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DOMINIQUE BOURN

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BAER SUMS AND FIBERED ASPECTS OF MAL'CEV OPERATIONS

by Dominique BOURN

RÉSUMÉ. Le sens géométrique des axiomes des lois de Mal'cev est ici réaffirmé de sorte que, dans le cas associatif, la classique action de groupe associée en découle naturellement dans le cadre général des catégories exactes au sens de Barr, lorsque le support est global. Cette action est définie par l'intermédiaire d'un foncteur *direction d* que l'on montre être une cofibration préservant les produits et l'objet final. On est alors dans la plaisante situation où toute structure de groupe sur un objet X de la base détermine canoniquement une structure monoïdale fermée sur la fibre au-dessus de X. On retrouve ainsi de manière conceptuelle la construction de Baer de la somme de deux extensions du groupe Q à noyaux abéliens déterminant la même structure de Q-module. Par ailleurs cette même cofibration *d* permet de préciser le lien entre catégories Naturellement Mal'cev et catégories essentiellement affines. On précise enfin la structure de la direction lorsque le support n'est pas global.

I. Introduction: the Chasles relation of a Mal'cev operation.

Let p be a ternary operation on a set X.

Definition 1. Let us call the *Chasles relation associated with p*, and denote by Ch(X,p) or simpler Ch(X), the relation R on X×X defined by:

$$(x,t)\mathrm{R}(y,z) \text{ if and only if } t=p(x,y,z).$$

This relation is represented by the following graph:

$$\mathrm{X}\times\mathrm{X}\times\mathrm{X} \underset{p_2}{\overset{g}{\rightrightarrows}} \mathrm{X}$$

where

$$g(x,y,z)=(x,p(x,y,z)) \quad\text{and}\quad p_2(x,y,z)=(y,z).$$

Then, clearly, this relation is:

reflexive	iff	$p(x,x,y) = y$	(1)
symmetric	iff	$(x,y,p(y,x,z)) = z$	(3)
transitive	iff	$p(x,y,p(y,z,u))=p(x,z,u)$	(4)

Obviously (1) and (4) imply (3).

Furthermore, the diagonal s_0: $X \to X{\times}X$ is an equivalence class of R iff

$$p(x,y,y) = x. \qquad (2)$$

In other words, the Chasles relation associated with a Mal'cev operation, i.e. satisfying (1) and (2) [9], which is right associative, i.e. satisfying (4), is an equivalence relation on X×X with the diagonal as an equivalence class. The axiom (3) has been extensively studied in [5] under the name of Big Dipper Identity. The axiom (4) is, *modulo* a twisting of variables, half of the axiom (3) in [7], and the Chasles relation essentially the same as the *geometric relation*.

The aim of this paper is to establish some properties of this equivalence relation and of its quotient. The behavior of this quotient will appear remarkably good in the general context of Barr exact categories, thanks to the following observation:

This equivalence relation is actually discretely fibered above the coarse relation gr(X) on X (for which all the elements of X are equivalent) by the first projection p_0: $X{\times}X \to X$, since the following square is a pullback:

$$\begin{array}{ccc} X{\times}X{\times}X & \xrightarrow{\;p_2\;} & X{\times}X \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_0} \\ X{\times}X & \xrightarrow{\;p_1\;} & X \end{array}$$

Now, the condition (2) is equivalent to the fact that the section s_0: $X \to X{\times}X$ of p_0 is a morphism of relation between gr(X) and Ch(X)

This observation will allow us to prove that the quotient functor d, called the *direction* of this relation, when it is restricted to the associative Mal'cev operations, is actually a cofibration preserving the products and the terminal object. As an application, we shall obtain a converse link between the notions of Naturally Mal'cev [6] and essentially affine [3] categories, as well as a closed monoidal structure on the associative and commutative Mal'cev operations having the same direction which will give a conceptual insight into the classical group structure on the extensions with abelian kernel.

II. The slice protodivision.

The previous map g appears naturally as an internal binary operation on the first projection p_0: X×X → X, seen as an object in the slice category Sets/X, with:

$$(x,y)\backslash(x,z) = (x,p(x,y,z)).$$

The condition (1) means that (x,x) is a left unit element, the condition (2) that this operation satisfies the quadratic neutrality:

$$(x,y)\backslash(x,y) = (x,x).$$

In other words, this binary operation is thus a left hand side protodivision, see [4] for instance.

Lemma 1. *The condition* (3) *implies that the protodivision is left regular, or equivalently that:*

$$a = b \quad \text{iff} \quad p(x,y,a) = p(x,y,b).$$

The fibred division. In presence of the Chasles relation, this protodivision will be fibred by the first projection p_0 when:

$$p(x',p(x',x,y),p(x',x,z)) = p(x',x,p(x,y,z)) \qquad (*)$$

which, thanks to (4), is equivalent to:

$$p(x',p(x',x,y),p(x',x,z)) = p(x',y,z) \qquad (5)$$

Proposition 1. *Given* (4), *then the protodivision is fibered above the coarse relation* gr(X) *iff it satisfies the cancellation rule which makes the protodivision a regular division.*

Proof: The cancellation rule, see [4] , for the protodivision means that:

$$((x,y)\backslash(x,z))\backslash((x,y)\backslash(x,t)) = (x,z)\backslash(x,t).$$

But the first term is: $(x,p(x,p(x,y,z),p(x,y,t)))$, and the second one is: $(x,p(x,z,t))$ ¤

Now, classically, a protodivision is a division if and only if it corresponds to a group structure, see [4]. Thus, in presence of the conditions (1), (2), (5), the first projection p_0 is endowed, as an object in the category Sets/X, with a group structure given by:

$$(x,y)*(x,t) = ((x,y)\backslash(x,x))\backslash(x,t) = (x,p(x,p(x,y,x),t)$$

The condition (5) for a Mal'cev operation is not straightforward, and the plain associativity axiom is more usual:

$$p(p(x,y,z),u,v) = p(x,y,p(z,u,v)) \qquad (6)$$

Proposition 2. *Given* (1) *and* (2), *then* (6) *is equivalent to* (4) *and* (5), *i.e. to the fact that the division is fibered.*

Proof: (6) implies (4) since:

$$p(x,y,p(y,z,t)) = p(p(x,y,y),z,t) = p(x,z,t).$$

Let us show that (6) implies (5). We have:

$$p(p(x,y,z),x,p(x,p(x,y,z),p(x,y,t))) = p(x,y,t) \quad \text{and}$$
$$p(p(x,y,z),x,p(x,z,t)) = p(p(p(x,y,z),x,x),z,t) = p(p(x,y,z),z,t) = p(x,y,p(z,z,t)) = p(x,y,t).$$

Then, by Lemma 1, we obtain (5).
Conversely, let us suppose (4) and (5). Then:

$$p(x,p(x,y,z),p(p(x,y,z),u,v)) = p(x,u,v)$$

by (4), and:

$$p(x,p(x,y,z),p(x,y,p(z,u,v))) = p(x,z,p(z,u,v))[\text{by (5)}] = p(x,u,v).$$

Whence, again by Lemma 1, we have:

$$p(p(x,y,z),u,v) = p(x,y,p(z,u,v)). \qquad ¤$$

Corollary 2. *From* (6), *we can derive the following identity:*

$$p(x,p(y,z,t),u) = p(p(x,t,z),y,u) \qquad (7).$$

Proof: We have:

$$p(p(y,z,t),x,p(x,p(y,z,t),u)) = u$$

and:

$$p(p(y,z,t),x,p(p(x,t,z),y,u)) = p(p(p(y,z,t),x,p(x,z,t)),y,u)$$
$$= p(p(p(y,z,t),z,t),y,u) = p(y,y,u) = u \; . \qquad ¤$$

Remark. Now we can check that:

$$p(x,p(x,y,x),t) = p(p(x,x,y),x,t) = p(y,x,t)$$

and that the group law $*$ on the projection p_0 can be more naturally written:

$$(x,y)*(x,t) = (x,p(y,x,t)).$$

The commutativity. The previous group structure on the first projection p_0 is abelian (see [4]) if and only if the division satisfies:

$$((x,y)\backslash(x,z))\backslash(x,z) = (x,y), \text{ i.e. if and only if: } p(x,p(x,y,z),z) = y. \qquad (8)$$

Certainly, the condition (8) is not straightforward, but:

Proposition 3. *Given* (1), (2) *and* (6), *then* (8) *is equivalent to the plain commutativity:*

$$p(x,y,z) = p(z,y,x). \qquad (9)$$

Proof: Let us suppose (8), then: $p(y,x,p(x,y,z)) = z$ and:

$$p(y,x,p(z,y,x)) = p(y,p(y,z,x),x) \text{ [by (7)]} = z \text{ [by (8)]}$$

Let us suppose (9), then:

$$p(x,p(x,y,z),z) = p(p(x,z,y),x,z) \text{ [by (7)]} = p(p(y,z,x),x,z) \text{ [by (9)]} = y. \quad ¤$$

Of course, these results which, for sake of simplicity, have been written in Sets, do hold actually in any left exact category E.

III. The direction of X and its canonical action.

From the previous observations, we shall easily derive the classical action associated with an associative Mal'cev operation, and this in the general context of any Barr exact category [1]. So, from now on, we shall suppose that E is a left exact and Barr exact category.

First, let us suppose that p satisfies (1) and (4), i.e. that the Chasles relation is an equivalence relation.

Definition 2. We shall call *direction of the ternary operation p* and denote by $d(X,p)$ or simply by $d(X)$ the quotient of $X{\times}X$ by Ch(X), i.e. the cokernel of g and p_2.

The Chasles relation being fibrered above gr(X), the following square is then a pullback ([1], 6.10), which makes $X{\times}X$ isomorphic to $X{\times}dX$

$$\begin{array}{ccc} X{\times}X & \xrightarrow{\;q(X)\;} & d(X) \\ {\scriptstyle p_0}\downarrow & & \downarrow \\ X & \longrightarrow & 1 \end{array}$$

When furthermore p satisfies (2) and the object X has a global support (i.e. the terminal map with domain X is a regular epimorphism), then the object $d(X)$ is canonically pointed by the factorisation of the diagonal through the quotients.

Main Remark. 1. The left hand side division on the first projection p_0 being discretely fibered in presence of the axiom (5) (or (6)), then, the quotient functor d preserving the pullbacks of discrete fibrations and the object X *having a global support*, the object $d(X)$ is itself endowed with a left hand side division which determines a group structure on $d(X)$. This group structure is essentially the same as the one given in [8] in a much more intricate context, in view of the applications the author had in mind.

2. The quotient map $q(X)$ consequently preserves the fibered operation $*$ and so we obtain the classical *Chasles identities*:

$$q(x,x) = 1 \quad \text{and} \quad q(x,y).q(y,z) = q(x,z)$$

since we have:

$$q(x,y).q(y,z) = q(x,y).q(x,p(x,y,z)) = q((x,y)*(x,p(x,y,z)) \\ = q(x,p(x,p(x,y,x),p(x,y,z)) = q(x,p(x,x,z)) = q(x,z).$$

3. Thus the map $q(\mathrm{X})$ is underlying to an internal functor q: gr(X) $\to$ K(d(X),1), where K(d(X),1) is the internal groupoid (with only one object) associated to the group structure d(X).

Proposition 4. *This group structure of* d(X) *acts on* X *in a simply transitive way.*

Proof: The action a(X) is defined in the following way:

$$\begin{array}{ccc} \mathrm{X}\times d(\mathrm{X}) \sim \mathrm{X}\times\mathrm{X} & \longrightarrow & \mathrm{X} \\ (x,c) \sim (x,y) & \dashrightarrow & x*c = y \end{array}$$

Moreover the isomorphism X×d(X) ~ X×X means that this action is simply transitive. ¤

As a corollary, we have:

Corollary 4. *Let* E *be a Barr exact category with products,* X *an object of* E *with a global support and* C *a group object in* E. *Then the three following conditions are equivalent:*

1. *There is an associative Mal'cev operation on* X *and* d(X) ≅ C.
2. *There is a simply transitive action of the group* C *on* X.
3. *There is a discrete fibration* q: gr(X) → K(C,1).

Proof: We showed 1 ⇒ 2. Conversely it is clear that a simply transitive action of a group C on X determines an associative Mal'cev operation on X:

$$\begin{array}{ccc} \mathrm{X}\times\mathrm{X}\times\mathrm{X} \sim \mathrm{X}\times\mathrm{X}\times\mathrm{C} & \longrightarrow & \mathrm{X} \\ (x,y,z) \sim (x,y,c) & \dashrightarrow & p(x,y,z) = x*c. \end{array}$$

The equivalence 2 ⇔ 3 is classical, see [2] for instance. ¤

Remarks. 1. Given a group C, the canonical action on itself determines the classical associated Mal'cev operation on C defined by $p(a,b,c) = a.b^{-1}.c$.

2. Of course, when moreover the Mal'cev law p is commutative (9), then the group d(X) is abelian. The converse is true as well.

3. We can consequently expect, from the classical result in affine geometry, that the associativity and commutativity of p imply the autonomous condition (10):

$$p(p(x,x',x''),p(y,y',y''),p(z,z',z''))=p(p(x,y,z),p(x',y',z'),p(x'',y'',z''))$$

Proposition 5. *Given* (1) *and* (2), *then* (6) *and* (9) *are equivalent to* (10).

Proof: From (6), Corollary 2 and (9),we get:

$$p(p(x,y,z),t,u)=p(x,y,p(z,t,u))=p(x,p(y,z,t),u)$$

and the fact that for any quintuple of elements the result of the operation does not depend on the brackets. On the other hand, according to (9), this result is not affected by any permutation of the elements in odd position or any permutation of the elements in even position. Now:

$$\begin{aligned}p(p(x,x',x''),p(y,y',y''),p(z,z',z''))&=p(p(x,p(y,y',y''),x''),x',p(z,z',z''))\\&=p(p(x,y,p(y',y'',x'')),x',p(z,z',z'')),\\p(x,y,p(p(y',y'',x''),x',p(z,z',z'')))&=p(x,y,p(z,x',p(p(y',y'',x''),z',z'')))\\&=p(x,y,p(z,x',p(p(y',z',x''),y'',z''))),\\p(x,y,p(z,x',p(y',z',p(x'',y'',z''))))&=p(p(x,y,z),x',p(y',z',p(x'',y'',z'')))\\&=p(p(x,y,z),p(x',y',z'),p(x'',y'',z'')).\end{aligned}$$

The converse is done in [6]. ¤

IV. Properties of the direction functor d.

Let us denote by E_g the full subcategory of E whose objects have a global support. Then this subcategory is stable by products, the inclusion functor $E_g \to E$ is a discrete cofibration, i.e. every morphism with its domain in E_g has its codomain in E_g, and is discretely fibred on the regular epimorphisms, i.e. every regular epimorphism with its codomain in E_g has its domain in E_g. Consequently E_g admits pullbacks along the regular epimorphisms and in particular along the split epimorphisms. The kernel equivalence of any map in E_g is in E_g too. Thus E_g is Barr exact with products.

Let us denote by RM(E) the category whose objects are the objects X of E endowed with a right associative Mal'cev operation p (i.e. satisfying 1, 2 and 4). Then RM(E) is again left exact and Barr exact.

Proposition 6. *The direction functor d:* RM(E) $\rightarrow$ E *preserves the products and the regular epimorphisms. When it is restricted to* RM(E_g), *it reflects the isomorphisms, preserves the pullbacks when they exist and consequently reflects them. Moreover, when two maps h, h' from* X *to* Y *such that $d(h) = d(h')$ have their kernel with a global support, they are equal. As a consequence the direction functor d:* RM(E_g) $\rightarrow$ E_g *is exact in the sense of Barr.*

Proof: Clearly Ch(X×X',p×p') is isomorphic to Ch(X,p)×Ch(X',p') and the quotient preserves the products in a Barr exact category ([1], 2.15). If h: X $\rightarrow$ X' is a morphism in RM(E) which is a regular epimorphism, then $h\times h$: X×X $\rightarrow$ X'×X' is again a regular epimorphism, and consequently so is $q(X').h\times h = d(h).q(X)$. Thus $d(h)$ is a regular epimorphism. Now let h be a morphism in RM(E_g) such that $d(h)$ is an isomorphism. Then the following outside square is a pullback:

$$\begin{array}{ccccc} X\times X & \xrightarrow{h\times h} & X'\times X' & \xrightarrow{q(X')} & d(X') \\ {\scriptstyle p_0}\downarrow & & {\scriptstyle p_0}\downarrow & & \downarrow \\ X & \xrightarrow{h} & X' & \longrightarrow & 1 \end{array}$$

being up to isomorphism equal to the following one:

$$\begin{array}{ccc} X\times X & \longrightarrow & d(X) \\ {\scriptstyle p_0}\downarrow & & \downarrow \\ X & \longrightarrow & 1 \end{array}$$

Consequently the previous left hand square is a pullback and according to ([1], 6.10), the following square is a pullback and h an isomorphism:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array}$$

Now let the following left hand square be a pullback in RM(E_g), then certainly the right hand one is a pullback too:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow f' \\ Z & \xrightarrow{f} & T \end{array} \qquad \begin{array}{ccc} X\times X & \longrightarrow & Y\times Y \\ \downarrow & & \downarrow f'\times f' \\ Z\times Z & \xrightarrow{f\times f} & T\times T \end{array}$$

But this last one is up to isomorphism the following one:

$$\begin{array}{ccc} X\times d(X) & \longrightarrow & Y\times d(Y) \\ \downarrow & & \downarrow f'\times d(f') \\ Z\times d(Z) & \xrightarrow[f\times d(f)]{} & T\times d(T) \end{array}$$

Let us denote by U the domain of the pullback of $d(f')$ along $d(f)$ and k: $d(X) \to U$ the induced factorisation, then $X\times k$: $X\times d(X) \to X\times U$ is an isomorphism and, X having a global support, the map k is itself an isomorphism (it is a consequence of ([1], 6.10)). Now let h, h': $X \to Y$ be two maps in RM(E). The kernel K of h and h' is obtained by the following pullback:

$$\begin{array}{ccc} K & \xrightarrow{k} & X \\ \downarrow & & \downarrow [h,h'] \\ Y & \xrightarrow[s_0]{} & Y\times Y \end{array}$$

If K has a global support, then it is also the case for X and Y, and this pullback is preserved by the functor d. But $d(k)$ is an isomorphism since $d(h) = d(h')$ and consequently k is itself an isomorphism, and we have: $h = h'$. ¤

Let AM(E) and AutM(E) be the full subcategories of RM(E) whose objects are respectively the associative and the autonomous Mal'cev operations. Then the restrictions of d to AM(E_g) and AutM(E_g) take their values in Gp(E) and Ab(E), the categories of internal groups in E and of internal abelian groups in E respectively. The previous remark about the canonical action of a group C on itself and the associated Mal'cev operation makes Gp(E) equivalent to the coslice category 1\AM(E) whose objects are the objects of AM(E) endowed with a map $1 \to X$, and Ab(E) equivalent to the coslice category 1\AutM(E).

Corollary 6. *If* E *is left exact and Barr exact, the category* AM(E_g) *is protomodular.*

Proof: The category AM(E_g) has pullbacks of split epimorphisms along any map and the functor d: AM(E_g) $\to$ Gp(E) preserves them. Furthermore it reflects the isomorphims. But Gp(E) is protomodular [3]. Thus the category AM(E_g) is protomodular. ¤

Now, here is our main result:

Theorem 7. *If* E *is left exact and Barr exact, the direction functor* d: AM(E_g) $\to$ Gp(E) *is a pseudo-cofibration. Moreover, the functor* d *reflecting the isomorphisms, every map in* AM(E_g) *is cocartesian and every fibre is a groupoid.*

Proof: A pseudo-cofibration guaranties the existence of a cocartesian map above a given map with domain d(X) but only up to isomorphism. Let (X,p) be an object in AM(E_g) and h: $d(X) \to G$ a group homomorphism. Let us consider the following equivalence relation R on X×G:

$$(x,f)R(y,g) \quad \text{iff} \quad f.g^{-1} = h \circ q(x,y) .$$

It is internally represented by the following graph:

$$X\times X\times G \underset{d}{\overset{c}{\rightrightarrows}} X\times G$$

where:

$$c(x,y,g) = (x, h\circ q(x,y).g) \text{ and } d(x,y,g) = (y,g) .$$

This equivalence relation is actually fibred above gr(X) by the projection p_X since the following square is a pullback:

$$\begin{array}{ccc} X\times X\times G & \xrightarrow{\quad d \quad} & X\times G \\ {\scriptstyle p_{X\times X}}\downarrow & & \downarrow{\scriptstyle p_X} \\ X\times X & \xrightarrow{\quad p_1 \quad} & X \end{array}$$

Let us denote by $h[X]$ the quotient of this relation R and r the associated regular epimorphism: $X\times G \to h[X]$. Now, on the one hand $h[X]\times G$ is the quotient of $R\times G$, and on the other hand the fibered aspect of the situation makes $h[X]\times h[X]$ the quotient of the following equivalence relation $R'=R\times_{gr(X)}R$:

$$X\times X\times G\times G \underset{d'}{\overset{c'}{\rightrightarrows}} X\times G\times G$$

where

$$c'(x,y,g,t) = (x, h\circ q(x,y).g, h\circ q(x,y).t) \text{ and } d'(x,y,g,t) = (y,g,t).$$

Now the isomorphism

$$\gamma\colon X\times G\times G \to X\times G\times G \quad \text{defined by} \quad \gamma(x,g,t) = (x,g,g.t)$$

determines a morphism between the relations $R\times G$ and R' which actually determines a simply transitive action of the group G on the relation R above gr(X), makes their quotients $h[X]\times G$ and $h[X]\times h[X]$ isomorphic and gives back a simply transitive action of G on $h[X]$. Consequently there is an associative Mal'cev operation on $h[X]$ whose associated direction is isomorphic to G. It is easy to check that the map

$$h' = r.[I_X, \omega]\colon X \to X\times G \to h[X],$$

where ω is the constant map on the unit of G, has its direction equal to h up to isomorphism. Now let u: $X \to Y$ be a morphism in $AM(E_g)$, such that there is a map w: $G \to d(Y)$ in Gp(E) satisfying: $w.h = d(u)$. The map $a(Y).u\times w$: $X\times G \to Y\times d(Y) \to Y$ coequalises the relation R and determines a map v: $h[X] \to Y$ in $AM(E_g)$ such that $v.h' = u$ and $d(v) = w$. It is the only one since:

$$\begin{aligned} v'.r = v'.a(h[X]).h'\times G = a(Y).v'\times d(v').h'\times G = a(Y).v'\times w.h'\times G \\ = a(Y).(v'.h')\times w = a(Y).u\times w, \end{aligned}$$

for any other arrow v' in $AM(E_g)$ satisfying the same identities. ¤

We shall call $h[X]$ the extension of X along the homomorphism h. Clearly the canonical Mal'cev operation on a group G is nothing but the extension along the initial map $1 \to G$ in Gp(E).

V. Naturally Mal'cev and essentially affine categories.

The functor d will allow us to clarify the relationship between the Naturally Mal'cev categories [6] and the essentially affine categories [3].

A *Naturally Mal'cev category* [6] is a category in which each object is endowed with a natural Mal'cev operation. The naturality of this operation makes it necessarily autonomous. Of course any additive category or any slice of additive category is Naturally Mal'cev, in the same way as the categories AutM(E) and AutM(E_g) for any category E. When E is Naturally Mal'cev, then clearly AutM(E) is equivalent to E.

Example. Another example is given by the following observation. Let Gp be the category of ordinary groups and Q a specific group, then the slice category Gp/Q is protomodular [3] and consequently any object in it has at most one Mal'cev operation which is necessarily autonomous. A homomorphism h: $G \to Q$ seen as an object of Gp/Q, has a Mal'cev operation if and only if its kernel is abelian. Let us denote by M(Gp/Q) the full subcategory of Gp/Q of such objects. It is the same category as AutM(Gp/Q) and is consequently Naturally Mal'cev .

Remark. Given an object X in AutM(E), then its Chasles relation is not only in E but actually in AutM(E). Indeed let us suppose that:

$$(x,t)R(y,z),\ (x',t')R(y',z') \text{ and } (x'',t'')R(y'',z'').$$

Then:

$$p(t,t',t'') = p(p(x,y,z),p(x',y',z'),p(x'',y'',z'')) = p(p(x,x',x''),p(y,y',y''),p(z,z',z''))$$

and thus

$$(p(x,x',x''),p(t,t',t''))R(p(y,y',y''),p(z,z',z'')).$$

Therefore the object $d(X)$ and the map $q(X)$: $X \times X \to d(X)$ are again in AutM(E), and consequently the canonical action $a(X)$: $X \times d(X) \to X$ too.

Moreover $d(q(X))$: $d(X \times X) = d(X) \times d(X) \rightarrow d(X)$ is nothing but $p_1 - p_0$, and $d(a(X))$: $d(X \times d(X)) = d(X) \times d(X) \rightarrow d(X)$ is nothing but $p_0 + p_1$.

Let us recall now what is an essentially affine category [3]. For that, let us denote by Pt(E) the category whose objects are the split epimorphisms in E with a given splitting and the maps the commutative squares between them, and by c the functor: Pt(E) $\rightarrow$ E assigning to each split epimorphism its codomain. This functor is a fibration as soon as E is left exact. The category E is said *essentially affine* when the fibration c is trivial, i.e. when every change of base functor is an equivalence [3]. Every additive category as well as every slice and coslice of an additive category is essentially affine.

The fact that the fibration c is trivial implies that it is an additive fibration [3] and the fact that it is an additive fibration is equivalent to the fact that E is Naturally Mal'cev [4]. Consequently any essentially affine category is Naturally Mal'cev. The converse is not true in general, since the variety AutMal of autonomous Mal'cev operations is Naturally Mal'cev as a category, but not essentially affine because of the empty set. On the contrary the category AutMal_g of autonomous Mal'cev operations on non empty sets is essentially affine. This observation is actually very general.

Proposition 8. *If* E *is a left exact and Barr exact category, then* $\text{AutM}(E_g)$ *is essentially affine. If* E *is moreover Naturally Mal'cev, then* E_g *is essentially affine.*

Proof: Let f: X $\rightarrow$ Y be a split (by s) epimorphism and h: Y $\rightarrow$ Z be a morphism in $\text{AutM}(E_g)$, then the following diagram can be completed into a pullback in Ab(E) since it is essentially affine

$$\begin{array}{ccc} d(X) & \longrightarrow & A \\ {\scriptstyle d(f)}\downarrow & & \downarrow{\scriptstyle \phi} \\ d(Y) & \underset{d(h)}{\longrightarrow} & d(Z) \end{array}$$

where ϕ is split by σ. Now, according to Theorem 7, the split monomorphism σ determines a cocartesian split monomorphism s': $Z \to W$ above it in AutM(E_g),and the split monomorphism s being also cocartesian, we can complete the square:

$$\begin{array}{ccc} X & \xrightarrow{h'} & W \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s'} \\ Y & \xrightarrow[h]{} & Z \end{array}$$

The map s' is split by a f' above ϕ and we have $f'.h' = h.f$ because s is cocartesian and the equality holds through the functor d. Now the following square is a pullback since, according to Proposition 6, the functor d reflects pullbacks.

$$\begin{array}{ccc} X & \xrightarrow{h'} & W \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[h]{} & Z \end{array}$$

When moreover E is Naturally Mal'çev, then E is equivalent to AutM(E) and E_g to AutM(E_g). Thus E_g is essentially affine. ¤

Remark. 1. Of course there are essentially affine categories, the slices of any additive category for instance, where all the objects have not necessarily a global support.

2. When X is an autonomous Mal'cev operation, the previous proposition makes the following diagram a pushout in AutM(E):

$$\begin{array}{ccc} X\times X & \xrightarrow{q} & d(X) \\ {\scriptstyle s_0}\uparrow & & \uparrow \\ X & \longrightarrow & 1 \end{array}$$

which is another way to characterize the direction.

VI. The Baer sums.

We have now, up to equivalence, a cofibration d: $AM(E_g) \to Gp(E)$ preserving the products as well as the terminal object, and such that any map in $AM(E_g)$ is cocartesian. Let us momentarily consider a regular cofibration d: $U \to V$ with the same properties. Then we are in the situation where any group object G in V will induce canonically a closed monoidal structure on the fibre $d^{-1}(G)$ above G. More precisely:

Theorem 9. *Let* d: $U \to V$ *be a cofibration preserving the products as well as the terminal object. Let us suppose morever that the cocartesian maps are stable by product. Then any monoid structure on an object* G *in* V *induces a canonical monoidal structure on the fibre* $d^{-1}(G)$. *When the monoid* G *is abelian, then the monoidal structure is symmetric. If, furthermore, the terminal maps and the diagonals are cocartesian, then, when* G *is a group, the monoidal structure is closed.*

Proof: Let us denote by m: $G \times G \to G$ the monoid law in V and by u: $1 \to G$ its unit. When two objects X and Y in U are above G, then $X \times Y$ is above $G \times G$. Now let us denote $X \otimes Y$ the codomain of the cocartesian map $\mu_{X,Y}$ with domain $X \times Y$ above the map m and I the codomain of the cocartesian map υ with domain the terminal object 1 of U above the unit u. Then clearly the unit and associativity axioms on G determine isomorphisms:

$$X \otimes I \sim X, \quad I \otimes Y \sim Y \quad \text{and} \quad (X \otimes Y) \otimes Z \sim X \otimes (Y \otimes Z)$$

which makes the fibre $d^{-1}(G)$ a monoidal category.

For instance, let us describe the first one. The right unit axiom of the monoid says that: $x.1 = x$. Internally speaking, we have: $m.I_G \times u = I_G$. Thus the two maps $\mu_{X,I}.I_X \times \upsilon$ and I_X in U are cocartesian above the same map and consequently their codomains are isomorphic.Checking the coherence axiom is straightforward. Now when G is abelian, we have $m.\tau_{G,G} = m$, where $\tau_{G,G}$ is the twisting isomorphism. Whence an isomorphism from $X \otimes Y$ to $Y \otimes X$ since $\mu_{Y,X}.\tau_{X,Y}$ and $\mu_{X,Y}$ are cocartesian on the same map.

When G is a group, let us denote X\Y the codomain of the cocartesian map with domain X×Y above the division map d: G×G → G representing the operation $d(x,y)= x^{-1}.y$. Then the equality $x.(x^{-1}.y) = y$ produces an isomorphism X⊗(X\Y) ~ Y and the equality $x^{-1}.(x.y) = y$ another one .X\ (X⊗Y) ~ Y which make X\Y a closure for X⊗Y. ¤

Corollary 9. *The "fibres" of the pseudo-cofibration d*: AutM(E_g) → Ab(E) *are closed monoidal symmetric, and the change of base functors are monoidal.*

Proof: Let us consider the pseudo-cofibration d: AM(E_g) → Gp(E). Now a group object in Gp(E) is necessarily abelian. Let us denote it A. Then the "fibre" of A is in AutM(E_g) (Corollary 4) and closed monoidal symmetric (Theorem 9). The remainder is straightforward. ¤

Example: the Baer sums. The category Gp/Q, as defined in Section V, is a left exact and Barr exact category, and $(\mathrm{Gp/Q})_g$ has the epimorphisms h: G → Q as objects, so that the category M($(\mathrm{Gp/Q})_g$) has the extensions with codomain Q and abelian kernels as objects. Moreover the category Ab(Gp/Q) is well known to be equivalent to the category Mod(Q) of Q-modules via the construction of the semi-direct product. In this case, the direction functor d: M($(\mathrm{Gp/Q})_g$) → Mod(Q) is nothing but the functor which assigns to any epimorphism h the classical canonical Q-module structure on its abelian kernel. According to Corollary 9, its fibres are closed monoidal symmetric groupoids.

Indeed, given a Q-module A, an extension with codomain Q, seen as an object in the category M($(\mathrm{Gp/Q})_g$), whose direction is A is nothing but a A-torsor on Q. The previous description of the tensor product is just a conceptual way to describe the tensor product of A-torsors in a Barr exact category as given in [2]. The interpretation of the cohomology of groups in the sense of Eilenberg-Mac Lane, in the same [2], asserts that this tensor product coincides with the classical Baer sum of two group extensions determining the same Q-module A. Consequently the associated group structures on these extensions is nothing but the group of the connected components of the closed monoidal symmetric groupoid d^{-1}(A).

Whence now the following reasonable definition:

Definition 3. Given a Barr exact category E with products, and X,Y two associative and commutative Mal'cev operations with global supports and same direction A, we shall call the *Baer sum* of X and Y the tensor product X⊗Y given in Corollary 9, i.e. the codomain of the cocartesian map with domain X×Y above the map p_0+p_1: A×A → A.

Remark. When E is Sets, then E_g is the category of non-empty sets. Then the fibres of *d* are connected groupoids. This is why, in this situation, the Baer sum was somehow unapparent.

VII. The structure of *d*(X) when the support is not global.

Let us suppose now that *p* is an associative Mal'cev operation on an object X in the Barr exact category E, but with the support of X no more global. Thus *d*(X) is no more pointed, since the terminal object 1 is no more the quotient of the coarse relation grX. However the slice division on Ch(X) does pass on the direction *d*(X) as a binary operation everywhere defined, which, of course, satisfies the cancellation rule:

(1) $$(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma) = \beta\backslash\gamma$$

Now, from the following identity in X×X:

$$((x,y)\backslash(x,y))\backslash(x,z) = (x,x)\backslash(x,z) = (x,z)$$

we get the weak quadratic neutrality

(2) $$(\alpha\backslash\alpha)\backslash\beta = \beta,$$

and from the following one:

$$((x,y)\backslash(x,y)) = (x,x) = (x,z)\backslash(x,z)$$

we get the quadratic constancy

(3) $$\alpha\backslash\alpha = \beta\backslash\beta .$$

Definition 4. Let us call a *metagroup* any set C (or any object C in a category E with product) endowed with a binary operation satisfying (1), (2) and (3).

Remark. The category of metagroups in Sets is clearly equivalent to the category of groups, plus the empty set. In other words, a non-empty metagroup in Sets is a group.

Thus the structure of the direction $d(X)$ is, in any circumstance, the structure of a metagroup.

Definition 5. A *metagroup action* of C on an object X is an external operation: $C \times X \rightarrow X$: $(\alpha, m) \rightsquigarrow \alpha \backslash m$ such that:

1. $(\alpha\backslash\beta)\backslash(\alpha\backslash m) = \beta\backslash m$,
2. $(\alpha\backslash\alpha)\backslash m = m$.

The pullback diagram after the Definition 2 means that the metagroup $d(X)$ acts on X, in a simply transitive way, meaning that for any pair (m,n) of objects in X, there is a unique α in C such that $\alpha\backslash m = n$. We shall denote this α by $\overrightarrow{mn}$, and consequently get $\overrightarrow{mn}\backslash m = n$.

But we have also the converse:

Proposition 10. *Any simply transitive action of the metagroup* C *on* X *determines an associative Mal'cev operation on* X *whose direction is isomorphic to* C.

Proof: Let us set $p(m,n,t) = \overrightarrow{nt}\backslash m$. We have then immediately $p(m,m,n) = n$. Let us now observe that $\overrightarrow{mn}\backslash\overrightarrow{mt} = \overrightarrow{nt}$. Indeed:

$$(\overrightarrow{mn}\backslash\overrightarrow{mt})\backslash n = (\overrightarrow{mn}\backslash\overrightarrow{mt})\backslash(\overrightarrow{mn}\backslash m) = \overrightarrow{mt}\backslash m = t.$$

We have therefore $\overrightarrow{nn}\backslash m = (\overrightarrow{mn}\backslash\overrightarrow{mn})\backslash m = m$ by (2) and consequently we get $\overrightarrow{nn} = \overrightarrow{mm}$. Thus

$$p(m,n,n) = \overrightarrow{nn}\backslash m = \overrightarrow{mm}\backslash m = m\,.$$

Let us check the associativity :

$$p(p(m,n,t),u,v)) \text{ is } \overrightarrow{uv}\backslash(\overrightarrow{nt}\backslash m),$$

$$p(m,n,p(t,u,v)) \text{ is } \overrightarrow{nq}\backslash m \text{ where } q = \overrightarrow{uv}\backslash t\,.$$

But this last equality means $\overrightarrow{tq} = \overrightarrow{uv}$, and the associativity will derive from: $\overrightarrow{tq}\backslash(\overrightarrow{nt}\backslash m) = \overrightarrow{nq}\backslash m$. But

$$\vec{tq}\backslash(\vec{nt}\backslash m) = (\vec{nt}\backslash\vec{nq})\backslash(\vec{nt}\backslash m) = \vec{nq}\backslash m \ .$$

The Chasles relation is given by

$$(m,n) \sim (m',n') \quad \text{iff} \quad n = p(m,m',n') = \vec{m'n'}\backslash m,$$

which means $\vec{m'n'} = \vec{mn}$. Thus $d(X)$ is isomorphic to C. Now to check that the two metagroup structures are themselves isomorphic is straightforward. ¤

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Département de Mathématiques
Université du Littoral Côte d'Opale
1 Quai Freycinet
B.P. 5526
59379 Dunkerque. France