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ON BRAIDINGS, SYLLAPSES AND SYMMETRIES

by Sjoed CRANS

RÉSUMÉ. A partir du produit tensoriel des **Gray**-catégories, je définis le concept d'un tas (pluriel: teisi) de dimension 4, qui généralise celui d'une **Gray**-catégorie. J'en déduis quelques indications pour une généralisation éventuelle aux dimensions supérieures. Le premier résultat est que les teisi de dimension 4 avec un seul objet et une seule flèche sont les 2-catégories (semistrictes) monoïdales tressées avec des $\tilde{R}$s triviaux. Ensuite, je combine l'idée qu'une 2-catégorie sylleptique devrait être un tas de dimension 5 avec un seul objet, une seule flèche et une seule 2-flèche, avec les indications ci-dessus. Le deuxième résultat est que ceci donne une notion de syllepsis qui est équivalente à celle de Day et Street. De même, une 2-catégorie symétrique devrait être un tas de dimension 6 avec un seul objet, une seule flèche, une seule 2-flèche et une seule 3-flèche, et le troisième résultat est que ceci donne une notion de symétrie qui est encore équivalente à celle de Day et Street. Ces deux résultats finaux s'étendent facilement à des 2-catégories monoïdales tressées et sylleptiques quelque peu plus faibles.

1 Introduction

Braided monoidal categories were introduced by Joyal and Street [22], with motivation from homotopy theory and from higher-dimensional category theory. Braidings are given by algebraic data $R_{A,B} : A \otimes B \to B \otimes A$, satisfying the same axioms as a symmetry but without the requirement that $R_{B,A} \circ R_{A,B} = \mathrm{id}_{A \otimes B}$. Braidings can also be interpreted geometrically, with $R_{A,B}$ being viewed as the crossing of two strings:

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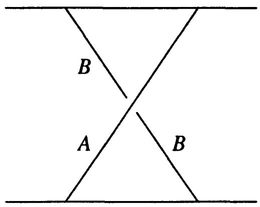

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Joyal and Street [20] have shown how such labeled braids on strings can be used as a graphical notation for calculating with braidings. Applications of braided monoidal categories abound, see for example [14] and [26].

A group (or monoid) can be seen as a category with one object $*$, the elements of the group becoming arrows in the category, and multiplication in the group becoming composition of arrows. Similarly, a monoidal category can be seen as a bicategory with one object, with the tensor product becoming composition. Gordon, Power and Street [15, Corollary 8.7] have shown that braided monoidal categories are precisely tricategories with one object and one arrow, which gives yet another way of interpreting braidings, namely as coming from 0-composition of 2-arrows in a tricategory.

One dimension up, there have been several attempts at defining braided monoidal bicategories. The general strategy here is to invoke the coherence theorem for tricategories [15], which implies that it is sufficient to define braidings on semistrict monoidal 2-categories. The first attempt was by Kapranov and Voevodsky [24, 23], who gave a long list of data and axioms. However, their definition contains several inaccuracies and errors, which was noted by Baez and Neuchl [5], who also gave an improved definition. I further improved the definition by adding axioms for the unit [10]. Day and Street [13], using different terminology, also gave a definition of braided monoidal 2-categories, but with fewer unit axioms.

Kapranov and Voevodsky motivated their definition by referring to their MAIN PRINCIPLE OF CATEGORY THEORY [24, p. 179], that in any category it is unnatural and undesirable to speak about equality of two objects. This PRINCIPLE also motivated more recent developments in the theory of weak n-categories [6, 3, 17], which have now made it possible to define braided monoidal bicategories more conceptually, as tetracategories [32] with one object and one arrow.

In practical situations, having the coherence data around all the time is often undesirable. This is already clear from braided monoidal categories:

if the associativity and unit axioms are not the main interest, one can use a coherence theorem which allows one to restrict to strict braided monoidal categories. This approach has been used very successfully by Joyal and Street [20]. Similarly, one would hope for a coherence theorem for braided monoidal bicategories, which should be a special case of a coherence theorem for tetracategories. Obviously, such a theorem will not involve (strict) 4-categories, as even in dimension 3 not every tricategory is triequivalent to a 3-category. But the fact that every tricategory *is* triequivalent to a **Gray**-category [15] gives strong evidence that the semistrict 4-dimensional categorical structures to which tetracategories will be "tetraequivalent" will have *compositions that are dimension raising.*

Exactly with the application of defining "semistrict 4-categories" in mind, I introduced a tensor product for **Gray**-categories [11]. For **Gray**-categories $\mathbb{C}$ and $\mathbb{D}$, their tensor product $\mathbb{C}\otimes\mathbb{D}$ has as generators expressions $c\otimes d$ of dimension $p+q$ for $c\in C_p$ and $d\in D_q$, for $p+q\leq 3$. The faces of such a generator $c\otimes d$ are composites of generators $c'\otimes d'$ for some specific faces c' and d' of c and d respectively. There are three kinds of relations: naturality, functoriality, and interchange relations. Enriching with respect to the resulting monoidal category **Gray-Cat** gives a 4-dimensional categorical structure in which composition is dimension raising, but *which otherwise is as strict as possible.* I call these structures 4-*dimensional teisi,*[1] and my conjecture is that every weak 4-category is weak 4-equivalent to a 4-dimensional tas.

Now some terminology. A **Gray**-category with only identity 3-arrows is a 2-category, and a 4-dimensional tas with only identity 4-arrows is a **Gray**-category. So it is sensible to define a 2-dimensional tas to be a 2-category, and a 3-dimensional tas to be a **Gray**-category. A strict monoidal category is a 2-category with one object, and a strict braided monoidal category is a **Gray**-category with one object and one arrow. Define a *monoidal* 2*D tas* to be a 3D tas with one object, and a *braided* 2*D tas* to be a 4D tas with one object and one arrow.

What does this imply for braided monoidal 2-categories? In the "weak" point of view, there are coherence constraints wherever possible, in particular, there are 2-isomorphisms

[1] *Tas*, plural teisi (pronounced TAY-see), is Welsh for "stack".

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "R_{A,B\otimes C}"] \arrow[dr, "R_{A,B}\otimes C"'] & {} \arrow[d, phantom, "\Downarrow"] & B\otimes C\otimes A \\
& B\otimes A\otimes C \arrow[ur, "B\otimes R_{A,C}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "R_{A\otimes B,C}"] \arrow[dr, "A\otimes R_{B,C}"'] & {} \arrow[d, phantom, "\Downarrow"] & C\otimes A\otimes B \\
& A\otimes C\otimes B \arrow[ur, "R_{A,C}\otimes B"'] &
\end{tikzcd}$$

In the "teisi" point of view, both braiding and tensor are dimension raising compositions, but the triangles above are for functoriality of the braiding in the tensor, which *hence* are required to be identity 2-arrows. Also, in the "weak" view the braiding is an equivalence, whereas in the "teisi" view it is an isomorphism. The first result of this paper is that these are the only differences between (semistrict) braided monoidal 2-categories (as defined in [10]) and braided 2D teisi. The interpretation of this is that the main obstacles for proving the conjecture above will be the weakness of functoriality and the weakness of invertibility. I should mention here that Baez and Neuchl [5, p. 242] (as corrected by me [10, p. 206]) have shown that either one of the functoriality triangles above can be made into an identity, but it is essential to the proof that the other one is not.

Defining monoidal 2D teisi as 3D teisi with one object involves a shift of dimension: the arrows, 2-arrows and 3-arrows of the 3D tas $\mathbb{C}$ become the objects, arrows and 2-arrows of a 2D tas which will be called the *looping* of $\mathbb{C}$, and denoted $\Omega(\mathbb{C})$, and which is in fact $\mathbb{C}(*,*)$, and 0-composition in $\mathbb{C}$ becomes extra structure on $\Omega(\mathbb{C})$, which could be called (-1)-composition, but is more customarily referred to as tensor. For braided 2D teisi, there is a double shift: the 2-, 3- and 4-arrows of the 4D tas $\mathbb{C}$ are the objects, arrows and 2-arrows of the 2D tas $\Omega^2(\mathbb{C}) = \mathbb{C}(*,*)(\mathrm{id}_*, \mathrm{id}_*)$, 1-composition in $\mathbb{C}$ becomes (-1)-composition, i.e., tensor, in $\Omega^2(\mathbb{C})$, and 0-composition in $\mathbb{C}$ becomes (-2)-composition, or 2-*tensor*, in $\Omega^2(\mathbb{C})$, more customarily referred to as braiding. The extra structure on a 2D tas given by a tensor and a braiding will be called 2-*monoidal* structure; thus '2-monoidal' is synonymous to 'braided'. Looping has an obvious inverse which will be called *delooping*, and which reconsiders a 2D tas $\mathbb{C}$ with a monoidal structure as a 3D tas $\Sigma(\mathbb{C})$ which happens to have one object, and a 2D tas $\mathbb{C}$ with 2-monoidal structure as a 4D tas $\Sigma^2(\mathbb{C})$ which happens to have one object and one arrow.

Having developed this general setup for 2-monoidal structures on 2D teisi, it is obvious that a *k-monoidal nD tas* should be a $(k+n)$D tas having one i-arrow for every $i < k$. The problem with this is that the notion of n-dimensional tas has not been defined yet. The definition of 4-dimensional tas gives some indications for these hypothetical higher-dimensional teisi, in particular, composition should be dimension raising, via the tensor product of globes [9, Section 3-5], but the combinatorics are too daunting at present for a general, rigorous treatment. I could probably define 5-dimensional teisi, but I will refrain from doing so, for reasons of space, clarity of exposition, and because I don't see much point in doing dimension 5, 6, etc., because ultimately the general, n-dimensional, notion will be needed anyway. Instead, I use the indications above to investigate 3- and 4-monoidal structures on 2D teisi. This will be more immediately useful, for example in the theory of 2-tangles [4] and in quantum field theory [2], and gives feedback which should be valuable for the further development of a theory of higher-dimensional teisi.

So far there have been two attempts at defining higher monoidal structures on braided monoidal 2-categories. The first attempt was by Breen [7], who defined weakly and strongly involutory monoidal 2-categories. The second attempt was by Day and Street [13], who, in their different terminology, defined what I have called sylleptic and symmetric monoidal 2-categories [10]. It is clear that the terms 'weakly involutory' and 'strongly involutory' for 3- and 4-monoidal are inappropriate. I would like to use 'sylleptic' as synonymous to '3-monoidal', but at the moment the latter is only a heuristic term, so in this paper I define the notion of *sylleptic 2D tas* separately, based on the indications for higher-dimensional teisi above. The second result of this paper is that, ignoring the difference between braided monoidal 2-categories and braided 2D teisi, this notion of syllepsis is equivalent to Day and Street's. The new notion of syllepsis actually extends easily to somewhat weaker braided monoidal 2-categories, where it is *a priori* more complicated than Day and Street's, but nonetheless still equivalent to it.

I do not want to use 'symmetric' as synonymous to '4-monoidal'. The reason for this is as follows. The indications for higher-dimensional teisi imply that the k-tensor of two objects of a k-monoidal nD tas should give an element of dimension $k-1$. If there are only identity $(k-1)$-arrows, i.e., $n < k-1$, then k-tensor can be considered not as extra structure but as giving an axiom. In other words, for a $(k-1)$-monoidal $(k-2)$D tas, being k-monoidal is a *property*. If $n < k-2$ then the axiom for k-tensor is always satisfied, as it states that some composite of identities is to be equal to some other composite of identities. So this "proves" the Stabilization "Theorem" for teisi, that the forgetful functor from $(k+1)$-monoidal nD teisi to k-monoidal nD teisi is an isomorphism for $k \geq n+2$ (compare Baez and Dolan's Stabilization Hypothesis for weak n-categories [2, p. 6089]). It also suggests the definition of an *ω-monoidal nD tas* as an nD tas which is k-monoidal for every k. I do want to use 'symmetric' as synonymous to 'ω-monoidal'. The stabilization "theorem" then implies that an $(n+2)$-monoidal nD tas is indeed symmetric. Note that for $n = 1$ this is in line with the usual terminology. In this paper I define, for $n = 2$, the notion of *symmetric 2D tas* separately, based on the indications for higher-dimensional teisi above. The third result of this paper is that, again ignoring the difference between braided monoidal 2-categories and braided 2D teisi, this notion of symmetry is equivalent to Day and Street's. As for syllepses, the new notion of symmetry actually extends easily to somewhat weaker sylleptic monoidal 2-categories, where it is *a priori* considerably more complicated than Day and Street's, but nonetheless still equivalent to it.

The first interpretation of these final two results lies in the combinatorics of higher-dimensional teisi. The calculations leading to the definitions of syllepsis and symmetry illustrate the intricate interactions that result from dimension raising, and exemplify some problems that will be encountered in a general, rigorous, higher-dimensional treatment. As such, they give a first glimpse, albeit only up to dimension 6, of what a theory for higher-dimensional teisi might look like.

The second interpretation of these results lies in the difference between weak n-categories and higher-dimensional teisi. In weak n-categories, all compositions are dimension preserving, and higher-dimensional elements are introduced by coherence conditions. Tentative low-dimensional calculations suggest that it should be possible to obtain the dimension raising aspect from the coherence constraints for interchange. This could, in the general case, be a first step in a proof of a coherence theorem for weak n-categories. In this context, the weaker versions of the above results might be useful, as they suggest what dimension raising might mean in a weaker situation. I do not know, however, how to extend Day and Street's [13] motivation for braidings, syllepses and symmetries, in terms of monoidal and braided structures on the homomorphism 'tensor', to dimension raising in weak 5-categories, and clearly more work remains to be done here.

These results, even in their "weak" form, are not relevant to the difference between "weak teisi" and teisi, as the coherence data used for the "weaker" syllepsis and symmetry do not generally occur for 0-composition in a tas.

Hence, the third interpretation of these results lies in the syllepsis and the symmetry itself. Their definition via composition in a tas, i.e., as dimension raising via the tensor product of globes, gives certain expressions for their faces. It is only because they are composition of elements whose faces are appropriate (weak) identities that these expressions can be simplified. So these two results also hold for such elements in a tas with more objects, but they are non-trivial even after a coherence theorem for weak n-categories has been proven.

This paper is organized as follows. In section 2 I define 4-dimensional teisi, and give some indications for higher-dimensional teisi. In section 3 I give a formal treatment of k-monoidal structures on teisi. Then in section x, for $4 \leq x \leq 7$, I use the indications above to investigate $(x-3)$-monoidal 2D teisi, and compare these to monoidal, braided monoidal, sylleptic monoidal and symmetric monoidal 2-categories respectively.

The higher-dimensional diagrams in this paper are to be interpreted as parity structures [31], checking sources and targets of cells; the 2-dimensional diagrams (and the 3-dimensional diagrams giving an equality between 2-dimensional diagrams) are to be interpreted as pasting schemes [18], describing composites (and equalities between such) in a 2-category.

2 Medium dimensional teisi

n-dimensional teisi are intended to be higher-dimensional categorical structures appropriate for a coherence theorem for weak n-categories. In this section I will give precise definitions up to dimension 4, and a heuristic approach for higher dimensions.

2.1 Dimension ≤ 3

Because of the coherence theorems for bi- [29] and tricategories [15], low-dimensional teisi are familiar categorical structures.

Definition 2.1 A 0-*dimensional tas* is a set.
A 1-*dimensional tas* is a category.
A 2-*dimensional tas* is a 2-category. ◇

Recall that **Gray** is the monoidal category of 2-categories and 2-functors with tensor product the pseudo-version of Gray's tensor product of 2-categories [16, 15].

Definition 2.2 A 3-*dimensional tas* is a **Gray**-category. ◇

I will abbreviate this to '3D tas', and reserve '3-tas' for an $(\boldsymbol{\omega}\text{-}\mathbf{Cat})_{\otimes}$-category.

Definition 2.3 A *functor* between n-dimensional teisi is, for $n = 0$ a function, for $n = 1$ a functor, for $n = 2$ a 2-functor, and for $n = 3$ a **Gray**-functor. ◇

This use of 'functor' is consistent with Kelly's [27].

For $n \leq 3$, nD teisi and functors form a category, which will be denoted by n**D-Teisi**.

2.2 Dimension 4

There is no coherence theorem for tetracategories yet, and no familiar 4-dimensional categorical structures except 4-categories, which are too strict.

3D-Teisi is a monoidal category, with tensor product the pseudo version of the tensor product of **Gray**-categories defined in [11].

Definition 2.4 A 4-*dimensional tas* is a 3**D-Teisi**-category.
A *functor* between 4-dimensional teisi is a 3**D-Teisi**-functor. ◇

One can unpack the definition of 4D tas to obtain an explicit list of operations and laws.

Lemma 2.5 *A* 4*-dimensional tas consists of collections C_0 of objects, C_1 of arrows, C_2 of* 2*-arrows, C_3 of* 3*-arrows and C_4 of* 4*-arrows, together with*

- *functions $s_n, t_n : C_i \to C_n$ for all $0 \leq n < i \leq 4$, also denoted d_n^- and d_n^+ and called n-*source *and n-*target*,*
- *functions $\#_n : C_{n+1}\, {}_{s_n}\!\times_{t_n} C_{n+1} \to C_{n+1}$ for all $0 \leq n < 4$, called* vertical composition*,*
- *functions $\#_n : C_i\, {}_{s_n}\!\times_{t_n} C_{n+1} \to C_i$ and $\#_n : C_{n+1}\, {}_{s_n}\!\times_{t_n} C_i \to C_i$ for all $0 \leq n \leq 2$, $n+1 < i < 4$, called* whiskering*,*
- *functions $\#_n : C_q\, {}_{s_n}\!\times_{t_n} C_p \to C_{p+q-n-1}$ for all $0 \leq n \leq 1$, $p,q > n+1$, $p+q-n-1 \leq 4$, called* horizontal composition*, and*
- *functions* $\mathrm{id}_- : C_i \to C_{i+1}$ *for all $0 \leq i \leq 3$, called* identity*,*

such that:

(i) $\mathbb{C}$ *is a* 4*-truncated globular set,*

(ii) for every $C, C' \in C_0$, the collection of elements of $\mathbb{C}$ with 0*-source C and* 0*-target C' forms a* 3*-dimensional tas $\mathbb{C}(C,C')$, with n-composition in $\mathbb{C}(C,C')$ given by $\#_{n+1}$ and identities given by* id_-*,*

(iii) for every $g : C' \to C''$ in C_1 and every C and $C''' \in C_0$, $-\#_0 g$ is a functor $\mathbb{C}(C'',C''') \to \mathbb{C}(C',C''')$ and $g\#_0-$ is a functor $\mathbb{C}(C,C') \to \mathbb{C}(C,C'')$,

(iv) for every $C \in C_0$,

$$s_0(\mathrm{id}_C) \;=\; C \;=\; t_0(\mathrm{id}_C),$$

(v) for every $C' \in C_0$ and every C and $C'' \in C_0$, $-\#_0 \mathrm{id}_{C'}$ is equal to the identity functor $\mathbb{C}(C',C'') \to \mathbb{C}(C',C'')$ and $\mathrm{id}_{C'}\#_0-$ is equal to the identity functor $\mathbb{C}(C,C') \to \mathbb{C}(C,C')$,

(vi) a. for every γ: $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ *in C_2 and* δ : $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ *in C_2,*

$$\begin{aligned} s_2(\delta \#_0 \gamma) &= (g' \#_0 \gamma) \#_1 (\delta \#_0 f) \\ t_2(\delta \#_0 \gamma) &= (\delta \#_0 f') \#_1 (g \#_0 \gamma), \end{aligned}$$

and $\delta \#_0 \gamma$ *is an iso-3-arrow,*

b. for every $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\gamma' \overset{3}{\Lleftarrow} \gamma"] & C' \end{tikzcd}$ *in* C_3 *and* $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ *in* C_2,

$$\begin{aligned} s_3(\delta \#_0 \varphi) &= ((\delta \#_0 f') \#_1 (g \#_0 \varphi)) \#_2 (\delta \#_0 \gamma) \\ t_3(\delta \#_0 \varphi) &= (\delta \#_0 \gamma') \#_2 ((g' \#_0 \varphi) \#_1 (\delta \#_0 f)), \end{aligned}$$

and $\delta \#_0 \varphi$ *is an iso-4-arrow,*

c. for every $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ *in* C_2 *and* $\psi :$ $\begin{tikzcd} C' \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\delta' \overset{3}{\Lleftarrow} \delta"] & C'' \end{tikzcd}$ *in* C_3,

$$\begin{aligned} s_3(\psi \#_0 \gamma) &= (\delta' \#_0 \gamma) \#_2 ((g' \#_0 \gamma) \#_1 (\psi \#_0 f)) \\ t_3(\psi \#_0 \gamma) &= ((\psi \#_0 f') \#_1 (g \#_0 \gamma)) \#_2 (\delta \#_0 \gamma), \end{aligned}$$

and $\psi \#_0 \gamma$ *is an iso-4-arrow,*

(vii) a. for every $\Gamma :$ $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "f'"'] \arrow[r, phantom, "\gamma' \; \varphi \overset{4}{\Lleftarrow} \varphi' \; \gamma"] & C' \end{tikzcd}$ *in* C_4 *and* $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ *in* C_2,

$$\begin{gathered} (((g' \#_0 \Gamma) \#_1 (\delta \#_0 f)) \#_2 (\delta \#_0 \gamma')) \#_3 (\delta \#_0 \varphi) \\ = \\ (\delta \#_0 \varphi') \#_3 ((\delta \#_0 \gamma) \#_2 ((\delta \#_0 f') \#_1 (g \#_0 \Gamma))), \end{gathered}$$

b. for every $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ *in* C_2 *and* $\Delta :$ $\begin{tikzcd} C' \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "g'"'] \arrow[r, phantom, "\delta' \; \psi \overset{4}{\Lleftarrow} \psi' \; \delta"] & C'' \end{tikzcd}$ *in* C_4,

$$
\begin{gathered}
(\psi' \#_0 \gamma) \#_3 ((\delta' \#_0 \gamma) \#_2 ((g' \#_0 \gamma) \#_1 (\Delta \#_0 f))) \\
= \\
(((\Delta \#_0 f') \#_1 (g \#_0 \gamma)) \#_2 (\delta \#_0 \gamma)) \#_3 (\psi \#_0 \gamma),
\end{gathered}
$$

c. *for every* $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\gamma' \overset{3}{\Lleftarrow} \gamma"] & C' \end{tikzcd}$ *in* C_3 *and* $\psi :$ $\begin{tikzcd} C' \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\delta' \overset{3}{\Lleftarrow} \delta"] & C'' \end{tikzcd}$ *in* C_3,

$$
\begin{gathered}
(((\psi \#_0 f') \#_1 (g \#_0 \gamma)) \#_2 (\delta \#_0 \varphi)) \#_3 \\
(((\psi \#_0 f') \#_1 (g \#_0 \varphi)) \#_2 (\delta \#_0 \gamma)) \#_3 \\
(((\delta' \#_0 f') \#_1 (g \#_0 \varphi)) \#_2 (\psi \#_0 \gamma)) \\
= \\
(\psi \#_0 \gamma') \#_2 ((g' \#_0 \varphi) \#_1 (\delta \#_0 f))) \#_3 \\
((\delta' \#_0 \gamma') \#_2 ((g' \#_0 \varphi) \#_1 (\psi \#_0 f))^{-1}) \#_3 \\
((\delta' \#_0 \varphi) \#_2 ((g' \#_0 \gamma) \#_1 (\psi \#_0 f))),
\end{gathered}
$$

(viii) a. *for every* $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3ex, "\Downarrow\gamma'"] & C' \end{tikzcd}$ *and* $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ *in* $\mathbb{C}$,

$$
\delta \#_0 (\gamma' \#_1 \gamma) \;=\; ((\delta \#_0 \gamma') \#_1 (g \#_0 \gamma)) \#_2 ((g' \#_0 \gamma') \#_1 (\delta \#_0 \gamma)),
$$

b. *for every* $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ *and* $\begin{tikzcd} C' \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\delta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\delta'"] & C'' \end{tikzcd}$ *in* $\mathbb{C}$,

$$
(\delta' \#_1 \delta) \#_0 \gamma \;=\; ((\delta' \#_0 f') \#_1 (\delta \#_0 \gamma)) \#_2 ((\delta' \#_0 \gamma) \#_1 (\delta \#_0 f)),
$$

c. *for every* $\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\gamma' \overset{\varphi'}{\Lleftarrow} \gamma' \overset{\varphi}{\Lleftarrow} \gamma"] & C' \end{tikzcd}$ *and* $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ *in* $\mathbb{C}$,

$$\delta \#_0 (\varphi' \#_2 \varphi) \quad = \quad \begin{array}{c} ((\delta \#_0 \varphi') \#_2 ((g' \#_0 \varphi) \#_1 (\delta \#_0 f))) \\ \#_3 \\ (((\delta \#_0 f) \#_1 (g \#_0 \varphi')) \#_2 (\delta \#_0 \varphi)), \end{array}$$

d. for every $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=4, "\gamma' \overset{\varphi}{\Rrightarrow} \gamma"] \arrow[r, phantom, shift right=4, "\Downarrow\gamma'"] & C' \end{tikzcd}$ *and* $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ *in* $\mathbb{C}$,

$$\delta \#_0 (\gamma' \#_1 \varphi) \quad = \quad \begin{array}{c} (((\delta \#_0 \gamma'') \#_1 (g \#_0 \gamma')) \#_2 ((g' \#_0 \gamma'') \#_1 (\delta \#_0 \varphi))) \\ \#_3 \\ (((\delta \#_0 \gamma'') \#_1 (g \#_0 \varphi)) \#_2 ((g' \#_0 \gamma'') \#_1 (\delta \#_0 \gamma))), \end{array}$$

e. for every $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=4, "\Downarrow\gamma"] \arrow[r, phantom, shift right=4, "\gamma' \overset{\varphi'}{\Rrightarrow} \gamma"] & C' \end{tikzcd}$ *and* $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ *in* $\mathbb{C}$,

$$\delta \#_0 (\varphi' \#_1 \gamma) \quad = \quad \begin{array}{c} (((\delta \#_0 \gamma'') \#_1 (g \#_0 \gamma)) \#_2 ((g \#_0 \varphi') \#_1 (\delta \#_0 \gamma))^{-1}) \\ \#_3 \\ (((\delta \#_0 \varphi') \#_1 (g \#_0 \gamma)) \#_2 ((g' \#_0 \gamma') \#_1 (\delta \#_0 \gamma))), \end{array}$$

f. for every $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "f'"'] \arrow[r, phantom, "\gamma \overset{3}{\Rrightarrow} \gamma"] & C' \end{tikzcd}$ *and* $\begin{tikzcd} C' \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=4, "\Downarrow\delta"] \arrow[r, phantom, shift right=4, "\Downarrow\delta'"] & C'' \end{tikzcd}$ *in* $\mathbb{C}$,

$$(\delta' \#_1 \delta) \#_0 \varphi \quad = \quad \begin{array}{c} (((\delta' \#_0 f') \#_1 (\delta' \#_0 \gamma')) \#_2 ((\delta' \#_0 \varphi) \#_1 (\delta \#_0 f))) \\ \#_3 \\ (((\delta' \#_0 f') \#_1 (\delta' \#_0 \varphi)) \#_2 ((\delta' \#_0 \gamma) \#_1 (\delta \#_0 f))), \end{array}$$

g.-j. Analogous to c.-f.,

(ix) for every $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=4, "\Downarrow\gamma"] \arrow[r, phantom, shift right=4, "\Downarrow\gamma'"] & C' \end{tikzcd}$ *and* $\begin{tikzcd} C' \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=4, "\Downarrow\delta"] \arrow[r, phantom, shift right=4, "\Downarrow\delta'"] & C'' \end{tikzcd}$ *in* $\mathbb{C}$,

$$\begin{array}{c} ((\delta' \#_0 f'') \#_1 (\delta \#_0 \gamma') \#_1 (g \#_0 \gamma)) \\ \#_2 \\ ((\delta' \#_0 \gamma') \#_1 (\delta \#_0 \gamma)) \\ \#_2 \\ ((g'' \#_0 \gamma') \#_1 (\delta' \#_0 \gamma) \#_1 (\delta \#_0 f)) \end{array} \quad = \quad \mathrm{id}_{(\delta' \#_1 \delta) \#_0 (\gamma' \#_1 \gamma)} \,,$$

(x) *for every* $c \in \mathbb{C}(C,C')_p$, $c' \in \mathbb{C}(C',C'')_q$ *and* $c'' \in \mathbb{C}(C'',C''')_r$ *with* $p+q+r \leq 3$,

$$(c'' \#_0 c') \#_0 c \quad = \quad c'' \#_0 (c' \#_0 c),$$

(xi) *for every* $c \in \mathbb{C}(C,C')_p$, $c' \in \mathbb{C}(C',C'')_q$ *with* $p,q > 0$, $p \leq 2$, $p+q \leq 3$,

$$c' \#_0 \mathrm{id}_c \quad = \quad \mathrm{id}_{c' \#_0 c},$$

and for every $c \in \mathbb{C}(C,C')_p$, $c' \in \mathbb{C}(C',C'')_q$ *with* $p,q > 0$, $q \leq 2$, $p+q \leq 3$,

$$\mathrm{id}_{c'} \#_0 c \quad = \quad \mathrm{id}_{c' \#_0 c}\,.$$

□

In this lemma, condition (ii) gives the "local" structure, conditions (iii) and (v) give the behaviour of "whiskering", condition (vi) gives the faces of a 0-composite, condition (vii) gives the *naturality axioms*, which describe behaviour of 0-composition with respect to higher dimensional cells, conditions (viii) and (xi) give the *functoriality* axioms, which describe behaviour of horizontal composition with respect to composition and identity, condition (ix) gives the *interchange* axiom, which describes the behaviour of horizontal composition with respect to composition *in both variables at the same time*, and condition (x) gives that $\#_0$ is *associative*.

Observe that the higher-dimensional functoriality axioms, conditions (viii) c.-j., depend on the lower-dimensional functoriality axioms, that the interchange axiom depends on those as well, and that the associativity axiom, (x), depends on those and on lower-dimensional instances of itself.

2.3 Higher dimensions

I will now not describe 5-dimensional teisi in elementary terms, define a tensor product, define 6-dimensional teisi, and so on to higher dimensions, for two reasons. The first reason is that this would involve increasingly complex equations, which would quickly become unmanageable. The second, related, reason is that I don't see much point in doing dimension 5, 6, etc., because ultimately the general, n-dimensional, notion will be needed anyway.

Instead, the elementary description of 4-dimensional teisi above gives some indications for the -hypothetical- general, higher-dimensional notion of ω-tas: it should have a dimension raising composition, which should satisfy functoriality, interchange, and associativity axioms. More precisely, but still heuristic and incomplete, one arrives at the following:
an ω-*tas* $\mathbb{C}$ should be a graded set $(C_i)_{i\in\omega}$ together with

- operations $s_n, t_n : A_i \to A_n$, called *n-source* and *n-target* respectively, for all $i > n$, also denoted d_n^- and d_n^+,
- operations $\#_n : C_q\, {}_{s_n}\!\times_{t_n} C_p \to C_{p+q-n-1}$, called *n-composition*, for every n, (with the convention that if $n > m$ then $\#_n$ binds stronger,) and
- operations $\mathrm{id}_- : C_i \to C_{i+1}$, called *identity*, (with the convention that $\mathrm{id}^k_- : C_i \to C_{i+k}$ denotes the composite of k id_-'s,)

such that

(i) $d^\beta_m d^\alpha_n = d^\beta_m$ for all m, n with $m < n$, and $\alpha, \beta = \pm$,

(ii)
$$
\begin{array}{rcll}
d^\alpha_m(c' \#_n c) & = & d^\alpha_m(c) & \text{if } m < n \\
 & = & d^-_n(c) & \text{if } m = n \text{ and } \alpha = - \\
 & = & d^+_n(c') & \text{if } m = n \text{ and } \alpha = + \\
 & = & d^\alpha_m(c') \#_n c & \text{for } c \in C_{n+1} \text{ and } c' \in C \ (c \in C \text{ and } c' \in C_{n+1} \text{ similar) if } m > n, \text{ this is called } \textit{vertical} \text{ composition} \\
 & \text{is} & \text{“governed by } d^\alpha_{m-n-1}(2_{p-n-1} \otimes 2_{q-n-1})\text{”} & \text{for } c \in C_p,\ c' \in C_q \text{ with } p, q > n+1 \text{ if } m > n, \text{ this is called } \textit{horizontal} \text{ composition}
\end{array}
$$

for all m, n, and $\alpha = \pm$,

(iii) (functoriality of $\#_m$ in $\#_n$) $c'' \#_n c' \#_m c$ “=” $(c'' \#_m c) \#_{p+(n-m)-1} (c' \#_m c)$, for $c \in C_p$, $c', c'' \in C$ and all m, n with $n > m$ ($c'' \#_m c' \#_n c$ similar),

(iv) (functoriality of $\#_m$ in $\#_n$ in both variables *at the same time* / interchange) $\mathrm{id}^{n-m}_{d' \#_n d \#_m c' \#_n c}$ “=” $(d' \#_m c') \#_n (d \#_m c)$ for $c, c', d, d' \in C$ and all m,n with $n > m$,

(v) (associativity) $(c'' \#_n c') \#_n c = c'' \#_n (c' \#_n c)$ for $c, c', c'' \in C$ and all n,

(vi) $d_m^\alpha(\mathrm{id}_c) = d_m^\alpha(c)$ for $c \in C_p$ with $p \geq m$ and all m,

(vii) $$\begin{aligned} c' \#_n \mathrm{id}_c &= c' && \text{for } c \in C_{n+1} \text{ and } c' \in C \\ &= \mathrm{id}_{c' \#_n c} && \text{for } c \in C_p \text{ and } c' \in C \text{ with } p > n+1 \end{aligned}$$
for all n ($\mathrm{id}_{c'} \#_n c$ similar). ◇

In an ω-tas there are two kinds of composition. A composition is vertical when the dimension of (at least) one of the elements to be composed is exactly one higher than the direction of composition (this includes what has previously been called *whiskering*), and horizontal otherwise, i.e., when the dimension of (each of) the elements is at least two higher than the direction of composition. The dimension of a vertical composite is the maximum of the dimensions of the elements, while the dimension of a horizontal composite is *higher* than the maximum of the dimensions of the elements.

Vertical composition can be seen as a special case of horizontal composition, namely by removing the condition that $p, q > n+1$ from the latter.

In horizontal composition in an ω-tas being "governed by $2_{p-n-1} \otimes 2_{q-n-1}$", 2_i denotes the free ω-tas on one i-dimensional element. The phrase indicates that the m-source and m-target of $c' \#_n c$ contain certain $d_{q'}^{\beta'}(c') \#_n d_{p'}^{\alpha'}(c)$'s for $p' + q' \leq m$, positioned relative to each other according to the tensor product of globes [9, Section 3-5]. Composition being "governed by $2_{p-n-1} \otimes 2_{q-n-1}$", rather than "by $2_{q-n-1} \otimes 2_{p-n-1}$", makes that the domain of a composition, vertical or horizontal, is a composite involving the domain of the first factor. It also gives formulae of a form which is familiar from homological algebra, with exactly the same sign conventions.

The quotation marks indicate that these statements are not precise, but are heuristic statements standing for as yet unknown more complex assertions involving large compositions involving the indicated composites. Apart from the combinatorial complexity, there are three conceptual problems preventing a rigorous definition.

Firstly, I do know that the m-source (target) of a horizontal composite $c' \#_n c$ will be a composite of $d_{q'}^{\beta'}(c') \#_n d_{p'}^{\alpha'}(c)$'s for $p' + q' \leq m$, where

$\alpha' = -$ $(+)$ if $p' < p$, $\beta' = (-)^{p'+1}$ $((-)^{p'})$ if $q' < q$ [9, Lemma 3-5.5]. I do not know how take the other horizontal compositions occurring in this configuration into account. This problem is deeper than just finding out where these horizontal compositions occur. Because horizontal composition raises dimension, horizontal composites of low dimensional elements occur alongside high dimensional elements, and it transpires that if a consistent choice for the direction of a horizontal composite is made, some of these horizontal composites of low dimensional elements run in the wrong direction. This means that to make sense of a composition like this, a horizontal composition needs to result in elements running in *both* directions. There are various possible solutions to this: just having the two elements with no relation, requiring the elements to be equivalences of some kind, or requiring them to be isomorphisms, as I have done for 4-dimensional teisi. Whatever solution is chosen, one still needs a way to determine which direction is to be used when.

Secondly, horizontal composites can themselves be horizontally composable with other elements, or even with other horizontal composites. This can result in the composite for a source or target of a horizontal composition being of higher dimension than the horizontal composite itself. A way needs to be found to deal with this as well.

Thirdly, the axioms should imply all other possible functoriality and associativity conditions, e.g., $(c''' \#_n c'' \#_n c') \#_n c = (c''' \#_n c'') \#_n (c' \#_n c) = c''' \#_n (c'' \#_n c' \#_n c)$, and combinations of those, e.g., *middle eight interchange*, and many more. It is not at all clear what would be a minimal set of axioms.

An example of the first conceptual problem already occurs in 4-dimensional teisi, and more pronounced on pages and . An example of the second conceptual problem occurs on page , and of the third conceptual problem on page .

Going back to finite dimensions,
an *n-dimensional tas* is an ω-tas $\mathbb{C}$ for which $C_i = C_n$ and $\mathrm{id}_- = \mathrm{id}_{C_n} : C_i \to C_{i+1}$ for all $i \geq n$. $\diamond$

In dimensions ≤ 4 this "agrees" with the definitions above.

In the sequel I will refer to nD and ω-teisi, but it should be remembered that at this point in time such statements are only mathematically precise for $n \leq 4$, and merely heuristic, incomplete, motivational arguments for $5 \leq n \leq \omega$.

Let $\mathbb{C}$ and $\mathbb{D}$ be ω-teisi or nD teisi. A *functor* $\mathbb{C} \to \mathbb{D}$ is a morphism of globular sets $f : C \to D$ respecting all n-compositions and identities, i.e., for which $f(c' \#_n c) = f(c') \#_n f(c)$ and $f(\mathrm{id}_c) = \mathrm{id}_{f(c)}$ for $c, c' \in \mathbb{C}$ and all n. $\diamond$

Again, in dimensions ≤ 4 this "agrees" with the definitions above.

The category of ω-teisi and functors will be denoted by **ω-Teisi**, and the category of nD teisi and functors by n**D-Teisi**.

2.4 Between the dimensions

It follows from the definition of nD teisi that an nD tas whose i-arrows for $k < i \leq n$ are identities is precisely a kD tas.

There is no "underlying nD tas functor", i.e., a right adjoint to the inclusion n**D-Teisi** $\hookrightarrow$ $(n+i)$**D-Teisi** for $0 < i \leq \infty$, because truncation doesn't result in an nD tas. There *is* a left adjoint, given by factoring out $(n+1)$-arrows of the $(n+i)$D or ω-tas.

2.5 Localization

In an ω-tas, composition is *dimension invariant*, i.e., the data and axioms depend on the dimension of the elements only relative to the direction of composition. This makes it possible to *localize*.

Definition 2.6 Let c and c' be k-arrows of an ω-tas $\mathbb{C}$ satisfying $d^{\alpha}_{k-1}(c) = d^{\alpha}_{k-1}(c')$ for $\alpha = \pm$. $\mathbb{C}(c, c')$, the *localization of* $\mathbb{C}$ *at* c *and* c', is the globular set having as i-dimensional elements $d \in C_{i+k+1}$ for which $s_k(d) = c$ and $t_k(d) = c'$, with m-faces given by d^{α}_{m+k+1} in $\mathbb{C}$. $\diamond$

Proposition 2.7 *Let c and c' be k-arrows of an ω-tas $\mathbb{C}$ satisfying $d^{\alpha}_{k-1}(c) = d^{\alpha}_{k-1}(c')$ for $\alpha = \pm$. Then the globular set $\mathbb{C}(c, c')$, with n-composition given by $(n+k+1)$-composition in $\mathbb{C}$, and with identity given by identity in $\mathbb{C}$, is an ω-tas.*

If $\mathbb{C}$ is an nD tas, then the ω-tas $\mathbb{C}(c, c')$ is an $(n-k)$D tas. $\square$

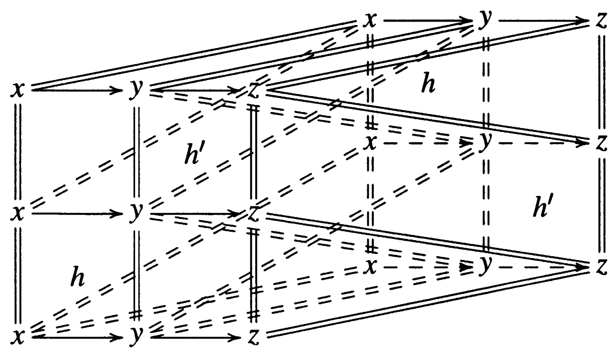
If $c = c'$, I will mention c only once, denoting $\mathbb{C}(c, c')$ by $\mathbb{C}_c$. More about $\mathbb{C}_c$ later on, see page .

2.6 The enriched viewpoint

There should be a tensor product of ω-teisi such that n-composition in $\mathbb{C}$ is given by a functor $\mathbb{C}(c,c') \otimes \mathbb{C}(c',c'') \to \mathbb{C}(c,c'')$. For composition in 4-dimensional teisi this is, by definition, the case.

2.7 Fundamental iso-teisi

A major motivation for the study of ω-teisi is that an example of an ω-tas should be the "fundamental ω-iso-tas". For a topological space X, the n-arrows of its fundamental ω-iso-tas $\Pi(X)$ should be certain equivalence classes of n-dimensional homotopies, i.e., continuous functions $I^n \to X$ which are constant on appropriate faces, with the other faces giving its sources and targets. Composition should be given by some sort of "modified juxtaposition of cubes", in particular, horizontal composition of 2-arrows h and h' should be represented by



.

The fundamental nD iso-tas has been defined up to dimension 2, where it is defined via the 2-categorical nerve. More about this example later on, see page 22.

3 k-monoidal teisi

k-monoidal nD teisi are intended to be the higher-dimensional generalization of strict monoidal categories and so on. Because of this, they should be applicable to tangles [21], higher-dimensional quantum integrable systems [24], topological quantum field theory [5] and representation theory [13], and to connected homotopy types [19, 8] and homotopy of k-fold loop spaces [1, 30].

In this section I will give a formal general treatment of k-monoidal teisi, rather than restricting myself to 4D teisi, because I think that the infinite dimensional approach gives more coherent view. But it should be remembered that statements about nD teisi are only mathematically precise for $n \leq 4$, and are merely heuristic, incomplete, motivational arguments for $5 \leq n \leq \omega$.

3.1 Delooping and looping

Definition 3.1 A *pointed* ω*-tas* is an ω-tas $\mathbb{C}$ together with an object $C \in \mathbb{C}$, the *base point.* $\boldsymbol{\omega}$**-Teisi**$_*$ (nD-**Teisi**$_*$) is the category consisting of pointed ω-teisi (nD teisi) and base point preserving functors. ◇

There is an underlying functor $U : \boldsymbol{\omega}\textbf{-Teisi}_* \to \boldsymbol{\omega}\textbf{-Teisi}$ forgetting the base point.

Definition 3.2 For $k > 0$, a k*-monoidal* ω*-tas* is an ω-tas having one $(k-1)$-arrow. $\bigotimes^k$-$\boldsymbol{\omega}$**-Teisi** is the full subcategory of $\boldsymbol{\omega}$**-Teisi** consisting of k-monoidal ω-teisi. $\bigotimes^0\text{-}\boldsymbol{\omega}\textbf{-Teisi} = \boldsymbol{\omega}\textbf{-Teisi}_*$, the category of pointed ω-teisi and base point preserving functors.

For $k > 0$, a *k-monoidal nD tas* is a $(k+n)$D tas having one $(k-1)$-arrow. $\bigotimes^k$-n**D-Teisi** is the full subcategory of $(k+n)$**D-Teisi** consisting of k-monoidal nD teisi. $\bigotimes^0\text{-}n\textbf{D-Teisi} = n\textbf{D-Teisi}_*$, the category of pointed nD-teisi and base point preserving functors. ◇

1-monoidal is also called *monoidal,* 2-monoidal is also called *braided,* and 3-monoidal is also called *sylleptic.*

Because a k-monoidal ω-tas has only one object, $\bigotimes^k$-$\boldsymbol{\omega}$**-Teisi** is also the full subcategory of $\boldsymbol{\omega}$**-Teisi**$_*$ consisting of k-monoidal ω-teisi. Either inclusion will be denoted by Σ^k. $\Sigma^k(\mathbb{C})$ is called the *(total) delooping of* $\mathbb{C}$.

The key point of this definition is that k-monoidal ω-teisi are considered as ω-teisi with extra structure. This is more pronounced in the finite dimensional case: a k-monoidal nD tas is actually a $(k+n)$D tas where the dimensions below k are regarded as k-monoidal structure. To emphasize this interpretation, the dimensions will be renumbered: if $\mathbb{C}$ is an ω-tas with one $(k-1)$-arrow, the i-arrows of $\mathbb{C}$ will be seen as the $(i-k)$-arrows of the corresponding k-monoidal ω-tas, and the n-compositions of $\mathbb{C}$ will be seen as $(n-k)$-compositions of the k-monoidal ω-tas. $(-n)$-composition is also called *n-tensor*, and will also be denoted by $\otimes_n$. So a k-monoidal ω-tas is an ω-tas which also has one $(-i)$-arrow for every $0 < i \leq k$, and extra operations $\otimes_n$ for every $0 < n \leq k$. This *reindexing* results in a *dichotomy of structures*: one can see $\#_{k-n}$ as composition in the ω-tas with one $(k-1)$-arrow *or* as the n-tensor on the k-monoidal ω-tas.

Definition 3.3 An *ω-monoidal ω-tas* is an ω-tas which is k-monoidal for every k. $\bigotimes^{\omega}$**-ω-Teisi** is the subcategory of **ω-Teisi** consisting of ω-monoidal ω-teisi and functors which preserve all k-monoidal structures.

An *ω-monoidal nD tas* is an nD tas which is k-monoidal for every k. $\bigotimes^{\omega}$-n**D-Teisi** is the subcategory of n**D-Teisi** consisting of ω-monoidal nD teisi and functors which preserve all k-monoidal structures. ◇

ω-monoidal is also called *symmetric*.

Some low dimensional examples. A monoidal set is a monoid, and Σ reconsiders it as a category with one object. Similarly, a monoidal category is more commonly known as a strict monoidal category, and Σ reinterprets it as a 2-category with one object. For $k \geq 2$, a k-monoidal set is actually a commutative monoid, which is exactly the reason why the higher homotopy groups are abelian. A 2-monoidal category is indeed a braided (strict) monoidal category, which justifies my terminology, and which was the way braided monoidal categories were originally conceived [22, Section 5]. Σ^2 reconsiders it as a **Gray**-category with one arrow. A 3-monoidal category is a symmetric strict monoidal category.

If X is a k-connected topological space, then its "fundamental ω-iso-tas" $\Pi(X)$ should be (weakly equivalent to) an ω-tas with one k-arrow. Joyal [19] has proven that k-monoidal 2D teisi classify k-connected $(2+k)$-types, and a similar result is expected in the general case.

Another example is the Tangle Hypothesis for teisi, that n-tangles in $n+k$ dimensions should form a k-monoidal nD tas, in fact the free k-monoidal nD tas "with duals" (whatever that may mean) on one object (compare Baez and Dolan's Tangle Hypothesis for weak n-categories [2, p. 6095]). There is also an infinite version, namely that n-tangles in infinitely many dimensions should form an ω-monoidal nD tas, in fact the free ω-monoidal nD tas "with duals" on one object.

A non-example is **ω-Teisi** itself: it is not monoidal because a tensor product of ω-teisi will not be *strictly* associative (compare the situation for 3D teisi [11, Section 7.1].

There is an inclusion

$$\Sigma^i : \bigotimes^k\text{-}\boldsymbol{\omega}\textbf{-Teisi} \hookrightarrow \bigotimes^{k-i}\text{-}\boldsymbol{\omega}\textbf{-Teisi}$$

for every k and $0 \leq i \leq k$. For an ω-tas $\mathbb{C}$ with a k-monoidal structure, $\Sigma^i(\mathbb{C})$ is the same data considered as an ω-tas which happens to have one i-arrow, with a $(k-i)$-monoidal structure. $\Sigma^i(\mathbb{C})$ is called the *i-th delooping of* $\mathbb{C}$.

For finite dimensional teisi,

$$\Sigma^i : \bigotimes^k\text{-}n\textbf{D-Teisi} \hookrightarrow \bigotimes^{k-i}\text{-}(n+i)\textbf{D-Teisi}.$$

Observe that $\Sigma^i\Sigma^j = \Sigma^{i+j}$ for all $i, j \geq 0$, $i + j \leq k$. So Σ^i is indeed applying Σ i times. Also, the notation Σ^k is unambiguous, i.e., the k-th delooping is equal to the total one.

There are also functors

$$U : \bigotimes^k\text{-}\boldsymbol{\omega}\textbf{-Teisi} \longrightarrow \bigotimes^j\text{-}\boldsymbol{\omega}\textbf{-Teisi}$$

for $j < k$ which simply forget $\#_{-n}$ for all $n > j$. Note that as an ω-tas with k-monoidal structure is canonically pointed, namely by id_*, or, which is the same thing, by the unit object, this also makes sense for $j = 0$.

For finite dimensional teisi,

$$U : \bigotimes^k\text{-}n\textbf{D-Teisi} \longrightarrow \bigotimes^j\text{-}n\textbf{D-Teisi}.$$

Lemma 3.4 $\bigotimes^\omega$**-ω-Teisi** *is the limit of the sequence*

$$\cdots \longrightarrow \bigotimes^k\text{-}\boldsymbol{\omega}\textbf{-Teisi} \xrightarrow{U} \bigotimes^{k-1}\text{-}\boldsymbol{\omega}\textbf{-Teisi} \longrightarrow \cdots \longrightarrow \boldsymbol{\omega}\textbf{-Teisi}_*.$$

□

Having interpreted Σ^k as reconsidering an ω-tas with extra structure as an ω-tas with one $(k-1)$-arrow, there is also a converse reconsideration. This converse can be defined more generally.

Definition 3.5 Let $k > 0$ and let $(\mathbb{C}, C)$ be a pointed ω-tas. $\Omega^k(\mathbb{C}, C)$ is the sub-ω-tas of $\mathbb{C}$ having as only $(k-1)$-arrow id_C^{k-1}. ◇

Ω^k is a functor $\boldsymbol{\omega}\textbf{-Teisi}_* \to \boldsymbol{\omega}\textbf{-Teisi}_*$, which can and will be considered as a functor $\boldsymbol{\omega}$**-Teisi**$_*$ $\to \bigotimes^k$-$\boldsymbol{\omega}$**-Teisi**. $\Omega^k(\mathbb{C})$ is called the *k-th loop* ω*-tas of* $\mathbb{C}$.

There is an intimate relation between localization of an ω-tas at an identity and looping, which would be an alternative way to define $\Omega^k(\mathbb{C})$.

Lemma 3.6 *Let* $k > 0$ *and let* $(\mathbb{C}, C)$ *be a pointed* ω*-tas. Then* $U(\Omega^k(\mathbb{C},C)) = \mathbb{C}(\mathrm{id}_C^{k-1}, \mathrm{id}_C^{k-1}) = \mathbb{C}_{\mathrm{id}_C^{k-1}}$. □

So the ω-tas obtained by localizing at an identity has extra structure, coming from $\#_i$ for $i < k$.

Looping of ω-teisi should be closely connected to looping in topology, via a diagram

$$\begin{tikzcd}
\mathbf{Top}_* \arrow[r, "\Pi"] \arrow[dd, "\Omega^k"'] & \boldsymbol{\omega}\textbf{-Teisi}_* \arrow[d, "\Omega^k"] \\
& \bigotimes^k\text{-}\boldsymbol{\omega}\textbf{-Teisi} \arrow[d, "U"] \\
\mathbf{Top}_* \arrow[r, "\Pi"'] & \boldsymbol{\omega}\textbf{-Teisi}_*
\end{tikzcd}$$

which should commute up to weak equivalence. Consequently, k-monoidal ω-teisi should be related to k-fold loop spaces. This will not solve the recognition problem for k-fold loop spaces though, as the problem of determining whether a given ω-tas is k-monoidal is probably equally difficult.

By restricting to appropriate subcategories, there are functors

$$\Omega^i : \bigotimes^k\text{-}\boldsymbol{\omega}\textbf{-Teisi} \longrightarrow \bigotimes^{k+i}\text{-}\boldsymbol{\omega}\textbf{-Teisi}$$

for every k and $i \geq 0$.

For finite dimensional teisi,

$$\Omega^i : \bigotimes^k\text{-}n\mathbf{D}\textbf{-Teisi} \longrightarrow \bigotimes^{k+i}\text{-}(n-i)\mathbf{D}\textbf{-Teisi}.$$

Observe that $\Omega^i\Omega^j = \Omega^{i+j}$ for all $i, j \geq 0$. So Ω^i is indeed applying Ω i times, and the notation Ω^k is unambiguous.

To justify the interpretation of Ω^i as a converse to Σ^i, observe that $\Omega^i\Sigma^i = \mathrm{id}_{\bigotimes^k\text{-}\boldsymbol{\omega}\textbf{-Teisi}}$, and that $\Sigma^i\Omega^i(\mathbb{C})$ is a sub-ω-tas of $\mathbb{C}$ which is equal to $\mathbb{C}$ precisely when $\mathbb{C}$ has one $(i-1)$-arrow.

Proposition 3.7 *Σ^i is left adjoint right inverse to Ω^i.*

Proof. The unit of the adjunction is indeed the identity, and the counit is the inclusion just mentioned. One of the triangular identities follows from the remark above and the other one holds because Ω^i of the inclusion $\Sigma^i\Omega^i(\mathbb{C}) \hookrightarrow \mathbb{C}$ is the identity. □

This way, $\bigotimes^{k+i}$-**ω-Teisi** becomes a coreflective subcategory of $\bigotimes^k$-**ω-Teisi**.

Kapranov and Voevodsky wrongly state [24, p. 203] that the left inverse to $\Sigma : \bigotimes^1$-1D-**Teisi** $\to$ 2D-**Teisi** is, in their notation, $\mathrm{Hom}(\mathbb{C},\mathbb{C})$, instead of the correct $\mathbb{C}(*,*)$.

3.2 Suspension

I will now temporarily work with finite dimensional teisi as this makes the distinction between Σ^{k-j} and U clearer.

There is also a converse to $U : \bigotimes^k$-nD-**Teisi** $\to \bigotimes^j$-nD-**Teisi**: it has a left adjoint $S^{k-j} : \bigotimes^j$-nD-**Teisi** $\to \bigotimes^k$-nD-**Teisi**, the *free k-monoidal nD tas on a j-monoidal nD tas*, also called the *$(k-j)$-th suspension.*

U commutes both with Σ^i and Ω^i for $i \leq j$: the forgetting and the reconsidering don't hurt each other, as long as there is enough monoidal structure present to do both consecutively. So the two squares in the diagram

$$\begin{tikzcd}
\bigotimes^k\text{-}n\mathbf{D}\text{-}\mathbf{Teisi} \arrow[r, "U"] \arrow[d, shift right, "\Sigma^i"'] & \bigotimes^j\text{-}n\mathbf{D}\text{-}\mathbf{Teisi} \arrow[d, shift right, "\Sigma^i"'] \\
\bigotimes^{k-i}\text{-}(n+i)\mathbf{D}\text{-}\mathbf{Teisi} \arrow[r, "U"'] \arrow[u, shift right, "\Omega^i"'] & \bigotimes^{j-i}\text{-}(n+i)\mathbf{D}\text{-}\mathbf{Teisi} \arrow[u, shift right, "\Omega^i"']
\end{tikzcd}$$

commute.

Σ^i and S^{k-j} are left adjoints to commuting functors, so commute up to isomorphism themselves. I do not know whether S^{k-j} commutes with Ω^i, nor do I know whether it is reasonable to expect this to be the case.

The relation between topological looping and looping of ω-teisi should extend to a similar relation between suspensions, via a diagram

$$
\begin{tikzcd}
\mathbf{Top}_* \arrow[r, "\Pi"] \arrow[dd, "S^k"'] & \boldsymbol{\omega}\textbf{-Teisi}_* \arrow[d, "S^k"] \\
& \bigotimes^k\textbf{-}\boldsymbol{\omega}\textbf{-Teisi} \arrow[d, "\Sigma^k"] \\
\mathbf{Top}_* \arrow[r, "\Pi"'] & \boldsymbol{\omega}\textbf{-Teisi}_*
\end{tikzcd}
$$

which should commute up to weak equivalence.

3.3 Stabilization

The lowest dimensional structure in a k-monoidal nD tas comes from $(-k)$-composition of objects, resulting in an element of dimension $0+0-(-k)-1=k-1$. If there are only identity $(k-1)$-arrows, i.e., $n<k-1$, then $(-k)$-composition can be considered not as giving extra structure but as giving an axiom. In other words, for a $(k-1)$-monoidal $(k-2)$D tas, being k-monoidal is a *property*.

If $n<k-2$ then the axiom for $(-k)$-composition is always satisfied, as it states some composition of identities to be equal to some other composition of identities, which is always true provided the boundaries match up, which is the case here. So, an $(n+2)$-monoidal nD tas is automatically a k-monoidal nD tas for any $k>n+2$, and $U: \bigotimes^k\textbf{-}n\textbf{D-Teisi} \to \bigotimes^{n+2}\textbf{-}n\textbf{D-Teisi}$ is the identity. So this "proves" the Stabilization Hypothesis for teisi (compare Baez and Dolan's Stabilization Hypothesis for weak n-categories [2, p. 6089]):

Theorem 3.8 (Stabilization) $S: \bigotimes^k\textbf{-}n\textbf{D-Teisi} \to \bigotimes^{k+1}\textbf{-}n\textbf{D-Teisi}$ *is an isomorphism for* $k \geq n+2$. □

It follows that an $(n+2)$-monoidal nD tas is a symmetric nD tas. This justifies my terminology: a symmetric strict monoidal category is indeed a symmetric 1D tas.

The relation between topological suspension and suspension of k-monoidal ω-teisi should also relate the stabilization theorem with its topological counterpart, which might eventually give a "post-modern algebraic" formulation of parts of stable homotopy theory.

3.4 Two-pointed suspension

For completeness, there is also a two-pointed delooping, looping and suspension. As I will not compare one-pointed and two-pointed I will use the same notation.

Definition 3.9 Let $k > 0$. A 2_k-*pointed* ω-*tas* is an ω-tas $\mathbb{C}$ with two objects $c, c' \in C_k$ satisfying $s_{k-1}(c) = s_{k-1}(c')$ and $t_{k-1}(c) = t_{k-1}(c')$. $\diamond$

Definition 3.10 Let $k > 0$. A k-*trivial* ω-*tas* is an ω-tas $\mathbb{C}$ which is equal to 2_ω up to dimension k, and for every $c \in \mathbb{C}$ of dimension at least $k+1$, $s_k(c) = d_k^-$ and $t_k(c) = d_k^+$. $\diamond$

Definition 3.11 Let $k > 0$ and let $(\mathbb{C}, c, c')$ be a 2_k-pointed ω-tas. $\Omega^k(\mathbb{C}, c, c')$ is the sub-ω-tas of $\mathbb{C}$ consisting of the elements d satisfying $s_k(d) = c$ and $t_k(d) = c'$. $\diamond$

So $\Omega^k(\mathbb{C}, c, c') = \mathbb{C}(c, c')$.

There is no canonical choice for two points in dimension $k+1$, so in order to iterate Ω^k this information needs to be added. Also, $\Omega^k(\mathbb{C})$ has no extra structure in itself. However, there is an action of $\mathbb{C}_c$ on the left and of $\mathbb{C}_{c'}$ on the right.

There is an inclusion Σ^k, and there are also Ω^i and Σ^i, all as before.

There are also functors U forgetting the lower dimensional pairs of points of a k-trivial ω-tas. The left adjoint S to U is particularly easy to describe: it just adds two points at each lower dimension required. In fact, S is inverse to U, and hence both commute with Σ and Ω.

4 What is tensor?

Recall that, by definition 3.2, a *monoidal 2D tas* is a 3D tas with one object. The main point is that this is to be considered as a 2D tas with extra structure. For completeness, and to fix some notation, I will now investigate the notion of monoidal 2D tas in more detail. Of course, a monoidal 2D tas is just a (semistrict) monoidal 2-category (as defined in [10]).

4.1 The tensor

The tensor on a 2D tas $\mathbb{C}$ is defined as 0-composition in $\Sigma(\mathbb{C})$, i.e., $A \otimes B = B \#_0 A$:

$$* \xrightarrow{A} * \xrightarrow{B} * \quad ,$$

and $f \otimes g = g \#_0 f$:

$$\begin{tikzcd} * \arrow[r, bend left, "A"] \arrow[r, bend right, "A'"'] \arrow[r, phantom, "\Downarrow f"] & * \arrow[r, bend left, "B"] \arrow[r, bend right, "B'"'] \arrow[r, phantom, "\Downarrow g"] & * \end{tikzcd} \quad .$$

0-composition of 2-arrows in $\Sigma(\mathbb{C})$ is given as a functor $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, which means that after reindexing, $f \otimes g$ is given in $\mathbb{C}$ by

$$\begin{tikzcd} & A \otimes B' \arrow[dr, "f \otimes B'"] & \\ A \otimes B \arrow[ur, "A \otimes g"] \arrow[dr, "f \otimes B"'] \arrow[rr, phantom, "\Downarrow f \otimes g"] & & A' \otimes B' \\ & A' \otimes B \arrow[ur, "A' \otimes g"'] & \end{tikzcd} \quad .$$

That $A \otimes B$ equals $B \#_0 A$, rather than the other way around, makes that the conventions for the tensor in a monoidal tas and for the tensor product of globes agree.

Kapranov and Voevodsky and Baez and Neuchl use $\otimes_{f,g}$ for $f \otimes g$, while Day and Street use $c_{f,g}$ for (the inverse of) the invertible 2-arrow $f \otimes g$, writing $f \otimes g$ for its domain.

4.2 Functoriality

Functoriality of $\#_0$ in $\#_n$ in $\Sigma(\mathbb{C})$ is, after reindexing, precisely 2-functoriality of $\otimes$ in each of its variables.

4.3 Associativity

Associativity of $\#_0$ in $\Sigma(\mathbb{C})$ is, after reindexing, precisely the equation $- \otimes (? \otimes !) = (- \otimes ?) \otimes !$ in $\mathbb{C}$, which is equation 2.1 of [10].

4.4 Unit

For $*$ the unique object of $\Sigma(\mathbb{C})$, id_* is the identity for $\#_0$ in $\Sigma(\mathbb{C})$. After reindexing, this gives an object I of $\mathbb{C}$ satisfying $I \otimes - = -$ and $- \otimes I = -$, which are equations 2.2 and 2.3 of [10].

5 What is braiding?

Recall that, by definition 3.2, a *braided 2D tas* is a 4D tas with one object and one arrow. The main point is that this is to be considered as a 2D tas with extra structure. I will now investigate the notion of braided 2D tas in more detail. The conclusion will be that braided 2D teisi differ from (semistrict) braided monoidal 2-categories (as defined in [10]) in precisely two points: in the weakness of functoriality of the braiding in the tensor, and in the weakness of invertibility of the braiding.

5.1 The braiding

The braiding on a 2D tas $\mathbb{C}$ is defined as 0-composition in $\Sigma^2(\mathbb{C})$, i.e., $R_{A,B} = A \#_0 B$:

$$\begin{tikzcd} * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow B"] & * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] & * \end{tikzcd} .$$

0-composition of 2-arrows in $\Sigma^2(\mathbb{C})$ is given as a functor $\Sigma(\mathbb{C}) \otimes \Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$, which means that after reindexing, $R_{A,B}$ is given in $\Sigma(\mathbb{C})$ by

$$\begin{tikzcd} & R_{*,*} \arrow[dr, "R_{*,B}"] & \\ R_{*,*} \arrow[ur, "R_{A,*}"] \arrow[dr, "R_{*,B}"'] \arrow[rr, phantom, "\Downarrow R_{A,B}"] & & R_{*,*} \\ & R_{*,*} \arrow[ur, "R_{A,*}"'] & \end{tikzcd} ,$$

and remembering that in $\Sigma^2(\mathbb{C})$ id_* is the identity for $\#_0$, this becomes

$$\begin{tikzcd} & * \arrow[dr, "B"] & \\ * \arrow[ur, "A"] \arrow[dr, "B"'] \arrow[rr, phantom, "\Downarrow R_{A,B}"] & & * \\ & * \arrow[ur, "A"'] & \end{tikzcd} \quad .$$

Reindexing once more, and using the interpretation of tensor as $\#_0$ in $\Sigma(\mathbb{C})$, $R_{A,B}$ is given in $\mathbb{C}$ by

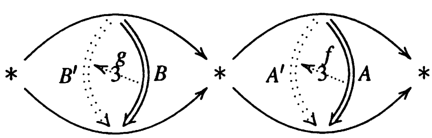
$$AB \xrightarrow{R_{A,B}} BA \quad .$$

That $R_{A,B}$ equals $A \#_0 B$, rather than the other way around, was necessary to make $R_{A,B}$ come out $AB \to BA$, rather than the other way around.

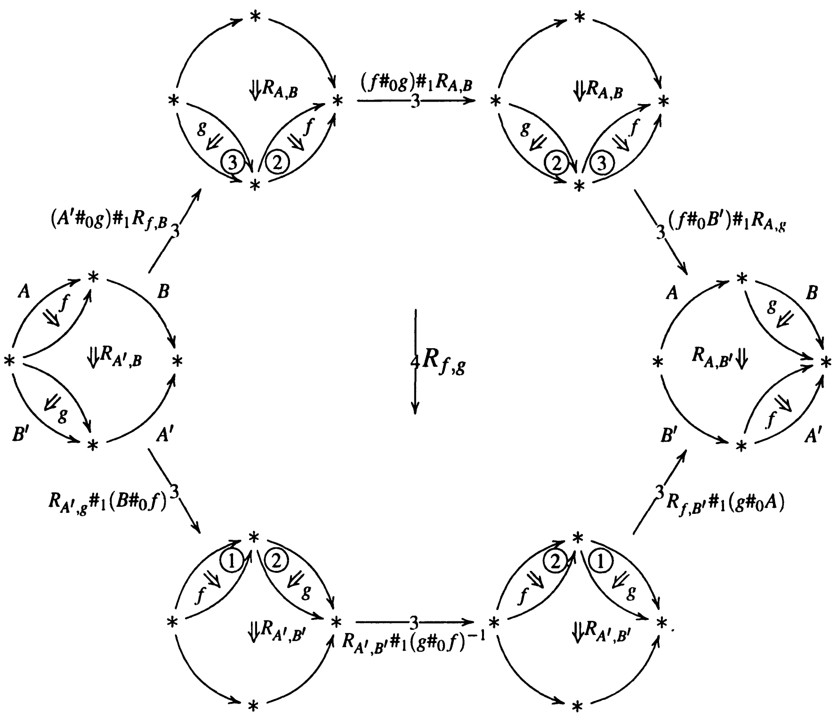
Condition (vi) a. of lemma 2.5 for 4D teisi with one arrow requires $R_{A,B}$ to be an isomorphism. Comparing this to the definition of braiding in [10], one sees that there the braiding is only required to be an equivalence. My conjecture is that a 4-dimensional coherence theorem will take care of this difference.

5.2 Naturality

Similarly, $R_{f,g}$ is defined as $f \#_0 g$ in $\Sigma^2(\mathbb{C})$:



0-composition of 3-arrows in $\Sigma^2(\mathbb{C})$ is "governed by $2_2 \otimes 2_2$", which means that after reindexing and taking into account the extra horizontal compositions occurring in there, $R_{f,g}$ is given in $\Sigma(\mathbb{C})$ by



Reindexing once more, $R_{f,g}$ is given in $\mathbb{C}$ by

$$
\begin{tikzcd}
 & A'B \arrow[rr, "R_{A',B}"] \arrow[d, phantom, "\Downarrow R_{f,B}"] & & BA' \arrow[dr, "gA'"] \arrow[dd, phantom, "\Downarrow gf"] & \\
 & & & & \\
AB \arrow[ur, "fB"] \arrow[rr, "R_{A,B}"] \arrow[dr, "Ag"'] & & BA \arrow[uur, "Bf"] \arrow[dr, "gA"] & & B'A' \\
 & & & & \\
 & AB' \arrow[rr, "R_{A,B'}"'] \arrow[u, phantom, "\Swarrow R_{A,g}"] & & B'A \arrow[ur, "B'f"'] &
\end{tikzcd}
\xrightarrow[3]{R_{f,g}}
\begin{tikzcd}
 & A'B \arrow[rr, "R_{A',B}"] \arrow[drr, "A'g"] & & BA' \arrow[dr, "gA'"] \arrow[d, phantom, "\Swarrow R_{A',g}"] & \\
 & & & & \\
AB \arrow[ur, "fB"] \arrow[dr, "Ag"'] & & A'B' \arrow[rr, "R_{A',B'}"] & & B'A' \\
 & & & & \\
 & AB' \arrow[ur, "fB'"] \arrow[rr, "R_{A,B'}"'] \arrow[uu, phantom, "\Downarrow (fg)^{-1}"] & & B'A \arrow[ur, "B'f"'] \arrow[u, phantom, "\Searrow R_{f,B'}"] &
\end{tikzcd}
.
$$

Because $\mathbb{C}$ is a 2D tas there are no other 3-arrows than identities, so $R_{f,g}$ is interpreted as an *axiom*, requiring that its domain and codomain be equal. This axiom corresponds to condition (vii) c. of lemma 2.5 for 4D teisi with one arrow. It is part of the statement, in the definition of braiding in [10], that $R_{-,?}$ is pseudo-natural.

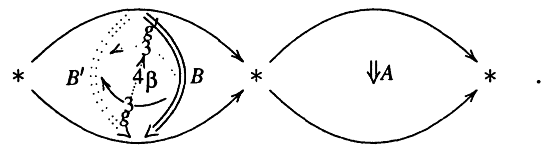
The 2-arrow fg runs in the wrong direction. This is rectified by interpreting the composite which is the codomain of $R_{f,g}$ as involving the inverse of fg.

Kapranov and Voevodsky and Baez and Neuchl have $R_{A,g}$ running in the opposite direction. This will have implications later on, see page .

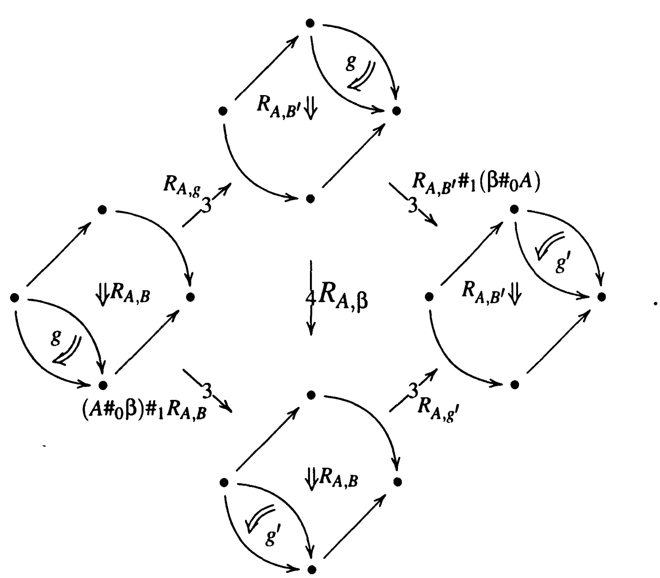
The cube for $R_{f,g}$ is almost exactly the cube referred to by the hieroglyph $(\rightarrow \otimes \rightarrow)$ of Kapranov and Voevodsky. Their choice of directions gives a decomposition of the cube, along gA and fB', in two parts in which there are no inverses, which is nice, but these parts are of unequal size, which is not so nice.

Day and Street have a 2-arrow they call $\rho_{f,g}$, which is the composite of either side of the above cube decomposed along Bf and fB'. This decomposition is sort of forced upon them by their definition of $f \otimes g$.

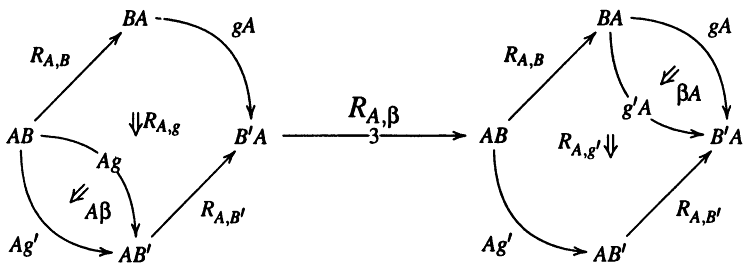
$R_{A,\beta}$ is defined as $A \#_0 \beta$ in $\Sigma^2(\mathbb{C})$:



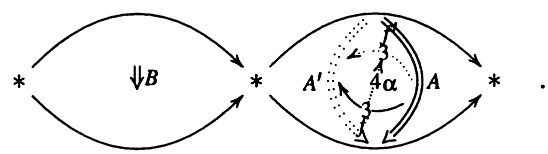
After reindexing, $R_{A,\beta}$ is given in $\Sigma(\mathbb{C})$ by



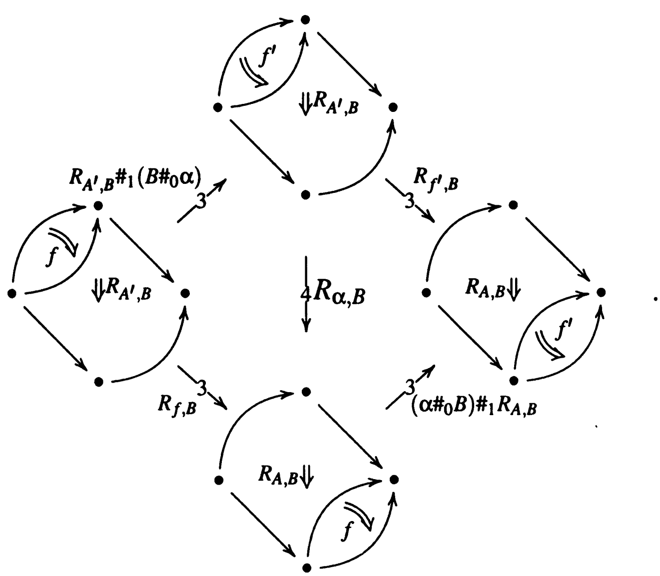
Reindexing once more, $R_{A,\beta}$ is given in $\mathbb{C}$ by



Similarly, $R_{\alpha,B}$ is defined as $\alpha\#_0 B$ in $\Sigma^2(\mathbb{C})$:



After reindexing, $R_{\alpha,B}$ is given in $\Sigma(\mathbb{C})$ by



Reindexing once more, $R_{\alpha,B}$ is given in $\mathbb{C}$ by

$$\begin{tikzcd}
 & A'B \arrow[dr, "R_{A',B}"] & & & & A'B \arrow[dr, "R_{A',B}"] & \\
AB \arrow[ur, bend left=50, "fB"] \arrow[ur, bend right, "f'B"'] \arrow[ur, phantom, "\Downarrow\alpha B"] \arrow[dr, "R_{A,B}"'] \arrow[r, phantom, "\Downarrow R_{f',B}"] & & BA' \arrow[rr, "R_{\alpha,B}", "3"'] & & AB \arrow[ur, bend left, "fB"] \arrow[dr, "R_{A,B}"'] \arrow[r, phantom, "R_{f,B}\Downarrow"] & & BA' \\
 & BA \arrow[ur, bend right, "Bf'"'] & & & & BA \arrow[ur, bend left, "Bf"] \arrow[ur, bend right, "Bf'"'] \arrow[ur, phantom, "\Downarrow B\alpha"] &
\end{tikzcd} .$$

$R_{A,\beta}$ and $R_{\alpha,B}$ are again interpreted as axioms requiring that their domain and codomain be equal. These axioms correspond to conditions (vii) a. and (vii) b. of lemma 2.5 for 4D teisi with one arrow. They are also part of the statement, in the definition of braiding in [10], that $R_{-,?}$ is pseudo-natural.

Kapranov and Voevodsky suggest that "in the definition of a 2-braiding it is the laxness which gives rise to the most meaningful data" [24, p. 180]. This is not the case: it is the horizontal composition which does that.

Kapranov and Voevodsky also suggest that "the possibility of actually drawing pictures of 3-dimensional polytopes [...] is lost in higher dimensions" [24, p. 180]. This is also not the case: the faces of the 4-dimensional $R_{f,g}$ above are 3-dimensional polytopes. And there is a practical obstruction to drawing pictures of *higher*-dimensional polytopes in that the polytopes become too big to handle, but there is no theoretical one, at least not for polytopes arising from higher-dimensional teisi.

Kapranov and Voevodsky furthermore suggest that "our 2-braidings can be seen as 'braidings of the usual braidings' " [23, p. 1]. This is not the case either: the usual braiding on a monoidal category corresponds, via reindexing, to the tensor of *arrows* in a monoidal 2-category, and hence braidings of the usual braiding correspond to $R_{f,g}$ from the braiding on a monoidal 2-category, and results in a symmetry if the 2-category has one object, i.e., if the monoidal 2-category is a braided monoidal category. The "2-braidings" $R_{A,B}$, for objects A and B of the monoidal 2-category, do not occur at all in this braided monoidal category.

5.3 Functoriality in tensor

In $\Sigma^2(\mathbb{C})$, functoriality of $\#_0$ in $\#_1$ in the second variable relates the two ways of composing the diagram

$$\begin{tikzcd} * \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=3ex, "\Downarrow B"] \arrow[r, phantom, shift right=3ex, "\Downarrow C"] & * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] & * \end{tikzcd} .$$

It corresponds to functoriality in one variable of the tensor product of the **Gray**-categories $\Sigma(\mathbb{C})$ and $\Sigma(\mathbb{C})$, so after reindexing, this relation is given in $\Sigma(\mathbb{C})$ by

$$\begin{tikzcd}[column sep=small] & * \arrow[dr, "B"] & & \\ * \arrow[ur, "A"] \arrow[dr, "B"'] & & * \arrow[dr, "C"] & \\ & * \arrow[dr, "C"'] \arrow[ur, phantom, "\Downarrow R_{A,BC}"] & & * \\ & & * \arrow[ur, "A"'] & \end{tikzcd} = \begin{tikzcd}[column sep=small] & * \arrow[dr, "B"] \arrow[d, phantom, "\Downarrow R_{A,B}"] & & \\ * \arrow[ur, "A"] \arrow[dr, "B"'] & {} & * \arrow[dr, "C"] \arrow[d, phantom, "\Downarrow R_{A,C}"] & \\ & * \arrow[ur, "A"] \arrow[dr, "C"'] & {} & * \\ & & * \arrow[ur, "A"'] & \end{tikzcd} .$$

Reindexing once more, this relation is given in $\mathbb{C}$ by

$$\begin{tikzcd} A \otimes B \otimes C \arrow[rr, "R_{A,B\otimes C}"] \arrow[dr, "R_{A,B} \otimes C"'] & \arrow[d, phantom, "\|"] & B \otimes C \otimes A \\ & B \otimes A \otimes C \arrow[ur, "B \otimes R_{A,C}"'] & \end{tikzcd} .$$

Similarly, in $\Sigma^2(\mathbb{C})$, functoriality of $\#_0$ in $\#_1$ in the first variable relates the two ways of composing the diagram

$$\begin{tikzcd} * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow C"] & * \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=3ex, "\Downarrow A"] \arrow[r, phantom, shift right=3ex, "\Downarrow B"] & * \end{tikzcd} .$$

After reindexing, this relation is given in $\Sigma(\mathbb{C})$ by

$$\begin{tikzcd}
& & * \arrow[dr, "C"] & \\
& * \arrow[ur, "B"] \arrow[dr, phantom, "\Downarrow R_{AB,C}"] & & * \\
* \arrow[ur, "A"] \arrow[dr, "C"'] & & * \arrow[ur, "B"'] & \\
& * \arrow[ur, "A"'] & &
\end{tikzcd}
=
\begin{tikzcd}
& & * \arrow[dr, "C"] \arrow[dd, phantom, "\Downarrow R_{B,C}"] & \\
& * \arrow[ur, "B"] \arrow[dr, "C"] \arrow[dd, phantom, "\Downarrow R_{A,C}"] & & * \\
* \arrow[ur, "A"] \arrow[dr, "C"'] & & * \arrow[ur, "B"'] & \\
& * \arrow[ur, "A"'] & &
\end{tikzcd} .$$

Reindexing once more, this relation is given in $\mathbb{C}$ by

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "R_{A\otimes B,C}"] \arrow[dr, "A\otimes R_{B,C}"'] & & C\otimes A\otimes B \\
& A\otimes C\otimes B \arrow[ur, "R_{A,C}\otimes B"'] \arrow[u, phantom, "\Vert"] &
\end{tikzcd} .$$

These two conditions correspond to conditions (viii) a. and (viii) b. of lemma 2.5 for 4D teisi with one arrow. Comparing them to the definition of braiding in [10], one sees that there these identities hold only op to isomorphism. My conjecture is that a 4-dimensional coherence theorem will take care of this difference.

The other parts of condition (viii) of lemma 2.5 for 4D teisi with one arrow are contained in the statements, in the definition of braiding in [10], that $R_{-,?}$ is pseudo-natural and that $\tilde{R}_{(-|?,!)}$ and $\tilde{R}_{(-,?|!)}$ are modifications.

If one opts for functoriality of the braiding in the tensor up to specified isomorphisms $\tilde{R}_{(A|B,C)}$ and $\tilde{R}_{(A,B|C)}$, these isomorphisms have to satisfy further, "coherence", conditions. The diagram

$$\begin{tikzcd}[column sep=large]
* \arrow[r, bend left=80] \arrow[r, bend left=30] \arrow[r, bend right=30] \arrow[r, bend right=80] \arrow[r, phantom, "\Downarrow B", yshift=5ex] \arrow[r, phantom, "\Downarrow C"] \arrow[r, phantom, "\Downarrow D", yshift=-5ex] & * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] & *
\end{tikzcd}$$

in $\Sigma^2(\mathbb{C})$ gives, after reindexing,

$$\begin{tikzcd}
ABCD \arrow[r, "R_{A,BCD}"] \arrow[d] \arrow[dr] \arrow[d, phantom, shift left=4ex, "\Leftarrow"] & BCDA \\
BACD \arrow[r] \arrow[ur, dashed] & BCAD \arrow[u] \arrow[u, phantom, shift right=4ex, "\Downarrow"]
\end{tikzcd},$$

which is equation 2.7 of [10], and the diagram

$$\begin{tikzcd}
* \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow D"] & * \arrow[r, bend left=80] \arrow[r, bend left=25] \arrow[r, bend right=25] \arrow[r, bend right=80] \arrow[r, phantom, shift left=4ex, "\Downarrow A"] \arrow[r, phantom, "\Downarrow B"] \arrow[r, phantom, shift right=4ex, "\Downarrow C"] & *
\end{tikzcd}$$

in $\Sigma^2(\mathbb{C})$ gives, after reindexing,

$$\begin{tikzcd}
ABCD \arrow[r, "R_{ABC,D}"] \arrow[d] \arrow[dr] \arrow[d, phantom, shift left=4ex, "\Leftarrow"] & DABC \\
ABDC \arrow[r] \arrow[ur, dashed] & ADBC \arrow[u] \arrow[u, phantom, shift right=4ex, "\Downarrow"]
\end{tikzcd},$$

which is equation 2.8 of [10].

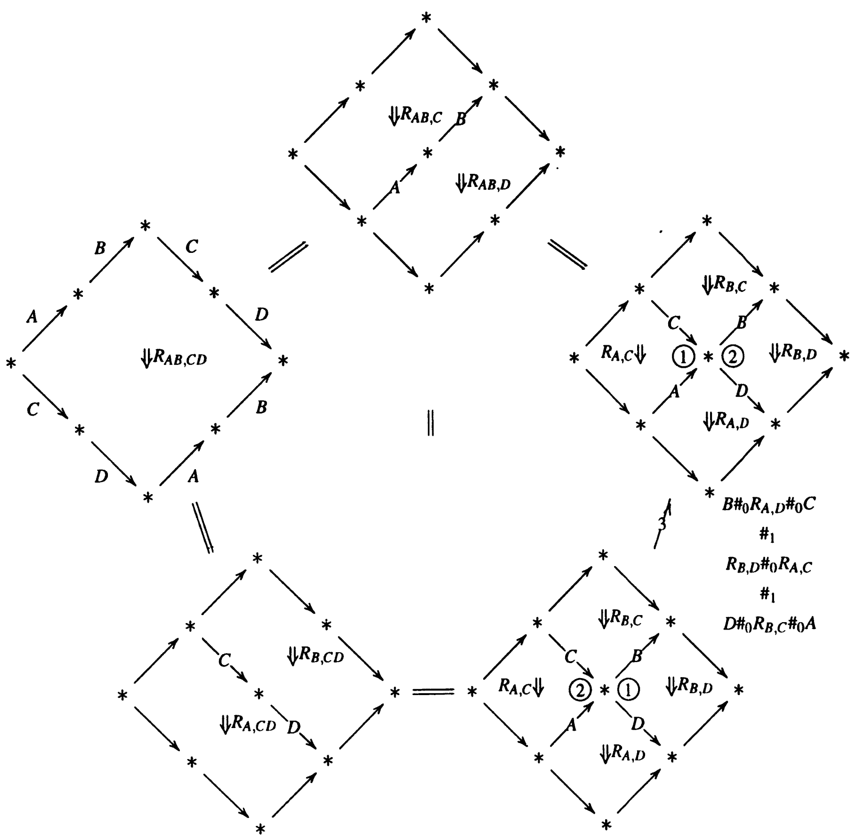
Kapranov and Voevodsky and Baez and Neuchl don't make the conceptual distinction between the pseudo-naturality of the braiding and the weakness of functoriality of the braiding in the tensor, attributing both to Kapranov and Voevodsky's MAIN PRINCIPLE OF CATEGORY THEORY. But the first one just comes from the horizontal composition of teisi, while the second one is a conscious *choice*. This conception also perpetrates the terminology: there is nothing "pseudo" about the braiding, and there is nothing "un-strict" about a semistrict monoidal 2-category.

5.4 Interchange

In $\Sigma^2(\mathbb{C})$, functoriality of $\#_0$ in $\#_1$ in both variables at the same time relates the two ways of composing the diagram

$$\begin{tikzcd} * \arrow[r, bend left=60, ""{name=U1}] \arrow[r, ""{name=M1}] \arrow[r, bend right=60, ""{name=L1}] & * \arrow[r, bend left=60, ""{name=U2}] \arrow[r, ""{name=M2}] \arrow[r, bend right=60, ""{name=L2}] & * \arrow[from=U1, to=M1, phantom, "\Downarrow C"] \arrow[from=M1, to=L1, phantom, "\Downarrow D"] \arrow[from=U2, to=M2, phantom, "\Downarrow A"] \arrow[from=M2, to=L2, phantom, "\Downarrow B"] \end{tikzcd} .$$

This corresponds to interchange for the tensor product of **Gray**-categories, so after reindexing and taking into account the extra horizontal composition occurring in there, this relation is given in $\Sigma(\mathbb{C})$ by



Reindexing once more, this relation is given in $\mathbb{C}$ by

$$
\begin{tikzcd}
ABCD \arrow[rrrr, "R_{AB,CD}"] \arrow[dd] \arrow[ddrr] \arrow[dr, dashed] & & & & CDAB \\
& ACDB \arrow[urrr, dashed] \arrow[drrr, dashed] & & & \\
ACBD \arrow[ur, dashed] \arrow[drr] \arrow[rr, phantom, "R_{A,C}R_{B,D}\Downarrow"] & & & & CADB \arrow[uu] \\
& & CABD \arrow[uurr] \arrow[urr] & &
\end{tikzcd}
\tag{5.1}
$$

which is equation 2.6 of [10] once the $\tilde{R}$'s (not) being identities is taken into account. This equation corresponds to condition (ix) of lemma 2.5 for 4D teisi with one arrow.

Day and Street have a 2-arrow they call $\varpi_{W,X,Y,Z}$, which is actually equal to the part of the above diagram between *ACBD* and *CDAB*, i.e., it is the composite

$$
\begin{tikzcd}
& WXZY \arrow[dr] \arrow[drr] & & \\
WXYZ \arrow[ur] \arrow[dr] \arrow[rr, phantom, "R_{W,X}R_{Y,Z}\Downarrow"] & & XWZY \arrow[r] \arrow[ur, phantom, "\Downarrow\tilde{R}_{(W|X,Z)}Y"] \arrow[dr, phantom, "\Downarrow X\tilde{R}^{-1}_{(W,Y|Z)}"] & XZWY \\
& XWYZ \arrow[ur] \arrow[urr] & &
\end{tikzcd}
.
$$

After reindexing, it is the composite

$$
\begin{tikzcd}
* \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r] \arrow[r, phantom, shift left=3, "\Downarrow X"] \arrow[r, phantom, shift right=3, "\Downarrow Z"] & * \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r] \arrow[r, phantom, shift left=3, "\Downarrow W"] \arrow[r, phantom, shift right=3, "\Downarrow Y"] & *
\end{tikzcd}
$$

in $\Sigma^2(\mathbb{C})$ with the extra condition that X is only tensored with Y in this order, i.e., X left of, or before, Y. The axiom for ϖ is, after reindexing, nothing but the composite

$$\begin{tikzcd}[column sep=huge]
* \arrow[r, bend left=80] \arrow[r, bend left=25] \arrow[r, bend right=25] \arrow[r, bend right=80] \arrow[r, phantom, shift left=4ex, "\Downarrow X"] \arrow[r, phantom, "\Downarrow Y"] \arrow[r, phantom, shift right=4ex, "\Downarrow Z"] & * \arrow[r, bend left=80] \arrow[r, bend left=25] \arrow[r, bend right=25] \arrow[r, bend right=80] \arrow[r, phantom, shift left=4ex, "\Downarrow W"] \arrow[r, phantom, "\Downarrow U"] \arrow[r, phantom, shift right=4ex, "\Downarrow V"] & *
\end{tikzcd}$$

in $\Sigma^2(\mathbb{C})$ with the stipulation that X is left of U and Y is left of V.

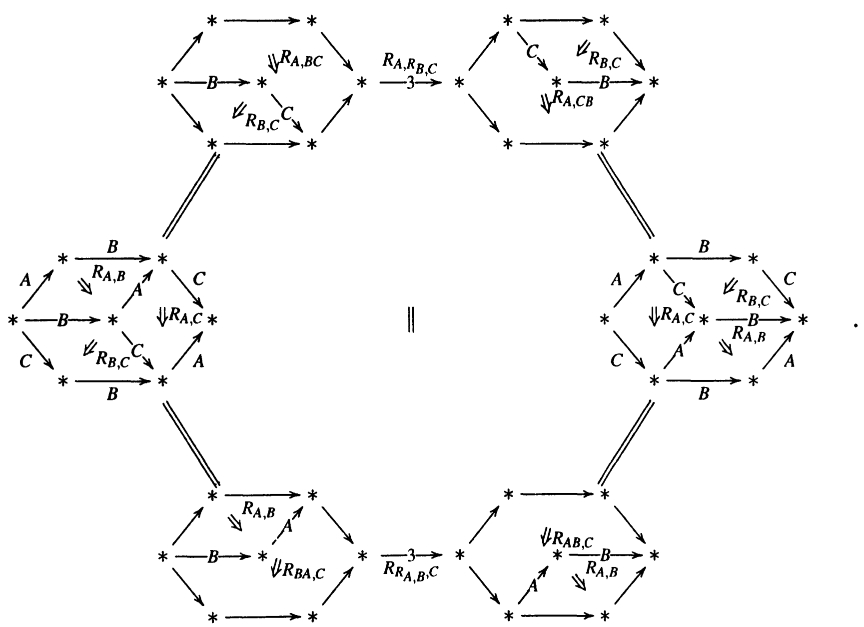
Kapranov and Voevodsky's resultohedra $N_{p,q}$ correspond to the braiding of the tensor of p objects with the tensor of q objects of $\mathbb{C}$. In particular, the shape of diagram 5.1 is the resultohedron $N_{2,2}$.

5.5 Associativity

In $\Sigma^2(\mathbb{C})$, associativity of $\#_0$ relates the two ways of composing the diagram

$$\begin{tikzcd}
* \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow C"] & * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow B"] & * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] & *
\end{tikzcd} .$$

This corresponds to associativity of the tensor product of **Gray**-categories, and after reindexing, this relation is given in $\Sigma(\mathbb{C})$ by



Reindexing once more, this relation is given in $\mathbb{C}$ by

$$
\begin{tikzcd}
 & BAC \arrow[r] \arrow[drr, dashed] & BCA \arrow[dr] & \\
ABC \arrow[ur] \arrow[urr] \arrow[dr] \arrow[drr, dashed] \arrow[rrr, phantom, "R_{R_{A,B},C} \Rrightarrow R_{A,R_{B,C}}"] & & & CBA \\
 & ACB \arrow[r] \arrow[urr] & CAB \arrow[ur] &
\end{tikzcd}
,
$$

which is equation 2.7 of [10], again taking into account the $\tilde{R}$'s not being identities. This equation corresponds to condition (x) of lemma 2.5 with $p = q = r = 1$ for 4D teisi with one arrow.

Kapranov and Voevodsky and Baez and Neuchl having $R_{A,g}$ running in the opposite direction implies that in the above diagram they also have $R_{A,R_{B,C}}$ running in the opposite direction, which explains why I *don't* need to take its inverse here.

Kapranov and Voevodsky call the two sides of the above diagram S^+ and S^-, but don't require any relation between them. Baez and Neuchl correct that, but everyone fails to see that the equation $S^+ = S^-$ is just good old-fashioned associativity.

5.6 Yang-Baxter and Zamolodchikov

The Yang-Baxter equation is the equation that is satisfied by the arrows $R_{A,B}$, $R_{B,C}$ and $R_{A,C}$ in a braided monoidal category $\mathbb{C}$, for A, B and C objects of $\mathbb{C}$. This equation can be proven from the axioms for a braided monoidal category in two, essentially different, ways. In a braided (monoidal) 2-category, these proofs give rise to composites of 2-arrows. The associativity axiom exactly states that these two composites are equal.

If $S_{A,B,C}$ is defined as any one of the two composite 2-arrows arising from the Yang-Baxter equation, which are equal by associativity, the Zamolodchikov equation is the equation that is satisfied by $S_{A,B,C}$, $S_{A,B,D}$, $S_{A,C,D}$, and $S_{B,C,D}$ in a braided (monoidal) 2-category $\mathbb{C}$, for A, B, C and D objects of $\mathbb{C}$. This equation can be proven from the axioms for a braided monoidal 2-

category in three, essentially different, ways. In a braided ω-tas these proofs give rise to composites of 3-arrows. The reindexation of the diagram

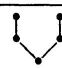
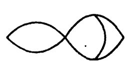
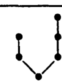
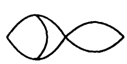
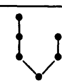
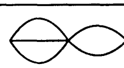
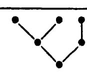
$$\begin{tikzcd} * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow D"] & * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow C"] & * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow B"] & * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] & * \end{tikzcd}$$

in $\Sigma^2(\mathbb{C})$ relates these composites, similar to, but more complicated than, the previous diagram relating the composites arising from the proofs of the Yang-Baxter equation.

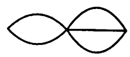
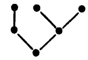
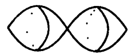
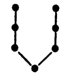
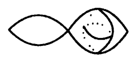
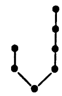
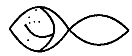
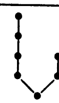
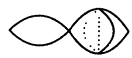
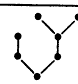
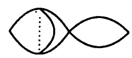
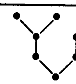
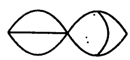
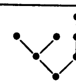
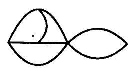
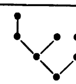
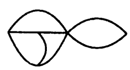
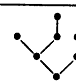
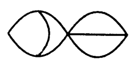
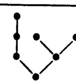
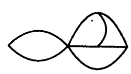
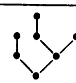
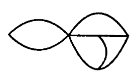
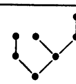
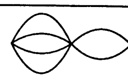
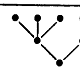
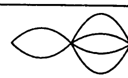
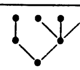
More detail about the Yang-Baxter and Zamolodchikov equations from the ω-teisi point of view and higher-dimensional generalizations will be given in another paper [12].

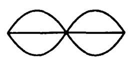
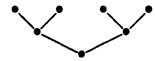
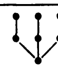
5.7 Hieroglyphs

I consider Kapranov and Voevodsky's hieroglyphic notation [25], which is also adapted by Baez and Neuchl [5], as somewhat inappropriate: it fails to distinguish between the different compositions, by overloading the tensor product symbol, and, partly because of that, the resulting expressions are not easily interpretable. I think a more conceptual and transparent, and hence much better, notation is generated by labeling a composition or axiom by the diagram in $\Sigma^2(\mathbb{C})$ which gives the relative position of the elements involved in the composition or axiom. The hieroglyphs are compared in the following table, which also includes the corresponding trees of Batanin [6]:

Name	tensor hieroglyph	globe hieroglyph[2]	tree hieroglyph
$R_{A,B}$	$\bullet \otimes \bullet$		
$R_{f,B}$	$\rightarrow \otimes \bullet$		
$R_{A,g}$	$\bullet \otimes \rightarrow$		
$\tilde{R}_{(A\|B,C)}$	$\bullet \otimes (\bullet \otimes \bullet)$		
			(table continues next page)

[2]Of course, one needs a good, logically structured drawing language, such as XY-pic, to produce these hieroglyphs.

Name	tensor hieroglyph	globe hieroglyph	tree hieroglyph
$\tilde{R}_{(A,B\|C)}$	$(\bullet\otimes\bullet)\otimes\bullet$		
$R_{f,g}$	$\rightarrow\otimes\rightarrow$		
$R_{\alpha,B}$	$\Downarrow\otimes\bullet$		
$R_{A,\beta}$	$\bullet\otimes\Downarrow$		
	$\rightarrow\rightarrow\otimes\bullet$		
	$\bullet\otimes\rightarrow\rightarrow$		
	$\rightarrow\otimes(\bullet\otimes\bullet)$		
	$\bullet\otimes(\rightarrow\otimes\bullet)$		
	$\bullet\otimes(\bullet\otimes\rightarrow)$		
	$(\bullet\otimes\bullet)\otimes\rightarrow$		
	$(\rightarrow\otimes\bullet)\otimes\bullet$		
	$(\bullet\otimes\rightarrow)\otimes\bullet$		
2.4	$\bullet\otimes(\bullet\otimes\bullet\otimes\bullet)$		
2.5	$(\bullet\otimes\bullet\otimes\bullet)\otimes\bullet$		
			(table continues next page)

Name	tensor hieroglyph	globe hieroglyph	tree hieroglyph
2.6	$(\bullet\otimes\bullet)\otimes(\bullet\otimes\bullet)$		
2.7	$S^+ = S^-$		

5.8 Unit axioms

For $*$ the unique object of $\Sigma^2(\mathbb{C})$, id^2_* is the identity for $\#_0$ in $\Sigma^2(\mathbb{C})$. After reindexing, this gives the equations $R_{I,-} = \mathrm{id}_-$ and $R_{-,I} = \mathrm{id}_-$ in $\mathbb{C}$, which are equations 2.8 and 2.9 of [10]. These equations correspond to axiom (xi) with $p = 0$ and $q = 0$ resp. for 4D teisi with one arrow. The remaining equations of [10] relate the unit with the weakness of functoriality of the braiding in the tensor.

Day and Street only have the equations $R_{I,A} = \mathrm{id}_A$ and $R_{A,I} = \mathrm{id}_A$.

5.9 Conclusion

Theorem 5.1 *A braided* 2*D tas is precisely a braided monoidal* 2*-category (as defined in [10]) for which* $R_{A,B}$ *is an isomorphism and* $\tilde{R}_{(A|B,C)}$ *and* $\tilde{R}_{(A,B|C)}$ *are identities for all* A, B *and* C. □

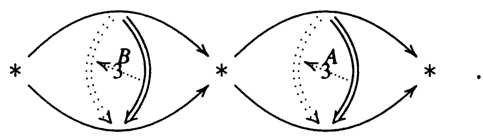
6 What is syllepsis?

From now on I will have to do things slightly different, as 5-dimensional teisi have not been defined yet. Therefore, I will investigate the heuristic notion of a 5D tas with one 2-arrow as motivation for a more *ad hoc* definition of a sylleptic 2D tas. Ignoring the difference between braided 2D teisi and (semistrict) braided monoidal 2-categories, sylleptic 2D teisi will turn out to be equivalent to (semistrict) sylleptic monoidal 2-categories (as defined in [10]).

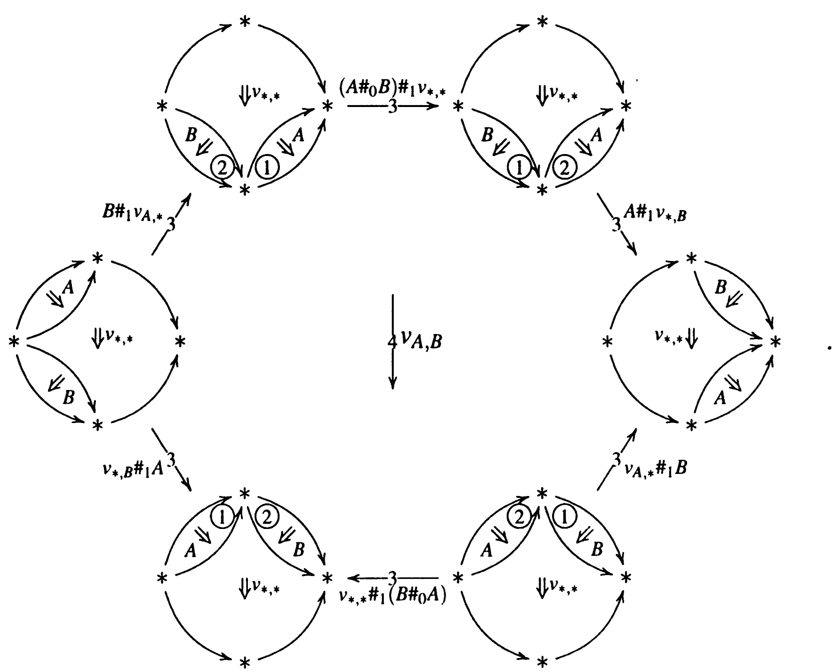
From now on I will draw arrows in diagrams in their canonical direction, rather than mess around with inverses.

6.1 The syllepsis

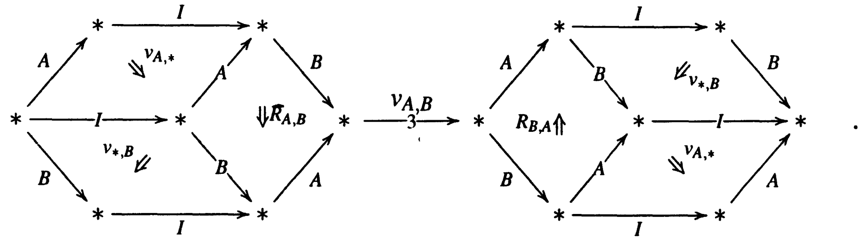
The syllepsis on a 2D tas $\mathbb{C}$ should be defined as 0-composition in $\Sigma^3(\mathbb{C})$, i.e., $\nu_{A,B} = A \#_0 B$:



0-composition of 3-arrows in $\Sigma^3(\mathbb{C})$ should be "governed by $2_2 \otimes 2_2$", which means that after reindexing and taking into account the extra horizontal compositions occurring in there, $\nu_{A,B}$ should be given in $\Sigma^2(\mathbb{C})$ by



Reindexing once more, and remembering that in $\Sigma^3(\mathbb{C})$ $\mathrm{id}^2_* \#_0 \mathrm{id}^2_* = \mathrm{id}^3_*$, $\nu_{A,B}$ should be given in $\Sigma(\mathbb{C})$ by



Reindexing once again, $v_{A,B}$ should be given in $\mathbb{C}$ by

$$\begin{tikzcd}
& IAB \arrow[rr, "IR_{A,B}"] & & IBA \arrow[dr, "v_{*,B}A"] & \\
AIB \arrow[ur, "v_{A,*}B"] \arrow[dr, "Av_{*,B}"'] & & \Downarrow v_{A,B} & & BIA \\
& ABI & & BAI \arrow[ll, "R_{B,A}I"] \arrow[ur, "Bv_{A,*}"'] &
\end{tikzcd},$$

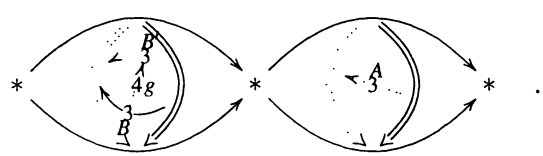
and remembering that in $\Sigma^3(\mathbb{C})$ id^3_* is the identity for $\#_2$ and id^2_* is the identity for $\#_0$, this becomes

$$\begin{tikzcd}
AB \arrow[r, bend left, "R_{A,B}"] \arrow[r, phantom, "\Downarrow v_{A,B}"] & BA \arrow[l, bend left, "R_{B,A}"]
\end{tikzcd}.$$

That $v_{A,B}$ equals $A\,\#_0\,B$, rather than the other way around, was necessary to make $v_{A,B}$ come out with domain $R_{A,B}$, rather than $R_{B,A}$.

6.2 Naturality

$v_{A,g}$ should be defined as $A\,\#_0\,g$ in $\Sigma^3(\mathbb{C})$:

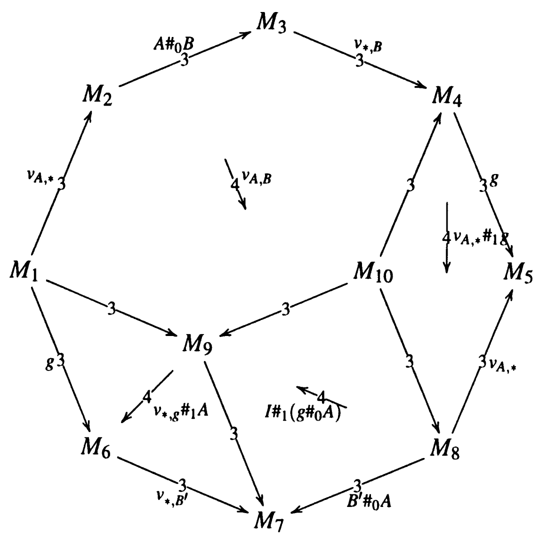


After reindexing, and remembering that in $\Sigma^3(\mathbb{C})$ $\mathrm{id}^2_* \#_0 \mathrm{id}^2_* = \mathrm{id}^3_*$, $v_{A,g}$ should be given in $\Sigma^2(\mathbb{C})$ by

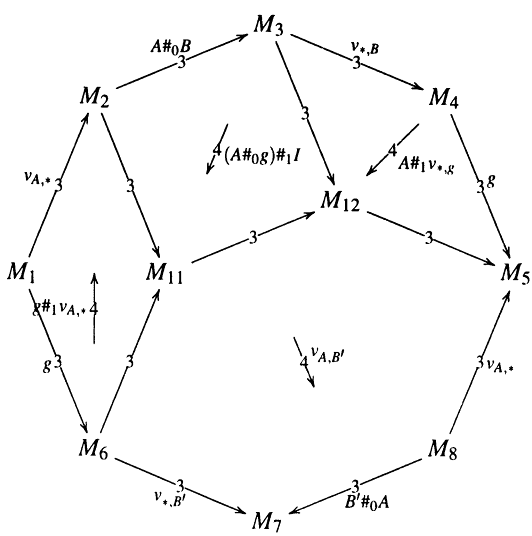
$$L_1 \xrightarrow[5]{v_{A,g}} L_2$$

where the L_s are given by:

L_1:

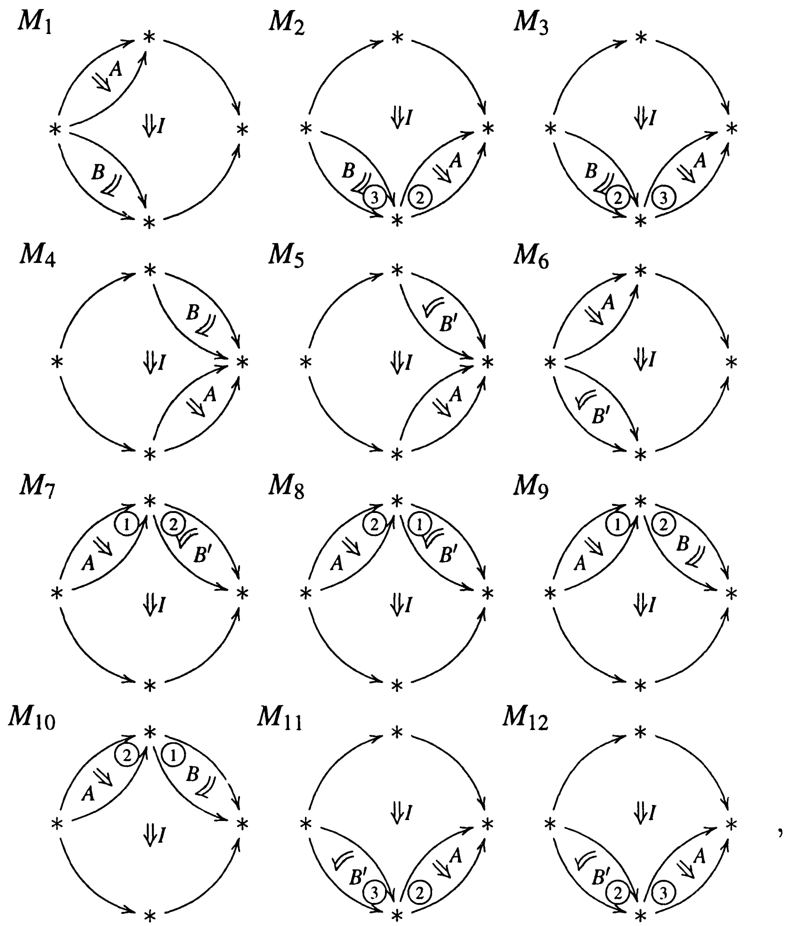


L_2:



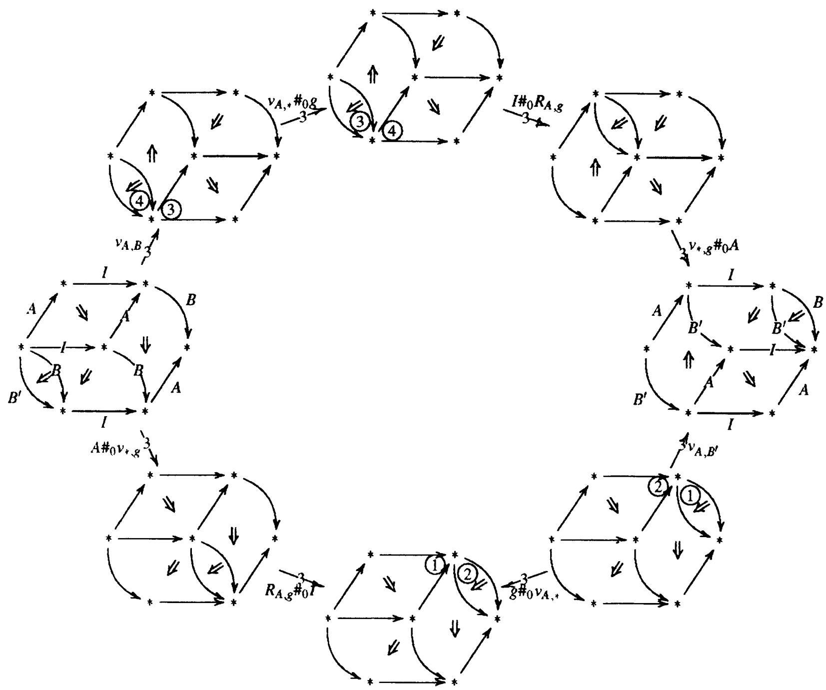
,

and where the M_t are as follows:



with the crossbar near the arrowheads in the diagram indicating that one or more elements in the boundary have "flipped over" to the other side, i.e., an element which is "canonically" in the domain is now in the codomain, or vice versa, or both.

Reindexing once more, $v_{A,g}$ should be given in $\Sigma(\mathbb{C})$ by

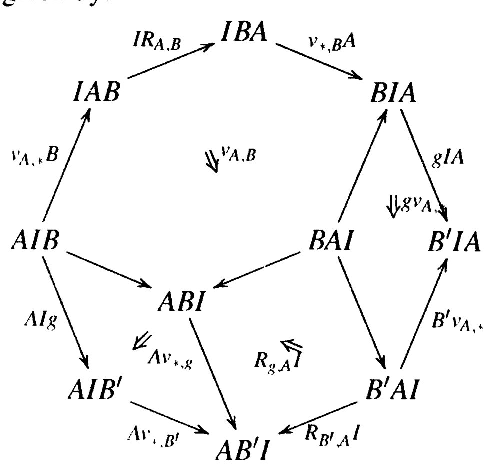


Reindexing once again, $v_{A,g}$ should be given in $\mathbb{C}$ by

$$N_1 \xrightarrow[3]{v_{A,g}} N_2$$

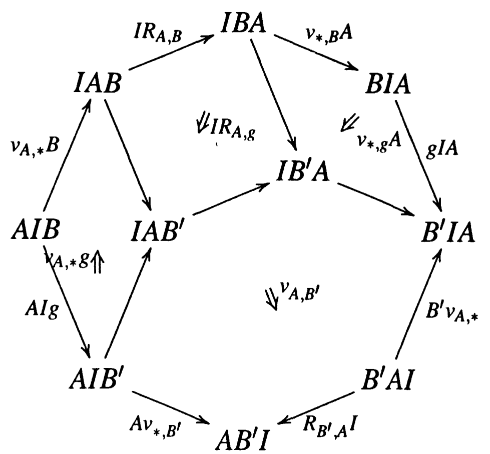
where the N_ν are given by:

N_1:



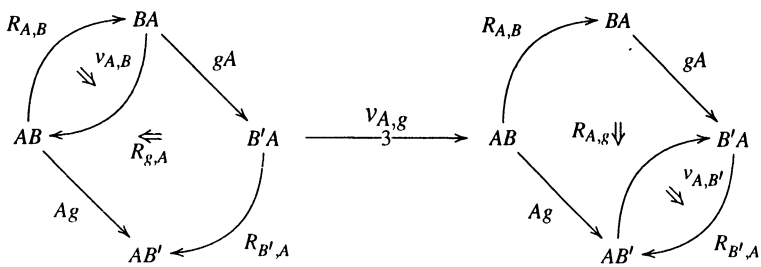
N_2:

Mmmm



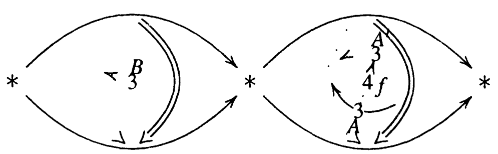
,

and remembering that in $\Sigma^3(\mathbb{C})$ id_*^3 is the identity for $\#_2$ and id_*^2 is the identity for $\#_0$, this becomes



.

Similarly, $\nu_{f,B}$ should be defined as $f\,\#_0\,B$ in $\Sigma^3(\mathbb{C})$:

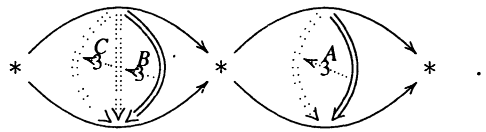


.

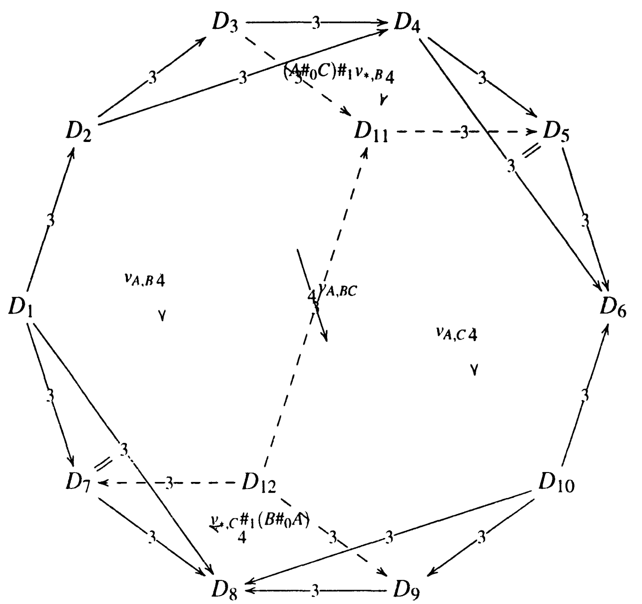
Reindexation of this is left to the reader.

6.3 Functoriality in tensor

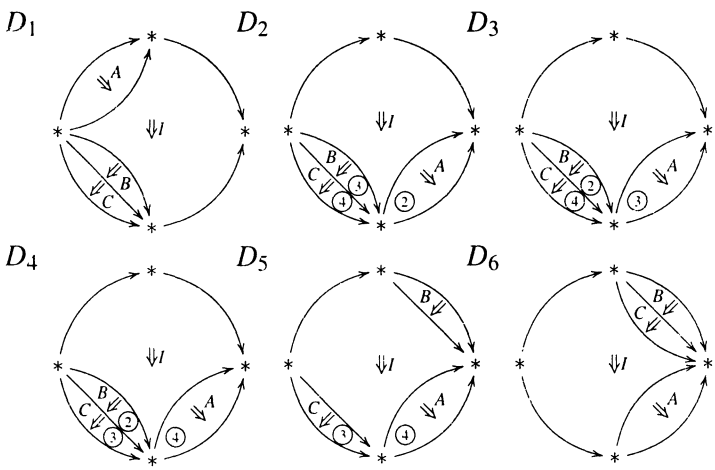
In $\Sigma^3(\mathbb{C})$, functoriality of $\#_0$ in $\#_2$ in the second variable relates the two ways of composing the diagram

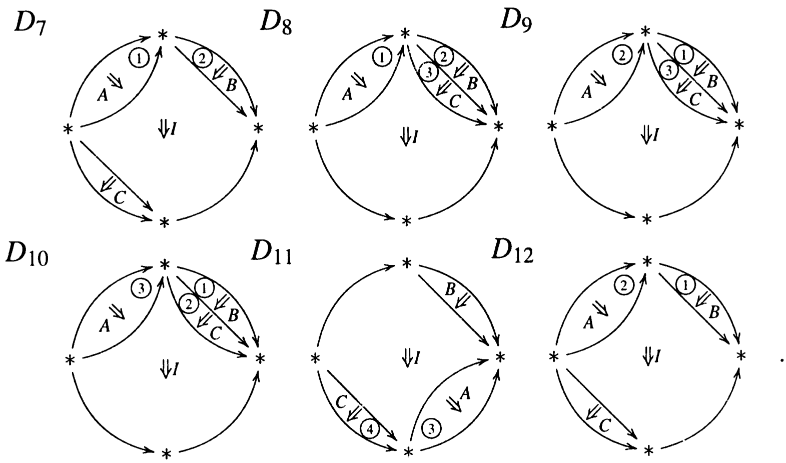


After reindexing, this relation should be given in $\Sigma^2(\mathbb{C})$ by

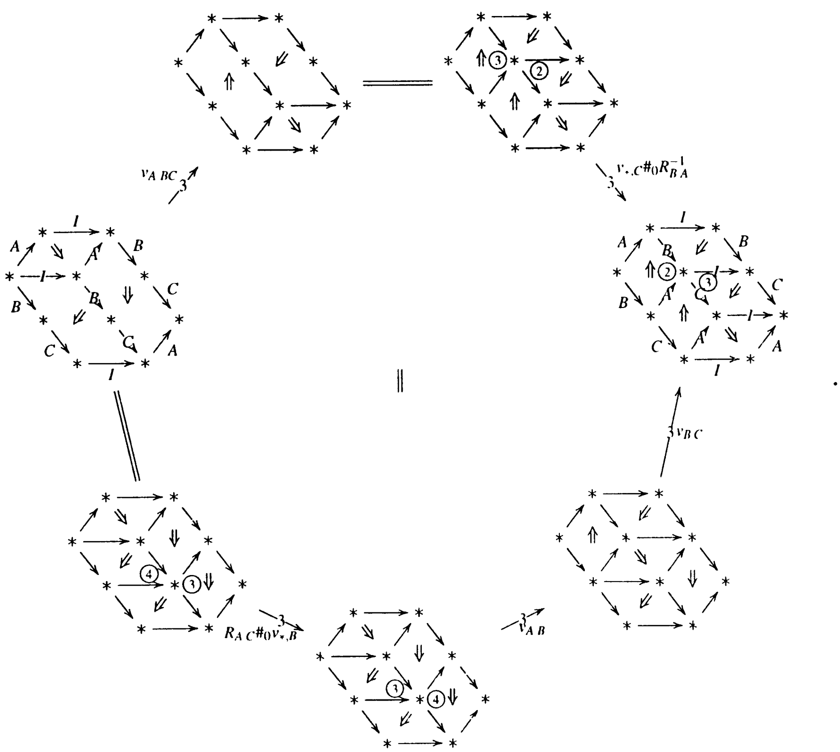


where the D_l are as follows:

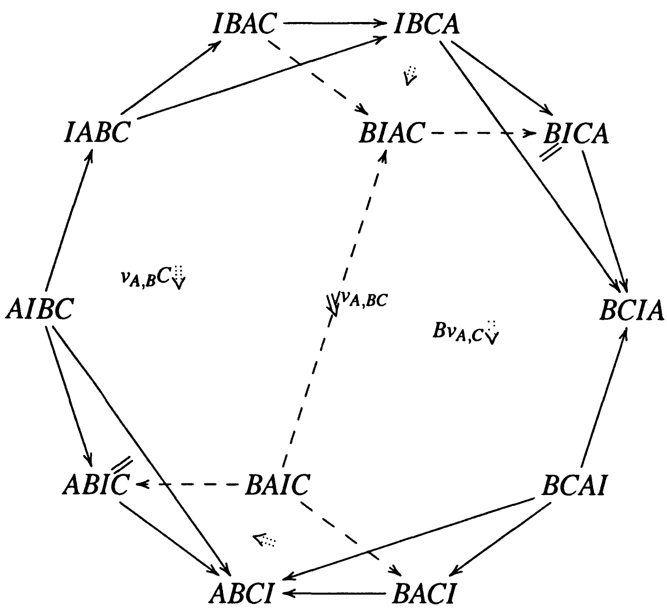




Reindexing once more, this relation is given in $\Sigma(\mathbb{C})$ by

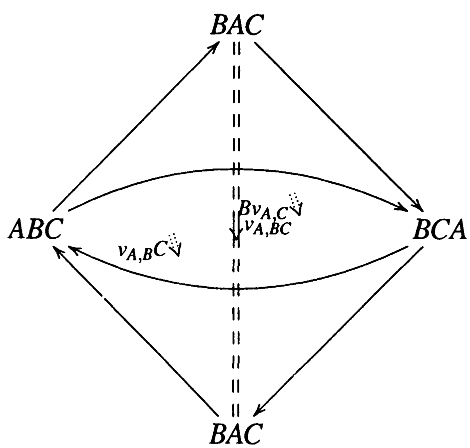


Reindexing once again, this relation is given in $\mathbb{C}$ by



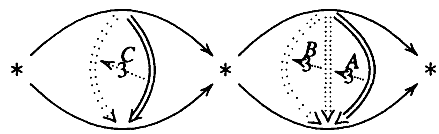
,

and remembering that in $\Sigma^3(\mathbb{C})$ id^3_* is the identity for $\#_2$ and id^2_* is the identity for $\#_0$, this becomes



.

Similarly, in $\Sigma^3(\mathbb{C})$, functoriality of $\#_0$ in $\#_2$ in the first variable relates the two ways of composing the diagram



Reindexation of this is left to the reader.

6.4 Functoriality in braiding

In $\Sigma^3(\mathbb{C})$, functoriality of $\#_0$ in $\#_3$ in the second variable relates two composites of 6-arrows. This relation is trivially satisfied as the only 6-arrows in $\Sigma^3(\mathbb{C})$ are identities. Functoriality of the syllepsis in the braiding becomes non-trivial for sylleptic 3- and higher-dimensional teisi.

6.5 Associativity

In $\Sigma^3(\mathbb{C})$, associativity of $\#_0$ relates two composites of 7-arrows. This relation is trivially satisfied as the only 7-arrows in $\Sigma^3(\mathbb{C})$ are identities. Associativity of the syllepsis becomes non-trivial for sylleptic 4- and higher-dimensional teisi.

The diagram for associativity of the syllepsis in a sylleptic 4D tas has already been investigated by me in another context [9, p. 153–163].

6.6 Unit axioms

In $\Sigma^3(\mathbb{C})$, id^3_* is the identity for $\#_0$. In $\mathbb{C}$ this means that syllepsis with I is the identity, i.e., $v_{A,I} = \mathrm{id}^2_A = v_{I,A}$ for all objects A of $\mathbb{C}$.

Day and Street note that these unit axioms follow from the other axioms. But their proof requires the triviality of $\tilde{R}_{(A|I,I)}$ and $\tilde{R}_{(I,I|A)}$, and the invertibility of $v_{A,I}$ and $v_{I,A}$ respectively. This suggests that it might be possible to economize on axiom (vii) for ω-teisi, by only requiring $c' \#_n \mathrm{id}_c = c'$ ($\mathrm{id}_{c'} \#_n c$ similar) for c of dimension n. This will not be the case in weaker situations, though.

6.7 Conclusion

Definition 6.1 A *sylleptic 2D tas* consists of a braided 2D tas $\mathbb{C}$ together with an iso-2-arrow

$$\begin{tikzcd}
AB \arrow[rr, bend left, "R_{A,B}"] \arrow[rr, bend right, "(R_{B,A})^{-1}"'] \arrow[rr, phantom, "\Downarrow\nu_{A,B}"] & & BA
\end{tikzcd}$$

for any objects $A, B \in \mathbb{C}$, such that the following diagrams commute:

$$\begin{tikzcd}
 & BA \arrow[dr] \arrow[d, phantom, "\gamma\Downarrow"] & \\
AB \arrow[ur, bend left] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow\alpha"] \arrow[dr] & {} \arrow[d, phantom, "\Downarrow\beta"] & B'A \\
 & AB' \arrow[ur, dashed] \arrow[ur, bend right] \arrow[ur, phantom, "\delta\Downarrow"'] &
\end{tikzcd} \tag{6.1}$$

where

$$\begin{aligned}
\alpha &= \nu_{A,B} & \qquad \gamma &= R_{A,g} \\
\beta &= (R_{B',A})^{-1} \#_0 R_{g,A} \#_0 (R_{B,A})^{-1} [3] & \qquad \delta &= \nu_{A,B'},
\end{aligned}$$

$$\begin{tikzcd}
 & A'B \arrow[dr, dashed] \arrow[dr, bend left] \arrow[dr, phantom, "\gamma\Downarrow"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
AB \arrow[ur] \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Downarrow\beta"] & {} \arrow[d, phantom, "\delta\Downarrow"] & BA' \\
 & BA \arrow[ur] &
\end{tikzcd} \tag{6.2}$$

where

$$\begin{aligned}
\alpha &= R_{f,B} & \qquad \gamma &= \nu_{A',B} \\
\beta &= \nu_{A,B} & \qquad \delta &= (R_{B,A})^{-1} \#_0 R_{B,f} \#_0 (R_{B,A'})^{-1} [4],
\end{aligned}$$

[3] This is the *mate* of $R_{g,A}$ in the sense of Kelly [28].

$$
\begin{tikzcd}
 & BAC \arrow[dr] \arrow[dd, equal] \arrow[dr, phantom, "\gamma\Downarrow", near end] & \\
ABC \arrow[ur] \arrow[dr] \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[r, phantom, "\beta\Downarrow"] & & BCA \\
 & BAC \arrow[ur] &
\end{tikzcd}
\tag{6.3}
$$

where

$$
\alpha = \nu_{A,BC} \qquad \begin{aligned} \beta &= \nu_{A,B}C \\ \gamma &= B\nu_{A,C}, \end{aligned}
$$

$$
\begin{tikzcd}
 & ACB \arrow[dr] \arrow[dd, equal] & \\
ABC \arrow[ur] \arrow[dr] \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[r, phantom, "\gamma\Downarrow", near start] & & CAB \arrow[l, phantom, "\beta\Downarrow"] \\
 & ACB \arrow[ur] &
\end{tikzcd}
\tag{6.4}
$$

where

$$
\alpha = \nu_{AB,C} \qquad \begin{aligned} \beta &= \nu_{A,C}B \\ \gamma &= A\nu_{B,C}, \end{aligned}
$$

and such that the following equations hold:

$$
\nu_{A,I} = \mathrm{id}_{\mathrm{id}_A} \tag{6.5}
$$

$$
\nu_{I,A} = \mathrm{id}_{\mathrm{id}_A}\,. \tag{6.6}
$$

◇

Theorem 6.2 *A sylleptic 2D tas is precisely a (semistrict) sylleptic monoidal 2-category (as defined in [10]) for which $R_{A,B}$ is an isomorphism and $\tilde{R}_{(A|B,C)}$*

[4]The mate of $R_{B,f}$.

and $\tilde{R}_{(A,B|C)}$ are identities for all A, B and C. □

This notion of syllepsis differs from the one in [10], as already indicated there, but they are obviously equivalent, as is indicated there too.

Even if the unit axioms for the tensor are weakened to (specified) isomorphisms and $*$ is the identity for the syllepsis only up to (specified) isomorphisms, with these isomorphisms satisfying suitable further coherence conditions, also with respect to one another (I will not give a precise definition here for reasons of space), then the hexagon diagram above can still be subdivided as follows:

$$\begin{tikzcd}
 & & IAB \arrow[r] & IBA \arrow[dr] \arrow[ddrr, two heads] & & \\
 & AB \arrow[ur, two heads] \arrow[rrr, "R_{A,B}"] \arrow[dd, equal] \arrow[ddrrr, phantom, "\Downarrow \nu_{A,B}"] & & & BA \arrow[dr] \arrow[dd, equal] & \\
AIB \arrow[uurr, two heads] \arrow[ur] \arrow[dr] \arrow[ddrr, two heads] & & & & & BIA \\
 & AB \arrow[dr, two heads] & & & BA \arrow[lll, "R_{B,A}"'] \arrow[ur, two heads] & \\
 & & ABI & BAI \arrow[l] \arrow[ur] \arrow[uurr, two heads] & &
\end{tikzcd},$$

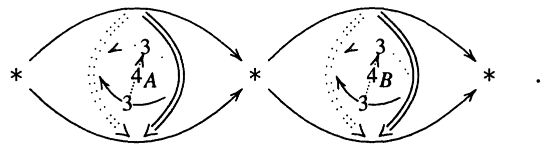
with coherence isomorphisms in the unlabeled regions. This implies that also for somewhat weaker braided monoidal 2-categories both notions of syllepsis are equivalent.

7 What is symmetry?

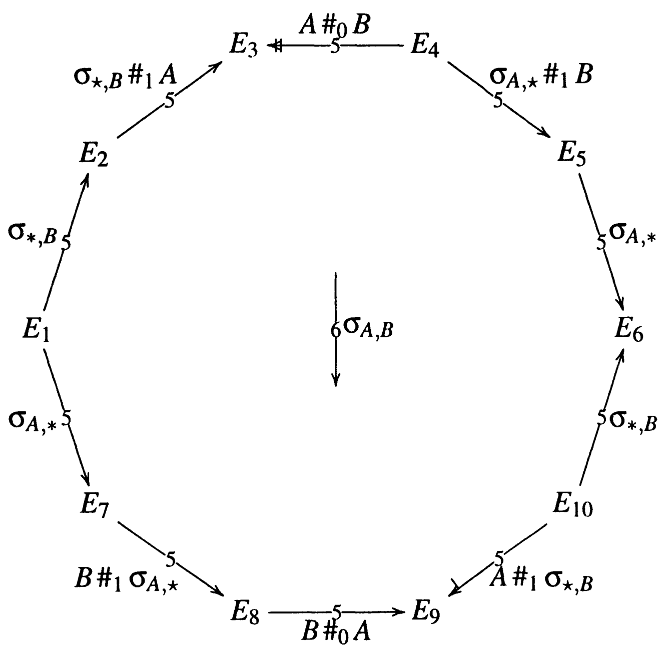
I will now investigate the heuristic notion of a 6D tas with one 3-arrow as motivation for an *ad hoc* definition of a 4-monoidal 2D tas, which, by stabilization, should be a symmetric 2D tas. Ignoring the difference between braided 2D teisi and (semistrict) braided monoidal 2-categories, as before, symmetric 2D teisi will indeed turn out to be equivalent to (semistrict) symmetric monoidal 2-categories (as defined in [10]).

7.1 The symmetry

The symmetry on a 2D tas $\mathbb{C}$ should be defined as 0-composition in $\Sigma^4(\mathbb{C})$, i.e., $\sigma_{A,B} = B\#_0 A$:

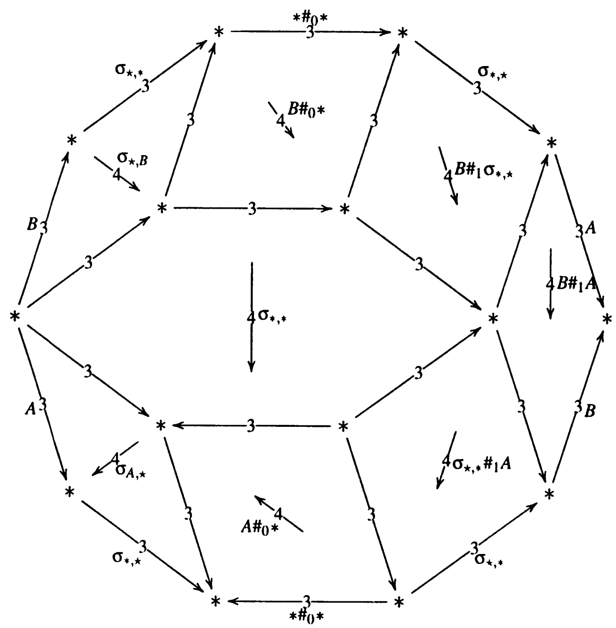


0-composition of 4-arrows in $\Sigma^4(\mathbb{C})$ should be "governed by $2_3 \otimes 2_3$", which means that after reindexing and taking into account the extra horizontal 0- and 1-compositions occurring in there, $\sigma_{A,B}$ should be given in $\Sigma^3(\mathbb{C})$ by

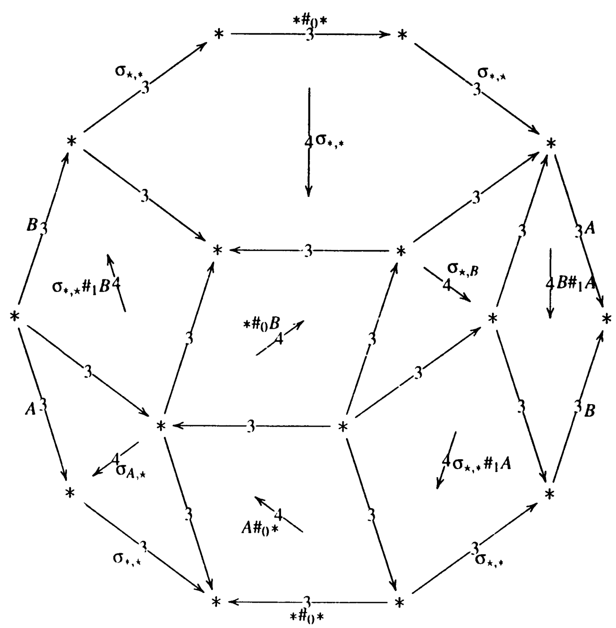


where the E_m are given by:

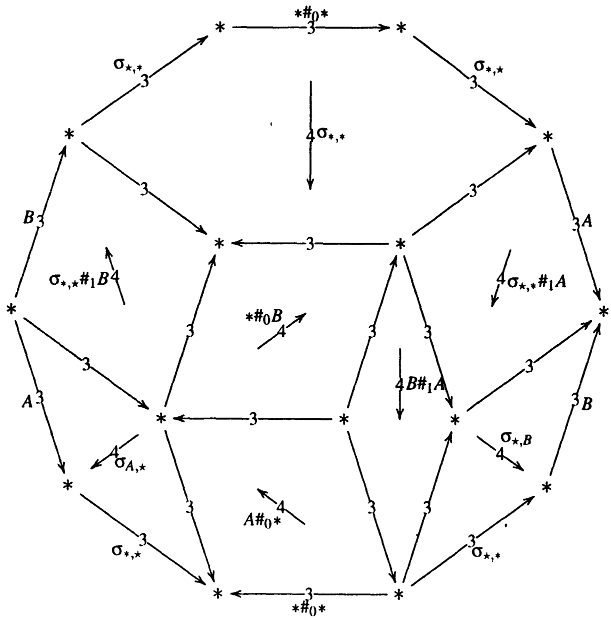
E_1:


$*\#_0*$
$\sigma_{*,*}$
$B\#_0*$
$\sigma_{*,B}$
$B\#_1\sigma_{*,*}$
B
A
$\sigma_{*,*}$
$B\#_1A$
A
$\sigma_{A,*}$
$\sigma_{*,*}\#_1A$
B
$A\#_0*$
$\sigma_{*,*}$
$\sigma_{*,*}$
$*\#_0*$

E_2:

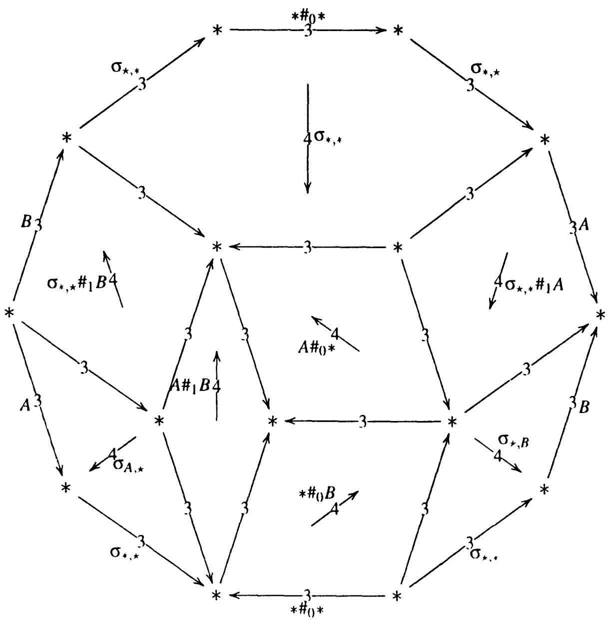

$*\#_0*$
$\sigma_{*,*}$
$\sigma_{*,*}$
$\sigma_{*,*}$
B
A
$\sigma_{*,*}\#_1B$
$\sigma_{*,B}$
$B\#_1A$
$*\#_0B$
A
B
$\sigma_{A,*}$
$\sigma_{*,*}\#_1A$
$A\#_0*$
$\sigma_{*,*}$
$\sigma_{*,*}$
$*\#_0*$

E_3:



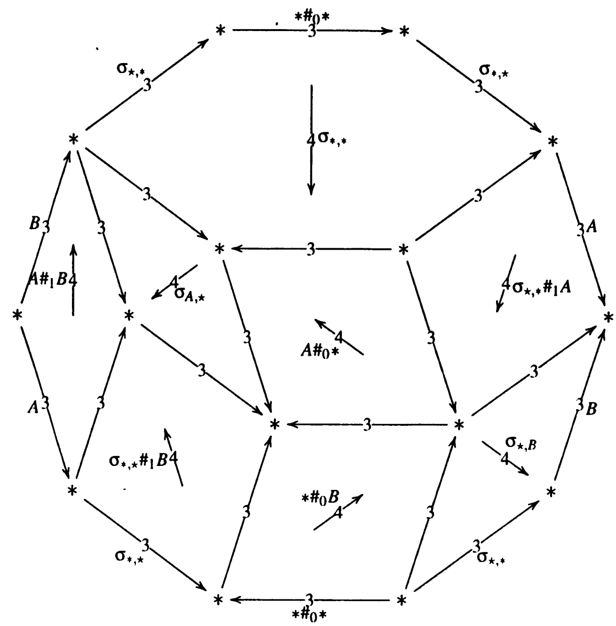
#0
σ*,*
σ*,*
σ*,*
B
A
σ*,*#1B
σ*,*#1A
*#0B
B#1A
A
B
σA,*
σ*,B
A#0*
σ*,*
σ*,*
#0

E_4:

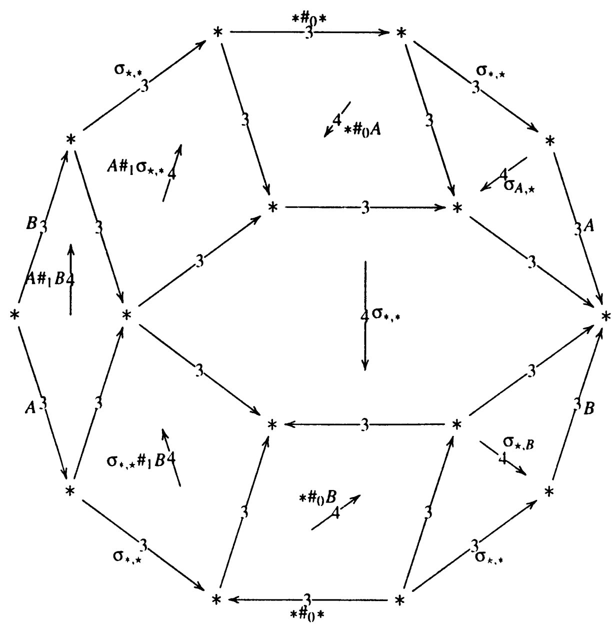


#0
σ*,*
σ*,*
σ*,*
B
A
σ*,*#1B
σ*,*#1A
A#0*
A#1B
A
B
σA,*
σ*,B
*#0B
σ*,*
σ*,*
#0

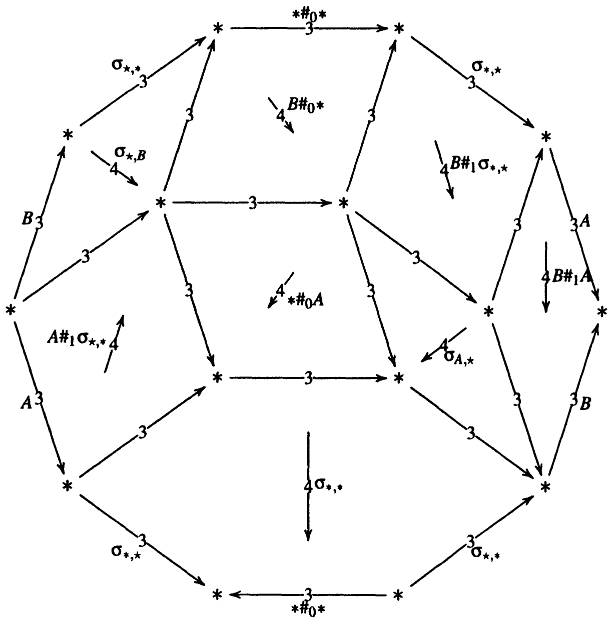
E_5:



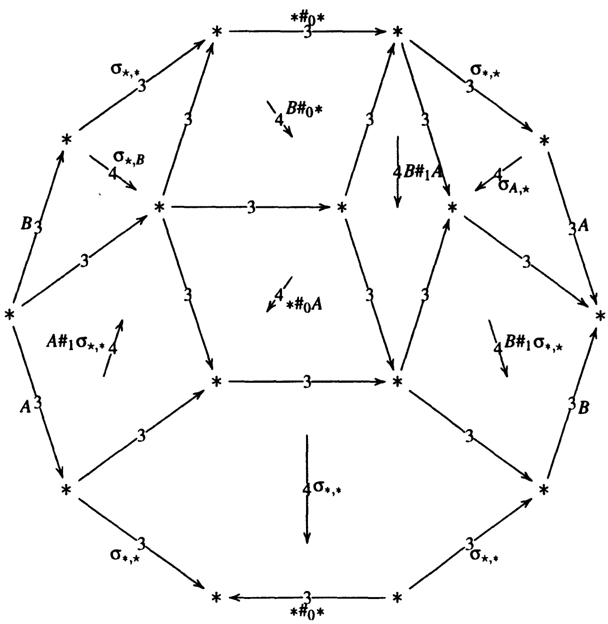
E_6:



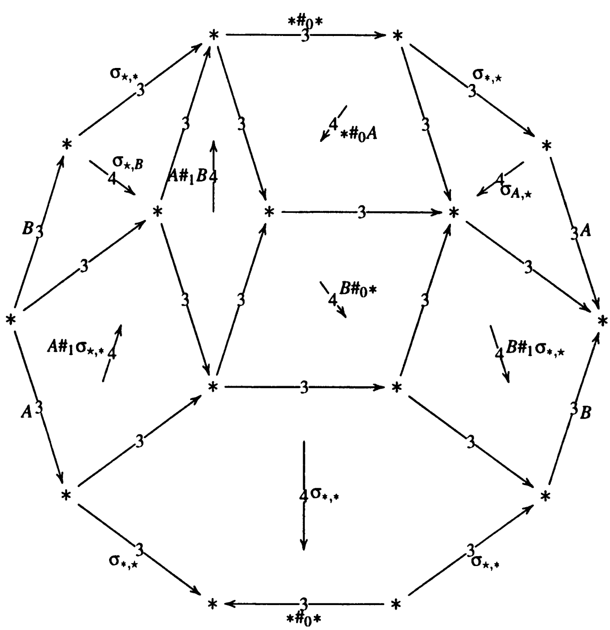
E_7:


#0
σ*,*
B#0*
σ*,B
B#1σ*,*
B
A
B#1A
*#0A
A#1σ*,*
σA,*
A
B
σ*,*
σ*,*
σ*,*
#0

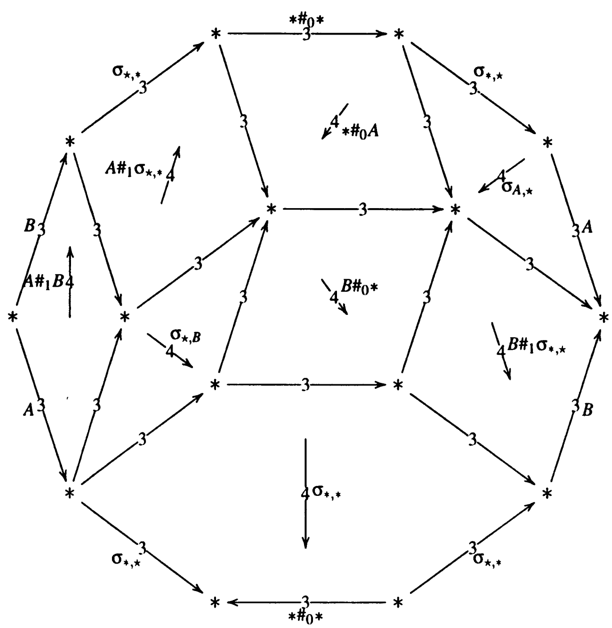
E_8:


#0
σ*,*
B#0*
σ*,*
σ*,B
B#1A
σA,*
B
A
*#0A
A#1σ*,*
B#1σ*,*
A
B
σ*,*
σ*,*
σ*,*
#0

E_9:



E_{10}:



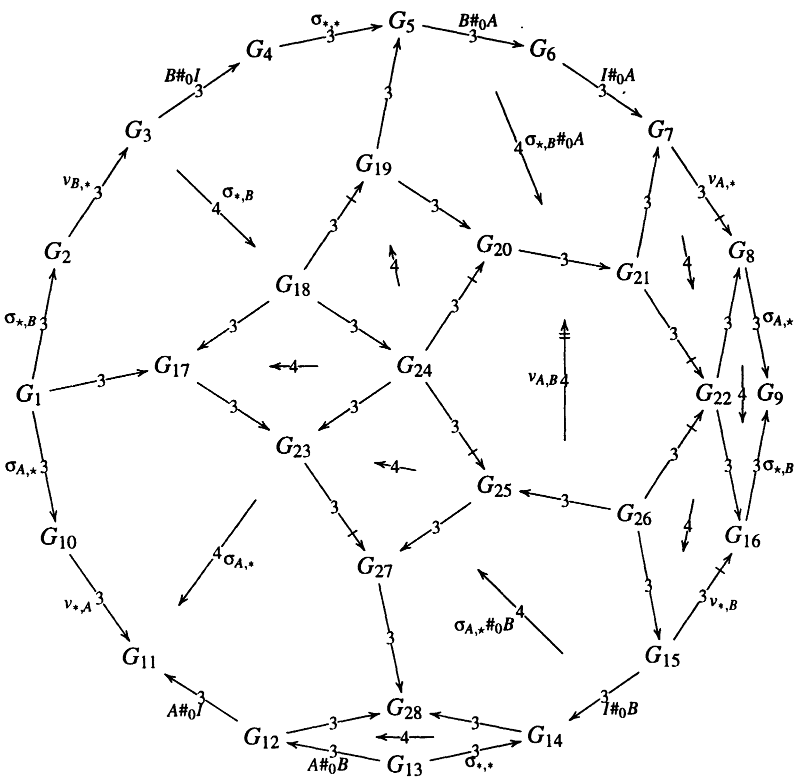
,

with the convention that $*$ is the unique 2-arrow and $\star$ is the unique arrow of

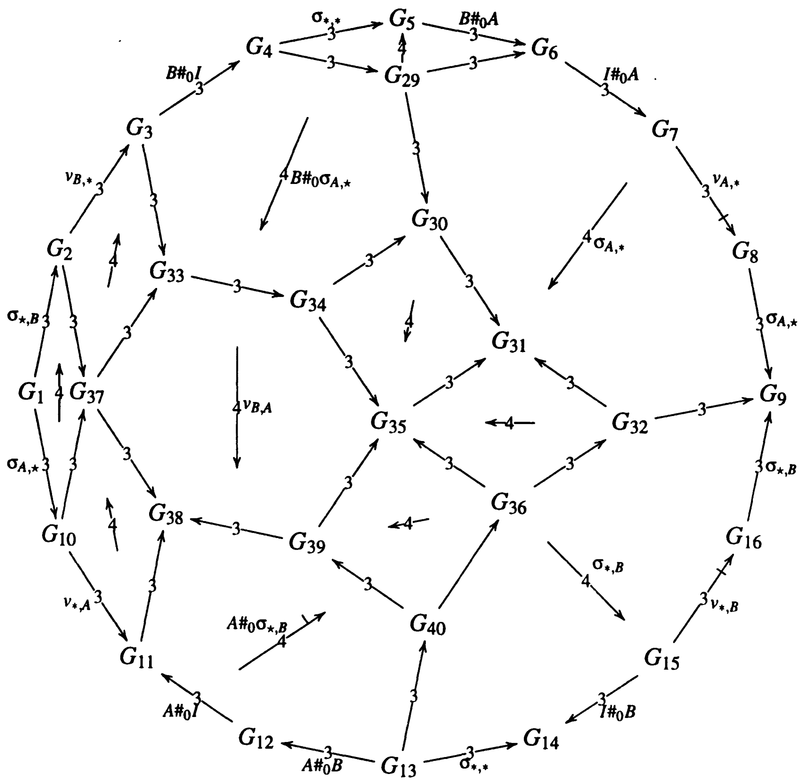
$\Sigma^3(\mathbb{C})$, and the crossbars near the arrowheads in the picture again indicating that one or more elements in the boundary have "flipped over" to the other side, i.e., an element which is "canonically" in the domain is now in the codomain, or vice versa, or both, and the double crossbar indicating that this happens only or as well for lower dimensional faces.
Reindexing once more, taking into account further horizontal compositions that occur, and remembering that in $\Sigma^4(\mathbb{C})$ $\mathrm{id}^3_* \#_0 \mathrm{id}^2_* = \mathrm{id}^2_* \#_0 \mathrm{id}^3_* = \mathrm{id}^4_*$ and $\mathrm{id}^3_* \#_1 \mathrm{id}^3_* = \mathrm{id}^4_*$, $\sigma_{A,B}$ should be given in $\Sigma^2(\mathbb{C})$ by

$$F_1 \xrightarrow[5]{\sigma_{A,B}} F_2$$

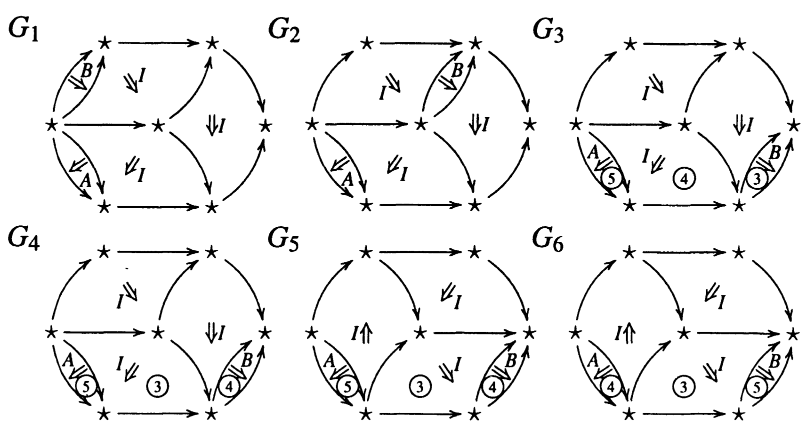
where the F_n are given by:
F_1:



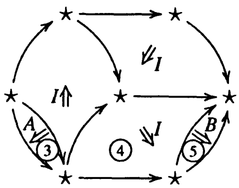
F_2:



and where the G_p are as follows:



G_7



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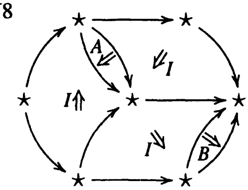

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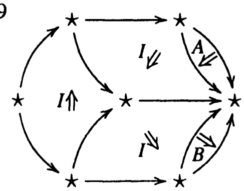

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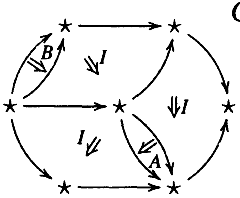

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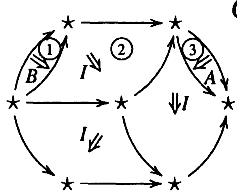

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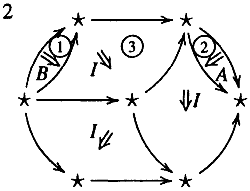

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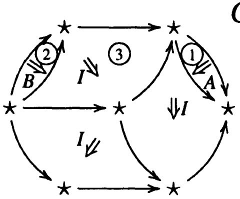

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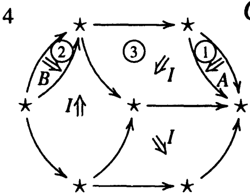

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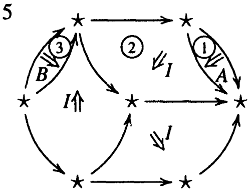
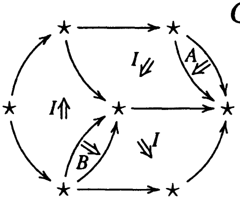


G_{16}



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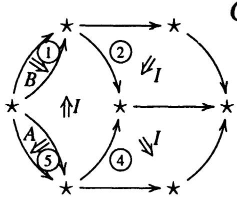

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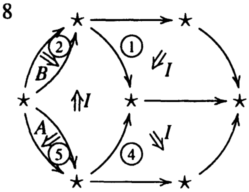
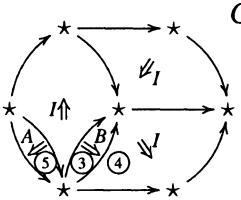
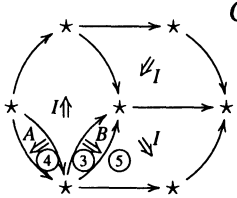


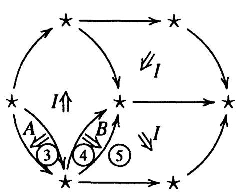
G_{19}



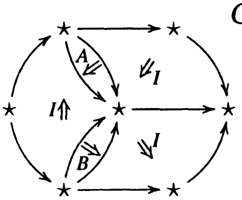
G_{20}



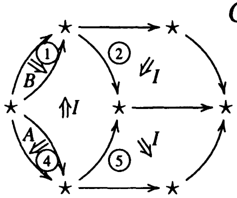
G_{21}



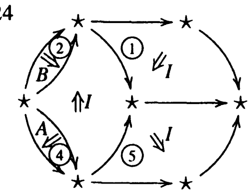
G_{22}

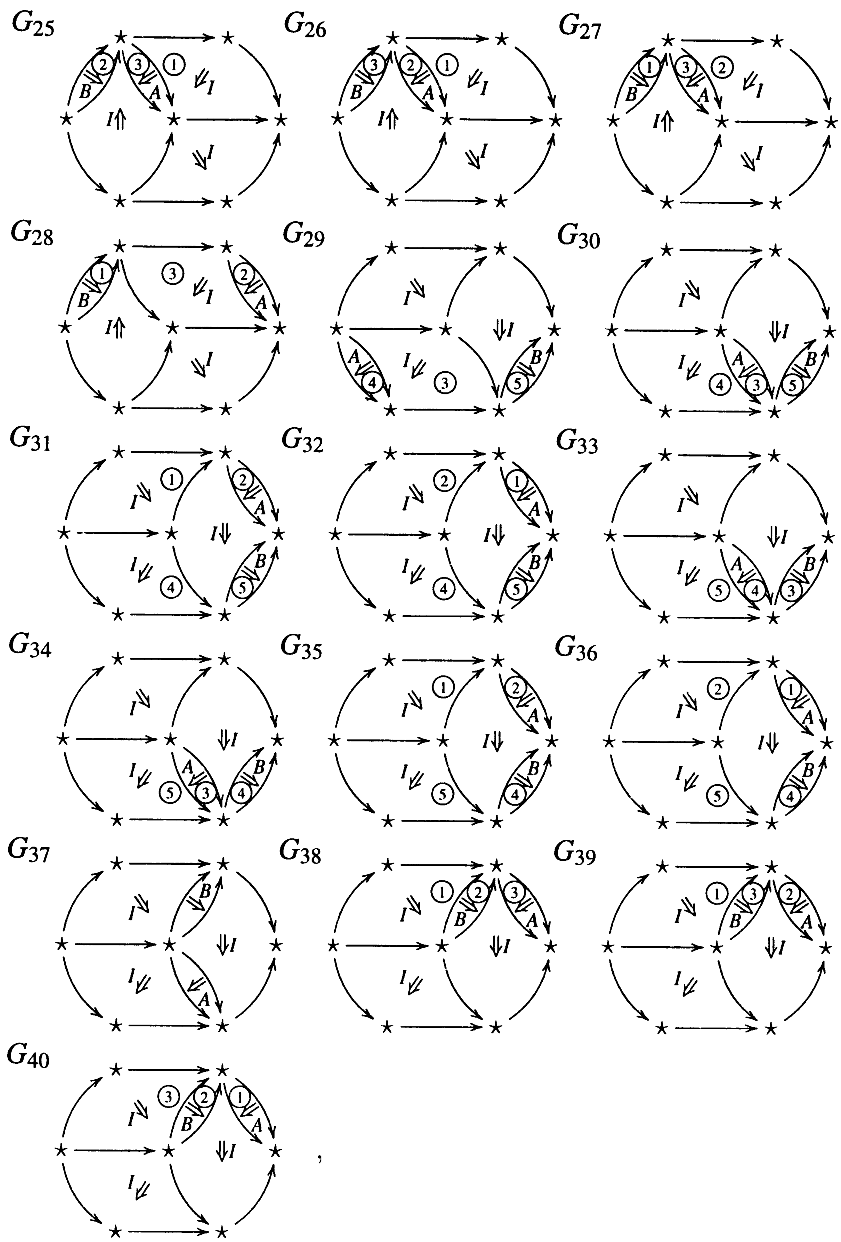


G_{23}



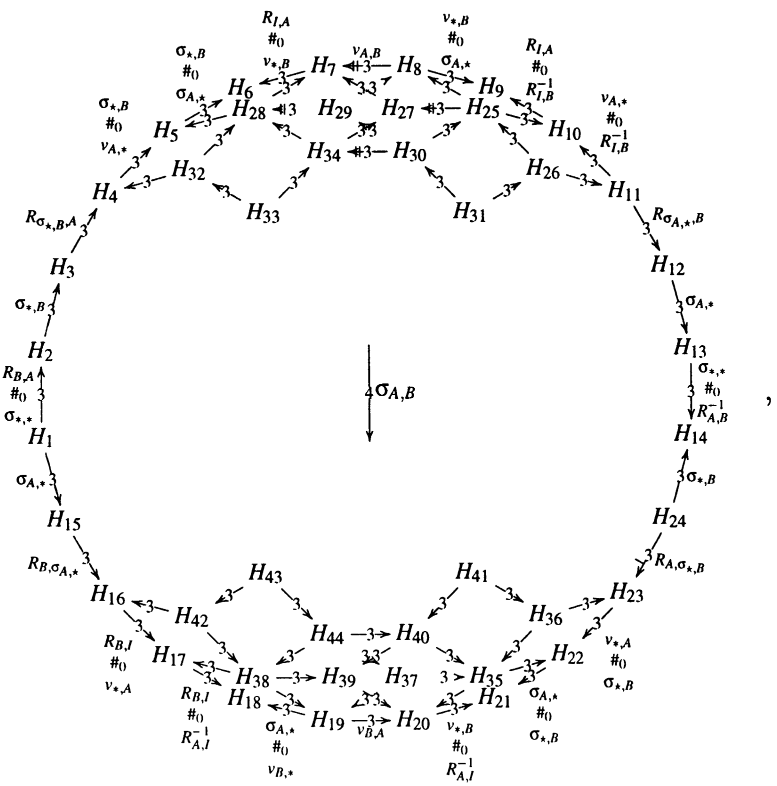
G_{24}





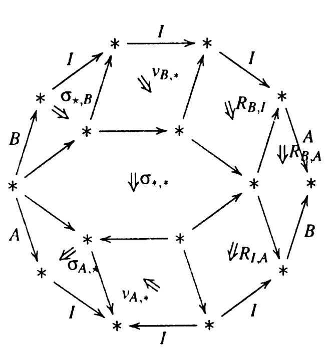
with $*$ being the unique arrow and $\star$ the unique object of $\Sigma^2(\mathbb{C})$, and the

crossbars as before.
Reindexing once again, $\sigma_{A,B}$ should be given in $\Sigma(\mathbb{C})$ by

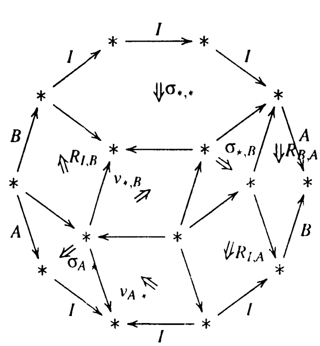


where the H_q are given by:

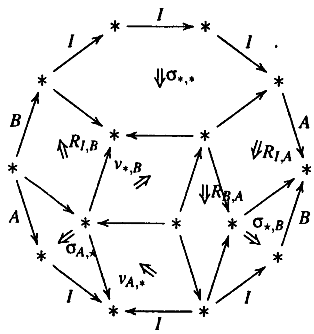
H_1
H_2



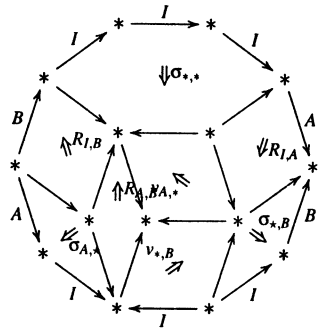
H_3



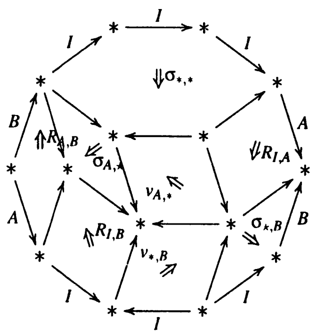
H_4
H_5
H_6
H_7
H_{27}
H_{28}
H_{32}
H_{33}
H_{34}



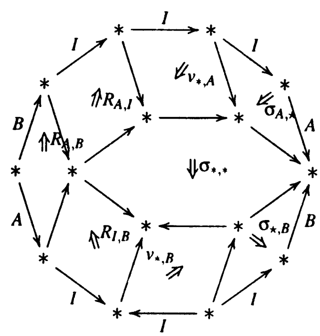
H_8
H_9
H_{10}
H_{11}
H_{25}
H_{26}
H_{29}
H_{30}
H_{31}



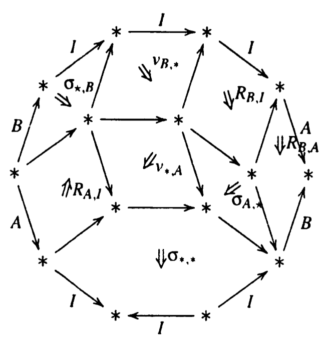
H_{12}



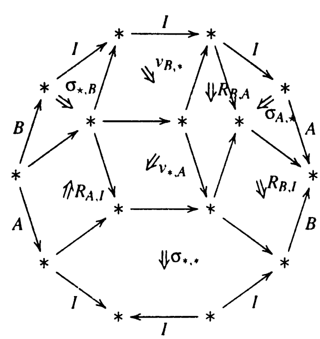
H_{13}
H_{14}



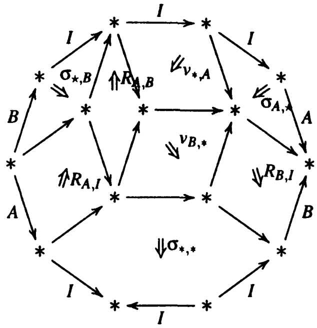
H_{15}



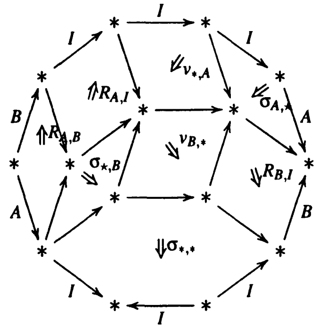
H_{16}
H_{17}
H_{18}
H_{19}
H_{37}
H_{38}
H_{42}
H_{43}
H_{44}



H_{20}
H_{21}
H_{22}
H_{23}
H_{35}
H_{36}
H_{39}
H_{40}
H_{41}



H_{24}

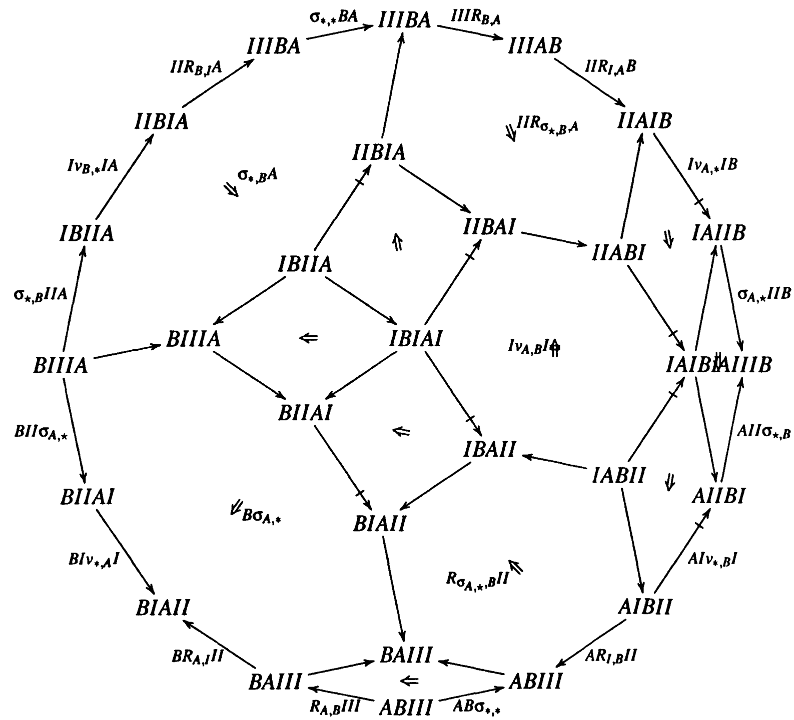


,

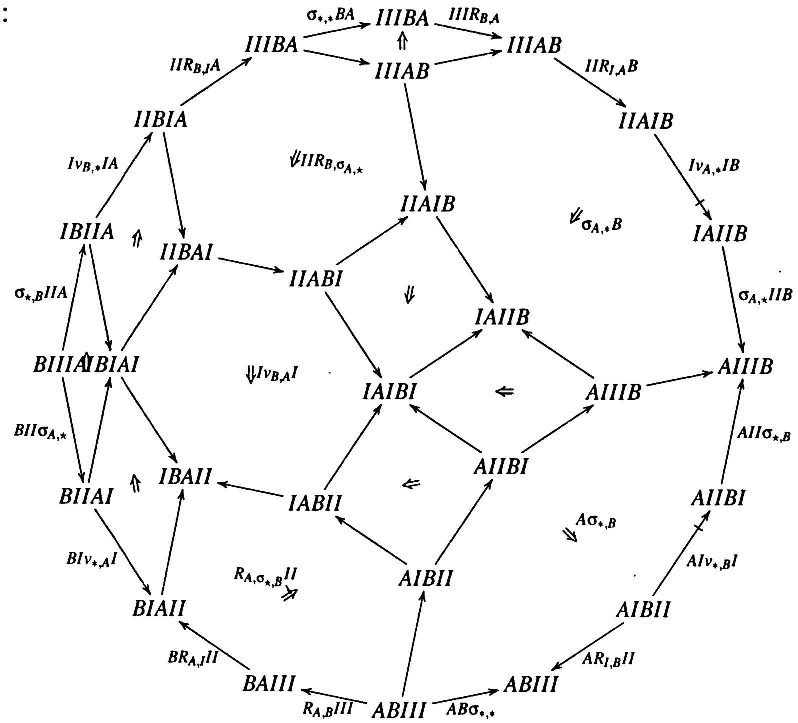
with $*$ being the unique object and $\star$ the unique (-1)-arrow of $\Sigma(\mathbb{C})$, and the crossbars as before.
Reindexing again once more, $\sigma_{A,B}$ should be given in $\mathbb{C}$ by

$$P_1 \xrightarrow[3]{\sigma_{A,B}} P_2$$

where the P_w are given by:
P_1:



P_2:



,

with $*$ being the unique (-1)-arrow and $\star$ the unique (-2)-arrow of $\Sigma(\mathbb{C})$, and the crossbars as before, and remembering that in $\Sigma^4(\mathbb{C})$ id^4_* is the identity for $\#_3$ and for $\#_2$, id^3_* is the identity for $\#_1$, and id^2_* is the identity for $\#_0$, this becomes

$$\begin{tikzcd} BA \arrow[rr, bend left, "R_{B,A}"] \arrow[rr, phantom, "\nu_{A,B}\Uparrow"] && AB \arrow[ll, bend left, "R_{A,B}"] \arrow[r, "\sigma_{A,B}", "3"'] & BA \arrow[rr, bend left, "R_{B,A}"] \arrow[rr, phantom, "\Downarrow\nu_{B,A}"] && AB \arrow[ll, bend left, "R_{A,B}"] \end{tikzcd} .$$

That $\sigma_{A,B}$ equals $B\#_0A$, rather than the other way around, was necessary to make $\sigma_{A,B}$ come out with domain $\nu_{A,B}$, rather than $\nu_{B,A}$. There is a pattern here when comparing $\#_{-k}$ in $\mathbb{C}$ with $\#_0$ in $\Sigma^k(\mathbb{C})$: $A \otimes B$ is given by $B\#_0A$, $R_{A,B}$ is given by $A\#_0B$, as is $\nu_{A,B}$, and $\sigma_{A,B}$ again by $B\#_0A$, i.e., $\#_{-k}$ runs in the opposite direction than $\#_{-k+1}$ only when k is even.The origin of this pattern is that the convention is such that, say, $R_{A,B}$ in the *domain* of $v_{A,B}$, rather than letting this being determined by the appropriate tensor products of globes.

7.2 Conclusion

Definition 7.1 A *symmetric 2D tas* is a sylleptic 2D tas $\mathbb{C}$ such that for any objects $A, B \in \mathbb{C}$ the following diagram commutes:

$$\begin{tikzcd} AB \arrow[r, bend left=50, "R_{A,B}"] \arrow[r, bend right=50, "(R_{B,A})^{-1}"'] \arrow[r, phantom, "\beta \qquad \alpha"] & BA \end{tikzcd} \tag{7.1}$$

where

$$\alpha = v_{A,B} \qquad\qquad \beta = (R_{B,A})^{-1} \#_0 v_{B,A} \#_0 R_{A,B}.$$

◇

Theorem 7.2 *A symmetric 2D tas is precisely a (semistrict) symmetric monoidal 2-category (as defined in [11]) for which $R_{A,B}$ is an isomorphism and $\tilde{R}_{(A|B,C)}$ and $\tilde{R}_{(A,B|C)}$ are identities for all A, B and C.* □

Even if the identity axioms are weakened to (specified) isomorphisms, with these isomorphisms satisfying suitable further coherence conditions, also with respect to one another (again, I will not give a precise definition here for reasons of space), then the big diagram for $\sigma_{A,B}$ in $\mathbb{C}$ can still be subdivided similar to the subdivision of the diagram for $v_{A,B}$ on page . This implies that also for somewhat weaker sylleptic monoidal 2-categories both notions of symmetry are equivalent.

Acknowledgement

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