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COMPLETION AND CLOSURE

by David HOLGATE

RESUME. La fermeture (ou, de façon synonyme la densité) a toujours joué un rôle important dans la théorie des complétions. S'appuyant sur des idées de Birkhoff, une fermeture est extraite de manière canonique d'un processus de complétion réflexive dans une catégorie. Cette fermeture caractérise la complétude et la complétion elle-même. La fermeture n'a pas seulement de bonnes propriétés internes, mais c'est la plus grande parmi les fermetures qui décrivent la complétion.

Le théorème principal montre que, équivalent aux descriptions fermeture/densité naturelle d'une complétion, est le simple fait marquant que les réflecteurs de complétion sont exactement ceux qui préservent les plongements. De tels réflecteurs peuvent être déduits de la fermeture elle-même. Le rôle de la préservation de la fermeture et du plongement jette alors une nouvelle lumière sur les exemples de complétion.

Introduction

Already in [Bir37] it was put forward that the property "completeness" is a closure property. The discussion there points out that given any "completing correspondence" an appropriate closure can be extracted which in turn describes completeness of systems – completely. The language of category theory, and in particular categorical closure operators, allows us to explore these ideas more rigorously in a contemporary setting.

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In a category with distinguished subobjects, any reflective completion procedure generates a closure operator that acts on these subobjects. This operator describes the natural closure and density associated with the completion. In particular:

- Complete objects are characterised as being absolutely closed.
- The closure describes the density with which any object is contained in its completion. Such dense containment in a complete object characterises the completion of an object.
- It is the largest, and the only idempotent and hereditary closure to characterise completeness.
- Dense subobjects provide the link between the uniqueness of completions and injective structures.
- The original completion can be retrieved from the closure.

This closure operator approach leads to the main theorem which provides the new insight that completion reflectors are characterised as being *exactly* those reflectors which preserve subobjects. (A property that Birkhoff understood to be essential for his "completing correspondences".) After looking at completions and the closures they induce, the final section explores which closures themselves induce completions. This final topic merits further study as part of a general theory of completeness relative to a closure operator.

As well as covering the standard examples in topology and algebra, light is shed on other completion procedures.

The author is grateful for the hospitality of the Universities of L'Aquila (Italy) and Bremen (Germany) during the writing of this paper. The referee's comments helped improve the exposition considerably, and we are also grateful for the referee's drawing our attention to [Rin71] which seems to be overlooked in many of the articles on categorical approaches to completeness.

Our standard reference for categorical matters is [AHS90]. For closure operators we refer to [DT95] or the introductory paper [DG87].

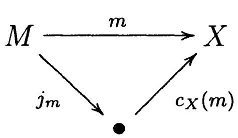
Fundamental to closure operators is the notion of *subobject.* To this end, we require the existence of a *proper factorisation structure* $(\mathcal{E}, \mathcal{M})$ for morphisms in a category $\mathcal{X}$ – i.e. a factorisation structure where $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ a class of monomorphisms. (e.g. the (*Surjection, Embedding*) factorisation structure in topological spaces that factors every continuous map through its image.) Pullbacks of $\mathcal{M}$-morphisms along any $\mathcal{X}$-morphism are assumed to exist (and are hence again in $\mathcal{M}$). Closure operators act on these $\mathcal{M}$-morphisms which represent subobjects in $\mathcal{X}$.

The $\mathcal{M}$-subobjects of a given $X \in Ob\mathcal{X}$ (i.e. those with codomain X) will be denoted $Sub(X)$. These are given the usual ordering, $m \leq n$ in $Sub(X)$ iff there is a morphism j with $nj = m$. (Note that j is uniquely determined, and also in $\mathcal{M}$.) Although $\leq$ is a preorder, we will write $m = n$ to denote $m \leq n$ and $n \leq m$, not distinguishing notationally between m and its $\leq$-equivalence class.

For a morphism $f : X \to Y$ in $\mathcal{X}$, the *image* of $m \in Sub(X)$ – denoted $f(m) \in Sub(Y)$ – is the $\mathcal{M}$-component of the $(\mathcal{E}, \mathcal{M})$-factorisation of the composition fm.

There are many descriptions of a closure operator c on $\mathcal{X}$ with respect to $\mathcal{M}$. Most intuitively, it is a family $\{c_X : Sub(X) \to Sub(X) \mid X \in Ob\mathcal{X}\}$ of *expansive* ($m \leq c_X(m)$ for every $m \in Sub(X)$, $X \in Ob\mathcal{X}$) *order preserving* ($m \leq n \Rightarrow c_X(m) \leq c_X(n)$ for all $m, n \in Sub(X)$, $X \in Ob\mathcal{X}$) assignments, such that every morphism $f : X \to Y$ in $\mathcal{X}$ is c-continuous ($f(c_X(m)) \leq c_Y(f(m))$ for every $m \in Sub(X)$). When clear from the context, the X subscript is usually omitted.

Forming the closure of any $m \in \mathcal{M}$ gives a factorisation.



m is *c-closed* if j_m is an isomorphism, and *c-dense* if $c_X(m)$ is an isomorphism. An $\mathcal{X}$-morphism $f : X \to Y$ is *c-dense* if m is c-dense in the $(\mathcal{E}, \mathcal{M})$ factorisation $f = me$ (i.e. $c(f(1_X)) = 1_Y$).

The operator c is *idempotent* if $c(m)$ is c-closed for every $m \in \mathcal{M}$, and c is *weakly hereditary* if j_m is c-dense for every $m \in \mathcal{M}$. A closure operator is *hereditary* if for any $m : M \to N$ and $n : N \to X$ in $\mathcal{M}$, $nc_N(m) = c_X(nm) \wedge n$ – the meet being formed relative to $\leq$ in $Sub(X)$.

The order on subobjects is extended pointwise to closure operators.

Lastly, for a given closure operator c, an object X in $\mathcal{X}$ is *c-Hausdorff* iff it satisfies the following:

$$\text{For every } M \overset{m}{\to} A \in \mathcal{M} \text{ and } u, v : A \to X \text{ such that } um = vm$$
$$\text{it follows that } uc(m) = vc(m)$$

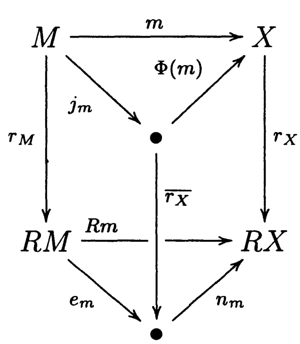
If the category has products, then this is equivalent to the familiar closed diagonal characterisation of Hausdorff spaces (cf. [CGT96]).

1. $\mathcal{M}$-reflectors and pullback closure

The functor $R : \mathcal{X} \to \mathcal{R}$ denotes a reflector from $\mathcal{X}$ to a full, isomorphism closed subcategory $\mathcal{R}$. For any $\mathcal{X}$-object X the reflection to RX is denoted $r_X : X \to RX$. If $\mathcal{A}$ is a class of $\mathcal{X}$-morphisms, R is termed an *$\mathcal{A}$-reflector* if $r_X \in \mathcal{A}$ for every $X \in Ob\mathcal{X}$.

Our paradigmatic example of an $\mathcal{M}$-reflector, and completion, will be the usual completion in the category of Hausdorff uniform spaces, with uniformly continuous maps. ($\mathcal{M}$ the class of uniform embeddings.) For a uniform space X, $r_X : X \to RX$ is the embedding of X into its Cauchy completion.

For any $f : X \to Y$ in $\mathcal{X}$, we have a commutative square $Rfr_X = r_Y f$. Considering such a square for $M \overset{m}{\to} X \in \mathcal{M}$ we form the *pullback closure* $\Phi(m)$ by taking the pullback of n_m along r_X where $n_m e_m = Rm$ is the $(\mathcal{E}, \mathcal{M})$ factorisation of Rm in $\mathcal{X}$. (j_m is the unique fill-in morphism.)



(*)

This operator, and in particular its link to perfect morphisms is studied in [Hol96, Hol98b]. It is a particular case of a more general construction introduced in [DT95] Exercise 5.V.

The notation of the above paragraphs will be fixed throughout.

1.1 Definition. For any closure operator c on $\mathcal{X}$ with respect to $\mathcal{M}$, the *absolutely c-closed objects* are those

$$\mathcal{A}_c := \{X \in Ob\mathcal{X} \mid \text{every } X \overset{m}{\to} Y \in \mathcal{M} \text{ with domain } X \text{ is } c\text{-closed}\},$$

while the *c-dense $\mathcal{M}$-morphisms* will be denoted

$$\mathcal{D}_c := \{c\text{-}dense\} \cap \mathcal{M}.$$

No notational distinction is made between the object class $\mathcal{A}_c$ and the corresponding full subcategory of $\mathcal{X}$.

1.2 Theorem. *Let $R : \mathcal{X} \to \mathcal{R}$ be an $\mathcal{M}$-reflector, and Φ the associated pullback closure. The following hold.*
(a) Equivalent for an object X in $\mathcal{X}$ are:
(i) $X \in Ob\mathcal{R}$.
(ii) X is absolutely Φ-closed.
(b) $R : \mathcal{X} \to \mathcal{R}$ is a $(\{\Phi\text{-dense}\} \cap \mathcal{M})$-reflector.
(c) Every Φ-closed $\mathcal{M}$-subobject of an $\mathcal{R}$-object is again in $\mathcal{R}$.
(d) Every $X \in Ob\mathcal{X}$ is Φ-Hausdorff.
(e) $\{\Phi\text{-dense}\} \subseteq Epi\mathcal{X}$

Proof. (a) Let $X \in Ob\mathcal{R}$ and construct $\Phi(m)$ for $X \overset{m}{\to} Y \in \mathcal{M}$. (Use notation analogous to that in the diagram $(*)$.) $Rm = r_Y m r_X^{-1} \in \mathcal{M}$ so e_m, and thus $\overline{r_Y} j_m$ are isomorphisms. From this it follows that m is Φ-closed.

On the other hand, $r_X \in \mathcal{D}_\Phi$ for any X since $Rr_X \in Iso\mathcal{X}$, so if $X \in \mathcal{A}_\Phi$ then r_X is also Φ-closed, hence an isomorphism, and $X \in Ob\mathcal{R}$.

(b) Φ-density of r_X for any $X \in Ob\mathcal{X}$ is observed in (a).

(c) Let $M \overset{m}{\to} X \in Sub(X)$ be Φ-closed. In $(*)$, we see that r_X, $\overline{r_X}$ and j_m are isomorphisms. Thus r_M is epic and a section and $M \in Ob\mathcal{R}$.

(d) Recalling the definition of Hausdorff given above, take $X \in Ob\mathcal{X}$ and consider $M \overset{m}{\to} A \in \mathcal{M}$ and $u, v : A \to X$ such that $um = vm$.

$$\begin{aligned} um = vm \ &\Rightarrow RuRm = RvRm \\ &\Rightarrow Run_m = Rvn_m \text{ (Since } e_m \in Epi\mathcal{X}) \\ &\Rightarrow r_X u\Phi(m) = r_X v\Phi(m) \end{aligned}$$

Since r_X is a monomorphism, the result follows.

(e) Let $M \overset{m}{\to} X \in \mathcal{D}_\Phi$ and $u, v : X \to Y$ coincide on m. Since Y is Φ-Hausdorff $u\Phi(m) = v\Phi(m)$, so $u = v$ and $\mathcal{D}_\Phi \subseteq Epi\mathcal{X}$. Since $\mathcal{E}$ is a class of epimorphisms this property carries to all Φ-dense morphisms. $\square$

In summary: we have a a $(\{\Phi\text{-}dense\} \cap \mathcal{M})$-reflection to $\mathcal{R}$ (for us the complete objects) in a setting of Hausdorff separation. $\mathcal{R}$ is the class of absolutely Φ-closed objects and is closed under Φ-closed subobjects. (Absolute closure alone, however, is not sufficient to characterise $\mathcal{R}$. The H-closed spaces of topology provide a well-known counter example (cf. [HS68]).)

Part (e) of the Theorem raises the question of when Φ-density characterises epimorphisms in $\mathcal{X}$. A discussion of this, invoking instances when R preserves $\mathcal{E}$-morphisms is given in [Hol98a].

The following lemma occurs as Exercise 2.F(b) in [DT95].

1.3 Lemma. *For any closure operator c on $\mathcal{X}$ with respect to $\mathcal{M}$, if $M \xrightarrow{m} X \in \mathcal{M}$ is c-dense and $X \xrightarrow{e} Y \in \mathcal{E}$ then em is c-dense.*

Proof. Take e and m as above. We need to show that $c(e(m))$ is an isomorphism. Continuity of e and density of m yield

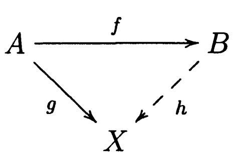
$$1_Y = e(1_X) = e(c(m)) \leq c(e(m)).$$

Whence the result. □

2. Completions

The notion of $\mathcal{M}$-reflector on its own is not enough to capture the idea of completion. $\mathcal{M}$-reflectors such as the Čech-Stone compactification of a Tychonoff space lie outside this realm. We consider two categorical approaches to completion theory – injectivity and uniqueness of completions. In conjunction with $\mathcal{M}$-reflectivity, these notions coincide.

2.1 Definition. Let $\mathcal{A}$ be a class of $\mathcal{X}$-morphisms. An $\mathcal{X}$-object X is $\mathcal{A}$-*injective* if for any $\mathcal{A}$-morphism $f : A \to B$ and morphism $g : A \to X$ there is an extension $h : B \to X$ such that $hf = g$.



The class of $\mathcal{A}$-injective objects will be denoted *Inj($\mathcal{A}$)*.

A reflector $R : \mathcal{X} \to \mathcal{R}$ is called *$\mathcal{A}$-subfirm* if it is an $\mathcal{A}$-reflector, and $f \in \mathcal{A} \Rightarrow Rf \in Iso\mathcal{X}$. It is called *$\mathcal{A}$-firm* if in fact $f \in \mathcal{A} \Leftrightarrow Rf \in Iso\mathcal{X}$. (This terminology is essentially that of [BG92].)

In *injectivity theory* the *Inj($\mathcal{A}$)* models the complete objects and $\mathcal{A}$ the "dense embeddings". When completions exist, every object has a (necessarily unique) injective hull. Notably such injective hulls need not be reflections. The literature on injectivity is vast. Key references include [Mar64], [BB67] and the more recent summary in [AHS90] Chapter 9.

Subfirm reflectors model the uniqueness of completions. For such reflectors, $r_X : X \to RX$ is the *unique* (up to isomorphism) "dense embedding" of X into a complete object – i.e. $\mathcal{R}$-object. This approach began with [Bac73] and is continued in [BGH92] and [BG92] where the links to injectivity are further illucidated. A more systematic and in-depth discussion of the correspondence between a reflective subcategory $\mathcal{R}$ and the morphism class $\{f \in Mor\mathcal{R} \mid Rf \in Iso\mathcal{R}\}$ is given in [CHK85].

2.2 Definition. An $\mathcal{M}$-reflector $R : \mathcal{X} \to \mathcal{R}$ is called a *completion reflector* if $\mathcal{R} \subseteq Inj(\{\Phi\text{-}dense\} \cap \mathcal{M})$, where Φ is the pullback closure induced by R.

For a given X, the reflection $r_X : X \to RX$ will be termed the *completion* of X.

Note that the reverse inclusion to that in the definition is always true since for an $\mathcal{M}$-reflector any $r_X \in \{\Phi\text{-}dense\} \cap \mathcal{M}$. Thus for a completion reflector, $\mathcal{R} = Inj(\{\Phi\text{-}dense\} \cap \mathcal{M})$

2.3 Proposition. *If $R : \mathcal{X} \to \mathcal{R}$ is a completion reflector, then R preserves $\mathcal{M}$-morphisms.*

Proof. Factorise $Rm = n_m e_m$ for $M \xrightarrow{m} X \in \mathcal{M}$. Since $n_m e_m r_M = Rm r_M = r_X m \in \mathcal{M}$ it follows that $e_m r_M \in \mathcal{M}$, and so by Lemma 1.3 $e_m r_M \in \mathcal{D}_\Phi$.

Because R is a completion reflector, $RM \in Inj(\mathcal{D}_\Phi)$, so there is an extension h such that $h e_m r_M = r_M$. Thus e_m is an isomorphism and $Rm \in \mathcal{M}$. □

The preservation of subobjects has always been associated with completion theory. (cf. for example [Rin71], [BGH92] Section 2, and more fundamentally [Bir37].) What makes the present definition of completion attractive is that out of the natural closure/density definition arises the closure free *characterisation* that a completion reflector is simply an $\mathcal{M}$-reflector that preserves $\mathcal{M}$-morphisms.

2.4 Theorem. *For an $\mathcal{M}$-reflector $R : \mathcal{X} \to \mathcal{R}$, with Φ the induced pullback closure, the following are equivalent.*
(a) R is a completion reflector.
(b) R preserves $\mathcal{M}$-morphisms.
(c) $R : \mathcal{X} \to \mathcal{R}$ is a $(\{\Phi\text{-dense}\} \cap \mathcal{M})$-subfirm reflector.
(d) $R : \mathcal{X} \to \mathcal{R}$ is a $(\{\Phi\text{-dense}\} \cap \mathcal{M})$-firm reflector.
(e) $\{\Phi\text{-dense}\} \cap \mathcal{M}$ is the class of $\mathcal{M}$-essential morphisms.

Proof. (a) $\Rightarrow$ (b) is Proposition 2.3. To see (b) $\Rightarrow$ (c), form $\Phi(m)$ for $m \in \mathcal{D}_\Phi$. Noting that $Rm \in \mathcal{M}$ and $Rm\overline{r_X}\Phi(m)^{-1} = r_X \in \mathcal{D}_\Phi$ we have $Rm \in \mathcal{D}_\Phi$. But Rm is trivially Φ-closed, hence an isomorphism and R is $\mathcal{D}_\Phi$-subfirm. (We already know from Theorem 1.2 that R is a $\mathcal{D}_\Phi$-reflector.)

If Rf is an isomorphism for $f \in Mor\mathcal{X}$, then clearly $f \in \mathcal{M}$ and is furthermore Φ-dense. Thus (d) follows from (c).

For (d) $\Rightarrow$ (a), let $X \in Ob\mathcal{R}$ and $f : A \to B$ be a Φ-dense $\mathcal{M}$-morphism. By the reflection property, any $g : A \to X$ has an extension $g^* : RA \to X$ such that $g^* r_A = g$. But Rf is an isomorphism, and so $h := g^*(Rf)^{-1} r_B$ provides the extension ensuring $X \in Inj(\mathcal{D}_\Phi)$.

To conclude, (b) and (c) together imply (e) since if $m \in \mathcal{D}_\Phi$ and $fm \in \mathcal{M}$, then $RfRm = R(fm) \in \mathcal{M} \Rightarrow Rf = R(fm)(Rm)^{-1} \in \mathcal{M}$ whence $f \in \mathcal{M}$. Trivially (e) $\Rightarrow$ (b) since each $r_X \in \{\Phi\text{-dense}\} \cap \mathcal{M}$. □

Apart from providing a closure free characterisation of completion reflectors, the theorem shows that the $(\{\Phi\text{-}dense\} \cap \mathcal{M})$-morphisms characterise complete objects both via injectivity and the following uniqueness property that comes from the firmness of R: for any $X \in Ob\mathcal{X}$ the completion $r_X : X \to RX$ is the unique (up to isomorphism) $(\{\Phi\text{-}dense\} \cap \mathcal{M})$-morphism from X into $\mathcal{R}$.

2.5 Proposition. *If $R : \mathcal{X} \to \mathcal{R}$ is a completion reflector, then Φ is an idempotent, hereditary closure operator.*

Proof. By the above result $\Phi(m)$ is simply the pullback of Rm along r_X for any $M \overset{m}{\to} X \in \mathcal{M}$. Rm is always Φ-closed and thus its pullback $\Phi(m)$ is too – i.e. Φ is idempotent.

Since the composition of two pullback squares yields a larger pullback square, Φ-closed maps are closed under composition. We conclude from [DT95] Theorem 2.4 that Φ is weakly hereditary.

Since R is $\mathcal{D}_\Phi$-firm, $m \in \mathcal{D}_\Phi \Leftrightarrow Rm$ is an isomorphism. From this we see that for n, $m \in \mathcal{M}$, $nm \in \mathcal{D}_\Phi \Rightarrow m \in \mathcal{D}_\Phi$. Heredity of Φ then follows from [DT95] Theorem 2.5. □

The following coalesces a number of observations made thus far:

2.6 Theorem. *Let $R : \mathcal{X} \to \mathcal{R}$ be a completion reflector and Φ the induced pullback closure. The following are equivalent for an $\mathcal{X}$-object X.*
(a) $X \in Ob\mathcal{R}$.
(b) X is absolutely Φ-closed.
(c) $X \in Inj(\{\Phi\text{-dense}\} \cap \mathcal{M})$.
(d) Every Φ-dense $\mathcal{M}$-morphism $d : X \to Y$ with domain X is an isomorphism.
(e) Every Φ-dense $\mathcal{M}$-morphism $d : Z \to X$ with codomain X is isomorphic to r_Z.
(f) Every Φ-closed $\mathcal{M}$-subobject of X is in $Ob\mathcal{R}$.

The present notion of completion and associated closure present a tidy theory. It should be noted that natural examples of *non-reflective completions* exist, especially in order theoretic structures. These are beyond the present scope and deserve further investigation from a closure perspective.

In [BGH92], *epimorphic embeddings* are used as an abstraction of dense embeddings. The present theory encompasses that of [BGH92], and their $\mathcal{S}$-firm epireflections are completions in our sense. In those examples, $(\{\Phi\text{-}dense\} \cap \mathcal{M})$ is the class of epimorphic $\mathcal{M}$-morphisms. There are examples, however, where $(\{\Phi\text{-}dense\} \cap \mathcal{M})$ is strictly contained in $Epi\mathcal{X} \cap \mathcal{M}$. Such examples fall outside of the scope of [BGH92], and it is here where the naturality of the present theory is particularly apparent.

2.7 Example. *Realcompactification in Alexandroff spaces.* Let $\mathcal{X}$ be the category of Alexandroff spaces (also called zero set spaces) as defined in [Hag74] ([Gor71]). Let $(\mathcal{E}, \mathcal{M})$ be the (*Surjection, Embedding*) factorisation structure on $\mathcal{X}$, and $\mathcal{R}$ be the full subcategory of realcompact Alexandroff spaces.

It is shown in [Gil81] that the reflector $R : \mathcal{X} \to \mathcal{R}$ is an $\mathcal{M}$-reflector with each $r_X : X \to RX$ an essential embedding. Thus R preserves $\mathcal{M}$-morphisms, and is a completion reflector.

[Gil81] also shows that R is *not* $(Epi\mathcal{X} \cap \mathcal{M})$-firm. Thus the Φ-dense morphisms are *strictly* contained in $Epi\mathcal{X}$. Moreover, since the epimorphisms in $\mathcal{X}$ are exactly the dense morphisms (with respect to the underlying topological closure), Φ is an idempotent, hereditary closure strictly smaller than the usual topological closure. This closure characterises the realcompact Alexandroff spaces as per Theorem 2.6. We do not know a concise, explicit description of the closure.

2.8 Remark. Simple or direct reflections (cf. [CHK85, BGH97]) induce (*Anti-perfect, Perfect*) factorisation structures on the category $\mathcal{X}$. Any completion reflector (in fact any $\mathcal{M}$-reflector whose pullback closure is hereditary) is a direct reflector. This makes available a number of results regarding factorisation structures.

3. Completion closures

We turn our attention to answering: Which closure operators characterise completion reflectors? If a completion reflector is characterised by a closure operator, how does the induced pullback closure relate to this original closure?

3.1 Definition. A closure operator c on $\mathcal{X}$ with respect to $\mathcal{M}$ is a *completion closure* if the absolutely c-closed objects $\mathcal{A}_c$ are $(\{c\text{-}dense\} \cap \mathcal{M})$-subfirmly reflective in $\mathcal{X}$.

3.2 Remark. For a completion closure c, the following immediate observations are made:

- c-dense $\mathcal{M}$-morphisms are epimorphic. Thus since $\mathcal{E} \subseteq Epi\mathcal{X}$, $\{c\text{-}dense\} \subseteq Epi\mathcal{X}$.
- If Φ is the pullback closure derived from the reflector to $\mathcal{A}_c$, $\{c\text{-}dense\} \subseteq \{\Phi\text{-}dense\}$. (Since if $m \in \mathcal{M}$ is c-dense then $Rm \in Iso\mathcal{X}$.)
- $Inj(\{\Phi\text{-}dense\} \cap \mathcal{M}) \subseteq Inj(\{c\text{-}dense\} \cap \mathcal{M})$.

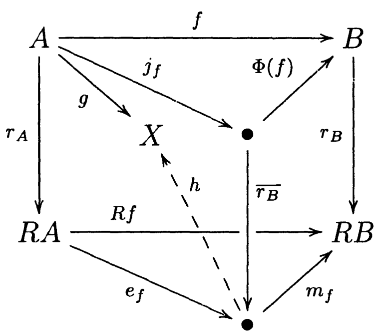
By Theorems 2.4 and 2.6, Φ induced by a completion reflector is indeed a completion closure operator.

3.3 Proposition *Let c be a completion closure and Φ the pullback closure derived from the reflector $R : \mathcal{X} \to \mathcal{A}_c$, then*

$$\mathcal{A}_c = Inj(\{c\text{-}dense\} \cap \mathcal{M}) = Inj(\{\Phi\text{-}dense\} \cap \mathcal{M}) = \mathcal{A}_\Phi.$$

Proof. The first equality follows since any $\mathcal{A}$-subfirmly reflective subcategory is equal to $Inj(\mathcal{A})$ ([BG92] Theorem 1.4). $\mathcal{A}_\Phi = Ob\mathcal{R}$ by Theorem 1.2. All that remains is to show $Inj(\mathcal{D}_c) \subseteq Inj(\mathcal{D}_\Phi)$.

Consider $X \in Inj(\mathcal{D}_c)$, $A \xrightarrow{f} B \in \mathcal{D}_\Phi$ and $g : A \to X$.



By Lemma 1.3 $e_f r_A \in \mathcal{D}_c$. Thus by the $\mathcal{D}_c$-injectivity of X, there is an extension h of g through $e_f r_A$ and $h\overline{r_B}\Phi(f)^{-1}$ is the extension we require. □

3.4 Corollary. *If c is a completion closure then the reflector $R : \mathcal{X} \to \mathcal{A}_c$ is a completion reflector.*

3.5 Proposition. *Let c be a completion closure, and Φ the pullback closure operator induced by the completion reflector $R : \mathcal{X} \to \mathcal{A}_c$, then $c \leq \Phi$.*

Proof. As a completion reflector R is $\mathcal{M}$-preserving, so for any $m \in \mathcal{M}$, $Rm \in \{c\text{-}closed\} \cap \mathcal{M}$. $\Phi(m)$, being the pullback of Rm, is thus c-closed, whence $c \leq \Phi$. □

Thus Φ forms a hull amongst the completion closure operators. What characterises this hull is its idempotence and heredity.

3.6 Lemma. *If c is an hereditary closure operator on $\mathcal{X}$ with respect to $\mathcal{M}$, and $R : \mathcal{X} \to \mathcal{R}$ is a ($\{c$-dense$\} \cap \mathcal{M}$)-reflector, then $\{\Phi$-dense$\} \subseteq \{c$-dense$\}$.*

Proof. Let $m : M \to X$ be Φ-dense. (In the notation of $(*)$) r_M is c-dense, hence also $e_m r_M = \overline{r_X} j_m \in \mathcal{M}$. Thus by [DT95] Theorem 2.5, j_m is c-dense because c is hereditary. But $\Phi(m)$ is an isomorphism, so the result follows. □

3.7 Corollary. *If c is an idempotent, hereditary completion closure operator on $\mathcal{X}$ with respect to $\mathcal{M}$, and Φ is the pullback closure operator induced by $R : \mathcal{X} \to \mathcal{A}_c$, then $c = \Phi$.*

Proof. If c is hereditary, then combining Lemma 3.6 and the second point of Remark 3.2, we conclude that $\{c$-*dense*$\} = \{\Phi$-*dense*$\}$. The idempotence of c then ensures that $c = \Phi$. □

3.8 Proposition. *Let* Compln *be the collection of completion reflectors in $\mathcal{X}$ ordered $R \leq R' \Leftrightarrow Ob\mathcal{R} \subseteq Ob\mathcal{R}'$. Let* ComplnClos *be the collection of all completion closures on $\mathcal{X}$ with respect to $\mathcal{M}$, with the usual closure operator ordering.*

There is a Galois correspondence

$$\text{Compln} \underset{\rho}{\overset{\phi}{\rightleftarrows}} \text{ComplnClos}$$

where $\phi(R)$ is the pullback closure induced by the completion reflector R and $\rho(c)$ is the completion reflector $R : \mathcal{X} \to \mathcal{A}_c$ associated with the completion closure c.

Proof. Take two completion reflectors $R \leq R'$. Since $Ob\mathcal{R} \subseteq Ob\mathcal{R}'$, $\Phi(m)$ is Φ'-closed for any $m \in \mathcal{M}$. Thus $\Phi'(m) \leq \Phi(m)$, and the assignment ϕ reverses order. If two completion closures are ordered $c \leq c'$, then clearly the $\mathcal{A}_{c'} \subseteq \mathcal{A}_c$ and ρ reverses order.

By Proposition 3.5, any completion closure $c \leq \phi \cdot \rho(c)$. Trivially $R = \rho \cdot \phi(R)$ for any completion reflector R, and the result follows. □

Remark. At this point, the assignment ρ is simply given by the definition of a completion closure. In the next section we provide a description of ρ.

The following example demonstrates that while $R = \rho \cdot \phi(R)$, Φ is not in general the *only* completion closure associated with a given completion reflector.

3.9 Example. Let $\mathcal{X}$ be the category with object class $\mathbb{N} \cup \{\infty\}$ with the obvious ordering. For two objects m, n in $\mathcal{X}$, there is a morphism $f : m \to n$ iff $m \leq n$, and then there is only one such morphism. Take $(\mathcal{E}, \mathcal{M}) = (Iso\mathcal{X}, Mor\mathcal{X})$.

For any $k \in \mathbb{N}$ define the closure operator c^k as follows:

$$c^k_m(n) = \begin{cases} \min\{n+k, m\} & \text{for } n \leq m \text{ in } \mathbb{N} \\ \infty & \text{iff } m = \infty \end{cases}$$

Take the reflector $R : \mathcal{X} \to \{\infty\}$, where for $m \in Ob\mathcal{X}$ the reflection is the unique $f : m \to \infty$. Trivially, R is a c^k-dense reflector to $\{\infty\}$, which is also the only absolutely c^k-closed object in $\mathcal{X}$. Since $R[Mor\mathcal{X}] = \{1_\infty\}$, R is $\mathcal{A}$-subfirm for any morphism class $\mathcal{A}$ containing the reflections. These facts together tell us that for every $k \in \mathbb{N}$, c^k is a completion closure, with associated completion reflector $R : \mathcal{X} \to \{\infty\}$.

Each c^k is weakly hereditary, but none is hereditary (consider $m < m + k + 1 < \infty$). Also, no c^k is idempotent. The pullback closure Φ is the indiscrete closure and is the only completion closure that is both idempotent and hereditary. In addition note that $\{\Phi\text{-}dense\} \cap \mathcal{M} = Mor\mathcal{X}$ and $Inj(\{\Phi\text{-}dense\} \cap \mathcal{M}) = \{\infty\}$.

It is worth noting that we could define another set of operators $\hat{c}^k$ for every $k \in \mathbb{N}$ by $\hat{c}^k_m(n) := \max\{c^k_m(n) - 1, n\}$. (Taking $\infty - 1 = \infty$.) None of these $\hat{c}^k$ is even weakly hereditary, yet all the points of the above example apply.

4. Retrieving the completion from a closure

Here we aim to identify the distinctive properties of a completion closure operator c. The $\{c\text{-}dense\} \cap \mathcal{M}$ subfirmness or injectivity property that distinguishes the associated completion reflector is revealed in the fact that every $\mathcal{X}$-object has a *"largest" dense extension.*

4.1 Definition. Let $\mathcal{D}$ be a class of epimorphisms in $\mathcal{X}$. We preorder $\mathcal{D}$ by: $e \sqsubseteq_\mathcal{M} f$ for e, $f \in \mathcal{D}$ iff there is an $m \in \mathcal{M}$ such that $me = f$. (Note that e and f necessarily have common domain.)

4.2 Proposition. *Let c be a completion closure, $R : \mathcal{X} \to \mathcal{A}_c$ the associated completion reflector. Let $X \in Ob\mathcal{X}$, with completion $r_X : X \to RX$. If $X \xrightarrow{d} Y \in (\{c\text{-dense}\} \cap \mathcal{M})$, then $d \sqsubseteq_{\mathcal{M}} r_X$.*

Proof. First recall that $\{c\text{-dense}\} \cap \mathcal{M}$ is indeed a class of epimorphisms containing r_X. If $X \xrightarrow{d} Y \in \{c\text{-dense}\} \cap \mathcal{M}$, then the morphism $(Rd)^{-1}r_Y$ renders $d \sqsubseteq_{\mathcal{M}} r_X$. □

4.3 Crucial properties for a completion closure. Let c be a closure operator on $\mathcal{X}$ with respect to $\mathcal{M}$. Consider the following properties:

1. $\{c\text{-dense}\} \subseteq Epi\mathcal{X}$.
2. The absolutely c-closed objects, $\mathcal{A}_c$, are closed under products and c-closed $\mathcal{M}$-subobjects.
3. For every $X \in Ob\mathcal{X}$, the family of all $(\{c\text{-dense}\} \cap \mathcal{M})$-morphisms with domain X has an upper bound with respect to $\sqsubseteq_{\mathcal{M}}$.

We will refer to these as properties 1 – 3 in the paragraphs below.

Let c be a completion closure. By Remark 3.2 and Proposition 4.2, c satisfies properties 1 and 3. Since $\mathcal{A}_c$ is reflective, it is closed under products. To verify the rest of property 2, let $m : M \to A$ be a c-closed subobject of $A \in \mathcal{A}_c$ and take the reflection $r_M : M \to RM$. There is an extension $m^* : RM \to A$ such that $m^*r_M = m$. Since r_M is c-dense and m is c-closed, the commutative square $m1_M = m^*r_M$ has a diagonal ([DT95] Corollary 2.4) which renders r_M an isomorphism and $M \in \mathcal{A}_c$.

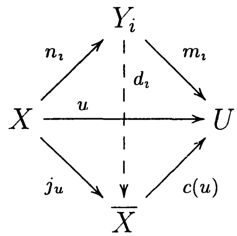
These observations prove necessity in the following result.

4.4 Theorem. *Let $\mathcal{X}$ be cowellpowered with products. An idempotent, weakly hereditary closure operator c on $\mathcal{X}$ with respect to $\mathcal{M}$ is a completion closure operator iff it satisfies properties 1 – 3.*

For an $\mathcal{X}$-object X, the reflection to the absolutely c-closed objects is the $\sqsubseteq_{\mathcal{M}}$-maximum of all $(\{c\text{-dense}\} \cap \mathcal{M})$-morphism with domain X.

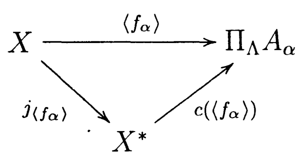
Proof. We show sufficiency. Let $u : X \to U$ be an $\sqsubseteq_{\mathcal{M}}$-upper bound for $(n_i : X \to Y_i)_{i \in I}$ – all $\mathcal{D}_c$-morphisms with domain X. For every $i \in I$, there is an $m_i \in \mathcal{M}$ such that $m_i n_i = u$. As a consequence, $u \in \mathcal{M}$.

Take the c-closure of u.



For every $i \in I$, there is a diagonal d_i for the square $m_i n_i \doteq c(u) j_u$. (Since $c(u)$ is c-closed and each n_i is c-dense.) It follows that $j_u : X \to \overline{X}$ is also an $\sqsubseteq_{\mathcal{M}}$- upper bound for $(n_i)_{i \in I}$. Since $j_u \in \mathcal{D}_c$ it is the *least upper bound.* Note in addition that $\overline{X} \in \mathcal{A}_c$. (Take the c-closure of $\overline{X} \stackrel{m}{\to} Y \in \mathcal{M}$. The composition $j_m j_u \in \{d_i \mid i \in I\}$, thus $j_m j_u \sqsubseteq_{\mathcal{M}} j_u$ and j_u is an isomorphism.)

Now we construct *the reflection from X to $\mathcal{A}_c$.* Since $\mathcal{X}$ is cowellpowered and $\{c\text{-}dense\} \subseteq Epi\mathcal{X}$ we can take a representative set of c-dense morphisms $(f_\alpha : X \to A_\alpha)_{\alpha \in \Lambda}$, where each $A_\alpha \in \mathcal{A}_c$. (This is non-empty by the existence of j_u.) Form the product and induced morphism $\langle f_\alpha \rangle : X \to \Pi_\Lambda A_\alpha$. By assumption $\Pi_\Lambda A_\alpha \in \mathcal{A}_c$. Because $\mathcal{E} \subseteq Epi\mathcal{X}$, the fact that there is an $\alpha_0 \in \Lambda$ for which $f_{\alpha_0} \cong j_u \in \mathcal{M}$ is enough to ensure that $\langle f_\alpha \rangle \in \mathcal{M}$. Take the c-closure of $\langle f_\alpha \rangle$.



It is now routine to check that $j_{\langle f_\alpha \rangle} : X \to X^*$ is the reflection to $\mathcal{A}_c$.

The upper bound and the reflection are isomorphic: Since $j_{\langle f_\alpha \rangle}$ is c-dense $j_{\langle f_\alpha \rangle} \sqsubseteq_{\mathcal{M}} j_u$, giving $m \in \mathcal{M}$ with $m j_{\langle f_\alpha \rangle} = j_u$. As j_u is c-dense, so is m, but $X^* \in \mathcal{A}_c$ which means that m is also c-closed and hence an isomorphism.

Lastly, we put $r_X : X \to RX := j_u : X \to \overline{X}$ and must verify that the reflector we have constructed is $(\{c\text{-}dense\} \cap \mathcal{M})$-subfirm. If $d : X \to Y$ is a $(\{c\text{-}dense\} \cap \mathcal{M})$ morphism, then we can construct $r_X : X \to RX$, $r_Y : Y \to RY$ and the extension $Rd : RX \to RY$ such that $Rd r_X = r_Y d$.

But $r_Y d \sqsubseteq_{\mathcal{M}} r_X$, so there is an $m \in \mathcal{M}$ with $mr_Y d = r_X$. One easily checks that m is an isomorphism, hence also Rd and the proof is complete. □

4.5 Remark. Without the insistence of idempotence and weak heredity, extra conditions would have to be imposed for the proof above to hold. Of course, any completion reflector has at least one idempotent, weakly hereditary closure associated with it – the pullback operator.

5. Examples of completions

The articles [Rin71], [BGH92] and [BG92] contain further examples of completion reflectors. We restrict our attention to standard examples and those which are new in the present context.

5.1 Standard examples in topology. The *usual completion* in each of the categories of *metric spaces, Hausdorff uniform spaces,* and *Hausdorff proximity spaces* (with non-expansive, uniformly continuous and proximally continuous maps respectively) is a completion as defined here. In each case $\mathcal{M}$ is the class of embeddings and the pullback closure is the underlying Kuratowski topological closure.

The corresponding results about absolute closure of complete spaces, the uniqueness of completions and the preservation of embeddings are well known.

In the category $\mathbf{Top}_0$ of T_0 *topological spaces* with continuous maps, the *sober* spaces (those for which every closed-irreducible set is a point closure) form a reflective subcategory, the reflection to which is a completion. This was first revealed in [Hof76] where it is in essence shown that the reflector is firm for epimorphic embeddings. The pullback closure in this case is the b-closure which also describes the epimorphisms in $\mathbf{Top}_0$ (cf. [Bar68]). ($\mathcal{M}$ is again the class of embeddings.)

5.2 Standard examples in Algebra. The reflection to the *Abelian groups* in the category of *cancellative Abelian monoids* (with homomorphisms preserving identity element) is a completion. Here, $\mathcal{M}$ is the class of injective homomorphisms. Writing the operation multiplicatively, the pullback closure of a submonoid N of M is

$$\Phi(N) = \{m \in M \mid \exists\ a, b \in N \text{ with } a = b \cdot m\}.$$

The Abelian groups are exactly those monoids which cannot be extended by this closure. Since the epimorphisms in the Abelian groups are surjective, the epimorphisms in the larger category are the Φ-dense homomorphisms.

In [BGH92] it is pointed out that the *Boolean algebras* are firmly epi-embedding reflective in the category of *bounded distributive lattices* (with homomorphisms preserving top and bottom). Thus the reflection is a completion with $\mathcal{M}$ the class of injective homomorphisms (argue as in 5.3 below). The completion is the embedding of a lattice into its Boolean envelope. For a sublattice M of L, the pullback closure is

$$\Phi(M) = \{l \in L \mid \exists\, m \in M \text{ with } l \wedge m = 0 \text{ and } l \vee m = 1\} \cup M.$$

(Add those complements of elements of M which exist in L.) Clearly the Boolean algebras are absolutely closed with respect to this operator. Φ-density characterises the epimorphisms in the bounded distributive lattices.

5.3 T_0 Quasi-uniform spaces. Details of the *bicompletion* reflection in the category $\mathbf{QUnif}_0$ of quasi-uniform spaces with T_0 first topology (and quasi-uniformly continuous maps) can be found in [FL82]. It was Császár who first demonstrated that the bicomplete spaces (those for which the join of the quasi-uniformity and its inverse generates a complete uniformity) are firmly epi-embedding reflective in $\mathbf{QUnif}_0$.

Let $\mathcal{M}$ be the class of quasi-uniform embeddings and Φ the pullback closure generated by the bicompletion. Being an $(Epi\mathcal{X} \cap \mathcal{M})$-firm reflector, $(Epi\mathcal{X} \cap \mathcal{M}) \subseteq (\Phi\text{-}dense \cap \mathcal{M})$. We know the reverse inclusion is true for any $\mathcal{M}$-reflector, so we conclude that $\{\Phi\text{-}dense\} = Epi\mathcal{X}$ in $\mathbf{QUnif}_0$, and the bicompletion is indeed a completion in the present sense. Moreover it is known ([Hol92] and [DK96]) that the epimorphisms are exactly those which are dense in the join topology. Hence Φ is precisely the Kuratowski closure in the join topology. Bicomplete spaces are absolutely closed with respect to this closure.

5.4 Hausdorff topological groups. Let $\mathbf{TopGrp}_0$ be the category of Hausdorff topological groups with continuous homomorphisms, $\mathcal{M}$ the class of topological embeddings amongst these homomorphisms. In [BG92] it is demonstrated that the *central completion* of a topological group constitutes a firm dense-$\mathcal{M}$-reflector (dense in the underlying topology) to the subcategory of centrally complete topological groups.

At the level of uniform spaces, this reflector acts exactly as the uniform completion reflector. Hence it preserves the embeddings and is a completion reflector in the present sense. Moreover since this is a dense-$\mathcal{M}$-firm reflector, we conclude that Φ is the underlying topological closure. Like example 2.7, this is a case where the Φ-dense morphisms are strictly contained in the epimorphisms which differ from the dense morphisms ([Usp94]).

5.5 MacNeille completion in partially ordered sets. (Thanks to Marcel Erné) In [Ern91] (Corollary 3.3) it is shown that in the category of partially ordered sets with cut-stable monotone maps, the MacNeille completion provides a reflection to the full subcategory of complete lattices with complete homomorphisms. (Complete homomorphisms preserve all joins and meets)

Let $\mathcal{M}$ be the class of cut-stable embeddings. We conclude from [Ern91] Corollaries 3.2 and 3.3 that the MacNeille completion induces an $\mathcal{M}$-reflector. Moreover [Ern91] Proposition 3.4 proves directly that the reflector preserves $\mathcal{M}$-morphisms, and thus this is a completion reflector in the present sense. The pullback closure operator is simply the join closure which characterises complete lattices and the completion process in the given category.

5.6 Completion in nearness frames. (Joint work with Aleš Pultr.) In [BHP97], the authors consider the question of functoriality of the completion of a nearness frame. (A point free analogue of the fundamental work on completion done in [BH78].) They extend the usual completion of a uniform frame to provide a completion coreflection in the category of strong nearness frames and frame homomorphisms.

In this setting, $(\mathcal{E}, \mathcal{M})$ is the usual (*Surjection, Embedding*) factorisation structure for frame homomorphisms. Following the same argumentation as in the proof of Proposition 2.3 above, but in the dual formulation, we see that this completion coreflector preserves $\mathcal{E}$-morphisms. (Instead of Φ-density, simply use the properties of dense homomorphisms.) Thus, if we dualise the situation totally we have a completion reflector in the present sense. The pullback closure describes dense frame homomorphisms in the category of locales.

5.7 Non-examples. In any category where we choose $(\mathcal{E}, \mathcal{M})$ such that $Epi\mathcal{X} = \mathcal{E}$ we would conclude that for a completion reflector $\{\Phi\text{-}dense\} \cap \mathcal{M} \subseteq \mathcal{E}$. From this it follows that $\{\Phi\text{-}dense\} \cap \mathcal{M}$ is the class of isomorphisms and hence that the reflections are isomorphisms. Thus in such situations no non-trivial completion reflector exists.

For k the usual topological closure in the category of *Hausdorff spaces* and continuous maps, the absolutely k-closed objects are the *H-closed spaces.* It has been shown in [HS68] that these spaces are not closed under k-closed subobjects. Thus k cannot be a completion closure in this setting if we take $\mathcal{M}$ to be the embeddings.

Even in **Tych**, the category of Tychonoff spaces and continuous maps, k is not a completion closure for $\mathcal{M}$ the class of embeddings. This is most easily seen by the fact that a Tychonoff space may have non-homeomorphic compactifications. (The compact Hausdorff spaces being the absolutely k-closed spaces.) Note that this means that the Čech-Stone compactification does not preserve embeddings.

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