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## ISOMORPHISMS AND SPLITTING OF IDEMPOTENTS IN SEMICATEGORIES

by Lutz *SCHRÖDER*

**RESUME.** On démontre que les catégories libres engendrées par certains systèmes de générateurs et relations, appelés semicatégories, ne contiennent pas d'autres isomorphismes que ceux expressément spécifiés par les relations données. De plus, la condition que tout idempotent se scinde dans une catégorie peut être vérifiée dans une semicégorie qui engendre la catégorie.

In the construction of abstract categories (e.g. as algebraic theories or in sketches, cf. [6]), it is often natural to define just a few of the required compositions and leave the rest to a free completion process; however, this leads to a word problem which is quite intractable in the general case. As a formalization of this approach, composition graphs have been introduced in [9], and among these structures, (weak) semicategories have been characterized by the property that the 'naive' solution of the word problem is correct.

As shown below, categories generated by semicategories behave well in two other important respects: The generating process does not produce additional isomorphisms and, in a suitable sense, does not affect the property that idempotents split.

Categorical terminology generally follows [1], except that hom-classes  $hom(A, B)$  are not required to be sets.

### 1 Composition Graphs

To begin, some basic facts about composition graphs are briefly reviewed; details (not needed for the understanding of the present article) can be found in [9] or [8]. In this context, a *graph* is a class with

two unary operations  $d$  and  $c$ , called *domain* and *codomain*, such that  $c^2 = dc = c$  and  $d^2 = cd = d$ .

**Definition 1.1** A *composition graph* is a graph  $\mathbf{A}$  with a partial binary operation called *composition* and denoted by  $(f, g) \mapsto fg$ , subject to the condition that  $d(fg) = dg$ ,  $c(fg) = cf$ , and  $df = cg$  whenever  $fg$  is defined (these structures are slightly more general than C. EHRESMANN's multiplicative graphs, cf. [5]).

The nodes of  $\mathbf{A}$ , i.e. the elements of  $d[\mathbf{A}] = c[\mathbf{A}]$ , are called *objects* or *identities* (notation:  $\text{Ident}(\mathbf{A})$ ) and are denoted by capital letters  $A, B$  etc. or in the form  $id_A, id_B$  etc.; the elements of  $\mathbf{A}$  are called *morphisms*. A *functor* between composition graphs is a graph morphism  $F$  such that  $FfFg$  is defined and equal to  $F(fg)$  whenever  $fg$  is defined.

A composition graph is called *identitive* if the terms  $fdf$  and  $cff$  are equal to  $f$  whenever they are defined, and *strongly identitive* if, moreover, these terms are always defined. Note that each identitive composition graph  $\mathbf{A}$  can be made strongly identitive by defining additional compositions; the arising composition graph is called the *strongly identitive modification* of  $\mathbf{A}$ . A composite  $fg$  is called *pseudodefined* in  $\mathbf{A}$  if it is defined in this modification.

A *semicategory* is an identitive composition graph which is *associative* in the sense that, whenever  $fg$  and  $gh$  are defined, then  $f(gh)$  and  $(fg)h$  are defined and equal.

**Remark 1.2** Observe that strongly identitive semicategories are just categories. Moreover, strong identitivity is a coreflective property: calling a morphism  $f$  *strongly identitive* if  $fdf = f = cff$  are defined, the strongly identitive coreflection of a composition graph  $\mathbf{A}$  consists of those morphisms  $f$  in  $\mathbf{A}$  for which  $df, f$ , and  $cf$  are strongly identitive. Strongly identitive morphisms are stable under composition in semicategories. Thus, each semicategory  $\mathbf{A}$  has a 'CAT-coreflection', called the *kernel category* of  $\mathbf{A}$  and denoted by  $\mathbf{A}_*$ .

Each composition graph  $\mathbf{A}$  canonically generates a category: To begin, the *word category*  $\mathbf{W}(\mathbf{A})$  over  $\mathbf{A}$  is defined as the category with words of the form  $\mathbf{f} = (f_n, \dots, f_1)_A$  as morphisms, where  $n \geq 0$ ,  $df_{i+1} = cf_i$  in  $\mathbf{A}$  for  $i = 1, \dots, n-1$ , and  $df_1 = A$  (the index object  $A$  will mostly

be omitted). Composition is just concatenation of words; the empty words  $()_A$  serve as identities. The domain and codomain functions are defined as expected. The obvious length functor to the additive monoid of natural numbers is denoted by  $\lambda$ .

Now write  $\mathbf{f} \vdash \mathbf{g}$  if either  $\mathbf{f} = (f, g)$  and  $\mathbf{g} = (fg)$  for composable morphisms  $f$  and  $g$  in  $\mathbf{A}$ , or  $\mathbf{f} = (id_A)$  and  $\mathbf{g} = ()_A$  for some object  $A$  in  $\mathbf{A}$ ; moreover, let  $\mathbf{f} \vdash_c \mathbf{g}$  iff there exist words  $\mathbf{a}, \mathbf{b}, \mathbf{r}$  and  $\mathbf{l}$  such that  $\mathbf{a} \vdash \mathbf{b}$ ,  $\mathbf{f} = \mathbf{lar}$  and  $\mathbf{g} = \mathbf{lbr}$  (this is, in a suitable sense, the compatible hull of  $\vdash$ ). Let  $\succeq$  and  $\cong$  denote the preorder respectively the equivalence generated by  $\vdash_c$ ; then  $\succeq$  is a partial order, and  $\cong$  is the congruence on  $\mathbf{W}(\mathbf{A})$  generated by  $\vdash$ .

The free category  $\mathbf{A}^*$  over  $\mathbf{A}$  is now obtained as the quotient of  $\mathbf{W}(\mathbf{A})$  modulo  $\cong$ ; more precisely: The functor  $R_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}^*$  which maps  $f$  to the class of the word  $(f)$  is universal in the sense that each functor from  $\mathbf{A}$  into a category factors uniquely through  $R_{\mathbf{A}}$ . Although  $R_{\mathbf{A}}$  need not be injective, we will often write  $f$  instead of  $R_{\mathbf{A}}f$ . Of course, the formation of the strongly identitive modification of a composition graph does not affect the associated free category.

This construction (a similar version of which, valid for strongly identitive composition graphs, can already be found in [5]) is quite unpleasant in the general case; however, one has

**Theorem 1.3** *Let  $\mathbf{A}$  be a composition graph; equivalent are:*

- (i) *Each equivalence class of words in  $\mathbf{A}^*$  has a unique minimal representative w.r.t.  $\succeq$ .*
- (ii) *For each word  $\mathbf{f}$  in  $\mathbf{W}(\mathbf{A})$ , there exists a unique minimal word in the downset of  $\mathbf{f}$  (w.r.t.  $\succeq$ ).*
- (iii)  *$\mathbf{A}$  is identitive, and whenever  $fg$  and  $gh$  are defined, then  $fg = f$  and  $gh = h$ , or  $f(gh)$  and  $(fg)h$  are pseudodefined and equal.*

Note that condition (iii) is fulfilled in semicategories. Condition (i) implies in particular that  $R_{\mathbf{A}}$  is injective.

Moreover, the above theorem facilitates the construction of a free semicategory over a composition graph:

**Definition 1.4** A subobject of a composition graph  $\mathbf{A}$ , i.e. an injective functor  $\mathbf{B} \hookrightarrow \mathbf{A}$ , is called a *subgraph* or, if  $\mathbf{B}$  is a semicategory, a *subsemicategory*.

**Remark 1.5** The class of subsemicategories of a given composition graph is closed under intersections; hence, for any composition graph  $\mathbf{A}$ , there is a smallest subsemicategory  $\mathbf{A}^+$  of  $\mathbf{A}^*$  which contains the image of  $\mathbf{A}$  (with the composition induced by  $\mathbf{A}$ ).

$\mathbf{A}^+$ , respectively the corresponding codomain restriction of  $R_{\mathbf{A}}$ , has a universal property analogous to that of  $\mathbf{A}^*$ : If  $\mathbf{B}$  is a semicategory and  $F : \mathbf{A} \rightarrow \mathbf{B}$  is a functor, then the extension  $F^* : \mathbf{A}^* \rightarrow \mathbf{B}^*$  has a restriction  $\mathbf{A}^+ \rightarrow \mathbf{B}$ , since the preimage of  $\mathbf{B}$ , i.e. the pullback of  $R_{\mathbf{B}}$  along  $F^*$ , is a subsemicategory of  $\mathbf{A}^*$ .  $\mathbf{A}^+$  is called the *free semicategory* over  $\mathbf{A}$ .

This construction, in turn, leads to a generalization of the associativity law, the phrasing of which requires some terminology:

**Definition 1.6** The notion of a *composition term* in the variables  $X_n, \dots, X_1$  is defined inductively: A composition term in  $X$  is just the variable  $X$ , and a composition term in  $X_n, \dots, X_1$ ,  $n > 1$ , is any term of the form  $(t_2 t_1)$ , where  $t_1$  and  $t_2$  are composition terms in  $X_k, \dots, X_1$  respectively  $X_n, \dots, X_{k+1}$ ,  $1 \leq k < n$ . The empty term is regarded as being a composition term in 0 variables.

We will not distinguish notationally between a composition term  $t$  and its interpretation as a (partial) term function in a given composition graph  $\mathbf{A}$ . If  $\mathbf{f} = (f_n, \dots, f_1)$  is a word in  $\mathbf{W}(\mathbf{A})$ , then the value of the term function at  $(f_n, \dots, f_1)$  is denoted by  $t\mathbf{f}$  (if  $t$  is the empty term, then  $t()_{\mathbf{A}} = id_{\mathbf{A}}$ ).

**Definition 1.7** Let  $\mathbf{A}$  be a composition graph. A word  $\mathbf{f} = (f_n, \dots, f_1)$  in  $\mathbf{W}(\mathbf{A})$  is called *totally composable* (with result  $f$ ) if there exists a composition term  $t$  such that  $t\mathbf{f} = f$  is defined.

For totally composable words (or words that are equivalent to words of length 1), we will use informal manners of speech like ‘ $\mathbf{f}$  belongs to  $\mathbf{A}$ ’ and the like.

We are now ready to prove the generalized associativity law:

**Theorem 1.8** *Let  $\mathbf{A}$  be a semicategory, and let  $\mathbf{f} = \mathbf{f}_3\mathbf{f}_2\mathbf{f}_1$  be a word in  $\mathbf{W}(\mathbf{A})$ , where  $\mathbf{f}_2$  is nonempty, such that  $\mathbf{f}_3\mathbf{f}_2$  and  $\mathbf{f}_2\mathbf{f}_1$  are totally composable with results  $r$  and  $s$ , respectively. Then  $(r)\mathbf{f}_1$  and  $\mathbf{f}_3(s)$  (and  $\mathbf{f}$ ) are totally composable with the same result.*

PROOF: Let  $\mathbf{f} = (f_n, \dots, f_1)$  and  $\mathbf{f}_2 = (f_l, \dots, f_k)$ ; let  $t_1$  and  $t_2$  be composition terms in the variables  $X_l, \dots, X_1$  respectively  $X_n, \dots, X_k$  such that  $t_1(\mathbf{f}_2\mathbf{f}_1) = s$  and  $t_2(\mathbf{f}_3\mathbf{f}_2) = r$  are defined. Let  $\mathbf{B}$  be the free category over the graph

$$0 \xrightarrow{Y_1} 1 \xrightarrow{Y_2} \dots \xrightarrow{Y_{n-1}} n-1 \xrightarrow{Y_n} n$$

and let  $\mathbf{C}$  be the subgraph of  $\mathbf{B}$  with morphism set

$$\{Y_j \dots Y_i \mid t_1 \text{ or } t_2 \text{ contains a subterm in } X_j, \dots, X_i\}$$

and composition defined according to the term structure of  $t_1$  and  $t_2$ . By Remark 1.5, the functor  $F : \mathbf{C} \rightarrow \mathbf{A}$  which maps  $Y_i$  to  $f_i$ ,  $i = 1, \dots, n$ , extends to  $\mathbf{C}^+$ , where  $\mathbf{C}^+$  denotes the free semicategory over  $\mathbf{C}$ . Hence it suffices to show the statement for the word  $(Y_n, \dots, Y_1)$  in  $\mathbf{W}(\mathbf{C}^+)$ .

Now one has equivalent words

$$(Y_n \dots Y_k, Y_{k-1}, \dots, Y_1) \cong (Y_n, \dots, Y_1) \cong (Y_n, \dots, Y_{l+1}, Y_l \dots Y_1)$$

in  $\mathbf{W}(\mathbf{C}^+)$ ; by Theorem 1.3, there exists a lower bound  $\mathbf{h} = (h_m, \dots, h_1)$  (w.r.t.  $\succeq$ ) for these words. Since  $ch_1 \geq cY_l = l$  and  $dh_m \leq dY_k = k - 1 < l$ ,  $\mathbf{h}$  must have length 1; moreover, identities cannot appear in the reduction to  $\mathbf{h}$ . Thus, the above words are totally composable in  $\mathbf{C}^+$  with the same result.  $\square$

**Remark 1.9** In the above theorem, it is of course not sufficient to require that  $\mathbf{f}_3\mathbf{f}_2$  and  $\mathbf{f}_2\mathbf{f}_1$  be merely equivalent to words of length 1, as examples of the type  $\mathbf{f} = (f, id, g)$  show.

## 2 Isomorphisms

We will now show that the free category over a semicategory does not contain any new isomorphisms. The following lemma already suggests that the relationship between semicategories and isomorphisms is somewhat peculiar:

**Lemma 2.1** *Let  $\mathbf{A}$  be a semicategory, and let  $f$  be an isomorphism in  $\mathbf{A}$  (i.e. there exists  $g$  such that  $gf = df$  and  $fg = cf$  are defined). Then  $f$  belongs to the kernel category  $\mathbf{A}_*$  of  $\mathbf{A}$ .*

(The kernel category has been defined in Remark 1.2.) Note that this implies that all isomorphisms in  $\mathbf{A}$  are isomorphisms in  $\mathbf{A}_*$ .

PROOF: Since  $gf$  and  $fg$  are defined, so are  $fdf = f(gf)$  and  $cff = (fg)f$ ; similarly, it is seen that  $df$  and  $cf$  are strongly identitive.  $\square$

The main statement is now an easy corollary of Theorem 1.8:

**Theorem 2.2** *Let  $\mathbf{A}$  be a semicategory; then*

$$\text{Iso}(\mathbf{A}^*) = \text{Iso}(\mathbf{A}_*) \cup \text{Ident}(\mathbf{A}).$$

PROOF: Let  $\mathbf{f}$  and  $\mathbf{h}$  be nonempty minimal words in  $\mathbf{W}(\mathbf{A})$  such that  $\mathbf{fh} \cong ()$  and  $\mathbf{hf} \cong ()$  (i.e.  $\mathbf{f}$  ‘is’ an isomorphism, but not an identity, in  $\mathbf{A}^*$ ). We show by induction on the length of  $\mathbf{fh}$  that  $\mathbf{f}$  and  $\mathbf{h}$  ‘belong’ to  $\mathbf{A}$  and hence, by Lemma 2.1, to  $\mathbf{A}_*$ .

First assume that both  $\mathbf{hf}$  and  $\mathbf{fh}$  are totally composable. By Theorem 1.8, this implies that  $\mathbf{f} \preceq \mathbf{fhf}$  and  $\mathbf{h} \preceq \mathbf{hfh}$  are totally composable.

Otherwise, one of the reduction steps in, say,  $\mathbf{fh} \succeq ()$  consists in removing an identity; thus, one has proper subdivisions  $\mathbf{f} = \mathbf{f}_2\mathbf{f}_1$  and  $\mathbf{h} = \mathbf{h}_2\mathbf{h}_1$  such that  $\mathbf{f}_1\mathbf{h}_2 \cong ()$ . Then  $\mathbf{f}_1$  and  $\mathbf{h}_2$ , and hence also  $\mathbf{f}_2$  and  $\mathbf{h}_1$ , are mutually inverse isomorphisms in  $\mathbf{A}^*$  and hence belong to  $\mathbf{A}_*$  by the inductive assumption, in contradiction to the minimality of  $\mathbf{f}$  and  $\mathbf{h}$ .  $\square$

**Remark 2.3** The above theorem can be generalized by replacing semicategories with (strongly) identitive composition graphs  $\mathbf{A}$  that satisfy a weaker associativity law, namely that  $(fg)h$  and  $f(gh)$  are defined and equal whenever  $fg$  and  $gh$  are defined, provided that  $g$  is not an identity. Under this weaker condition, the class of isomorphisms in  $\mathbf{A}^*$  is the closure of the class of isomorphisms in  $\mathbf{A}$  under composition; Theorem 2.2 follows from this statement by Lemma 2.1. The proof uses the fact that the weaker associativity law admits a generalized version similar to Theorem 1.8, even though the free construction of Remark 1.5 does not carry over to this context.

A further generalization of the above theorem to composition graphs that satisfy the equivalent conditions of Theorem 1.3 can be invalidated by means of finite counterexamples.

### 3 Idempotents

Unlike isomorphisms, idempotents in the free category over a semicategory need not be immediately ‘visible’; however, the frequently required property that idempotents split (often referred to as Cauchy-completeness, in particular in the context of algebraic theories; cf. [2, 3, 4, 7]) is easily checked within the generating semicategory.

The following technical lemma formalizes the rather obvious insight that reductions of a concatenation of two minimal words can only occur at the boundary between the two components:

**Lemma 3.1** *Let  $\mathbf{A}$  be a composition graph; let  $k$  be a morphism in  $\mathbf{A}$ , and let  $\mathbf{f}$  and  $\mathbf{g}$  be non-empty minimal words in  $\mathbf{W}(\mathbf{A})$ . If  $\mathbf{f}(k)\mathbf{g} \succeq \mathbf{h}$ , where  $\mathbf{h}$  is nonempty, then there exists a morphism  $h$  in  $\mathbf{A}$  and subdivisions  $\mathbf{f} = \mathbf{f}_2\mathbf{f}_1$  and  $\mathbf{g} = \mathbf{g}_2\mathbf{g}_1$  such that*

$$\mathbf{h} = \mathbf{f}_2(h)\mathbf{g}_1 \quad \text{and} \quad \mathbf{f}_1(k)\mathbf{g}_2 \succeq (h)$$

(hence  $\lambda\mathbf{f}_1 + \lambda\mathbf{g}_2 = \lambda(\mathbf{f}(k)\mathbf{g}) - \lambda\mathbf{h}$ ).

PROOF: If  $\mathbf{f}(k)\mathbf{g} \vdash_c \mathbf{l}$ , then by minimality of  $\mathbf{f}$  and  $\mathbf{g}$ ,

$$\mathbf{l} = (f_n, \dots, f_2)(f_1k)\mathbf{g} \quad \text{or} \quad \mathbf{l} = \mathbf{f}(kg_m)(g_{m-1}, \dots, g_1),$$

where  $\mathbf{f} = (f_n, \dots, f_1)$ ,  $\mathbf{g} = (g_m, \dots, g_1)$ , and  $f_1k$  respectively  $kg_m$  are pseudodefined in  $\mathbf{A}$ . The claim follows from this observation by induction on the length of  $\mathbf{f}(k)\mathbf{g}$ .  $\square$

Furthermore, we need the following cancellability property:

**Lemma 3.2** *Let  $\mathbf{A}$  be a semicategory. If, for words  $\mathbf{f}$ ,  $\mathbf{g}$ , and  $\mathbf{h}$  in  $\mathbf{W}(\mathbf{A})$ ,  $\mathbf{f}\mathbf{g}$  is totally composable and  $\mathbf{h}\mathbf{f}$  is equivalent to an identity, then  $\mathbf{g}$  is totally composable.*



PROOF: Assume w.l.o.g. that  $\mathbf{f}$  and  $\mathbf{h}$  are minimal. We proceed by induction on the length of  $\mathbf{hf}$ .

Choose subdivisions  $\mathbf{h} = \mathbf{h}_2\mathbf{h}_1$  and  $\mathbf{f} = \mathbf{f}_2\mathbf{f}_1$ , where  $\mathbf{h}_1$  is nonempty and  $\mathbf{h}_1\mathbf{f}_2 \cong ()$ , with minimal  $\lambda\mathbf{h}_1$ . Then  $\mathbf{f}_2$  is nonempty, and  $\mathbf{h}_1\mathbf{f}_2$  is totally composable; hence,  $\mathbf{f}_1\mathbf{g} \preceq \mathbf{h}_1\mathbf{f}_2\mathbf{f}_1\mathbf{g}$  is totally composable by Theorem 1.8. Since  $\mathbf{h}_2\mathbf{f}_1 \cong \mathbf{h}_2\mathbf{h}_1\mathbf{f}_2\mathbf{f}_1 \cong ()$ , the inductive assumption, applied to  $\mathbf{h}_2$  and  $\mathbf{f}_1$ , yields that  $\mathbf{g}$  is totally composable.  $\square$

The envisaged result is now easily proved:

**Theorem 3.3** *Let  $\mathbf{A}$  be a semicategory. Equivalent are*

- (i) *Idempotents in  $\mathbf{A}^*$  split.*
- (ii) *Idempotents in  $\mathbf{A}$  split in  $\mathbf{A}^*$  (i.e. whenever  $hh = h$  is defined in  $\mathbf{A}$ , then there exist words  $\mathbf{p}$  and  $\mathbf{r}$  such that  $\mathbf{pr} \cong (h)$  and  $\mathbf{rp} \cong ()$ ).*
- (iii) *Idempotents in  $\mathbf{A}$  split (i.e. whenever  $hh = h$  is defined in  $\mathbf{A}$ , then there exist  $p$  and  $r$  such that  $pr = h$  and  $rp = id$  are defined in  $\mathbf{A}$ ).*

PROOF: (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (i): Let  $\mathbf{f} = (f_n, \dots, f_1)$  be a minimal word in  $\mathbf{W}(\mathbf{A})$  such that  $\mathbf{ff} \cong \mathbf{f}$ ; we have to show that there exist  $\mathbf{p}$  and  $\mathbf{r}$  such that  $\mathbf{f} \cong \mathbf{pr}$  and  $\mathbf{rp} \cong ()$ . By Theorem 1.3,  $\mathbf{ff} \succeq \mathbf{f}$ ; assuming w.l.o.g. that  $\mathbf{f}$  is nonempty, we have  $\mathbf{h}$  such that  $\mathbf{ff} \succeq \mathbf{h} \vdash_c \mathbf{f}$ . Applying the above lemma to  $\mathbf{f}(id)\mathbf{f} \succeq \mathbf{h}$  (note that  $\lambda(\mathbf{f}(id)\mathbf{f}) - \lambda\mathbf{h} = n$ ), we obtain a morphism  $h$  in  $\mathbf{A}$  and a subdivision  $\mathbf{f} = \mathbf{f}_2\mathbf{f}_1 = (f_n, \dots, f_1)(f_{t-1}, \dots, f_1)$  such that

$$\mathbf{h} = \mathbf{f}_2(h)\mathbf{f}_1$$

and

$$\mathbf{f}_1(id)\mathbf{f}_2 \succeq (h);$$

furthermore, we can assume w.l.o.g. that  $f_i h = f_i$  is pseudodefined in  $\mathbf{A}$ , and hence  $\mathbf{f}_2(h) \cong \mathbf{f}_2$ . Thus

$$(h, h) \cong \mathbf{f}_1\mathbf{f}_2(h) \cong \mathbf{f}_1\mathbf{f}_2 \cong (h);$$

by Theorem 1.3, this implies that  $hh = h$  is pseudodefined in  $\mathbf{A}$ . By (ii) (or trivially, if  $h$  is an identity), we have words  $\mathbf{p}$  and  $\mathbf{r}$  such that  $\mathbf{pr} \cong (h)$  and  $\mathbf{rp} \cong ()$ . Now

$$(\mathbf{f}_2\mathbf{p})(\mathbf{rf}_1) \cong \mathbf{f}_2(h)\mathbf{f}_1 \cong \mathbf{f}_2\mathbf{f}_1 \cong \mathbf{f}$$

and

$$(\mathbf{rf}_1)(\mathbf{f}_2\mathbf{p}) \cong \mathbf{r}(h)\mathbf{p} \cong \mathbf{rprp} \cong (),$$

i.e. we have the required splitting.

(iii) $\Rightarrow$ (ii): Trivial

(ii) $\Rightarrow$ (iii): Let minimal words  $\mathbf{r}$  and  $\mathbf{p}$  form a splitting of  $h$  as in the statement; we show by induction on the length of  $\mathbf{pr}$  that  $h$  splits in  $\mathbf{A}$ .

If  $\mathbf{pr}$  is totally composable, then, by Lemma 3.2 and its dual, so are  $\mathbf{r}$  and  $\mathbf{p}$ ; i.e. these words already provide the required splitting of  $h$  in  $\mathbf{A}$ .

Otherwise, there exist subdivisions  $\mathbf{p} = \mathbf{p}_2\mathbf{p}_1$  and  $\mathbf{r} = \mathbf{r}_2\mathbf{r}_1$  such that  $\mathbf{p}_1\mathbf{r}_2$  is nonempty and equivalent to an identity. Then  $\mathbf{p}_1$  and  $\mathbf{r}_2$  are mutually inverse isomorphisms in  $\mathbf{A}^*$ ; hence,  $\mathbf{p}_2$  and  $\mathbf{r}_1$  form a further splitting of  $h$ , so that  $h$  splits in  $\mathbf{A}$  by the inductive hypothesis.  $\square$

**Remark 3.4** The above theorem can be generalized by replacing semi-categories with (strongly) identitive composition graphs that satisfy the weakened associativity law formulated in Remark 2.3; the proof relies on a slightly modified version of Lemma 3.2. Furthermore, it is easily checked that the proof of the equivalence of conditions (i) and (ii) only uses the fact that  $\mathbf{A}$  satisfies the equivalent conditions of Theorem 1.3.

However, finite counterexamples show that, under the latter conditions, condition (ii) above does not in general imply condition (iii).

**Remark 3.5** It is often reasonable to introduce equations between words of arbitrary length into the description of a category (cf. [10]). However, a single equation of the type  $fgf = f$ , say, gives rise to an idempotent in the free category (namely,  $fg$ ) which is ‘invisible’ from the point of view of the generating system; it is easy to see that a generalization of the above theorem to such structures can be invalidated in this way.

We conclude with an example which shows that idempotents in the free category generated by a semicategory  $\mathbf{A}$  need not belong to  $\mathbf{A}$ .

**Example 3.6** Let  $\mathbf{A}$  be the composition graph with one object and morphisms  $f, g, h, p, r$ , and  $a$ , where composition is given by

$$\begin{array}{ll} gf = h & rh = r \\ fh = f & hp = p \\ hh = h & fp = a \\ pr = h & ar = f \\ rp = id & a id = a \\ pid = p & ga = p \\ idr = r & idid = id. \end{array}$$

A moderately tedious verification shows that  $\mathbf{A}$  is indeed a semicategory. The only nontrivial idempotent in  $\mathbf{A}$  splits as  $h = pr$ ; however,  $fg$  is idempotent in  $\mathbf{A}^*$ , since  $fgfg = fhg = fg$ , but does not belong to  $\mathbf{A}$ . (Of course,  $fg$  splits:  $(fp)(rg) = fhg = fg$  and  $(rg)(fp) = rhp = rp = id$ .)

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