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FREE CROSSED RESOLUTIONS FROM SIMPLICIAL RESOLUTIONS WITH GIVEN *CW*-BASIS

by A. MUTLU and T. PORTER

RESUME. Dans cet article, les Auteurs examinent la relation entre une CW-base pour un groupe simplicial, et des méthodes pour engendrer librement le complexe croisé associé. On examine en détail le cas des résolutions, en comparant les résolutions simpliciales libres et les résolutions croisées d'un groupe.

Introduction

When J.H.C. Whitehead wrote his famous papers on "Combinatorial Homotopy", [26], it would seem that his aim was to produce a combinatorial, and thus potentially constructive and computational, approach to homotopy theory, analogous to the combinatorial group theory developed earlier by Reidemeister and others. In those papers, he introduced CW-complexes and also the algebraic 'gadgets' he called homotopy systems, and which are now more often called free crossed complexes, [5], or totally free crossed chain complexes, [3].

Another algebraic model for a (connected) homotopy type is a simplicial group and again, there, one finds a notion of freeness. In both cases we have 'freeness', yet no easily defined category of things on which our objects are 'free'. Kan, [15], introduced the notion of a CW-basis for a free simplicial group and more recently, [9], R.A.Brown has introduced Peiffer-Whitehead word systems or extended group presentations as a means of presenting a 'homotopy system'. In both cases the aim is to use the 'generators' as a combinatorial way of controlling or manipulating the algebraic model, i.e. extending the 'yoga' of combinatorial group theory to higher dimensions.

There are ways of passing from a simplicial group to a crossed complex (see for example, [14]) and as these are all equivalent to a left adjoint, one expects freeness to be preserved, and it is, but this is not trivial. As we do not know on what type of thing the simplicial group is free, nor on what the crossed complex is free, the conclusion is not a simple consequence of left adjointness of some sort. The problem is that to construct the n^{th} level, you need some generators together with a map to the $(n-1)^{\text{st}}$ one, and of course you cannot do that until that level is constructed!

In this paper we apply methods from our earlier papers [20, 21], to examine the relationship between the notions of free basis for simplicial groups and that for crossed complexes. We have included a shortened proof of the result from [10] and [14], describing the passage from simplicial groups to crossed complexes, as this allows for a direct verification of freeness at the base of the crossed complex.

Although our results would seem to apply in general, we have restricted detailed attention to simplicial resolutions. This is partially since there are known problems of non-realizability of a homotopy system by a CW-complex (cf. Whitehead, [26], section 15, or R.A.Brown, [9], p.527) and hence by a free simplicial group with CW-basis. It thus seems prudent to understand these non-realizability results better from this simplicial viewpoint before attempting to look at the general case. Those results do not seem to disturb the general case in any significant way, but they leave them somewhat incomplete in the view they give of the general problem.

1 Preliminaries

We will denote the category of groups by $\mathfrak{Grp}$.

1.1 Simplicial Groups

A simplicial group **G** is a simplicial object in the category of groups. We will denote the category of simplicial groups by $\mathfrak{SimpGrp}$. We will only need a small amount of the extensive theory of simplicial groups here and would refer to the book by May [17] or the survey by Curtis [13] for information on the more 'classical' parts of the theory. We will assume a basic knowledge of the elementary homotopy of simplicial sets and simplicial groups, but will also refer to facts and concepts from earlier parts of this series of papers, [20, 21, 22]. If **G** is a simplicial group, then (NG, ∂) will be the corresponding Moore complex. Our conventions on this and related notions are given in [20].

1.2 Step By Step Constructions

This section is a brief résumé of how to construct simplicial resolutions. The work depends heavily on a variety of sources, mainly [1], [16] and [18]. André only treats commutative algebras in detail, but Keune [16] does discuss the general case quite clearly.

First recall the following notation and terminology which will be used in the construction of a simplicial resolution.

Let $[n]$ be the ordered set, $[n] = \{0 < 1 < \cdots < n\}$. We define the following maps: Firstly the injective monotone map $\delta_i^n : [n-1] \to [n]$ is given by

$$\delta_i^n(x) = \begin{cases} x & \text{if } x < i, \\ x+1 & \text{if } x \geq i, \end{cases}$$

for $0 \leq i \leq n \neq 0$. An increasing surjective monotone map $\alpha_i^n : [n+1] \to [n]$ is given by

$$\alpha_i^n(x) = \begin{cases} x & \text{if } x \leq i, \\ x-1 & \text{if } x > i, \end{cases}$$

for $0 \leq i \leq n$. We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \to [n]$.

Killing Elements in Homotopy Groups

The following section describes the 'step-by-step' construction due to André [1], that source however concentrates on simplicial algebras. We have adapted his treatment to handle simplicial groups.

We recall that if F and G are groups, a map

$$G \times F \to F$$

$$(g,f) \longmapsto {}^g f$$

is a left action if and only if for all $g, g' \in G, f, f' \in G$,

1. ${}^g(ff') = {}^g f \, {}^g f'$,
2. ${}^{gg'} f = {}^g({}^{g'} f)$,
3. ${}^1 f = f$.

In this case we say F is a G-group.

Let $\mathbf{G}$ be a simplicial group and let $k \geq 1$ be fixed. Suppose we are given a set Ω of elements $\Omega = \{x_\lambda : \lambda \in \Lambda\}$, $x_\lambda \in \pi_{k-1}(\mathbf{G})$, then we can choose a corresponding set of elements $\vartheta_\lambda \in NG_{k-1}$ so that $x_\lambda = \vartheta_\lambda \, \partial_k(NG_k)$. (If $k = 1$, then as $NG_0 = G_0$, the condition that $\vartheta_\lambda \in NG_0$ is immediate.) We want to 'kill' the elements in Ω.

We form a new simplicial group F_n where

1) F_n is the free G_n-group,

$$F_n = \coprod_{\lambda,t} G_n\{y_{\lambda,t}\} \text{ with } \lambda \in \Lambda \text{ and } \ t \in \{n,k\},$$

where $G_n\{y\} = G_n * < y >$, the free product of G_n and a free group generated by y.

2) For $0 \leq i \leq n$, the group homomorphism $s_i^n : F_n \to F_{n+1}$ is obtained from the homomorphism $s_i^n : G_n \to G_{n+1}$ with the relations

$$s_i^n(y_{\lambda,t}) = y_{\lambda,u} \quad \text{with} \quad u = t\alpha_i^n, \ \ t : [n] \to [k].$$

3) For $0 \leq i \leq n \neq 0$, the group homomorphism $d_i^n : F_n \to F_{n-1}$ is obtained from $d_i^n : G_n \to G_{n-1}$ with the relations

$$d_i^n(y_{\lambda,t}) = \begin{cases} y_{\lambda,u} & \text{if the map} \quad u = t\delta_i^n \quad \text{is surjective,} \\ t'(\vartheta_\lambda) & \text{if} \qquad u = \delta_k^k t', \\ 1 & \text{if} \qquad u = \delta_j^k t' \quad \text{with } \ j \neq k, \end{cases}$$

by extending multiplicatively.

We sometimes denote the $\mathbf{F}$ so constructed by $\mathbf{G}(\Omega)$.

Remark : In a 'step-by-step' construction of a simplicial resolution, (see below), there are thus the following properties: i) $F_n = G_n$ for $n < k$, ii) $F_k =$ a free G_k-group over a set of non-degenerate indeterminates, all of whose faces are the identity except the k^{th}, and iii) F_n is a free G_n-group on some degenerate elements for $n > k$.

We have immediately the following result, as expected.

Proposition 1.1 *The inclusion of simplicial groups* $\mathbf{G} \hookrightarrow \mathbf{F}$, *where* $\mathbf{F} = \mathbf{G}(\Omega)$, *induces a homomorphism*

$$\pi_n(\mathbf{G}) \longrightarrow \pi_n(\mathbf{F})$$

for each n, *which for* $n < k - 1$ *is an isomorphism,*

$$\pi_n(\mathbf{G}) \cong \pi_n(\mathbf{F})$$

and for $n = k-1$, *is an epimorphism with kernel generated by elements of the form* $\bar{\vartheta}_\lambda = \vartheta_\lambda \partial_k NG_k$, *where* $\Omega = \{x_\lambda : \lambda \in \Lambda\}$.

□

Constructing Simplicial Resolutions

The following result is essentially due to André [1].

Theorem 1.2 *If* G *is a group, then it has a free simplicial resolution* $\mathbb{F}$.

Proof: The repetition of the above construction will give us the simplicial resolution of a group. Although 'well known', we sketch the construction so as to establish some notation and terminology.

Let G be a group. The zero step of the construction consists of a choice of a free group F and a surjection $g : F \to G$ which gives an isomorphism $F/\mathrm{Ker}g \cong G$ as groups. Then we form the constant simplicial group $\mathbf{F}^{(0)}$ for which in every degree n, $F_n = F$ and $d_i^n = \mathrm{id} = s_j^n$ for all i, j. Thus $\mathbf{F}^{(0)} = \mathbf{K}(F, 0)$ and $\pi_0(\mathbf{F}^{(0)}) = F$. Now choose a set Ω^0 of normal generators of the normal subgroup $N = \mathrm{Ker}(F \xrightarrow{g} G)$, and obtain the simplicial group in which $F_1^{(1)} = F(\Omega^0)$ and for $n > 1$, $F_n^{(1)}$ is a free F_n-group over the degenerate elements as above. This simplicial group will be denoted by $\mathbf{F}^{(1)}$ and will be called the *1-skeleton of a simplicial resolution of the group G.*

The subsequent steps depend on the choice of sets, $\Omega^0, \Omega^1, \Omega^2, \ldots, \Omega^k, \ldots$ Let $\mathbf{F}^{(k)}$ be the simplicial group constructed after k steps, the k-skeleton of the resolution. The set Ω^k is formed by elements a of $F_k^{(k)}$ with $d_i^k(a) = 1$ for $0 \leq i \leq k$ and whose images $\bar{a}$ in $\pi_k(\mathbf{F}^{(k)})$ generate that module over $F_k^{(k)}$ and $\mathbf{F}^{(k+1)}$.

Finally we have inclusions of simplicial groups

$$\mathbf{F}^{(0)} \subseteq \mathbf{F}^{(1)} \subseteq \cdots \subseteq \mathbf{F}^{(k-1)} \subseteq \mathbf{F}^{(k)} \subseteq \cdots$$

and in passing to the inductive limit (colimit), we obtain an acyclic free simplicial group $\mathbf{F}$ with $D_n = F_n^{(k)}$ if $n \leq k$. $\mathbb{F} = (\mathbf{F}, g)$ is thus a simplicial resolution of the group G.

The proof of theorem is completed. □

Remark : A variant of the 'step-by-step' construction gives: if $\mathbf{G}$ is a simplicial group, then there exists a free simplicial group $\mathbf{F}$ and an epimorphism $\mathbf{F} \longrightarrow \mathbf{G}$ which induces isomorphisms on all homotopy groups. The details are omitted as they are well known.

Terminology : It is sometimes useful to write $\mathbb{F}^{(k)} = (\mathbf{F}^{(k)}, g)$ for the augmented simplicial group constructed at the k^{th} step. The data needed to go from $\mathbb{F}^{(k)}$ to $\mathbb{F}^{(k+1)}$ are more precisely a set Ω^k and a function $g^{(k)} : \Omega^k \longrightarrow F_k^{(k)}$ whose image is contained in $NF_k^{(k)}$ and which generates $\pi_k(\mathbb{F}^{(k)})$. (We often consider $g^{(k)}$ as being an inclusion and leave it out of the notation.) The pair $(\Omega^k, g^{(k)})$ is then called *k-dimensional construction data for the resolution* and the finite sequence $((\Omega^0, g^{(0)}), \ldots, (\Omega^{k-1}, g^{(k-1)}))$ is called a k^{th}*-level presentation* of the group G.

The key observation, which follows from the universal property of the construction, is a freeness statement:

Proposition 1.3 *Let $\mathbf{F}^{(k)}$ be a k-skeleton of a simplicial resolution of G and $(\Omega^k, g^{(k)})$ k-dimension construction data for $\mathbb{F}^{(k+1)}$. Suppose given a simplicial group morphism $\Theta : \mathbf{F}^{(k)} \longrightarrow \mathbf{G}$ such that $\Theta_*(g^{(k)}) = 0$, then Θ extends over $\mathbb{F}^{(k+1)}$.*

This freeness statement does not contain a uniqueness clause. That can be achieved by choosing a lift for $\Theta_k g^{(k)}$ to NG_{k+1}, a lift that must exist since $\Theta_*(\pi_k(\mathbb{F}^{(k)}))$ is trivial.

When handling combinatorially defined resolutions, rather than functorially defined ones, this proposition is as often as close to 'left adjointness' as is possible without entering the realm of homotopical algebra to an extent greater than is desirable for us here.

We have not talked here about the homotopy of simplicial group morphisms, and so will not discuss homotopy invariance of this construction for which one adapts the description given by André, [1], or Keune, [16].

1.3 CW-bases

We recall from [15] and [13] the following definitions.

Definition 1.4 *A simplicial group* $\mathbf{F}$ *will be called free if*
(a) F_n *is a free group with a given basis, for every integer* $n \geq 0$,
(b) *The bases are stable under all degeneracy operators, i.e., for every pair of integers* (i, n) *with* $0 \leq i \leq n$ *and every given generator* $x \in F_n$ *the element* $s_i(x)$ *is a given generator of* F_{n+1}.

Definition 1.5 *Let* $\mathbf{F}$ *be a free simplicial group (as above). A subset* $\mathfrak{F} \subset \mathbf{F}$ *will be called a* CW *– basis for* $\mathbf{F}$ *if*
(a) $\mathfrak{F}_n = \mathfrak{F} \cap F_n$ *freely generates* F_n *for all* $n \geq 0$,
(b) $\mathfrak{F}$ *is closed under degeneracies, i.e.,* $x \in \mathfrak{F}_n$ *implies* $s_i(x) \in \mathfrak{F}_{n+1}$ *for all* $0 \leq i \leq n$,
(c) *if* $x \in \mathfrak{F}_n$ *is non-degenerate, then* $d_i(x) = e_{n-1}$, *(*e_{n-1}*, the identity element of* F_{n-1}*) for all* $0 \leq i < n$.

Let $\mathbf{F}$ be a free simplicial group with given CW-basis, $\mathfrak{F}$, then $X_0 = \mathfrak{F}_0$ freely generates F_0, that is, $F_0 = F(X_0)$. In general, note that if $Y_n = \mathfrak{F}_n \setminus \bigcup_{i=0}^{n-1} s_i(\mathfrak{F}_i)$ then $Y_n \subseteq NF_n$.

1.4 Crossed Modules

J. H. C. Whitehead (1949) [26] described crossed modules in various contexts especially in his investigation into the group structure of relative homotopy groups.

Definition 1.6 *A* pre-crossed module of groups *consists of a group,* G_1, *a* G_1*-group* G_2, *and a group homomorphism* $\partial : G_2 \longrightarrow G_1$, *such that for all* $g_2 \in G_2, g_1 \in G_1$

$$CM1) \qquad \partial({}^{g_1}g_2) = g_1 \partial(g_2) {g_1}^{-1}.$$

This is a crossed module *if in addition, for all* $g_2, g_2' \in G_2$,
$CM2)$ $\quad {}^{\partial(g_2)}g_2' \;=\; g_2 g_2' (g_2)^{-1}.$

The second condition (CM2) is called *the Peiffer identity.* We denote such a crossed module by (G_2, G_1, ∂). Clearly any crossed module is a pre-crossed module.

1.5 Free Crossed Modules

The notion of a free crossed module was described by J. H. C. Whitehead [26]. We refer the reader to [7] for the construction of a free crossed module on a presentation and the proofs of the results below. The related notion of totally free (pre-)crossed module is discussed in [22].

Theorem 1.7 *A free crossed module G_1-module (G_2, G_1, ∂) exists on any function $f : S \to G_1$ with codomain G_1.*

Proof: See [7]. □

If (G_2, G_1, ∂) is a free crossed G_1-module on the trivial function

$$1 : S \to G_1,$$

then G_2 is a free G_1-module on the set S.

2 Crossed Complexes

2.1 Peiffer pairings and boundaries in the Moore complex

Firstly we recall from [20] the following result. Let **G** be a simplicial group with Moore complex **NG** and for $n \geq 1$, let D_n be the normal subgroup generated by the degenerate elements of n. If $G_n \neq D_n$, then

$$NG_n \cap D_n = N_n \cap D_n \quad \text{for all } n \geq 1,$$

where N_n is a normal subgroup in G_n generated by an explicitly given set of elements.

2.2 Crossed Complexes and Crossed Resolutions

The definition of a crossed complex (over a groupoid) was given by R. Brown and P. J. Higgins (1981) [5] generalising earlier work of Whitehead (1949) [26]. Crossed resolutions are discussed in several sources. A particularly useful one is the thesis of Tonks, [25], which handles constructions of crossed resolutions in some detail.

2.3 Peiffer-Whitehead word systems

R.A.Brown, [9], introduces a system of generators for a 'homotopy system' that he calls a *Peiffer-Whitehead word system.* His sets of generators are only in a finite number of dimensions whilst ours may need to be in an infinite set of levels to get a resolution, so his needs are not the same as ours, but nonetheless it seems worthwhile to include his definition as it provides a point of comparison with his work:

Definition 2.1 [9], p. 525

A Peiffer-Whitehead word system *or* extended group presentation *W consists of a finite list of finite sets $\langle W^{(1)}|W^{(2)}|\dots W^{(n)}\rangle$ together with* boundary homomorphisms $d_3, \dots, d_n$ *described as follows:*
$W^{(1)} = I_1$ is a set of indices;
$W^{(2)} = \{w^2_\beta|\beta \in I_2\}$ is a set of words representing elements of the free group $F = F(I_2)$;
$W^{(3)} = \{w^3_\gamma|\gamma \in I_3\}$ is a set of words representing elements of the free F-crossed module $C(I_2)$ with boundaries $\{c_\beta = \langle w^2_\beta\rangle|\beta \in I_2\}$;
$W^{(m)} = \{w^m_\mu|\mu \in I_m\}$ $(4 < m \leq n)$ is a set of words representing elements of the free $\mathbb{Z}G$-module $M_m = M_m(I_{m-1})$, where G is the group presented by $\langle W^{(1)}|W^{(2)}\rangle$;
$d_3 : C(I_2) \to F$ is a group homomorphism determined by $d_3(i_\beta) = \langle w^2_\beta\rangle$;
$d_4 : M_4(I_3) \to C(I_2)$ is a homomorphism determined by $d_4(i_\gamma) = \langle w^3_\gamma\rangle$;
$d_m : M_m(I_{m-1}) \to M_{m-1}$ $(4 < m \leq n)$ is a module homomorphism determined by $d_m(i_\lambda) = \langle w^{m-1}_\lambda\rangle$;
In addition all words must have trivial boundaries

$$d_m\langle w^m_\mu\rangle = identity \quad for\ \mu \in I_m, 3 \leq m \leq n.$$

Such a word system clearly specifies the generators of each level and their images in the next level down.

2.4 From Simplicial Groups to Crossed Complexes

P. Carrasco and A. M. Cegarra [11] defined

$$C_n(\mathbf{G}) = \frac{NG_n}{(NG_n \cap D_n)\ d_{n+1}(NG_{n+1} \cap D_{n+1})}$$

for a simplicial group $\mathbf{G}$. This gives a crossed complex $\mathfrak{C}(\mathbf{G})$ starting from the Moore complex $(\mathbf{NG},\ \partial)$ of $\mathbf{G}$. The map $\partial_n : C_n(\mathbf{G}) \to C_{n-1}(\mathbf{G})$ will be that induced by d_n^n. Their proof requires an understanding of hypercrossed complexes. P. J. Ehlers and T. Porter, [14], developed a more direct proof for simplicial groupoids. Here we will sketch a shorter argument showing that $\mathfrak{C}(\mathbf{G})$ is a crossed complex as we can use some of the ideas later on. This proof emphasises the role played by the various $F_{\alpha,\beta}$. These pairing operations on the Moore complex were introduced in [20] and [21]. They are defined by forming $[s_\alpha x, s_\beta y]$ and then projecting the result into the Moore complex. Detailed examples and calculations are given in these papers cited above. If $x \in NG_n$, we will write $\bar{x}$ for the corresponding element of $C_n(\mathbf{G})$.

Lemma 2.2 *The subgroup $(NG_n \cap D_n)\ d_{n+1}(NG_{n+1} \cap D_{n+1})$ is a normal subgroup in G_n.*

Proof: This is a routine use of the degeneracies. □

Proposition 2.3 *Let $\mathbf{G}$ be a simplicial group, then defining $C(\mathbf{G}) = (C_n(\mathbf{G}), \partial)$ as above yields a crossed complex.*

Proof: (i) For $n \geq 2$, $C_n(\mathbf{G})$ is abelian, in fact

$$\begin{aligned} F_{(n-1)(n)}(x,y) &= [s_{n-1}x,\ s_n y]\ [s_n y,\ s_n x] \\ d_{n+1}F_{(n-1)(n)}(x,y) &= [x,\ s_{n-1}d_n(y)]\ [y,x] \end{aligned}$$

is in $(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})$, so $d_{n+1}F_{(n-1)(n)}(x,y) \equiv 1 \bmod (NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})$ giving $\overline{xy} = \overline{yx}$.
(ii) For $x \in NG_n$, and $y \in NG_m$, taking $\alpha = (n, n-1, \ldots, m)$, and

$\beta = (m-1)$, it is easy to see that

$$F_{\alpha,\beta}(x_\alpha, y_\beta) = \prod_{k=0}^{n-m+1} [s_n s_{n-1} \ldots s_m x,\ s_{m-1+k} y]^{(-1)^k} . [s_n s_{n-1} \ldots s_m(x),\ s_n y]$$

$$d_{n+1} F_{\alpha,\beta}(x_\alpha, y_\beta) = \prod_{k=0}^{n-m} [s_n s_{n-1} \ldots s_m x,\ s_{m-1+k} y]^{(-1)^k} [s_{n-1} \ldots s_m(x),\ y].$$

This implies that $[s_m^{(n-m)}(x), y] \in (NG_n \cap D_n) d_{n+1}(NG_{n+1} \cap D_{n+1})$, (where $s_m^{(n-m)} x = s_{n-1} \ldots s_m x$) which shows that the actions of NG_m on NG_n defined by conjugation

$$^{\bar{x}}\bar{y} = \overline{s_m^{(n-m)}(x) y s_m^{(n-m)}(x)^{-1}}$$

via these degeneracies are trivial if $m \geq 1$. For $m = 1$, this gives $\alpha = (n, n-1, \ldots, 1)$, $\beta = (0)$ and

$$F_{(n,n-1\ldots,1)(0)}(x,y) = \prod_{k=0}^{n} [s_n s_{n-1} \ldots s_1 x,\ s_{0+k} y]^{(-1)^k} [s_n s_{n-1} \ldots s_1(x),\ s_n y],$$

where $x \in NG_1, y \in NG_n$, and it is easily checked that

$$d_{n+1} F_{(n,n-1\ldots,1)(0)}(x,y) = \prod_{k=0}^{n-1} [s_{n-1} \ldots s_1 x,\ s_{0+k} d_n y]^{(-1)^k} [s_n s_{n-1} \ldots s_1(x),\ s_n y].$$

Then $[s_n s_{n-1} \ldots s_1(x),\ s_n y] \equiv 1 \mod (NG_n \cap D_n) d_{n+1} NG_{n+1} \cap D_{n+1})$. This gives the following if $\bar{x} \in C_1$ then $\bar{x}$ and $\partial_1 \bar{x}$ acts on C_n in the same way and so $\partial_1 C_1$ acts trivially on C_n.

(iii) This axiom follows since

$$C_1(\mathbf{G}) = \frac{NG_1}{\partial_2(NG_2 \cap D_2)} = \frac{NG_1}{[\mathrm{Ker} d_1,\ \mathrm{Ker} d_0]}$$

and $[\mathrm{Ker} d_1, \mathrm{Ker} d_0]$ contains the Peiffer elements so $(C_1(\mathbf{G}), C_0(\mathbf{G}), \partial)$ is a crossed module.

(iv) By defining

$$\partial_n \bar{z} = \overline{d_n^n(z)} \quad \text{with} \quad z \in NG_n$$

one obtains a well defined map $\partial : C_n(\mathbf{G}) \longrightarrow C_{n-1}(\mathbf{G})$ satisfying $\partial\partial = 1$. □

One of the immediate consequences of the above is that if $\mathbb{G} = (\mathbf{G}, f)$ is a simplicial group augmented over a group G, then $\mathfrak{C} = (C(\mathbf{G}), f)$ is an augmented crossed complex over G. Moreover if $\mathbb{G}$ is exact at G_0, then $\mathfrak{C}$ is also exact at $C_0(\mathbf{G})$. Thus to study what happens to a resolution we need only consider the freeness and exactness in higher dimensions.

3 'Step-by-Step' Constructions and CW-bases

In this section, we describe the special case of the 'step-by-step' construction of a free simplicial resolution and its skeleton up to dimension 2 and will interpret this construction and see how that relates to other constructions such as that of a free crossed module.

Many of the observations that we will make, do apply more generally to arbitrary free simplicial groups with specified CW-basis, but our aim here is limited to examining resolutions in some detail. We first examine the relationship of a CW-basis to the step-by-step construction given earlier.

The 1-skeleton $\mathbf{F}^{(1)}$ of a free simplicial resolution of a group G was built by adding new indeterminates, for instance, in one to one correspondence with Ω^1 a set of generators for $\pi_1(F^{(0)})$, $F_1^1 = F_1^{(0)}(X_0) = F(s_0(X_0) \cup Y_1) \cong F(s_0(X_0)) * F(Y_1)$, where $*$ is free product, with the face maps and degeneracy map

$$F(s_0(X_0) \cup Y_1) \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} F(X_0) \xrightarrow{d_0^0} G$$

where $F(X_0) \xrightarrow{d_0^0} G$ is an augmentation map and s_0, d_0^1 and d_1^1 are given by

$$d_1^1(y_i) = b_i \in \operatorname{Ker} d_0^0, \quad d_0^1(y_i) = 1, \quad s_0(x_0) = s_0(x_0) \quad \text{for} \quad x_0 \in X_0.$$

We note that this makes $\langle X_0 \mid d_1 Y \rangle$ into a presentation of G in the ordinary sense. The 1-skeleton $\mathbf{F}^{(1)}$ looks like:

$$\ldots \quad F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1)) \underset{s_1,s_0}{\overset{d_0,d_1,d_2}{\rightrightarrows}} F(s_0(X_0) \cup (Y_1)) \underset{s_0}{\overset{d_1,d_0}{\rightrightarrows}} F(X_0).$$

Note that for $n > 1$, higher levels of $\mathbf{F}^{(1)}$ are generated by the degenerate elements.

Lemma 3.1 *We assume given the 1-skeleton* $\mathbf{F}^{(1)}$. *Let* d_0^1 *and* d_1^1 *be evaluation homomorphisms. Then*

i) $\mathrm{Ker} d_0^1 = \langle Y_1 \rangle$,

ii) $\mathrm{Ker} d_1^1 = \langle Z \rangle$,

where $Z = \{s_1(y)^{-1}s_0(y) : y \in Y_1\}$ *and* $\langle Y_1 \rangle$ *is normal closure of* Y_1.

Proof: Clear. □

Note $\pi_0(\mathbf{F}^{(1)}) \cong G$.

The link between the bottom step of a step-by-step construction of the resolution and that of a CW-basis $\mathfrak{F}$ is thus clear. The 2-skeleton gives the non-degenerate elements of the resulting CW-basis, $\mathfrak{F}_2$, and in general we can take $Y_n \cong \Omega^{n-1}$, and $\mathfrak{F}_n = Y_n \cup \bigcup s_i(\mathfrak{F}_{n-1})$. For both combinatorial and computational purposes, the way in which Y_n corresponds to Ω^{n-1} can be important and in general it is necessary to specify the function $g^{n-1} : \Omega^{n-1} \longrightarrow NF_n$ or its last face $d_n g^{n-1} : \Omega^{n-1} \longrightarrow NF_{n-1}$.

Remark: For homological and computational reasons, it is often useful also to specify the contracting homotopyon the underlying simplicial set of $\mathbf{F}$ and to build this into the resolution progresses. We will not discuss how to do this here however as it is not needed for our immediate purposes.

Before carrying on the 'step-by-step' construction of the free simplicial group, we will interpret the first homotopy group, $\pi_1(\mathbf{F}^{(1)})$, of $\mathbf{F}^{(1)}$ to find what it looks like.

For any simplicial group $\mathbf{F}$, if $\mathbf{F} = \mathbf{F}^{(1)}$, then,

$$\pi_1(\mathbf{F}) = \mathrm{Ker}(d_1 : \mathrm{Ker} d_0^1/[\mathrm{Ker} d_1^1,\ \mathrm{Ker} d_0^1] \longrightarrow F_0).$$

Indeed, by definition, the first homotopy group is

$$\pi_1(\mathbf{F}) = (\mathrm{Ker} d_0^1 \cap \mathrm{Ker} d_1^1)/d_2^2(\mathrm{Ker} d_0^2 \cap \mathrm{Ker} d_1^2).$$

By a lemma of Brown and Loday [8], see also [21], the denominator of this homotopy group isexactly

$$\partial_2(NF_2) = d_2^2(\mathrm{Ker}d_0^2 \cap \mathrm{Ker}d_1^2) = [\mathrm{Ker}d_0^1,\ \mathrm{Ker}d_1^1]$$

and the morphism

$$\delta : \mathrm{Ker}d_0^1/\partial_2(NF_2) \longrightarrow F_0,$$

where $\delta = d_1$ (restricted to $NF_1/\partial_2 NF_2$), is a crossed module. Here NF_0 acts on $NF_1/\partial_2 NF_2$ by conjugation via s_0, that is,

$$\begin{array}{ccc} NF_1/\partial_2 NF_2 \times NF_0 & \longrightarrow & NF_1/\partial_2 NF_2, \\ (x, \overline{y}) & \longmapsto & {}^x\overline{y} = \overline{s_0(x) y s_0(x)^{-1}}, \end{array}$$

where $\overline{y}$ denotes the corresponding element of $NF_1/\partial_2 NF_2$ whilst $y \in NF_1$.

$$\begin{aligned} \pi_1(\mathbf{F}) &= \mathrm{Ker}(\mathrm{Ker}d_0^1/\partial_2(NF_2) \longrightarrow F_0) \\ &= \mathrm{Ker}(NF_1/[\mathrm{Ker}d_1,\ \mathrm{Ker}d_0] \longrightarrow F_0). \end{aligned}$$

Proposition 3.2 *Given a presentation $P = \langle X_0 \mid R \rangle$ of a group G and $\mathbf{F}^{(1)}$, the 1-skeleton of the free simplicial group generated by this presentation, then*

$$\delta : NF_1^{(1)}/\partial_2(NF_2^{(1)}) \longrightarrow NF_0^{(1)}$$

is the free crossed module on $R \to F(X_0)$, and $\pi_2(\mathbf{F}^{(1)})$ is the module of identities of the presentation P.

Proof: Clear. □

Note that for the case of $\mathbf{F}^{(2)}$, if $x_i,\ x_j$ are in $NF_1^{(2)}$, then generators of the normal subgroup $NF_2^{(2)} \cap D_2$ are of the form $[s_1(x_i)^{-1}s_0(x_j),\ s_1(x_j)]$. We now will recall the next step of the construction of a free simplicial group. We take a set of generators $\Omega^1 = \{S_i\} \subset \pi_1(\mathbf{F}^{(1)})$ and kill off the elements in the homotopy group $\pi_1(\mathbf{F}^{(1)})$ by adding new indeterminates $Y_2 = \{y_i\}$ into $F_2^{(1)}$ where Y_2 is in $1-1$ correspondence with Ω^1 to establish

$$F_2^{(2)} = F_2^{(1)}(Y_2) = F(s_1 s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1) \cup Y_2)$$

together with

$$d_0^2(y_i) = 1, \quad d_1^2(y_i) = 1, \quad d_2^2(y_i) = S_i, \text{ mod } \partial_3 NF_3^{(2)}.$$

Hence the 2-skeleton $\mathbf{F}^{(2)}$ looks like

$$F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1) \cup Y_2) \underset{s_1,s_0}{\overset{d_0,d_1,d_2}{\rightleftarrows}} F(s_0(X_0) \cup (Y_1)) \underset{s_0}{\overset{d_1,d_0}{\rightleftarrows}} F(X_0),$$

and, of course,

$$\begin{aligned} F_2 &= F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1) \cup Y_2) \\ &\cong F(s_1s_0(X_0)) * F(s_0(Y_1)) * F(s_1(Y_1)) * F(Y_2). \end{aligned}$$

In $\mathbf{F}^{(2)}$, higher levels than dimension 2 are generated by degenerate elements.

This pattern, of course, continues to higher dimensions. We thus have in each dimension, k-dimensional construction data $(\Omega^k, g^{(k)})$ and a k^{th}-level presentation of the group, G. The various Ω^k thus provide us with a CW-basis for $\mathbf{F}$.

4 Free Crossed Resolutions

In this section we want to examine in slightly more detail this step-by-step construction through the perspective of the corresponding crossed complex, examining not only to see if $\mathfrak{C}(\mathbb{F})$ is a crossed resolution of a group G, but also how the homotopy type of $\mathfrak{C}(\mathbb{F}^{(k)})$ is constructed from $\mathfrak{C}(\mathbb{F}^{(k-1)})$. Knowledge of this process would seem essential if the construction of crossed resolutions is to be 'mechanised'. It also helps in the interpretation of homological invariants and their linkage with combinatorial properties of a presentation or of a higher level presentation of a group G.

As the analysis is applicable in greater generality, we start by looking at an arbitrary free simplicial group with chosen CW-basis.

A 'step-by-step' construction of a free simplicial group is constructed from simplicial group inclusions

$$\mathbf{F}^{(0)} \subseteq \mathbf{F}^{(1)} \subseteq \mathbf{F}^{(2)} \subseteq \dots$$

We take the functor $\mathfrak{C}$ which is described in Section 2.4, to see what $C_n(\mathbf{F}^{(k)})$ looks like, where $\mathbf{F}^{(k)}$ is the k-skeleton of that construction, concentrating our attention in low dimensions. For $k = 0$, we have the 0-skeleton $\mathbb{F}^{(0)}$ of the construction

$$\mathbb{F}^{(0)} : \ldots \quad F(X_0) \Longrightarrow F(X_0) \to F(X_0)/N.$$

Here $\mathbf{F}^{(0)}$ is the trivial simplicial group in which in every degree n, $F_n^{(0)} = F(X_0)$ and $d_i^n = \mathrm{id} = s_j^n$. It is easy to see that $C_0(\mathbf{F}^{(0)}) = F(X_0)$ as $NF_1 \cap D_1$ is trivial. The 1-skeleton is

$$\ldots\ F(s_1 s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1)) \underset{s_1, s_0}{\overset{d_0, d_1, d_2}{\rightleftarrows}} F(s_0(X_0) \cup Y_1) \underset{s_0}{\overset{d_1, d_0}{\rightleftarrows}} F(X_0),$$

and since $F_2^{(1)}$ is generated by degenerate elements, $F_2^{(1)} = D_2$, so the crossed complex term $C_1(\mathbf{F}^{(1)})$ is the following

$$\begin{aligned} C_1(\mathbf{F}^{(1)}) &= \frac{NF_1^{(1)}}{(NF_1^{(1)} \cap D_1)\partial_2(NF_2^{(1)} \cap D_2)}, \\ &= \frac{NF_1^{(1)}}{\partial_2(NF_2^{(1)} \cap D_2)} \qquad \text{since } NF_1 \cap D_1 = 1, \\ &= \frac{NF_1^{(1)}}{\partial_2(NF_2^{(1)})} \qquad \text{as } F_2^{(1)} = D_2. \end{aligned}$$

By Lemma 3.1 and the Brown-Loday lemma [8], we have $NF_1^{(1)} = \langle Y_1 \rangle$ and $\partial_2(NF_2^{(1)})$ is generated by the Peiffer elements, respectively. It then follows that

$$C_1(\mathbf{F}^{(1)}) = \langle Y_1 \rangle / P_1.$$

Here P_1 is the first dimensional Peiffer normal subgroup. The proof of Theorem 1.7 from [7] interprets within this context as showing that $\partial_1 : \langle Y_1 \rangle / P_1 \longrightarrow F(X_0)$ is the free crossed module on the presentation $\langle X_0 \mid d_1(Y_1) \rangle$ of $\pi_0(\mathbf{F})$.

Looking at the case 2, the 2-skeleton of the construction is

$$\dots F(s_1s_0(X_0)\cup s_0(Y_1)\cup s_1(Y_1)\cup Y_2) \underset{s_1,s_0}{\overset{d_0,d_1,d_2}{\rightrightarrows}} F(s_0(X_0)\cup Y_1) \underset{s_0}{\overset{d_1,d_0}{\rightrightarrows}} F(X_0).$$

As before $F_3^{(2)} = D_3$ as $F_3^{(2)}$ is generated by the degeneracy elements. Thus the second term of the crossed complex is

$$\begin{aligned} C_2(\mathbf{F}^{(2)}) &= \frac{NF_2^{(2)}}{(NF_2^{(2)}\cap D_2)\partial_3(NF_3^{(2)}\cap D_3)}, \\ &= \frac{NF_2^{(2)}}{(NF_2^{(2)}\cap D_2)\partial_3(NF_3^{(2)})} \qquad \text{as } F_3^{(2)} = D_3. \end{aligned}$$

If $x, y \in NF_1$, then $NF_2 \cap D_2$ is generated by the elements of the form $[s_1x^{-1}s_0x,\ s_1y]$ and in general, if $x,\ y \in NF_{n-1}$, then $s_{n-1}x^{-1}s_{n-2}x,\ s_{n-1}y] \in NF_n \cap D_n$. Now look at $\partial_3(NF_3^{(2)})$ in terms of the skeleton $\mathbf{F}^{(2)}$. In a similar way to the proof of Lemma 3.1 and as $d_0^2(y_i) = d_1^2(y_i) = 1\ y \in Y_2$, one can readily obtain that:

$$NF_2^{(2)} = \langle s_1(Y_1)\cup Y_2\rangle \cap \langle Z \cup Y_2\rangle.$$

where Z as in Lemma 3.1.

On the other hand, [21] shows that on writing $K_I = \bigcap_{i\in I}\mathrm{Ker}d_i$ for $I \subseteq [n-1]$,

$$\partial_3(NG_3^{(2)}) = \prod_{I,J}[K_I,\ K_J][K_{\{0,2\}},\ K_{\{0,1\}}][K_{\{1,2\}},\ K_{\{0,1\}}][K_{\{1,2\}},\ K_{\{0,2\}}],$$

where $I \cup J = [2]$, $I \cap J = \emptyset$, so this is generated by the following elements: for $x_i \in NF_1 = \mathrm{Ker}d_0$ and $y_1, y_2 \in NF_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1$ with $1 \leq i, j \leq n$,

$$\begin{aligned} &[s_0x_i{}^{-1}s_1s_0d_1x_i,\ y_1] && (1)\\ &[s_1x_i{}^{-1}s_0x_i,\ s_1d_2(y_1)y_1{}^{-1}] && (2)\\ &[x_is_1d_2x_i{}^{-1}s_0d_2x_i,\ s_1y_1] && (3)\\ &[y_1{}^{-1}s_1d_2y_1,\ y_2] && (4)\\ &[y_1s_1d_2y_1{}^{-1}s_0d_2y_1,\ y_2] && (5)\\ &[y_1s_1d_2y_1{}^{-1}s_0d_2y_1,\ s_1d_2(y_2)y_2{}^{-1}] && (6). \end{aligned}$$

The normal subgroup generated by these elements will be denoted by P_2 and will be called the second dimensional Peiffer normal subgroup. We thus in principle have not only an explicit presentation of $C_2(\mathbf{F}^{(2)})$ but a list of seven 'generic' moves analogous to the Peiffer moves introduced by Brown and Huebschmann, [7].

Writing $Q_2 = NF_2^{(2)} \cap D_2$, we get the second term of the crossed complex as follows

$$C_2(\mathbf{F}^{(2)}) = \frac{\langle s_1(Y_1) \cup Y_2 \rangle \cap \langle Z \cup Y_2 \rangle}{Q_2 \cdot P_2}$$

Proposition 4.1 *Let* $\mathbf{F}^{(2)}$ *be the 2-skeleton of a free simplicial group resolving* $G = F(X_0)/N$. *Then*

$$\mathfrak{C}^{(2)}: \quad NF_2^{(2)}/(Q_2 \cdot P_2) \xrightarrow{\partial_2} \langle Y_1 \rangle / P_1 \xrightarrow{\partial_1} F(X_0) \xrightarrow{g} F(X_0)/N \xrightarrow{f} 1$$

is the 2-skeleton of a free crossed resolution of G *where* ∂_2 *and* ∂_1 *are given respectively by: for* $y_1 \in \langle s_1(Y_1) \cup Y_2 \rangle \cap \langle Z \cup Y_2 \rangle$ *and* $x_i \in \langle Y_1 \rangle$,

$$\partial_2(y_1(Q_2 \cdot P_2)) = d_2(y_1)P_1 \text{ and } \partial_1(x_i P_1) = d_1(X_i).$$

where $NF_2^{(2)}$ *is* $(\langle s_1(Y_1) \cup Y_2 \rangle \cap \langle Z \cup Y_2 \rangle)$.

Proof: This follows immediately from the description of the 'step-by-step' construction of the free simplicial group. □

This result gives a combinatorial description of the $C_2(\mathbf{F}^{(2)})$ term and if we manipulate the elements of $s_1(Y_1) \cup Y_2$ and $Z \cup Y_2$, remembering that $Z = \{s_1(y)^{-1}s_0(y) : y \in Y_1\}$, we can identify the generators as elements in the module of identities of the presentation $\langle X_0 \mid d_1(Y_1) \rangle$. The elements of Y_2 map via d_2 to a set of generators of this module since, of course, that is how they were chosen.

To complete our analysis of the rôle of a CW-basis in a free simplicial resolution $\mathbb{F} = (\mathbf{F}, g)$ of a group G, we need to check that $(C(\mathbf{F}), C(g))$ is a free *crossed* resolution of G and to see what happens to the CW-basis in the 'conversion'.

First a proposition showing how homotopies behave under the functor, $\mathfrak{C}$.

Proposition 4.2 *Suppose* $\mathbf{f}_0, \mathbf{f}_1 : \mathbf{G} \to \mathbf{H}$ *are morphisms of simplicial groups and* $\mathbf{h} : \mathbf{f}_0 \simeq \mathbf{f}_1$ *is a homotopy between them. Then* $\mathbf{h}$ *induces a homotopy* $\mathfrak{C}(\mathbf{h}) : \pi(1) \otimes \mathfrak{C}(\mathbf{G}) \to \mathfrak{C}(\mathbf{H})$ *between* $\mathfrak{C}(\mathbf{f}_0)$ *and* $\mathfrak{C}(\mathbf{f}_1)$.

(Here the $\otimes$ is the tensor product of crossed complexes introduced by Brown and Higgins, [6], and $\pi(1) := \pi(\Delta^1)$ is the groupoid 'unit interval'. For more on the homotopy theory of crossed complexes, the simplicial category theory of the category of crossed complexes, etc., see [4] and [25].)

Proof:

The homotopy $\mathbf{h}$ can be realised as a morphism, $\mathbf{h} : \Delta[1]\bar{\otimes}\mathbf{G} \to \mathbf{H}$, where $\Delta[1]\bar{\otimes}$ is the simplicial tensor within the simplicially enriched category of simplicial groups (or groupoids) (see Quillen, [24], or the discussion in [20].) This is given as a colimit of copies of $\mathbf{G}$ by the construction outlined in [24].

The functor $\mathfrak{C}$ can be thought of in two equivalent ways. It is either the composite of the reflection onto the variety of simplicial group(oid) T-complexes (cf. [14]) followed by the equivalence between that and the category of crossed complexes, or alternatively it uses the Cegarra-Carrasco equivalence between simplicial groupoids and hypercrossed complexes of group(oid)s followed by the reflection onto the variety of crossed complexes within that category. (The advantage at this point in using groupoids is that $\pi(1)$ is naturally a groupoid, but this can be avoided if desired.) From either description it is clear that $\mathfrak{C}$ will preserve colimits and thus tensors with simplicial sets, thus

$$\mathfrak{C}(\Delta[1]\bar{\otimes}\mathbf{G}) \cong \Delta[1]\bar{\otimes}\mathfrak{C}(\mathbf{G}) \cong \pi(1) \otimes \mathfrak{C}(\mathbf{G}).$$

Composing $\mathfrak{C}(\mathbf{h})$ with these isomorphisms gives the result. □

Corollary 4.3 *If* $\mathbf{g} : \mathbf{F} \to \mathbf{K}(G, 0)$ *is a free simplicial resolution of* G, *then* $\mathfrak{C}(\mathbf{g}) : \mathfrak{C}(\mathbf{F}) \to \mathfrak{C}(\mathbf{K}(G, 0)) = G$ *is a free crossed resolution of* G.

Proof: The data on $\mathbf{g}$ can be specified by giving a homotopy between the identity on $\mathbf{F}$ and the map that 'squashes NF down to G' and then uses a section of the augmentation map, g_0, to yield a map back to NF_0. The corollary now follows from the previous result applied to this simplicial homotopy. □

To finish the comparison, we will show that each $C_n(\mathbf{F})$ is a free G-module on Y_n if $n \geq 2$. We start with $n = 2$ but in fact almost the same proof works in higher dimensions.

Suppose that M is a G-module and $\Theta : Y_2 \to M$ is a function, we want to prove that $C_2(\mathbf{F})$ is free on Y_2, so we need to extend Θ to a map on $C_2(\mathbf{F})$. Form the crossed complex

$$\ldots \to M \to 1 \to G$$

with M in dimension 2, G in dimension 0, all other levels being trivial and the action of G on M being the given one. This has an associated simplicial group $S(M,G)$ with $NS(M,G)$ this crossed complex. There is an obvious morphism, ϕ from $\mathbf{F}^{(1)}$ to $S(M,G)$, inducing the quotient morphism $g : F(X_0) \to F(X_0)/N \cong G$. As $\pi_1(S(M,G))$ is trivial, Proposition 1.3 applies to show ϕ extends over $\mathbf{F}^{(2)}$ also extending Θ. Now we use $\mathfrak{C}$ to pass back to crossed complexes to get

$$\mathfrak{C}(\phi) : \mathfrak{C}(\mathbf{F}^{(2)}) \to M$$

extending Θ. As $C_2(\mathbf{F}^{(2)}) \simeq C_2(\mathbf{F})$, this proves the claim that $C_2(\mathbf{F}^{(2)})$ is a free G-module on Y_2.

Of course, the only difference that is needed in dimension n is in the definition of $S(M,G)$, where M is placed in dimension n and $\Theta : Y_n \to M$ is given.

We have proved:

Proposition 4.4 *If $\mathbb{F}$ is a simplicial resolution of G given by a construction data sequence $\{(Y_i, g^{(i)}), i = 0, 1, \ldots\}$ and $\mathbf{F}^{(k)}$ is the corresponding k-skeleton, then if $k \geq 2$, $C_k(\mathbb{F}^{(k)})$ is a free G-module on Y_k.*

Summarising we get:

Theorem 4.5 *The 'step-by-step' construction of simplicial resolution of a group, G, yields a 'step-by-step' construction of a crossed resolution of G via the crossed complex construction, $\mathfrak{C}$.*

As a bonus for our method we also have given an explicit description of the crossed complex construction in low dimensions. The construction data to dimension n yields an n-dimensional word system in the sense of R.A.Brown. What is less clear, as we have mentioned before, is why the word system given by Whitehead (see [9], Example 2.2.3) would not seem to lift back to give construction data for a free simplicial group.

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