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LOCAL ENRICHMENTS OF CATEGORIES

by A. PULTR and W. THOLEN *

Résumé: Une classe \mathcal{F} d'objets d'une catégorie générale \mathcal{C} (que nous pouvons regarder comme le système des objets finis de \mathcal{C}) induit naturellement des topologies Hausdorff sur les hom-ensembles (A, B) . De cette façon, \mathcal{C} devient une **Haus**-catégorie. De plus, on a une **Loc**-catégorie $\tilde{\mathcal{C}}$ naturellement associée dont \mathcal{C} est le spectre; dans $\tilde{\mathcal{C}}$, un cadre $\tilde{\mathcal{C}}(A, B)$ peut être non-trivial également lorsque $\mathcal{C}(A, B) = \emptyset$.

Our aim is to show that the hom-sets $\mathcal{C}(A, B)$ of an arbitrary category \mathcal{C} with a given class \mathcal{F} of subobjects – we can think of them as of finite subobjects in \mathcal{C} – carry a zero-dimensional Hausdorff topology in which they are the spectra of naturally defined locales $\tilde{\mathcal{C}}(A, B)$. In fact, there is a **Loc**-category $\tilde{\mathcal{C}}$ for the category **Loc** of locales with its (non-closed) cartesian structure, whose class of objects is that of \mathcal{C} , and whose composition law is mapped to the composition in \mathcal{C} by the spectrum functor. Thus the composition in \mathcal{C} is continuous with respect to the constructed topologies; in other words, \mathcal{C} is a **Haus**-category, for the category **Haus** of Hausdorff spaces with the product structure. We note that $\tilde{\mathcal{C}}(A, B)$ may be non-trivial even if $\mathcal{C}(A, B) = \emptyset$.

The inspiration for this work came from Wraith's paper [13] on localic groups (the same idea was used in [4] for the construction of arbitrarily large localic groups without non-trivial points). There are close connections with forcing and classifying topos theory, as well as with theoretical computer science. In fact, the locale $\tilde{\mathcal{C}}(A, B)$ may be described in terms of a Grothendieck topology on (the opposite of)

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the preordered set of finite partial maps from A to B , and we note that the condition **Fin 2** in 1.2 below is related to Plotkin's physical feasibility thesis and the approximation by compact elements – see [10], [11]. However, in this paper we are trying to keep the exposition as elementary as possible, thus economizing on the categorical and localic techniques used. At the same time, the assumptions on the category \mathcal{C} and its finite-subject structure \mathcal{F} are kept at a minimal level. In particular, essentially no completeness or cocompleteness assumptions on \mathcal{C} are needed. This may primarily seem like a point of esthetics, but it turns out that it increases the range of relevant examples. For instance, any locally finitely presentable category \mathcal{C} with \mathcal{F} the finitely generated subobjects satisfies the axioms used in this paper (see 1.2, 3.1). Furthermore, we use the (non-complete) category of antireflexive posets to obtain fairly easy examples of non-spatial locales (see 3.2).

As usual, the category **Loc** of locales is the dual of the category **Frm** of frames (see 1.5; for more information about frames, see [5] or [12]). Ordinary categorical notions can be found in [9], and those concerning enriched categories in [1] and [8].

1. The localic structure of hom-sets

1.1. Throughout this paper we consider a category \mathcal{C} which comes equipped with a class \mathcal{F} of monomorphisms in \mathcal{C} . For every object A in \mathcal{C} , \mathcal{F}/A is the pre-ordered class of morphisms in \mathcal{F} with codomain A , to which we refer as the *finite subobjects* of A .

We call $m : F \rightarrow A$ in \mathcal{F} a *union* of a family $(m_i : F_i \rightarrow A)_{i \in I}$ in \mathcal{F} if m is a least upper bound of $(m_i)_{i \in I}$ in \mathcal{F}/A , and if the family $(j_i : F_i \rightarrow F)_{i \in I}$ with $m j_i = m_i$ ($i \in I$) is \mathcal{C} -*epic*, in the sense that $u j_i = v j_i$ ($i \in I$) implies $u = v$ for any pair of morphisms $u, v : F \rightarrow B$ in \mathcal{C} ; we write

$$m \cong \bigvee_{i \in I} m_i$$

in this case. The union of the empty family in \mathcal{F}/A is denoted by $0_A \rightarrow A$. Note that any hom-set $\mathcal{C}(0_A, B)$ contains at most one morphism.

By $|\mathcal{F}|$ we denote the class of objects which are domains of morphisms in \mathcal{F} . They are referred to as the *finite* objects in \mathcal{C} .

1.2. In what follows we shall consider the following conditions on \mathcal{C} and \mathcal{F} :

- Fin 1** (*Finite unions of finite subobjects exist and are finite*) For $(m_i : F_i \rightarrow A)_{i \in I}$ in \mathcal{F} with I finite, $\bigvee_{i \in I} m_i$ exists in \mathcal{F}/A . In addition, for the bottom element $0_A \rightarrow A$ in \mathcal{F}/A one has $\mathcal{C}(0_A, 0_B) \neq \emptyset$ whenever $\mathcal{C}(0_A, B) \neq \emptyset$.
- Fin 2** (*Morphisms can be defined by defining their finite restrictions*) For every $A \in \mathcal{C}$, the canonical cocone with vertex A over the domain functor $\mathcal{F}/A \rightarrow \mathcal{C}$ is a colimit in \mathcal{C} .
- Fin 3** (*Morphisms with finite domains have finite images*) Every \mathcal{C} -morphism $f : F \rightarrow A$ with $F \in |\mathcal{F}|$ can be factored as $f = me$ with $m \in \mathcal{F}$.

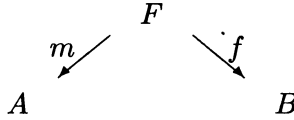
Note that the conditions **Fin 2** and **Fin 3** trivially hold when every identity morphism in \mathcal{C} belongs to \mathcal{F} (so that every object in \mathcal{C} is finite). In this case, however, our constructions below are of little interest (see 1.9 below). Note further that the additional requirement on 0_A in **Fin 1** and condition **Fin 3** will not be used in Section 1 in order to set up the localic structure.

1.3. In many of our examples, \mathcal{C} comes equipped with a proper $(\mathcal{E}, \mathcal{M})$ -factorization system for morphisms (see [2]) and \mathcal{F} is chosen as the class of all \mathcal{M} -morphisms with domain in a given iso-closed class $|\mathcal{F}|$ of objects in \mathcal{C} . If \mathcal{C} has finite coproducts then finite unions exist in \mathcal{M}/A , and conditions **Fin 1** – **Fin 3** can be equivalently described in more familiar terms:

- Fin 1'** For every $A \in \mathcal{C}$, \mathcal{F}/A is closed under finite unions in \mathcal{M}/A .
- Fin 2'** $|\mathcal{F}|$, as a full subcategory of \mathcal{C} , is dense in \mathcal{C} .
- Fin 3'** For every \mathcal{E} -morphism $e : G \rightarrow G$, $F \in |\mathcal{F}|$ implies $G \in |\mathcal{F}|$.

In the presence of **Fin 3'** note that **Fin 1'** holds in particular if $|\mathcal{F}|$ is closed under finite coproducts in \mathcal{C} .

1.4. For \mathcal{C} objects A and B , a pair (m, f) of \mathcal{C} -morphisms with common domain and $m \in \mathcal{F}$ is called a *finite partial morphism* from A to B ; m is its *domain*.

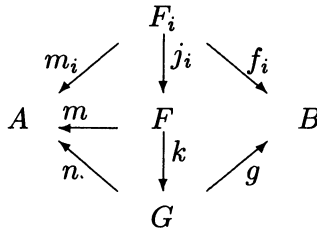


The set $\widehat{\mathcal{C}}(A, B)$ of all finite partial morphisms from A to B is pre-ordered by

$$(m, f) \leq (n, g) \quad \text{iff} \quad (\exists s)(ns = m \text{ and } gs = f).$$

Proposition: *Under condition **Fin 1**, every finite family in the set $\widehat{\mathcal{C}}(A, B)$ with an upper bound in $\widehat{\mathcal{C}}(A, B)$ has a least upper bound in $\widehat{\mathcal{C}}(A, B)$.*

PROOF: For the empty family in $\widehat{\mathcal{C}}(A, B)$ to have an upper bound means that $\widehat{\mathcal{C}}(A, B)$ is not empty. But then there is a (unique) morphism $0_A \rightarrow B$. $(0_A \rightarrow A, 0_A \rightarrow B)$ serves as a bottom element in $\widehat{\mathcal{C}}(A, B)$, i.e. as the least upper bound of the empty family. If two finite partial morphisms $(m_1, f_1), (m_2, f_2)$ have a common upper bound (n, g) with $m := m_1 \vee m_2$ one obtains commutative diagrams



Since (j_1, j_2) is epic, the morphism $f := gk$ does not depend on the choice of the upper bound (n, g) . It is then easy to see that (m, f) is a least upper bound of $(m_1, f_1), (m_2, f_2)$ in $\widehat{\mathcal{C}}(A, B)$; we write

$$(m, f) \cong (m_1, f_1) \vee (m_2, f_2). \quad \square$$

1.5. Recall that a *frame* is a complete lattice which satisfies the generalized distributive law

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i),$$

and that a *frame homomorphism* is a mapping preserving arbitrary joins (including the bottom 0) and finite meets (including the top 1).

The *free frame* $U\widehat{\mathcal{C}}(A, B)$ over $\widehat{\mathcal{C}}(A, B)^{\text{op}}$ (i.e. over $\widehat{\mathcal{C}}(A, B)$ provided with the order opposite to 1.4) can be defined as the set of all up-sets (= upwards closed sets) of $\widehat{\mathcal{C}}(A, B)$ ordered by inclusion. The canonical map

$$\uparrow : \widehat{\mathcal{C}}(A, B) \rightarrow U\widehat{\mathcal{C}}(A, B)$$

is universal with respect to the property that (the existing) finite joins are transformed into meets; any other such map into a frame L factors uniquely through a frame homomorphism $U\widehat{\mathcal{C}}(A, B) \rightarrow L$.

1.6. For a given pair of objects A, B in \mathcal{C} we want to describe the property that for ever finite subobject $m : F \rightarrow A$ there are “sufficiently many” partial morphisms from A to B with domain m . One way of expressing this directly would be the requirement that, for all $m : F \rightarrow A$,

$$\bigwedge \{(m, f) \mid f \in \mathcal{C}(F, B)\} \cong 0$$

holds in $\widehat{\mathcal{C}}(A, B)$ with 0 the bottom element of $\widehat{\mathcal{C}}(A, B) \neq \emptyset$ as constructed in 1.4. However, when thinking of $\widehat{\mathcal{C}}(A, B)$ as being embedded into $U\widehat{\mathcal{C}}(A, B)$ by the order reversing map \uparrow , this requirement would translate into

$$\bigvee \{\uparrow(m, f) \mid f \in \mathcal{C}(F, B)\} = 1,$$

with \bigvee given by the set-theoretic union and $1 = \widehat{\mathcal{C}}(A, B)$ the top element in $U\widehat{\mathcal{C}}(A, B)$. It turns out that we do not need to impose this identity as a condition on the initial data. It suffices to consider the appropriate quotient frame of $U\widehat{\mathcal{C}}(A, B)$ in which the desired identity holds.

Hence we consider the frame congruence \sim on $U\widehat{\mathcal{C}}(A, B)$ generated by

$$\bigvee \{\uparrow(m, f) \mid f \in \mathcal{C}(A, B)\} \sim 1 \quad \text{for all } (m : F \rightarrow A) \in \mathcal{F}.$$

Hence in the quotient frame

$$\widetilde{\mathcal{C}}(A, B) = U\widehat{\mathcal{C}}(A, B) / \sim$$

the identity

$$(*) \quad \bigvee \{[m, f] \mid f \in \mathcal{C}(F, B)\} = 1$$

holds for all m , with $[m, f]$ denoting the \sim -class of $\uparrow(m, f)$.

For any frame L , the maps $\varphi : \widehat{\mathcal{C}}(A, B) \rightarrow L$ which factor uniquely through the canonical map

$$\widehat{\mathcal{C}}(A, B) \rightarrow \widetilde{\mathcal{C}}(A, B), \quad (m, f) \mapsto [m, f]$$

by a frame homomorphism $\xi : \widetilde{\mathcal{C}}(A, B) \rightarrow L$, are exactly the maps φ which transform (the existing) finite joins into meets and satisfy the condition

$$(**) \quad \bigvee \{\varphi(m, f) \mid f \in \mathcal{C}(F, B)\} = 1$$

for every $m : F \rightarrow A$ in \mathcal{F} . The explicit description of ξ in terms of φ is

$$\xi([u]) = \bigvee \{\varphi(m, f) \mid (m, f) \in u\}$$

for every $u \in U\widehat{\mathcal{C}}(A, B)$.

1.7. For a frame L let ΣL be its *spectrum*, that is, the set of all frame homomorphisms $L \rightarrow 2 = \{0 < 1\}$. We want to determine $\Sigma\widetilde{\mathcal{C}}(A, B)$. First we observe that every (total) morphism $t : A \rightarrow B$ in \mathcal{C} defines a homomorphism $\xi_t : \widetilde{\mathcal{C}}(A, B) \rightarrow 2$ with

$$\xi_t([m, f]) = 1 \quad \text{iff} \quad tm = f.$$

In fact, the condition

$$\varphi_t(m, f) = 1 \quad \text{iff} \quad tm = f$$

defines a map $\varphi_t : \widehat{\mathcal{C}}(A, B) \rightarrow 2$ which maps (the existing) finite joins to meets: for the bottom element $(0_A \rightarrow A, 0_A \rightarrow B)$ in $\widehat{\mathcal{C}}(A, B) \neq \emptyset$ one has $(0_A \rightarrow B) = t(0_A \rightarrow A)$, hence φ_t maps it to 1; furthermore, if $(m_1, f_1), (m_2, f_2)$ have a common upper bound, the construction of their join (m, f) in 1.4 shows that

$$\begin{aligned} \varphi_t(m_1, f_1) = \varphi_t(m_2, f_2) = 1 & \quad \text{iff} \quad tm_1 = f_1 \text{ and } tm_2 = f_2 \\ & \quad \text{iff} \quad tm = f \\ & \quad \text{iff} \quad \varphi_t(m, f) = 1. \end{aligned}$$

Finally, φ_t obviously satisfies (**) of 1.6 since

$$\bigvee \{ \varphi_t(m, f) \mid f \in \mathcal{C}(F, B) \} \geq \varphi_t(m, tm) = 1,$$

hence the existence of ξ_t is shown.

Now one can easily prove:

Proposition: *Under conditions **Fin 1** and **Fin 2** the assignment $t \mapsto \xi_t$ defines a bijection*

$$\Phi_{A,B} : \mathcal{C}(A, B) \rightarrow \Sigma \widetilde{\mathcal{C}}(A, B).$$

PROOF: We must show that every frame homomorphism $\xi : \widetilde{\mathcal{C}}(A, B) \rightarrow 2$ can be written as $\xi = \xi_t$ for a unique t . For every $m : F \rightarrow A$ in \mathcal{F} , since one has

$$\bigvee \{ \xi([m, f]) \mid f \in \mathcal{C}(F, B) \} = 1$$

by (**) of 1.6, there is a morphism $f_m : F \rightarrow B$ with $\xi([m, f_m]) = 1$. Actually, f_m is uniquely determined since the implications

$$[m, f] \wedge [m, g] \neq 0 \quad \Rightarrow \quad \uparrow(m, f) \cap \uparrow(m, g) \neq \emptyset \quad \Rightarrow \quad f = g$$

hold. The f_m 's define a cocone: if $n \leq m$ in \mathcal{F}/A one has $ms = n$ for some s , hence $f_ms = f_n$ by the same argument as given before. Now the colimit property **Fin 2** yields a uniquely determined morphism $t : A \rightarrow B$ with $tm = f_m$ for all $m \in \mathcal{F}/A$; equivalently, the latter condition means $\xi = \xi_t$. \square

1.8. The spectrum ΣL of any frame carries a sober topology given by the open sets

$$V(a) = \{\xi \in \Sigma L \mid \xi(a) = 1\}, \quad a \in L.$$

For $L = \tilde{\mathcal{C}}(A, B)$ the sets $V([m, f])$ with $(m, f) \in \widehat{\mathcal{C}}(A, B)$ constitute a basis of this topology. Indeed, for every $u \in U\widehat{\mathcal{C}}(A, B)$,

$$V([u]) = \bigcup_{(m, f) \in u} V([m, f]).$$

Furthermore, if $(m_1, f_1), (m_2, f_2)$ have a common upper bound in $\widehat{\mathcal{C}}(A, B)$ then

$$V([m_1, f_1]) \cap V([m_2, f_2]) = V([m, f])$$

with $(m, f) = (m_1, f_1) \vee (m_2, f_2)$; otherwise the intersection is empty, since if there is a $\xi \in \Sigma\tilde{\mathcal{C}}(A, B)$ with $\xi([m_1, f_1]) = 1 = \xi([m_2, f_2])$, we must have $[m_1, f_1] \wedge [m_2, f_2] \neq 0$, hence $\uparrow(m_1, f_1) \cap \uparrow(m_2, f_2) \neq \emptyset$.

Corollary: *Under conditions **Fin 1** and **Fin 2** each hom-set $\mathcal{C}(A, B)$ carries a zero-dimensional Hausdorff topology with basic open sets*

$$W(m, f) = \{t \in \mathcal{C}(A, B) \mid tm = f\}, \quad (m, f) \in \widehat{\mathcal{C}}(A, B).$$

This topology makes $\Phi_{A, B}$ of 1.7 a homeomorphism.

PROOF: Obviously, since $W(m, f) = \Phi_{A, B}^{-1}(V[m, f])$, these sets form a basis of a topology of $\mathcal{C}(A, B)$ which makes $\Phi_{A, B} : \mathcal{C}(A, B) \rightarrow \Sigma\tilde{\mathcal{C}}(A, B)$ a homeomorphism. For $t_1 \neq t_2$ in $\mathcal{C}(A, B)$ one obtains from **Fin 2** that $m \in \mathcal{F}/A$ with $t_1 m \neq t_2 m$; trivially, the neighbourhoods

$W(m, t_i m)$ of the t_i are disjoint. Hence we have a T_2 -topology. Finally, each basic open set $W(m, f)$ is closed since, for $t \notin W(m, f)$, $W(m, tm)$ is a neighbourhood of t disjoint from $W(m, f)$. \square

1.9. We may summarize the previous construction by the following commutative diagram

$$\begin{array}{ccc}
 \tilde{\mathcal{C}}(A, B) & \xrightarrow{V} & \Omega\Sigma\tilde{\mathcal{C}}(A, B) \\
 \uparrow & & \uparrow \\
 \hat{\mathcal{C}}(A, B) & \xrightarrow{W} & \Omega\mathcal{C}(A, B)
 \end{array}$$

where the left vertical arrow is the natural map and the right vertical arrow is the isomorphism induced by $\Phi_{A,B}$, and ΩX is the frame of open sets of a topological space X .

Recall that a frame L is *spatial* if the map $V : L \rightarrow \Omega\Sigma L$ is an isomorphism. Hence we have:

Corollary: *If $\tilde{\mathcal{C}}(A, B)$ is spatial, then it is isomorphic to the frame of open sets in $\mathcal{C}(A, B)$. If, in addition, there is a \mathcal{C} -epimorphism in \mathcal{F}/A , then $\tilde{\mathcal{C}}(A, B)$ is isomorphic to the power set of $\mathcal{C}(A, B)$.*

PROOF: We must prove the second statement. In fact, if there is a \mathcal{C} -epimorphism $m : F \rightarrow A$ in \mathcal{F} then for every $t \in \mathcal{C}(A, B)$, $\{t\} = W(m, tm)$.

The assumption of spatiality in this Corollary is essential as we shall see in 3.3 below.

2. The localic composition rule

2.1. The coproduct of two frames L and K will be denoted by $L \otimes K$. Recall that it is generated by elements of the form

$$x \otimes y = i(x) \wedge j(y) \quad (x \in L, y \in K)$$

with i and j the canonical injections. Since i and j preserve finite meets and arbitrary joins, one can write

$$i(x) = x \otimes 1 \quad \text{and} \quad j(y) = 1 \otimes y$$

and one has the rules

$$\begin{aligned} x \otimes \left(\bigvee_{i \in I} y_i \right) &= \bigvee_{i \in I} (x \otimes y_i), & \left(\bigvee_{i \in I} x_i \right) \otimes y &= \bigvee_{i \in I} (x_i \otimes y), \\ (x_1 \otimes y_1) \wedge (x_2 \otimes y_2) &= (x_1 \wedge x_2) \otimes (y_1 \wedge y_2), \\ x \otimes 0 &= 0 \otimes y = 0, & 1 \otimes 1 &= 1. \end{aligned}$$

One also concludes that if $L = \{0 = 1\}$ is the one-element frame, then $L \otimes K \cong L \cong K \otimes L$ for every frame K .

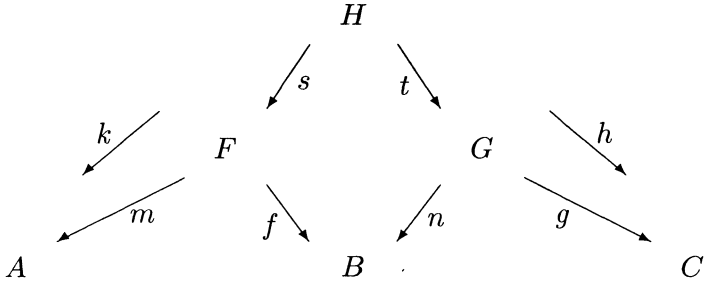
2.2. For finite partial morphisms $(m, f) \in \widehat{\mathcal{C}}(A, B)$, $(n, g) \in \widehat{\mathcal{C}}(B, C)$ and $(k, h) \in \widehat{\mathcal{C}}(A, C)$ we write

$$(k, h) \leq (n, g)(m, f)$$

if there are morphisms s and t such that

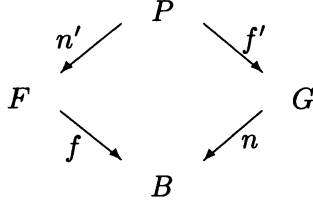
$$ms = k, \quad fs = nt, \quad gt = h,$$

as visualised by the following diagram



This way we have described what it means that the partial morphism (k, h) is a restriction of the composite $(n, g)(m, f)$ without actually

having defined the composite $(n, g)(m, f)$. However, if the pullback



exists in \mathcal{C} , with $n' \in \mathcal{F}$ and $mn' \in \mathcal{F}$, so that the composite can be defined as usual by

$$(n, g)(m, f) = (mn', gf'),$$

the \leq -relation introduced above is the order \leq of the preordered class $\widehat{\mathcal{C}}(A, B)$ of 1.4.

2.3. Proposition: *Under the conditions **Fin 1** and **Fin 2**, there is a frame homomorphism*

$$\mu_{ABC} : \widetilde{\mathcal{C}}(A, C) \rightarrow \widetilde{\mathcal{C}}(B, C) \otimes \widetilde{\mathcal{C}}(A, B)$$

with $\mu_{ABC}([k, h]) = \bigvee \{[n, g] \otimes [m, f] \mid (k, h) \leq (n, g)(m, f)\}$ for all $(k, h) \in \widehat{\mathcal{C}}(A, C)$.

PROOF: According to 1.6 it suffices to show that the map

$$\varphi_{ABC} : \widehat{\mathcal{C}}(A, C) \rightarrow \widetilde{\mathcal{C}}(B, C) \otimes \widetilde{\mathcal{C}}(A, B)$$

with φ_{ABC} the above join has the property that it transforms all existing finite joins into meets, and that

$$(**) \quad \bigvee \{\varphi_{ABC}(m, h) \mid h \in \mathcal{C}(F, C)\} = 1$$

holds for all $m : F \rightarrow A$ in \mathcal{F} .

First we point out that φ_{ABC} is order reversing. Next, for $\widehat{\mathcal{C}}(A, C) \neq \emptyset$ we must show that

$$\varphi_{ABC}(0_A \rightarrow A, 0_A \rightarrow C) = 1.$$

This trivially holds in the case $\widehat{\mathcal{C}}(A, B) = \emptyset$ or $\widehat{\mathcal{C}}(B, C) = \emptyset$ since then $\widetilde{\mathcal{C}}(A, B)$ or $\widetilde{\mathcal{C}}(B, C)$ and therefore $\widetilde{\mathcal{C}}(A, B) \otimes \widetilde{\mathcal{C}}(B, C)$ are one-element frames. Hence we may assume $\widehat{\mathcal{C}}(A, B) \neq \emptyset \neq \widehat{\mathcal{C}}(B, C)$ and have

$$\begin{aligned} \varphi_{ABC}(0_A \rightarrow A, 0_A \rightarrow C) &= \\ &= \bigvee \{ [n, g] \otimes [m, f] \mid (0_A \rightarrow A, 0_A \rightarrow C) \leq (n, g)(m, f) \} \geq \\ &\geq [0_B \rightarrow B, 0_B \rightarrow C] \otimes [0_A \rightarrow A, 0_A \rightarrow B] = 1 \otimes 1 = 1, \end{aligned}$$

since $(0_A \rightarrow A, 0_A \rightarrow C) \leq (0_B \rightarrow B, 0_B \rightarrow C)(0_A \rightarrow A, 0_A \rightarrow B)$ by **Fin 1**.

Let now $(k, h) \cong (k_1, h_1) \vee (k_2, h_2)$ hold in $\widehat{\mathcal{C}}(A, C)$. Since φ_{ABC} is order-reversing we have

$$\varphi_{ABC}(k, h) \leq \varphi_{ABC}(k_1, h_1) \wedge \varphi_{ABC}(k_2, h_2).$$

Using the distributive law and 2.1 we can present $\varphi_{ABC}(k_1, h_1) \wedge \varphi_{ABC}(k_2, h_2)$ as the join of all elements

$$([n_1, g_1] \wedge [n_2, g_2]) \otimes ([m_1, f_1] \wedge [m_2, f_2])$$

with $(k_i, h_i) \leq (n_i, g_i)(m_i, f_i)$, $i = 1, 2$. These “products” become 0 whenever (m_1, f_1) , (m_2, f_2) or (n_1, g_1) , (n_2, g_2) does not have an upper bound in $\widehat{\mathcal{C}}(A, B)$ resp. $\widehat{\mathcal{C}}(B, C)$ and hence they do not contribute to the join. In all other cases we have

$$(k_i, h_i) \leq (n, g)(m, f), \quad i = 1, 2,$$

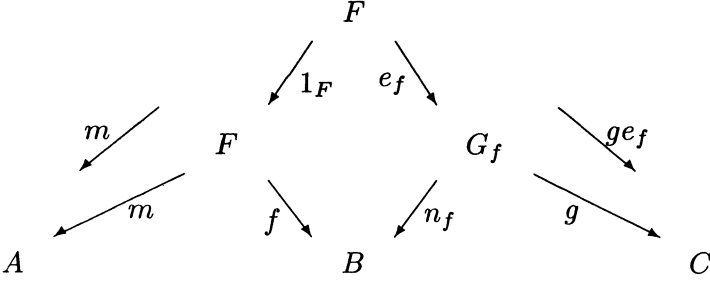
with $(m, f) \cong (m_1, f_1) \vee (m_2, f_2)$ and $(n, g) \cong (n_1, g_1) \vee (n_2, g_2)$. Thus, we obtain

$$\begin{aligned} \varphi_{ABC}(k_1, h_1) \wedge \varphi_{ABC}(k_2, h_2) &\leq \\ &\leq \bigvee \{ [n, g] \otimes [m, f] \mid (k_i, h_i) \leq (n, g)(m, f), \quad i = 1, 2 \} = \\ &= \bigvee \{ [n, g] \otimes [m, f] \mid (k, h) \leq (n, g)(m, f) \} = \varphi_{ABC}(k, h). \end{aligned}$$

Finally we verify (**). Consider $m : F \rightarrow A$ in \mathcal{F} . By **Fin 3** every $f : F \rightarrow B$ factors as $f = n_f e_f$ with an $n_f : G_g \rightarrow B$ in \mathcal{F} . Then

$$(n, g e_f) \leq (n_f, g)(m, f)$$

holds for every $g \in \mathcal{C}(G_f, C)$, as shown in the following the diagram



We obtain

$$\begin{aligned}
 & \bigvee \{ \varphi_{ABC}(m, h) \mid h \in \mathcal{C}(F, C) \} \geq \\
 & \geq \bigvee \{ \varphi_{ABC}(m, ge_f) \mid f \in \mathcal{C}(F, B), g \in \mathcal{C}(G_f, C) \} \geq \\
 & \geq \bigvee \{ [n_f, g] \otimes [m, f] \mid f \in \mathcal{C}(F, B), g \in \mathcal{C}(G_f, C) \} = \\
 & = \bigvee \{ 1 \otimes [m, f] \mid f \in \mathcal{C}(F, B) \} = \\
 & = 1 \otimes \bigvee \{ [m, f] \mid f \in \mathcal{C}(F, B) \} = 1 \otimes 1 = 1,
 \end{aligned}$$

since $\bigvee \{ [n_f, g] \mid g \in \mathcal{C}(G_f, C) \} = 1$ for all $f \in \mathcal{C}(F, B)$. □

2.4. Let

$$\iota_A : \tilde{\mathcal{C}}(A, A) \rightarrow 2$$

be the frame homomorphism ξ_{1_A} (see 1.7), so that

$$\iota_A([m, f]) = 1 \quad \text{iff} \quad m = f$$

for all $(m, f) \in \hat{\mathcal{C}}(A, A)$. With this notation one has:

Theorem: *Under conditions **Fin 1** and **Fin 3** there is a **Loc**-category $\tilde{\mathcal{C}}$ with class of objects $|\tilde{\mathcal{C}}| = |\mathcal{C}|$, with hom-objects $\tilde{\mathcal{C}}(A, B)$, composition law μ_{ABC} , and with identity elements ι_A .*

PROOF: We need to verify the associativity and the unit axioms. For $(m, f) \in \hat{\mathcal{C}}(A, B)$, $(n, g) \in \hat{\mathcal{C}}(B, C)$, $(k, h) \in \hat{\mathcal{C}}(C, D)$ and $(l, e) \in$

$\widehat{\mathcal{C}}(A, D)$ write $(l, e) \leq (k, h)(n, g)(m, f)$ if there are morphisms s, t, u such that $ms = l, fs = nt, gt = kr$ and $nr = e$. It is easy to see that the following statements are equivalent:

- (i) $(l, e) \leq (k, h)(n, g)(m, f)$,
- (ii) there is a $(u, x) \in \widehat{\mathcal{C}}(A, C)$ such that $(l, e) \leq (k, h)(u, x)$ and $(u, x) \leq (n, g)(m, f)$,
- (iii) there is a $(v, y) \in \widehat{\mathcal{C}}(B, D)$ such that $(l, e) \leq (v, y)(m, f)$ and $(v, y) \leq (k, h)(n, g)$.

Hence for all $(l, e) \in \widehat{\mathcal{C}}(A, D)$,

$$\begin{aligned}
 (1 \otimes \mu_{ABC})\mu_{ACD}([l, e]) &= \\
 &= (1 \otimes \mu_{ABC})(\bigvee\{[k, h] \otimes [u, x] \mid (l, e) \leq (k, h)(u, x)\}) = \\
 &= \bigvee\{[k, h] \otimes \mu_{ABC}([u, x]) \mid (l, e) \leq (k, h)(u, x)\} = \\
 &= \bigvee\{[k, h] \otimes [n, g] \otimes [m, f] \mid \text{condition (ii) holds}\} = \\
 &= \bigvee\{[k, h] \otimes [n, g] \otimes [m, f] \mid \text{condition (iii) holds}\} = \\
 &= (\mu_{BCD} \otimes 1)\mu_{ABC}([l, e]).
 \end{aligned}$$

Therefore the diagram

$$\begin{array}{ccc}
 \widetilde{\mathcal{C}}(A, D) & \xrightarrow{\mu_{ACD}} & \widetilde{\mathcal{C}}(C, D) \otimes \widetilde{\mathcal{C}}(A, C) \\
 \mu_{ABD} \downarrow & & \downarrow 1 \otimes \mu_{ABC} \\
 \widetilde{\mathcal{C}}(B, D) \otimes \widetilde{\mathcal{C}}(A, B) & \xrightarrow{\mu_{BCD} \otimes 1} & \widetilde{\mathcal{C}}(C, D) \otimes \widetilde{\mathcal{C}}(B, C) \otimes \widetilde{\mathcal{C}}(A, B)
 \end{array}$$

commutes. (For simplicity, we have neglected the associativity isomorphism of \otimes here.) Next we must show that

$$\widetilde{\mathcal{C}}(A, B) \xrightarrow{\mu_{ABB}} \widetilde{\mathcal{C}}(B, B) \otimes \widetilde{\mathcal{C}}(A, B) \xrightarrow{\iota_B \otimes 1} 2 \otimes \widetilde{\mathcal{C}}(A, B)$$

is the coproduct injection ($[u] \mapsto 1 \otimes [u]$). Indeed, for all $(m, f) \in \widetilde{\mathcal{C}}(A, B)$ one has

$$\begin{aligned}
 (\iota_B \otimes 1)\mu_{ABB}([m, f]) &= \\
 &= \bigvee\{\iota_B([n, g]) \otimes [m', f'] \mid (m, f) \leq (n, g)(m', f')\} = \\
 &= \bigvee\{1 \otimes [m', g'] \mid (m, f) \leq (n, n)(m', f')\}.
 \end{aligned}$$

But if $(m, f) \leq (n, n)(m', f')$, then $(m, f) \leq (m', f')$, hence

$$1 \otimes [m', f'] \leq 1 \otimes [m, f].$$

On the other hand, choosing for n an \mathcal{F} -morphism through which f factors, one has $(m, f) \leq (n, n)(m, f)$. Thus, the last join is $1 \otimes [m, f]$.
 \square

2.5. The spectrum functor

$$\Sigma : \mathbf{Frm}^{\text{op}} = \mathbf{Loc} \rightarrow \mathbf{Top}$$

preserves products (as a right adjoint). Up to isomorphism, Σ transforms the composition law and the identity elements into those of the **Set**-category $\tilde{\mathcal{C}}$:

Proposition *Under conditions **Fin 1**, **Fin 2** and **Fin 3**, and with Φ_{AB} as in 1.7, one has commutative diagrams*

$$\begin{array}{ccc} \mathcal{C}(B, C) \times \mathcal{C}(A, B) & \xrightarrow{\text{composition}} & \mathcal{C}(A, C) \\ \Phi_{BC} \times \Phi_{AB} \downarrow & & \downarrow \Phi_{AC} \\ \Sigma(\tilde{\mathcal{C}}(B, C) \otimes \tilde{\mathcal{C}}(A, B)) & \xrightarrow{\Sigma\mu} & \Sigma\tilde{\mathcal{C}}(A, C) \end{array}$$

and

$$\begin{array}{ccc} 1 & \xrightarrow{\text{identity}} & \mathcal{C}(A, A) \\ \cong \downarrow & & \downarrow \Phi_{AA} \\ \Sigma 2 & \xrightarrow{\Sigma\iota} & \Sigma\tilde{\mathcal{C}}(A, A) \end{array}$$

PROOF: Recall that, for a frame homomorphism $\alpha : L \rightarrow M$ and $\xi \in \Sigma M$, one has $(\Sigma\alpha)(\xi) = \xi\alpha$, and that the canonical morphism

$$\Sigma L \times \Sigma M \rightarrow \Sigma(L \otimes M)$$

maps $(\varepsilon, \eta) \in \Sigma L \times \Sigma M$ to the “point” ξ of $L \otimes M$ with $\xi(x \otimes y) = \varepsilon(x) \wedge \eta(y)$ for $(x, y) \in L \times M$.

Let $s : A \rightarrow B$ and $t : B \rightarrow C$ be \mathcal{C} -morphisms, and let

$$\xi \in \Sigma(\tilde{\mathcal{C}}(B, C) \otimes \tilde{\mathcal{C}}(A, B))$$

be the point to which (ξ_t, ξ_s) is mapped (see 1.7). Then, for all $(k, h) \in \widehat{\mathcal{C}}(A, C)$,

$$\begin{aligned} (\Sigma\mu_{ABC})(\xi)([k, h]) &= \xi(\mu_{ABC}([k, h])) = \\ &= \bigvee \{ \xi([n, g] \otimes [m, f]) \mid (k, h) \leq (n, g)(m, f) \} = \\ &= \bigvee \{ \xi_t([n, g]) \otimes \xi_s([m, f]) \mid (k, h) \leq (n, g)(m, f) \}. \end{aligned}$$

This join is 1 if and only if there are $m \in \mathcal{F}/A$ and $n \in \mathcal{F}/B$ such that

$$(k, h) \leq (n, tn)(m, sm).$$

It is easy to see that this latter condition is satisfied if and only if $tsk = h$. Hence

$$\xi(\mu_{ABC}([k, h])) = \xi_{ts}([k, h])$$

for all $(k, h) \in \widehat{\mathcal{C}}(A, C)$, so the first diagram is commutative.

The commutativity of the second diagram is trivial. \square

Corollary: The composition $\mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$ is continuous in the Hausdorff topology from 1.8.

3. Examples and remarks

3.1. Let \mathcal{C} be a locally finitely presentable category in the sense of Gabriel and Ulmer ([3]). Then \mathcal{C} is cocomplete and has (strong-epi, mono)-factorizations. With $|\mathcal{F}|$ the class of finitely generated objects of \mathcal{C} , it follows from 9.3 and 9.5 of [3] that conditions **Fin 1'**, **Fin 2'** and **Fin 3'** of 1.3 are satisfied.

In the special case that \mathcal{C} is a variety of (finitary) universal algebras, “finitely generated” is to be understood in the usual sense. Thus the hom-sets $\mathcal{C}(A, B)$ in the categories of groups, rings, etc., carry a zero-dimensional T_2 -topology with basic open sets

$$W(F, f) = \{t : A \rightarrow B \mid t|_F = f\},$$

with F a finitely generated subalgebra of A and $f : F \rightarrow B$ a homomorphism.

3.2. In the category \mathcal{C} of compactly generated Hausdorff spaces with its (strong-epi, mono)-factorization structure, one may choose for $|\mathcal{F}|$ the category of compact Hausdorff spaces. Again, conditions **Fin 1'**, **Fin 2'** and **Fin 3'** are easily seen to hold. The topology 1.8 on $\mathcal{C}(A, B)$ is generated by the open sets

$$W(K, f) = \{t : A \rightarrow B \mid t|_K = f\}$$

with K a compact subspace of A and $f : K \rightarrow B$ continuous. This topology is, in general, properly finer than the compact-open topology, but coarser than the discrete topology. It makes

$$((s, t) \mapsto s \cdot t) : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$$

continuous.

3.3. A frame is spatial if $V(a) \neq V(b)$ for $a \neq b$ in L (see 1.9). For $L = \tilde{\mathcal{C}}(A, B)$ we have

$$V([u]) = \{\xi \mid (\exists(m, f) \in u)(\xi([m, f]) = 1)\}.$$

Under conditions **Fin 1**, **Fin 2** the open sets $V([u])$ may be equivalently replaced by

$$W(u) = \{t \in \mathcal{C}(A, B) \mid (\exists m \in \mathcal{F}/A)((m, tm) \in u)\}.$$

Thus, $\tilde{\mathcal{C}}(A, B)$ is spatial if and only if

$$W(u) = W(v) \quad \Rightarrow \quad u \sim v$$

holds for all $u, v \in U\widehat{\mathcal{C}}(A, B)$ (note that “ \Leftarrow ” holds always). In particular, if $\mathcal{C}(A, B) = \emptyset$, we always have $W(u) = \emptyset = W(v)$; hence, $\widetilde{\mathcal{C}}(A, B)$ is spatial if and only if $\widetilde{\mathcal{C}}(A, B)$ is *trivial*, in the sense that $0=1$. Therefore, *any example with $\mathcal{C}(A, B) = \emptyset$ but $\widetilde{\mathcal{C}}(A, B)$ non-trivial gives an example of a non-spatial frame.*

We first derive a sufficient condition for $\widetilde{\mathcal{C}}(A, B)$ to be non-trivial. Recall that we may think of $\widetilde{\mathcal{C}}(A, B)$ as the set of elements in $U\widehat{\mathcal{C}}(A, B)$ which are *saturated* (in [4] this property was referred to as *coherent*) with respect to the relation

$$\bigvee \{ \uparrow(m, f) \mid f \in \mathcal{C}(F, B) \} \sim 1$$

for all $m : F \rightarrow A$ in \mathcal{F} , and that the map $[-] : U\widehat{\mathcal{C}}(A, B) \rightarrow \widetilde{\mathcal{C}}(A, B)$ is then given by $u \mapsto [u] = \bigwedge \{ v \mid v \text{ saturated, } u \leq v \}$. A necessary condition for $\widetilde{\mathcal{C}}(A, B)$ to be non-trivial is that $\widehat{\mathcal{C}}(A, B) \neq \emptyset$. In that case we have $1 = \uparrow(0_A \rightarrow A, 0_A \rightarrow B)$. Hence a sufficient condition for $1 \neq 0$ is that $0 = \emptyset$ be saturated; an easy translation of the definition of that condition gives equivalently:

$$(\forall m \in \mathcal{F}/A)(\forall (n, g) \in \widehat{\mathcal{C}}(A, B))(\exists (k, h) \in \widehat{\mathcal{C}}(A, B)) \\ \text{such that } (m \leq k \ \& \ (n, g) \leq (k, h)).$$

Certainly, given the solid arrows in the diagram

$$\begin{array}{ccccc}
 & & F & & \\
 & m \swarrow & \downarrow & & \\
 A & \xleftarrow{k} & H & \xrightarrow{\dots h \dots} & B \\
 & n \searrow & \uparrow s & \nearrow g & \\
 & & G & &
 \end{array}$$

with $k \cong m \vee n$ in \mathcal{F}/A , the dotted arrow h exists if B is injective with respect to the class $\{s \mid (\exists k \in \mathcal{F})(ks \in \mathcal{F})\}$. In the setting of 1.3 this gives:

Proposition: *Under conditions **Fin 1'**, **Fin 2'** and **Fin 3'**, the frame $\widetilde{\mathcal{C}}(A, B)$ is non-trivial if $\widehat{\mathcal{C}}(A, B) \neq \emptyset$ and if B is injective with respect to \mathcal{M} -morphisms with domain and codomain in $|\mathcal{F}|$. \square*

Now let \mathcal{C} be the category of antireflexive posets (i.e., sets with a binary transitive relation $<$ such that $x \not< x$) and of $<$ -preserving maps. Let $|\mathcal{F}|$ be the class of finite posets, and let \mathcal{M} be given by order-embeddings ($x < y \Leftrightarrow f(x) < f(y)$). Clearly, \mathbb{Q} , the set of rationals with the natural $<$, is injective with respect to order-embeddings of finite objects. Hence $\tilde{\mathcal{C}}(A, \mathbb{Q})$ is non-trivial for every A with $\hat{\mathcal{C}}(A, \mathbb{Q}) \neq \emptyset$. But $\mathcal{C}(A, B)$ may be empty, for instance for $A = \mathbb{R}$, the real line. Hence we have:

Corollary: $\tilde{\mathcal{C}}(\mathbb{R}, \mathbb{Q})$ is a non-spatial frame.

3.4. In the setting of 1.3, with a given subclass $|\mathcal{F}|$ of objects in \mathcal{C} , let us assume that we have two proper factorization systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$ such that **Fin 1'**, **Fin 2'** and **Fin 3'** hold in both cases. We show that the induced topologies τ_1, τ_2 on the hom-set $\mathcal{C}(A, B)$ actually coincide for any two objects A, B . Indeed, for $t \in W(m_1, f)$ with $m_1 \in \mathcal{M}_1$ one has $t \in W(m_2, tm_2) \subseteq W(m_1, f)$ with m_2 the \mathcal{M}_2 -part of an $(\mathcal{E}_2, \mathcal{M}_2)$ -factorization of m_1 . Hence τ_2 is finer than τ_1 ; symmetrically, τ_1 is finer than τ_2 . Therefore, when $|\mathcal{F}|$ is fixed, the topology of $\mathcal{C}(A, B)$ is independent of the choice of the proper factorization system.

3.5. In all previous examples, \mathcal{F} is given by the class $|\mathcal{F}|$ of finite objects and a proper factorization system $(\mathcal{E}, \mathcal{M})$. However, there are examples of a different type. For instance in **Set**, let \mathcal{F} be the class of those injections $m : F \rightarrow A$ for which $F = \emptyset$ or $A \setminus m(F)$ is finite (the case $F = \emptyset$ has to be included in order to satisfy **Fin 1**). Similarly, in the category of K -vector spaces, one may consider for \mathcal{F} the monomorphisms $m : F \rightarrow A$ for which $F = 0$ or $m(F)$ has finite codimension in A .

At this point we wish to emphasize that it is essential to consider empty unions in **Fin 1**, since this may seem unnatural in light of the examples above. For this consider again the category \mathcal{C} of antireflexive posets (see 3.3), but with \mathcal{F} the class of those order-embeddings $m : F \rightarrow A$ for which $F = \emptyset$ or $A \setminus m(F)$ is finite. Then the frame $\tilde{\mathcal{C}}(\mathbb{R}, \mathbb{Q})$ is trivial despite the facts that $\hat{\mathcal{C}}(\mathbb{R}, \mathbb{Q}) \neq \emptyset$ and that \mathbb{Q} is injective with respect to all \mathcal{F} -morphisms with non-void domain. In fact, in

general. the frame $\tilde{\mathcal{C}}(A, B)$ is certainly trivial if there is at least one finite subobject $m : F \rightarrow A$ with $\mathcal{C}(F, B) = \emptyset$ since then

$$0 = \bigvee \{[m, f] \mid f \in \mathcal{C}(A, B)\} = 1$$

holds in $\tilde{\mathcal{C}}(A, B)$.

3.6. Finally we want to show that the frame construction 1.5 is coherent with finitary algebraic structures that a particular object may carry. More precisely: suppose that $(a_i : B^{s_i} \rightarrow B)_{i \in I}$ is a system of operations on an object $B \in \mathcal{C}$. Since any representable functor $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$ preserves products, one has induced operations

$$\alpha_i : \mathcal{C}(A, B)^{s_i} \rightarrow \mathcal{C}(A, B)$$

which by 2.5 live in **Haus**. But in fact one even has:

Proposition: *Under the conditions **Fin 1**, **Fin 2** and **Fin 3** there are frame morphisms*

$$\tilde{\alpha}_i : \tilde{\mathcal{C}}(A, B) \rightarrow \bigotimes^{s_i} \tilde{\mathcal{C}}(A, B)$$

such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{C}(A, B)^{s_i} & \xrightarrow{\alpha_i} & \mathcal{C}(A, B) \\ \Phi_{A, B}^{s_i} \downarrow & & \downarrow \Phi_{A, B} \\ \Sigma \tilde{\mathcal{C}}(A, B)^{s_i} \cong \Sigma(\bigotimes^{s_i} \tilde{\mathcal{C}}(A, B)) & \xrightarrow{\Sigma \tilde{\alpha}_i} & \Sigma \tilde{\mathcal{C}}(A, B) \end{array}$$

PROOF: Omitting the subscripts we may define a homomorphism $\hat{\alpha} : \hat{\mathcal{C}}(A, B) \rightarrow \bigotimes^s \hat{\mathcal{C}}(A, B)$ by

$$\hat{\alpha}(m, f) = \bigvee \{[m, f_1] \otimes \cdots \otimes [m, f_s] \mid f_j \in \mathcal{C}(F, B), \alpha(f_1, \dots, f_s) = f\}$$

for $m : F \rightarrow A$ in \mathcal{F} . $f \in \mathcal{C}(F, B)$: here (f_1, \dots, f_s) is the morphism $F \rightarrow B^s$ with components f_j . It is easily seen that $\hat{\alpha}$ is order-reversing

and maps the bottom (if it exists) to the top. To see that the existing finite joins are mapped into meets take

$$\begin{aligned}
 \widehat{\alpha}(m, f) \wedge \widehat{\alpha}(n, g) &= \\
 &= \bigvee \left\{ \bigotimes_{i=1}^s [m, f_i] \wedge \bigotimes_{i=1}^s [n, g_i] \mid \alpha(f_1, \dots) = f, \alpha(g_1, \dots) = g \right\} = \\
 &= \bigvee \left\{ \bigotimes_{i=1}^s ([m, f_i] \wedge [n, g_i]) \mid \alpha(f_1, \dots) = f, \alpha(g_1, \dots) = g \right\}.
 \end{aligned}$$

Now one needs to consider only the case that all $[m, f_j] \wedge [n, g_j]$ are of the form $[k, h_j]$ with $k = m \vee n$. Then, with $h = \alpha(h_1, \dots, h_s)$, it follows that $(m, f) \vee (n, g) = (k, h)$ (see 1.4), and that

$$\widehat{\alpha}(m, f) \vee \widehat{\alpha}(n, g) \geq \widehat{\alpha}(k, h).$$

In order to show that $\widehat{\alpha}$ extends to a frame morphism $\widetilde{\alpha} : \widetilde{\mathcal{C}}(A, B) \rightarrow \bigotimes^s \widetilde{\mathcal{C}}(A, B)$, by 1.5 we just need to show that, for every $m \in \mathcal{F}/A$,

$$\bigvee \{ \widehat{\alpha}(m, f) \mid f \in \mathcal{C}(F, B) \} = 1.$$

This formula indeed holds true since the join is

$$\begin{aligned}
 &\bigvee \{ [m, f_1] \otimes \dots \otimes [m, f_s] \mid f_i \in \mathcal{C}(F, B) \} = \\
 &= \bigvee \{ [m, f_1] \mid f_1 \in \mathcal{C}(F, B) \} \otimes \dots \otimes \bigvee \{ [m, f_s] \mid f_s \in \mathcal{C}(F, B) \} = \\
 &= 1 \otimes \dots \otimes 1 = 1.
 \end{aligned}$$

Proving the commutativity of the diagram above is similar to the proof in 2.5. \square

A lengthy but routine check also gives that, if the a_i 's satisfy a system of equations, the same equations also hold with respect to the $\widetilde{\alpha}_i$'s in $\widetilde{\mathcal{C}}(A, B)$. Thus one has:

Corollary *For an algebraic theory \mathbb{A} in the sense of Lawvere, let the C -object B carry an \mathbb{A} -structure in \mathcal{C} . Then, for every C -object A ,*

$\tilde{\mathcal{C}}(A, B)$ carries an \mathbb{A} -structure in **Loc** which Σ maps to the canonical \mathbb{A} -structure on $\mathcal{C}(A, B)$. \square

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