$ will be said to be a *sectioned pairwise map from* (p,s) *to* (q,t). If, in this situation, $B = C$ and $k = 1_B$, then h will be said to be a *sectioned map over B.* Thus we have the concepts of $\mathcal{T}^0$*-pairwise map* and $\mathcal{T}^0$*-map over* B, respectively.

A sectioned pairwise map $<H, K>$, from $(p \times 1_I, s \times 1_I)$ to (q,t), will be called a *sectioned pairwise homotopy.* This agrees with the concept of a $\mathcal{T}^0$*-pairwise homotopy.* In the case where $B = C$ and $K: B \times I \to B$ denotes the projection, then we have the idea of a *sectioned homotopy over* B or $\mathcal{T}^0$*-homotopy over* B.

The corresponding concept of homotopy equivalence is *sectioned fibre homotopy equivalence*, *sectioned FHE* or $\mathcal{T}^0$*FHE.*

We will now consider situations in which $(p: X \to B,\ s: B \to X)$, and $(q: Y \to C,\ t: C \to Y)$ are sectioned overspaces, $h: X \times \{0\} \to Y$ is a map and $K: B \times I \to C$ is a homotopy such that $<h, K|(B \times \{0\})>$ is a sectioned pairwise map from $(p \times 1_{\{0\}}, s \times 1_{\{0\}})$ to (q,t). Let us now fix (q,t). Then (q,t) will be said to satisfy the *sectioned covering homotopy property* or *sectioned CHP* if, for all such choices of (p,s), h and K, there exists a homotopy $H: X \times I \to Y$ extending h and such that $<H, K>$ is a sectioned pairwise homotopy from $(p \times 1_I, s \times 1_I)$ to (q,t). Equivalently, (q,t) satisfies the $\mathcal{T}^0$*CHP.*

If, in the last definition, homotopies K are required to be stationary on the interval $[0, \frac{1}{2}]$, then (q,t) will be said to satisfy the *sectioned weak covering homotopy property*, the *sectioned WCHP* or the $\mathcal{T}^0$*WCHP.*

A sectioned overspace will be said to be a *sectioned fibration* if it satisfies the sectioned WCHP, i.e. if it is a $\mathcal{T}^0$*-fibration.*

The sectioned overspace (q, t) over C will be said to be *sectioned homotopy trivial* if it is sectioned FHE to a trivial sectioned space $(\pi_{Z,C}, \sigma_{Z,C})$, for some choice of a pointed space $(Z, *)$. This agrees, of course, with the concept of (q, t) being $\mathcal{T}^0$*-homotopy trivial.*

If there is a numerable cover $\mathcal{V}$ of C and, for each $V \in \mathcal{V}$, $(q|V, t|V)$ is sectioned homotopy trivial, then (q, t) will be said to be *sectioned locally homotopy trivial* or *sectioned LHT*. Thus we have the concept (q, t) being $\mathcal{T}^0 LHT$.

Let $C \in \mathcal{W}$. Then a sectioned overspace (q, t) over C is a sectioned fibration if and only if it is sectioned LHT [B1, thm.6.3]. It follows by [B1, prop.7.1] that, in such situations, the section t is continuous. *From this point on, the base spaces of fibrations will be assumed to be in $\mathcal{W}$, so we have the option of taking sectioned fibration to mean sectioned LHT. Furthermore, all sections of sectioned fibrations can be assumed to be continuous.*

Let $(F, *)$ be a given pointed space. We will now take $\mathcal{F}$ to be the subcategory $\mathcal{H}^0$ of $\mathcal{T}^0$ that consists of all pointed spaces that are pointed homotopy equivalent to $(F, *)$ and all pointed homotopy equivalences between such spaces. Then the *sectioned fibration with fibres of the pointed homotopy type of* $(F, *)$ and $\mathcal{H}^0$*-fibration* concepts coincide. We use $\mathcal{H}^0(F, *)$, of course, to denote the monoid of pointed self homotopy equivalences of $(F, *)$. The homotopies involved here are assumed to be families of pointed maps.

Theorem 8.1 : The Classification of Sectioned Fibrations.
Let $(F, *)$ be a given pointed space. Then there is a sectioned fibration $(p_{\mathcal{H}^0\mathcal{H}^0} : X_{\mathcal{H}^0\mathcal{H}^0} \to \mathbf{B}_{\mathcal{H}^0(F,*)},\ s_{\mathcal{H}^0\mathcal{H}^0} : \mathbf{B}_{\mathcal{H}^0(F,*)} \to X_{\mathcal{H}^0\mathcal{H}^0})$, with fibres of the pointed homotopy type of $(F, *)$, that is triple universal amongst sectioned fibrations with such fibres.

Proof. We saw, in [B3, ch.6, ex.(iii)], that $\mathcal{E} = \mathcal{T}^0$ carries the structure of a proper category of well enriched spaces, with A a one point space. Taking $\mathcal{F} = \mathcal{H}^0$, and recalling the equivalence of sectioned concepts and $\mathcal{T}^0$-concepts, the result follows via theorem 4.5.

The above result imposes neither topological conditions on F nor any condition on the type of basepoint used. For some purposes, however, it is useful to work with sectioned fibrations that have cofibrations as their distinguished sections.

Definition 8.2 *A sectioned overspace (q, t) will be said to be a well sectioned fibration if q is a Hurewicz fibration and t is a closed cofibration.*

We recall the discussion of such fibrations in [B1, p.145-147]. In particular, [B1, thm.7.4] shows that well sectioned fibrations satisfy the sectioned CHP and hence are sectioned fibrations in our previous sense.

A pointed space $(F, *)$ will be said to be *well pointed* if the inclusion of the point in the space is a cofibration. We recall, from [B1, p.129], that the spaces of our $\mathcal{T}$, i.e. compactly generated weak Hausdorff spaces, are T_1-spaces. So such inclusions of basepoints into well pointed spaces are closed cofibrations.

Let $\mathcal{T}^{w0}$ denote the category of well pointed spaces and pointed maps. A *$\mathcal{T}^{w0}$-fibration* is then a *sectioned fibration whose fibres are well pointed spaces.* According to [B1, cor.7.3], the fibres of well sectioned fibrations are in $\mathcal{T}^{w0}$, so well pointed fibrations are a class of $\mathcal{T}^{w0}$-fibrations. We will obtain an analogue of the previous classification result for sectioned fibrations, but using well pointed fibres. This will enable us to establish a classification theorem for well sectioned fibrations.

Let $(F, *)$ be a well pointed space. Then $\mathcal{H}^{w0}$ will denote the category of fibres in $\mathcal{T}^{w0}$ that contains $(F, *)$. An *$\mathcal{H}^{w0}$-fibration* is a *sectioned fibration whose fibres are well pointed spaces of the pointed homotopy type of $(F, *)$.*

Proposition 8.3 *Let $(F, *)$ be a well pointed space. Then there is a sectioned fibration $(q_{\mathcal{H}^0\mathcal{H}^0} : Y_{\mathcal{H}^0\mathcal{H}^0} \to \mathbf{B}_{\mathcal{H}^0(F,*)},\ t_{\mathcal{H}^0\mathcal{H}^0} : \mathbf{B}_{\mathcal{H}^0(F,*)} \to Y_{\mathcal{H}^0\mathcal{H}^0})$, with well pointed fibres of the pointed homotopy type of $(F, *)$, that is triple universal amongst sectioned fibrations with such fibres.*

Proof. The proof that $\mathcal{T}^{w0}$ carries the structure of a proper category of well enriched spaces follows the corresponding proof for $\mathcal{T}^0$, except that we also have to show that the corresponding cylinders and mapping cylinders are well pointed. Thus, if $(X, *)$ is a well pointed space, we have to show that the cylinder $X \times_{\mathcal{T}^{w0}} I = (X \times I) \,/\, (* \times I)$

is well pointed. If $(Y,*)$ is also a well pointed space and $f:X\to Y$ is a pointed map, we have to show that the pointed mapping cylinder $\mathcal{T}^{w0}MC(f) = Y\cup_f(X\times_{\mathcal{T}^{w0}}I)$ [B2, p.88] is well pointed. These conditions are easily verified, using parts of the proof of [Mn, thm.6.5.5]. Taking $\mathcal{F}=\mathcal{H}^{w0}$, the result follows by theorem 4.5.

Let $(q:Y\to C,\ t:C\to Y)$ be a sectioned fibration with fibres in $\mathcal{H}^0$. We define $Y^{\ominus}$ to be the unpointed mapping cylinder $MC(t)$ [B2, p.89] of t. Then $q^{\ominus}:Y^{\ominus}\to C$ will be the map defined by $q^{\ominus}(y)=q(y)$, where $y\in Y$, and $q^{\ominus}(c,u)=c$, where $c\in C$ and $u\in I$. We take $t^{\ominus}:C\to Y^{\ominus}$ to be the map with $t^{\ominus}(c)=(c,0)$, where $c\in C$.

Let us recall a construction of [Str, prop.2]: if f is a given map then it can be factored as a composite $p(f)h(f)$, where $h(f)$ is both a closed cofibration and a homotopy equivalence and $p(f)$ is a Hurewicz fibration. Given (q,t), as above, we define $(q^{\oplus}:Y^{\oplus}\to C,\ t^{\oplus}:C\to Y^{\oplus})$ to be the sectioned space over C specified by $(p(q^{\ominus}),h(q^{\ominus})t^{\ominus})$. We notice that $h(q^{\ominus}):Y^{\ominus}\to Y^{\oplus}$ is a map over C.

Proposition 8.4 *Let $(F,*)$ be a well pointed space, C be a numerably contractible space and $(q:Y\to C,\ t:C\to Y)$ be a sectioned fibration with fibres of the pointed homotopy type of F. Then $(q^{\oplus},t^{\oplus})$ is a well sectioned fibration that is sectioned FHE to (q,t).*

Proof. We recall that F' denotes F with a whisker grown at $*$. The point, at the isolated end of the whisker, will be the basepoint $*'$ of F'. A sectioned LHT structure for q, with fibre $(F,*)$, extends to give a sectioned LHT structure for $q^{\ominus}$, with fibre $(F',*')$. For if V is a subspace of C and there is a sectioned FHE $g:F\times V\to Y|V$ from the trivial sectioned space $(\pi_{F,V}:F\times V\to V,\ \sigma_{F,V}:V\to F\times V)$ to $(q|V,t|V)$, then $Y^{\ominus}|V$ is the mapping cylinder for $t|V:V\to Y|V$ and we can define a map $h:F'\times V\to Y^{\ominus}|V$ by $h(y,c)=g(y,c)$ and $h(u,c)=(c,u)$, where $y\in F$, $c\in V$ and $u\in I$. Then h is a sectioned FHE from the trivial sectioned space $(\pi_{F',V},\sigma_{F',V})$ to $(q^{\ominus}|V,t^{\ominus}|V)$.

There is a retraction $k:Y^{\ominus}\to Y$ defined by $k(y)=y$ and $k(c,u)=t(c)$, where $y\in Y$, $c\in C$ and $u\in I$. Recalling the last paragraph of [B1, p.146] and [B1, thm.5.5], we see that k is a sectioned FHE from $(q^{\ominus},t^{\ominus})$ to (q,t).

We know that $q^{\oplus}$ is a Hurewicz fibration and that $t^{\oplus}$, the composite of two cofibrations, is a cofibration. Now $(q^{\oplus}, t^{\oplus})$ is a sectioned space, so $t^{\oplus}(C) = (1_Y^{\oplus}, t^{\oplus}q^{\oplus})^{-1}(\Delta)$, where Δ denotes the diagonal subspace of $Y^{\oplus}$. The weak Hausdorffness of $Y^{\oplus}$ ensures that $t^{\oplus}(C)$ is closed. It follows that $(q^{\oplus}, t^{\oplus})$ is a well sectioned fibration.

The homotopy equivalence $h(q^{\ominus})$ is a sectioned FHE from $(q^{\ominus}, t^{\ominus})$ to $(q^{\oplus}, t^{\oplus})$ [B1, thm.5.5]. Hence (q, t) and $(q^{\oplus}, t^{\oplus})$ are sectioned FHE.

Proposition 8.5 *The class $\mathbb{S}\mathcal{T}^{w0}$fibn of well sectioned fibrations is a diffuse closed class of $\mathcal{T}^{w0}$-fibrations.*

Proof. This is immediate from [B1, prop.7.2] and proposition 8.4.

Theorem 8.6 : The Classification of Well Sectioned Fibrations. *Let $(F, *)$ be a well pointed space. Then there is a well sectioned fibration $(p_\omega : X_\omega \to B_\omega,\ s_\omega : B_\omega \to X_\omega)$, with fibres of the pointed homotopy type of $(F, *)$, that is triple universal amongst well sectioned fibrations with such fibres.*

In fact (p_ω, s_ω) is $((q_{\mathcal{H}^0\mathcal{H}^0})^{\oplus}, (t_{\mathcal{H}^0\mathcal{H}^0})^{\oplus})$, so the classifying space B_ω is the bar construction CW-complex $\mathbf{B}_{\mathcal{H}^0(F,*)}$.

Proof. We noted, in the proof of proposition 8.3, that the enriched structure on $\mathcal{T}^{w0}$ extends to a proper well enriched structure. Taking $\mathbb{S}\mathcal{T}^{w0}$*fibn* to be the class of well sectioned fibrations with the specified type of fibres, the result follows from theorem 4.7 and proposition 8.5.

Theorem 8.7 : Validity of Criteria for Sectioned and Well Sectioned Fibrations.

(i) The theory of sectioned fibrations and their classifying spaces, as presented in theorem 8.1, satisfies our criteria (1), (2) ... (5).

(ii) The theory of well sectioned fibrations and their classifying spaces, as presented in theorem 8.6, satisfies our criteria (1), (2) ... (5). Thus (6) is valid for well sectioned fibrations.

Proof. (i) This follows from (i) of theorem 6.6, since theorem 8.1 is a particular case of theorem 4.5.

(ii) This follows from (ii) of theorem 6.6, since theorem 8.6 is a particular case of theorem 4.7. With regard to the caveat stated in theorem

6.6 concerning additional assumptions on $\mathcal{E}$, our verification of the diffuse condition (proposition 8.4) requires that $(F, *)$ be a well pointed space. However, this condition always holds for fibres to well sectioned fibrations [B1, cor.7.3], so it is a minimal condition as required.

Conclusion concerning Objective A. We have proved a classification theorem for $\mathbb{S}\mathcal{F}$-fibrations (theorem 4.7) and shown that it meets of our criteria (1), (2) ... (5) (theorem 6.5(ii)). We have shown, in theorems 5.8, 6.7(ii), 7.3, 7.4(ii), 8.6 and 8.7(ii), that (6) is valid for all three types of classical fibration and in corollary 6.2 that (7) is satisfied. Hence Objective A has been achieved.

Addendum We remarked, in the last part of section 5, that the classification results for principal G-fibrations of [F] and this paper are complementary results. The question arises as to whether there is a theory of $\mathcal{F}$-fibrations that is complementary to the theory of this paper, in an analagous fashion. A recent paper, [PPP], develops just such a theory.

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