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FIBREWISE EXPONENTIAL LAWS IN A QUASITOPOS

by KYUNG CHAN MIN, YOUNG SUN KIM and JIN WON PARK

RESUME. Les auteurs obtiennent différents types de lois exponentielles relatives aux fibrations dans un quasi-topos C, telles

$$C_{ABD}(X\times Y,Z) \cong C_{ABD}(X,C_{BD}(Y,Z)) \text{ et } M_{XD}(X\times_B Y,Z) \cong M_B(X,C_{BD}(Y,Z)),$$

avec isomorphismes dans C pour des espaces sur différentes bases. Ils montrent qu'il existe un isomorphisme dans C entre l'espace des applications préservant les fibres $q \to r$ et l'espace des sections transversales à $q \bullet_1 r$. Les exemples des espaces de convergence, des espaces de convergence séquentiels et des espaces simpliciaux sont discutés, ainsi que le cas proche des espaces quasi-topologiques et des espaces compactement engendrés.

1. Introduction

In homotopy theory, exponential laws play an important role in the theory of fibrations. Several researchers have investigated the problem of finding convenient categories in which various such laws are valid [3-9,21,22]. For example, a theory of fibrewise exponential laws has been developed for fibrations with variable base spaces. This has been done in the context of the category of compactly generated spaces, with the assumption that the base spaces are weak Hausdorff [6,16,17]. In particular, given fibrations $q : Y \to B$, $r : Z \to D$, the construction and properties of a function space $C_{BD}(Y, Z)$ and an associated fibration $q \cdot r : C_{BD}(Y, Z) \to B \times D$ are investigated in [6,7,16] and [17]. In this case fibrewise exponential laws play a crucial role [6,7,16,17]. So far, for the study of covering homotopy property, compactly generated spaces and quasi-topological spaces have been used [3,6,16,17]. However, from a structural point of view, this work has not been carried out in a fully convenient category. The main reasons for these difficulties are that the category of compactly generated spaces is not a quasitopos and quasi-topological spaces do not form a category, but a quasi-category.

In 1986, J. Adámek and H. Herrlich [1] showed that a topological category $\mathcal{C}$ over a quasitopos is a quasitopos if and only if for any $B \in \mathcal{C}$, the comma category $\mathcal{C}_B$ is cartesian closed.

Thus it is natural to expect a convenient category to be a topological category that is also a quasitopos. Let $\mathcal{C}$ be a topological construct, i.e., a topological category over the category **Set** of sets such that every constant map between objects in $\mathcal{C}$ is a morphism in $\mathcal{C}$. H. Herrlich [11] showed that $\mathcal{C}$ is a quasitopos if and only if final epi-sinks in $\mathcal{C}$ are universal, i.e., pullbacks of final epi-sinks are final epi-sinks. This fact will play an essential role for our theory in this paper. It was on this basis that, in 1992, Min and Lee [13] obtained natural exponential laws in the category of convergence spaces over a base B. We note that a small-fibred topological construct which is a quasitopos is precisely a topological universe, as introduced by L. D. Nel [18].

In this paper, in a quasitopos $\mathcal{C}$, we obtain various types of fibrewise exponential laws including $C_{ABD}(X \times Y, Z) \simeq C_{ABD}(X, C_{BD}(Y, Z))$ and $M_{XD}(X \times_B Y, Z) \simeq M_B(X, C_{BD}(Y, Z))$ for spaces over various base spaces. Moreover, we show that there exists an isomorphism in $\mathcal{C}$, between the space of fibre preserving maps from q to r and the space of cross-sections to the function space $C_{BD}(Y, Z)$ over B. We should note that most of the known results, related to the above isomorphisms in $\mathcal{C}$, were proved only on the level of one-to-one correspondences between sets. As examples, we introduce the category of convergence spaces containing the category of topological spaces as a bireflective subcategory, the category of sequential convergence spaces, the category of simplicial spaces, quasitopoi similar to quasi topological spaces and compactly generated spaces. In particular we give internal descriptions of natural function spaces for convergence spaces, sequential convergence spaces and simplicial spaces with variable base spaces.

Notation. For a pair of sets A and B, we will use $\pi_1 : A \times B \longrightarrow A$ and $\pi_2 : A \times B \longrightarrow B$ to denote the projections on the first and second variables, respectively.

The letter j will be used to denote various inclusions, often of one function space into another function space.

For a map $p : X \to A$, the fibre X_a is the set $p^{-1}(a)$, where $a \in A$.

2. Function spaces and exponential laws I

Let $\mathcal{C}$ be a topological construct which is a quasitopos and $p : X \to A$ be a morphism in $\mathcal{C}$. Then we say that X is an *object over a base* A in $\mathcal{C}$. Let $p : X \to A$ and $q : Y \to B$ be morphisms in $\mathcal{C}$. A *fibre preserving map* from X to Y is a pair of morphisms $f_1 : X \to Y$ and $f_0 : A \to B$ in $\mathcal{C}$ such that $q \circ f_1 = f_0 \circ p$, i.e., the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[f_0]{} & B \end{array}$$

commutes. Of course, if p is surjective, then f_1 determines f_0. We write this map by $(f_1, f_0) : p \to q$.

For given $p : X \to A$ and $q : Y \to B$ in $\mathcal{C}$, let

$$C_{AB}(X,Y) = \bigcup_{a \in A,\, b \in B} C(X_a, Y_b)$$

as a set, where $C(X_a, Y_b)$ is the set of all morphisms from a fibre X_a to a fibre Y_b, and let $p \cdot q : C_{AB}(X,Y) \to A \times B$ a natural map in **Set**. We also define $p \cdot_1 q : C_{AB}(X,Y) \to A$ to be the function $\pi_1 \circ (p \cdot q)$. Let $Z \in \mathcal{C}$ and $f : Z \to C_{AB}(X,Y)$ be a function. Then $(p \cdot_1 q) \circ f : Z \to A$, so we can view $C_{AB}(X,Y)$ and Z as objects over A in **Set**. Then there is a function $1_X \times_A f : X \times_A Z \to X \times_A C_{AB}(X,Y)$, defined by $(1_X \times_A f)(x,z) = (x, f(z))$, where (x,z) is in $X \times_A Z$.

Let $ev : X \times_A C_{AB}(X,Y) \to Y$ be the natural evaluation function. Then we define S to be the set of functions $f : Z \to C_{AB}(X,Y)$ such that $(p \cdot q) \circ f$ and $ev \circ (1_X \times_A f)$ are morphisms in $\mathcal{C}$.

On $C_{AB}(X,Y)$, give the final structure in $\mathcal{C}$ with respect to the sink S. Then, by the property of the final structure on $C_{AB}(X,Y)$, it is easy to see that $p{\cdot}q : C_{AB}(X,Y) \to A \times B$ is a morphism. The function $p{\cdot}q : C_{AB}(X,Y) \to A \times B$ and the function $p{\cdot}_1 q : C_{AB}(X,Y) \to A$ are morphisms in $\mathcal{C}$. So we can take $C_{AB}(X,Y)$ to be either an object over $A \times B$ or an object over A, in the sense of the category $\mathcal{C}$. We will use $p{\cdot}_2 q$ to denote another morphism in $\mathcal{C}$, i.e., $\pi_2 \circ (p{\cdot}q) : C_{AB}(X,Y) \to B$.

Proposition 2.1. *Let $p : X \to A$ and $q : Y \to B$ be morphisms in $\mathcal{C}$. Then the evaluation map $ev : X \times_A C_{AB}(X,Y) \to Y$ is a fibre preserving map in $\mathcal{C}$.*

Proof. Since $\mathcal{C}$ is a topological construct which is a quasitopos and $\{f | f \in S\}$ is a final epi-sink in $\mathcal{C}$, $\{1_X \times f | f \in S\}$ is a final epi-sink in $\mathcal{C}$ by the cartesian closedness of $\mathcal{C}$. Hence so also is the pullback $\{1_X \times_A f | f \in S\}$, by the universality of the final epi-sink in $\mathcal{C}$. From the definition of S, we see that ev is a map in $\mathcal{C}$. Clearly ev is a fibre preserving map in $\mathcal{C}$.

Theorem 2.2. *Let $p : X \to A$, $q : Y \to B$ and $r : Z \to D$ be morphisms in $\mathcal{C}$. Then the map*

$$\phi : C_{ABD}(X \times Y, Z) \to C_{ABD}(X, C_{BD}(Y,Z)),$$

which is defined by $\phi(f)(x)(y) = f(x,y)$, is an isomorphism in $\mathcal{C}$, where $f \in C_{ABD}(X \times Y, Z), x \in X$ and $y \in Y$.

Proof. Let $f \in C(X_a \times Y_b, Z_d)$. Then $\phi(f) \in C(X_a, C(Y_b, Z_d))$, since $\mathcal{C}$ is a cartesian closed category. Hence it is easy to see that ϕ is bijective. From the following commutative diagram

$$\begin{array}{ccc} (X \times Y) \times_{A \times B} C_{ABD}(X \times Y, Z) & \xrightarrow{ev} & Z \\ {\scriptstyle 1 \times_{A \times B} \phi^{-1}} \uparrow & & \uparrow {\scriptstyle ev} \\ (X \times Y) \times_{A \times B} C_{ABD}(X, C_{BD}(Y,Z)) & \xrightarrow[1 \times_B ev]{} & Y \times_B C_{BD}(Y,Z), \end{array}$$

we can see that ϕ^{-1} is a morphism in $\mathcal{C}$.

Conversely, consider the following commutative diagram

$$\begin{array}{ccc} Y \times_B C_{BD}(Y,Z) & \xrightarrow{ev} & Z \\ {\scriptstyle 1\times_B \alpha}\uparrow & & \uparrow{\scriptstyle ev} \\ Y\times_B(X\times_A C_{ABD}(X\times Y,Z)) & \xrightarrow[\sim]{} & (X\times Y)\times_{A\times B} C_{ABD}(X\times Y,Z) \end{array}$$

where $\alpha(x,f)(y) = f(x,y)$. By the definition of $C_{BD}(Y,Z)$, α is a morphism in $\mathcal{C}$. Now from the following commutative diagram

$$\begin{array}{ccc} X\times_A C_{ABD}(X, C_{BD}(Y,Z)) & \xrightarrow{ev} & C_{BD}(Y,Z) \\ {\scriptstyle 1\times_A \phi}\uparrow & \nearrow{\scriptstyle \alpha} & \\ X\times_A C_{ABD}(X\times Y,Z) & & \end{array}$$

ϕ is a morphism in $\mathcal{C}$. Therefore ϕ is an isomorphism in $\mathcal{C}$.

For morphisms $p : X \to B$ and $q : Y \to B$ in $\mathcal{C}$, let

$$C_B(X,Y) = \bigcup_{b\in B} C(X_b, Y_b)$$

as a set, where $C(X_b, Y_b)$ is the set of all morphisms in $\mathcal{C}$, from the fibre X_b to the fibre Y_b, where $b \in B$. Further, let $(pq) : C_B(X,Y) \to B$ be the natural projection in **Set**. Let $S = \{f : Z \to C_B(X,Y) | ev \circ (1_X \times_B f)$ is a morphism in $\mathcal{C}$, where Z is an object over B in $\mathcal{C}$ and f a set map over $B\}$. On $C_B(X,Y)$, give the final structure in $\mathcal{C}$ with respect to the sink S.

Remark. It can be shown that the function space $C_B(X,Y)$ is the subspace of $C_{BB}(X,Y)$ by the definition of $C_{BB}(X,Y)$ and the catesian closedness of the category $\mathcal{C}_B$.

The exponential law in proposition 2.2 can be regarded as a generalization of the well known exponential law in $\mathcal{C}_B$ by the following corollary.

Corollary 2.3. *Let $p : X \to B, q : Y \to B$ and $r : Z \to B$ be morphisms in $\mathcal{C}$. Then there is an isomorphism*

$$\overline{\phi} : C_B(X \times_B Y, Z) \to C_B(X, C_B(Y, Z))$$

in $\mathcal{C}$.

Proof. It is enough to consider the following commutative diagram

$$\begin{array}{ccc} C_B(X \times_B Y, Z) & \xrightarrow{\overline{\phi}} & C_B(X, C_B(Y, Z)) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ C_{BBB}(X \times Y, Z) & \xrightarrow[\phi]{} & C_{BBB}(X, C_{BB}(Y, Z)) \end{array}$$

where $\overline{\phi}$ is the restriction and corestriction of ϕ.

Proposition 2.4. *Let $q : Y \to B$ and $r : Z \to D$ be morphisms in $\mathcal{C}$ and U, V be subspaces of B and D, respectively. Then the morphisms $q{\cdot}r|_{U\times V} : C_{BD}(Y, Z)|_{U\times V} \to U \times V$ and $(q|_U){\cdot}(r|_V) : C_{UV}(Y|_U, Z|_V) \to U \times V$ coincide.*

Proof. It is immediate that the two constructions agree at the set-function level. Let $1 : C_{BD}(Y, Z)|_{U\times V} \to C_{UV}(Y|_U, Z|_V)$ be an identity map. By the following commutative diagram

$$\begin{array}{ccc} Y \times_B C_{BD}(Y, Z) & \xrightarrow{ev} & Z \\ & & \uparrow{\scriptstyle j'} \\ {\scriptstyle j}\uparrow & & Z|_V \\ & & \uparrow{\scriptstyle ev} \\ Y|_U \times_U C_{BD}(Y, Z)|_{U\times V} & \xrightarrow[1_{Y|_U} \times_U 1]{} & Y|_U \times_U C_{UV}(Y|_U, Z|_V), \end{array}$$

it is easy to see that 1 is a morphism in $\mathcal{C}$. By a similar argument we can see that 1^{-1} is a morphism in $\mathcal{C}$.

Remark. Take $U = \{b\}$ and $V = \{d\}$. From proposition 2.4, the fibre $C_{BD}(Y, Z)_{(b,d)}$ of (b, d) is the function space $C(Y_b, Z_d)$ in $\mathcal{C}$.

Proposition 2.5. *Let $q : Y \to B, r_0 : Z_0 \to D$ and $r_1 : Z_1 \to D$ be morphisms in $\mathcal{C}$.*

(1) *If $f : Z_0 \to Z_1$ is a morphism over D then $f_\sharp : C_{BD}(Y, Z_0) \to C_{BD}(Y, Z_1)$, defined by $f_\sharp(g) = (f_d)g : Y_b \to (Z_1)_d$, where $f_d : (Z_0)_d \to (Z_1)_d$ is the restriction and corestriction of f, is a morphism over $B \times D$, where $g : Y_b \to (Z_0)_d$ with $b \in B$ and $d \in D$.*

(2) *The rule $f \to f_\sharp$ is functorial, i.e., $(1_{Z_0})_\sharp = 1_{C_{BD}(Y,Z_0)}$ and, if $h : Z_1 \to W$ is another morphism over D then $(hf)_\sharp = h_\sharp f_\sharp$.*

Proof. (1) By the following commutative diagram

$$\begin{array}{ccc} Y \times_B C_{BD}(Y, Z_1) & \xrightarrow{ev} & Z_1 \\ {\scriptstyle 1\times_B f_\sharp}\uparrow & & \uparrow{\scriptstyle f} \\ Y \times_B C_{BD}(Y, Z_0) & \xrightarrow[ev]{} & Z_0, \end{array}$$

$f_\sharp$ is a morphism.

(2) For $b \in B, d \in D$, and $g : Y_b \to (Z_0)_d$, $(1_{Z_0})_\sharp(g) = ((1_{Z_0})_d)g = 1_{C_{BD}(Y,Z_0)}(g)$, and $(hf)_\sharp(g) = ((hf)_d)g = (h_d)(f_d)g = h_d(f_\sharp(g)) = h_\sharp f_\sharp(g)$.

Proposition 2.6. *Let $q : Y \to B, r_0 : Z_0 \to D$ and $r_1 : Z_1 \to D$ be morphisms in $\mathcal{C}$. Suppose $Z_0 \times I$ is an object over D with a morphism $r_0 \circ \pi_1$. If $F : Z_0 \times I \to Z_1$ is a homotopy over D then $F_* : C_{BD}(Y, Z_0) \times I \to C_{BD}(Y, Z_1)$, defined by $F_*(g, t)(y) = F(g(y), t)$, where $g : Y_b \to (Z_0)_d$ for some $b \in B, d \in D, t \in I, y \in Y$ and $q(y) = b$, is a homotopy over $B \times D$.*

Proof. By the following commutative diagram

$$\begin{array}{ccc} Y \times_B C_{BD}(Y, Z_1) & \xrightarrow{ev} & Z_1 \\ {\scriptstyle 1\times_B F_*}\uparrow & & \uparrow{\scriptstyle F} \\ Y \times_B C_{BD}(Y, Z_0) \times I & \xrightarrow[ev\times 1]{} & Z_0 \times I, \end{array}$$

F_* is a morphism.

3. Function spaces and exponential laws II

Let $p : X \to B, q : Y \to B$ and $r : Z \to D$ be morphisms in $\mathcal{C}$. Now we obtain another type of exponential law. Let $M_{BD}(Y, Z) = \{(f_1, f_0) | (f_1, f_0) : q \to r\}$. We consider $M_{BD}(Y, Z)$ as a subspace of $C(Y, Z) \times C(B, D)$. For given spaces Y and Z over D in $\mathcal{C}$ with morphisms q and r, respectively, let $M_D(Y, Z) = \{f : Y \to Z | f$ is a morphism over $D\}$. We give $M_D(Y, Z)$ the subspace structure of $C(Y, Z)$. $M_D(Y, Z)$ can be considered as a subspace of $M_{DD}(Y, Z)$. In fact, $M_D(Y, Z)$ is isomorphic to a subspace of $M_{DD}(Y, Z)$ in which f_0 is fixed as 1_D.

Consider $M_{XD}(X \times_B Y, Z)$ and $M_B(X, C_{BD}(Y, Z))$. In this case, $X \times_B Y$ is considered as a space over X with natural morphism q_p, the pull-back of q along p, and $C_{BD}(Y, Z)$ as a space over B with morphism $q \cdot_1 r$. Define a function $\psi : M_{XD}(X \times_B Y, Z) \to M_B(X, C_{BD}(Y, Z))$ as follows. For $(f_1, f_0) \in M_{XD}(X \times_B Y, Z)$, the rule $\psi(f_1, f_0)(x)(y) = f_1(x, y)$ defines $\psi(f_1, f_0)(x)$ as a function from Y_b to Z_d where $p(x) = q(y) = b, f_0(x) = d$. For such $x \in X_b$, $\psi(f_1, f_0)(x)$ is the composite morphism $Y_b \cong \{x\} \times Y_b \xrightarrow{j} X_b \times Y_b \xrightarrow{(f_1)_b} Z_d$, where $(f_1)_b$ is the appropriate restriction and corestriction of f. Thus $\psi(f_1, f_0)$ is a function, from X to $C_{BD}(Y, Z)$, that is clearly over B. Moreover, $\psi(f_1, f_0)(x)(y) = f_1(x, y)$.

Lemma 3.1. *The map* $\psi : M_{XD}(X \times_B Y, Z) \to M_B(X, C_{BD}(Y, Z))$ *is a morphism in* $\mathcal{C}$.

Proof. Since $ev \circ (1_Y \times_B \psi(f_1, f_0)) = f_1$, $\psi(f_1, f_0) \in M_B(X, C_{BD}(Y, Z))$. Let $E = X \times_B (Y \times M_{XD}(X \times_B Y, Z))$. From the following commutative diagram

$$\begin{array}{ccc}
Y \times_B C_{BD}(Y, Z) & \xrightarrow{\quad ev \quad} & Z \\
 & & \uparrow{\scriptstyle ev} \\
\uparrow{\scriptstyle 1_Y \times_B \beta} & & X \times_B Y \times C(X \times_B Y, Z) \\
 & & \uparrow{\scriptstyle 1 \times \pi_1} \\
E & \xrightarrow[1 \times j]{\qquad} & X \times_B Y \times C(X \times_B Y, Z) \times C(X, D)
\end{array}$$

we have a morphism $\beta : X \times M_{XD}(X \times_B Y, Z) \to C_{BD}(Y, Z)$ over B, defined by $\beta(x, f_1, f_0)(y) = \psi(f_1, f_0)(x)(y)$. We note that $X \times_B (Y \times W)$

$= (X \times_B Y) \times W$ for any space W. Now since $ev \circ (1_X \times \psi) = \beta$, we see that $\psi : M_{XD}(X \times_B Y, Z) \to C(X, C_{BD}(Y, Z))$ is a morphism in $\mathcal{C}$. Hence the result follows.

Next consider the function $\varphi : M_B(X, C_{BD}(Y, Z)) \to C(X \times_B Y, Z) \times C(X, D)$ defined by $\varphi(f) = (f_1, f_0)$ for $f \in M_B(X, C_{BD}(Y, Z))$, where $f_1(x, y) = f(x)(y)$ and $f_0(x) = q \cdot_2 r(f(x))$.

Lemma 3.2. *The map*

$$\varphi : M_B(X, C_{BD}(Y, Z)) \to C(X \times_B Y, Z) \times C(X, D)$$

is a morphism in $\mathcal{C}$.

Proof. Since $f_0 = (q \cdot_2 r) \circ f$ and $f_1 = ev \circ (1_Y \times_B f)$, φ is well-defined. From the following commutative diagram

$$\begin{array}{ccc}
(X \times_B Y) \times C(X \times_B Y, Z) & \xrightarrow{ev} & Z \\
 & & \Big\uparrow{\scriptstyle ev} \\
{\scriptstyle 1\times(\pi_1\circ\varphi)}\Big\uparrow & & Y \times_B C_{BD}(Y, Z) \\
 & & \Big\uparrow{\scriptstyle (t\times 1)\circ(1\times_B ev)} \\
(X \times_B Y) \times M_B(X, C_{BD}(Y, Z)) & \xrightarrow[1\times j]{} & X \times_B (Y \times C(X, C_{BD}(Y, Z))),
\end{array}$$

where $t : X \times_B Y \to Y \times_B X$ is the switching morphism in $\mathcal{C}$, we have a morphism $\pi_1 \circ \varphi$. Moreover, from the following commutative diagram

$$\begin{array}{ccc}
X \times C(X, D) & \xrightarrow{ev} & D \\
 & & \Big\uparrow{\scriptstyle q\cdot_2 r} \\
{\scriptstyle 1\times(\pi_2\circ\varphi)}\Big\uparrow & & C_{BD}(Y, Z) \\
 & & \Big\uparrow{\scriptstyle ev} \\
X \times M_B(X, C_{BD}(Y, Z)) & \xrightarrow[1\times j]{} & X \times C(X, C_{BD}(Y, Z)),
\end{array}$$

we have a morphism $\pi_2 \circ \varphi$. Therefore φ is a morphism in $\mathcal{C}$.

Using the above two lemmas, we can prove the following theorem.

Theorem 3.3. *Let $p : X \to B$, $q : Y \to B$ and $r : Z \to D$ be morphisms in $\mathcal{C}$. Then*

$$\psi : M_{XD}(X \times_B Y, Z) \to M_B(X, C_{BD}(Y, Z))$$

which is defined by $\psi(f_1, f_0)(x)(y) = f_1(x, y)$ is an isomorphism in $\mathcal{C}$.

Proof. Note that the image of φ is contained in $M_{XD}(X \times_B Y, Z)$, i.e., $\varphi : M_B(X, C_{BD}(Y, Z)) \to M_{XD}(X \times_B Y, Z)$ is well defined. This function is the inverse of ψ. Hence the result follows.

As a special case of theorem 3.3., we can obtain the following well known exponential law for $\mathcal{C}_B$.

Corollary 3.4. *Let $p : X \to B$, $q : Y \to B$ and $r : Z \to B$ be morphisms in $\mathcal{C}$. Then there is an isomorphism*

$$\overline{\psi} : M_B(X \times_B Y, Z) \to M_B(X, C_B(Y, Z))$$

in $\mathcal{C}$.

Proof. It is enough to consider the following commutative diagram

$$\begin{array}{ccc} M_B(X \times_B Y, Z) & \xrightarrow{\overline{\psi}} & M_B(X, C_B(Y, Z)) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ M_{XB}(X \times_B Y, Z) & \xrightarrow[\psi]{} & M_B(X, C_{BB}(Y, Z)) \end{array}$$

where $\overline{\psi}$ is the restriction and corestriction of ψ. In this case, we consider $M_B(X \times_B Y, Z)$ as a subspace of $M_{XB}(X \times_B Y, Z)$ in which f_0 is fixed as p.

Let $p : X \to B, q : Y \to B$ and $r : Z \to D$ be morphisms in $\mathcal{C}$. Let $M^g_{XD}(X \times_B Y, Z) = \{(f, g) | (f, g) : q_p \to r\}$ for a given morphism $g : X \to D$ in $\mathcal{C}$. Then we have the following result which is the same type of fibred exponential law as is theorem 2.2 in [6].

Corollary 3.5. *The restriction*

$$\psi : M^g_{XD}(X \times_B Y, Z) \to M_{B\times D}(X, C_{BD}(Y,Z))$$

is an isomorphism in $\mathcal{C}$, where X is viewed as a space over $B \times D$ with the projection $p \times g : X \to B \times D$.

Proof. The proof is similar to that for corollary 3.4.

Corollary 3.6. *Let $r_0 : Z_0 \to D$ and $r_1 : Z_1 \to D$ be morphisms in $\mathcal{C}$. Then there is an isomorphism in $\mathcal{C}$ between the space of homotopies $H : Z_0 \times I \to Z_1$ such that $r_1 \circ H = \pi_1 \circ (r_0 \times 1_I)$ and the space of homotopies $H^0 : D \times I \to C_{DD}(Z_0, Z_1)$ over $D \times D$, that lift $(\pi_1, \pi_1) : D \times I \to D \times D$ over $r_0 \cdot r_1$, defined by $H(z,t) = H^0(d,t)(z)$ where $r_0(z) = d$ and $t \in I$.*

Proof. This follows from corollary 3.5 with $X = D \times I, B = D$ and $p = g = \pi_1 : D \times I \to D$, using the identification $Z_0 \times_D (D \times I) = Z_0 \times I$.

Corollary 3.7. *There is an isomorphism in $\mathcal{C}$*

$$M_{BD}(Y,Z) \cong M_B(B, C_{BD}(Y,Z)),$$

i.e., there is an isomorphism in $\mathcal{C}$ between the space of fibre preserving maps $(f_1, f_0) : q \to r$ and the space of cross-sections s to $q \cdot_1 r$.

Proof. Take p to be the identity on B and note that $B \times_B Y \cong Y$.

Assume that the fibres of q and r are all of the same homotopy type. Let $H(Y_b, Z_d)$ be the set of $\mathcal{C}$-homotopy equivalences of Y_b to Z_d. Let $H_{BD}(Y,Z)$ be the set $\bigcup_{b\in B, d\in D} H(Y_b, Z_d)$, and give it the subspace structure derived from $C_{BD}(Y,Z)$.

Corollary 3.8. *There is an isomorphism in $\mathcal{C}$ between the space of fibre preserving maps $(f_1, f_0) : q \to r$ whose restrictions to fibres are homotopy equivalences and the space of cross-sections s to $(q \cdot_1 r)|_{H_{BD}(Y,Z)} : H_{BD}(Y,Z) \to B$.*

Proof. This is immediate from corollary 3.7, since we are considering subspace structures on subspaces that correspond under the isomorphism of that corollary.

We now need a more specific notation for pull-back spaces. Let X be a space over B in $\mathcal{C}$ with morphism p and $f : A \to B$ be a morphism in $\mathcal{C}$. We write $A \times_f X$ for $\{(a,x) | f(a) = p(x)\}$, i.e., the space obtained by pulling p back along f.

Proposition 3.9. *Let $p : X \to B, f : A \to B$ and $g : A \to B$ be morphisms in $\mathcal{C}$. Consider A as a space over $B \times B$ with morphism (f,g). Then*

$$\lambda : M_A(A \times_f X, A \times_g X) \to M_{B \times B}(A, C_{BB}(X,X))$$

where $\lambda(h)(a) : X_{f(a)} \to X_{g(a)}$ is defined by $\lambda(h)(a)(x) = \pi_2(h(a,x))$, where $h \in M_A(A \times_f X, A \times_g X)$ and $(a,x) \in A \times_f X$, is an isomorphism in $\mathcal{C}$.

Proof. Consider $\gamma : A \times M_A(A \times_f X, A \times_g X) \to C_{BB}(X,X)$, where $\gamma(a,h) : X_{f(a)} \to X_{g(a)}$ is defined by $\gamma(a,h)(x) = \pi_2 \circ h(a,x)$. Let $E = \{(x,a,h) \in X \times (A \times M_A(A \times_f X, A \times_g X)) | p(x) = f(a)\}$. Then, from the following commutative diagram

$$\begin{array}{ccc}
X \times_B C_{BB}(X,X) & \xrightarrow{ev} & X \\
 & & \uparrow{\scriptstyle \pi_2} \\
 & & A \times_g X \\
{\scriptstyle 1 \times_B \gamma}\uparrow & & \uparrow{\scriptstyle ev} \\
 & & (A \times_f X) \times C(A \times_f X, A \times_g X) \\
 & & \uparrow{\scriptstyle 1 \times j} \\
E & \longrightarrow & (A \times_f X) \times M_A(A \times_f X, A \times_g X)
\end{array}$$

we can see that γ is a morphism in $\mathcal{C}$.

Moreover, by the following commutative diagram,

$$\begin{array}{ccc}
A \times M_{B \times B}(A, C_{BB}(X,X)) & \xrightarrow{ev} & C_{BB}(X,X) \\
{\scriptstyle 1 \times \lambda}\uparrow & \nearrow{\scriptstyle \gamma} & \\
A \times M_A(A \times_f X, A \times_g X) & &
\end{array}$$

it is shown that λ is a morphism in $\mathcal{C}$.

Conversely, consider the map $\mu : M_{B\times B}(A, C_{BB}(X,X)) \to M_A(A \times_f X, A \times_g X)$, where $\mu(k) : A \times_f X \to A \times_g X$ is defined by $\mu(k)(a,x) = (a, k(a)(x))$. Since $(p \cdot p) \circ k = (f,g)$, μ is well-defined. Consider the following commutative diagram,

$$\begin{array}{ccc} (A \times_f X) \times M_A(A \times_f X, A \times_g X) & \xrightarrow{ev} & A \times_g X \\ {\scriptstyle 1\times\mu}\uparrow & \nearrow{\scriptstyle \alpha} & \\ (A \times_f X) \times M_{B\times B}(A, C_{BB}(X,X)) & & \end{array}$$

where $\alpha((a,x),k) = (a, k(a)(x))$. By routine work with an exponential law in $\mathcal{C}$, it is easy to see that α is a morphism in $\mathcal{C}$ and hence μ is a morphism in $\mathcal{C}$. Clearly, μ is the inverse of λ.

4. Examples

We introduce some quasitopos including important spaces which play an essential role in homotopy theory.

(a) Convergence spaces :

Let X be a set. We use $\mathcal{F}(X)$ to denote the set of all filters on X and $\mathcal{P}(\mathcal{F}(X))$ to denote the power set of $\mathcal{F}(X)$. A *convergence structure* on X is a function $c : X \to \mathcal{P}(\mathcal{F}(X))$ satisfying the following axioms

(1) $\dot{x} \in c(x)$, where $\dot{x}$ is the filter generated by $\{x\}$,
(2) if $\mathcal{F} \in c(x)$ and $\mathcal{F} \subset \mathcal{G}$, then $\mathcal{G} \in c(x)$,
(3) if $\mathcal{F}, \mathcal{G} \in c(x)$, then $\mathcal{F} \cap \mathcal{G} \in c(x)$.

Such a pair (X, c) will be called a *convergence space.* The filter in $c(x)$ are said to be *convergent to* x. We usually write $\mathcal{F} \to x$ instead of $\mathcal{F} \in c(x)$. By a *continuous map* $f : X \to Y$ between convergence spaces is meant a function $f : X \to Y$ such that $f(\mathcal{F}) \to f(x)$ in Y whenever $\mathcal{F} \to x$ in X. The category **Conv** is formed by all convergence spaces and all continuous maps between them. It is well known that **Conv** is a concrete quasitopos (cf [18]) and it is a very useful category in various respects. In particular, it contains the category **Top** of topological spaces as a bireflective subcategory. In [13,14], the authors obtained fibrewise exponential laws in $\mathbf{Conv_B}$ and studied the category $\mathbf{Conv_B}$ extensively.

Let $p : X \longrightarrow A$ and $q : Y \longrightarrow B$ be continuous maps between convergence spaces. Define a convergence structure on $C_{AB}(X,Y)$ as follows. A filter $\mathcal{F}$ converges to f in $C_{AB}(X,Y)$, where $f \in C(X_a, Y_b)$ if and only if

(1) for any filter $\mathcal{A}$ in X which converges to $x \in X_a$, $(\mathcal{F} \cap \dot{f})(\mathcal{A} \cap \dot{x})$ converges to $f(x)$ in Y and

(2) $(p\cdot q)(\mathcal{F})$ converges to $(p\cdot q)(f)$ in $A \times B$, where, for $g \in C(X_a, X_b)$, $p \cdot q : C_{AB}(X,Y) \longrightarrow A \times B$ is defined by $(p \cdot q)(g) = (a,b)$.

By a routine work, we can show that $C_{AB}(X,Y)$ is a convergence space over $A \times B$ and moreover it is the natural function space for convergence spaces with variable base spaces constructed in Section 2. If $A = B = \{*\}$, then $C_{AB}(X,Y)$ is the natural function space in **Conv**.

Remark. P. I. Booth, P. R. Heath and R. A. Piccinini [7] introduced these types of exponential law in the category of compactly generated spaces. But, since the category of compactly generated spaces is not quasitopos, they showed only the one-to-one correspondence between two function spaces as a set. However, since the category **Conv** is a quasitopos, we have a natural homeomorphism between two function spaces of the convergence spaces over variable base spaces. Hence it is very useful to deal with the category of convergence spaces instead of the category of compactly generated spaces in studying the problems which are concerned with the exponential laws.

(b) Sequential convergence spaces :

A *sequential convergence space* is an ordered pair (X,ξ) of sets, where $\xi \subset X^N \times X$ is a specified relation between sequences $u \in X^N$ and point $p \in X$, subject to the following three Fréchet-Urysohn axioms. In what follows we will express the statement $((u_n),p) \in \xi$ by writing u_n *converges to* p in (X,ξ). As usual, a subsequence $(u_{s(n)})$ of (u_n) will be specified by a strictly increasing function $s : N \longrightarrow N$. The axioms are as follows

(1) If $u_n = p$ for all n, then u_n converges to p.
(2) If u_n converges to p and $(u_{s(n)})$ is a subsequence of (u_n), then $u_{s(n)}$ converges to p.
(3) If $(u_n) \in X^N$ is such that every subsequence $(u_{s(n)})$ has a further subsequence $(u_{s(t(n))})$, where $u_{s(t(n))}$ converges to p, then u_n converges to p.

A *sequentially continuous map* $f : (X, \xi) \to (Y, \eta)$ is a function $f : X \to Y$ such that $f(u_n)$ converges to $f(p)$ in (Y, η) whenever u_n converges to p in (X, ξ). The category **Seq** is formed by all sequential convergence spaces and sequentially continous maps. In [15], the category **Seq** is shown to be a concrete quasitopos. In [12], the categories of sequential spaces and sequential spaces with unique sequential limits were discussed to obtain convenient categories of fibre spaces.

Let $p : X \to A$ and $q : Y \to B$ be sequentially continuous maps between sequential convergence spaces. Define a sequential convergence structure on $C_{AB}(X, Y)$ as follows. A sequence f_n converges to f in $C_{AB}(X, Y)$, where $f \in C(X_a, Y_b)$ if and only if
(1) for any subsequence $f_{s(n)}$ of f_n and any sequence x_n in X which converges to $x \in X_a$, the sequence

$$\mathbf{f}_{s(n)}(x_n) = \begin{cases} f_{s(n)}(x_n) & \text{if } f_{s(n)}(x_n) \text{ can be defined} \\ f(x) & \text{otherwise} \end{cases}$$

converges to $f(x)$ in Y and
(2) the sequence $(p \cdot q)(f_n)$ converges to $(p \cdot q)(f)$, where $p \cdot q : C_{AB}(X, Y) \to A \times B$ is the map defined by $(p \cdot q)(g) = (a, b)$ for $g \in C(X_a, Y_b)$.

By a routine work, we know that $C_{AB}(X, Y)$ is a sequential convergence space over $A \times B$ and moreover it is the natural function space for sequential convergence spaces with variable base spaces constructed in Section 2. If $A = B = \{*\}$, then $C_{AB}(X, Y)$ is the natural function space in **Seq**.

(c) Simplicial complexes :

It is not necessary to recall the importance of simplicial complexes in algebraic topology. Let K be a set and $\underline{K}$ a set of subsets of K satisfying

(1) $\{k\} \in \underline{K}$ for each $k \in K$,
(2) $E \in \underline{K}$ implies that E is nonvoid and finite,
(3) $E \in \underline{K}$ and $F \subset E$ is nonvoid imply $F \in \underline{K}$.
Then $(K, \underline{K})$ is called a *simplicial complex* [20]. The elements of K are called *vertices* and the elements of $\underline{K}$ are called *simplexes.* Let $(K, \underline{K})$ and $(K', \underline{K}')$ be simplicial complexes. A map $f : K \to K'$ is called *simplicial*, provided $f[E] \in \underline{K}'$ for each $E \in \underline{K}$. **Simp** denotes the category of simplicial complexes and simplicial maps. This category **Simp** is also a usual candidate for constructing homotopy theory [2,10]. G. Preuß [19] showed that the category **Simp** is a concrete quasitopos. In [19], the stability of final coverings under pullbacks has proved particularly useful in the study of connection and factorization properties. Let $\mathbf{Simp}_n$ be the subcategory of **Simp** consisting of simplicial complexes with dimension less than or equal to n. He showed that the category $\mathbf{Simp}_n$ is a concrete quasitopos and a bicoreflective subcategory of **Simp**. Moreover, if $n < m$, then $\mathbf{Simp}_n$ is a proper subcategory of $\mathbf{Simp}_m$. Therefore, in many respects, the quasitopos **Simp** can be utilized using our categorical results in the previous section.

Let $p : X \to A$ and $q : Y \to B$ be simplicial maps between simplicial complexes. Define a simplicial structure on $C_{AB}(K, L)$ as follows. A subset S of $C_{AB}(K, L)$ is in $\underline{C_{AB}(K, L)}$ if and only if
(1) S is nonempty and finite
(2) $S(E_a) \in \underline{L}$ for any $E_a \in \underline{K_a}$ if $S(K_a) \neq \emptyset$
(3) $p \cdot q(S) \in \underline{A \times B}$, where $p \cdot q : C_{AB}(K, L) \to A \times B$ is the map defined by $p \cdot q(g) = (a, b)$ for $g \in C(K_a, L_b)$.

By a routine work, we can show that $(C_{AB}(K, L), \underline{C_{AB}(K, L)})$ is a simplicial complex over $A \times B$, which is the natural function space for simplicial complexes with variable base spaces constructed in section 2. If $A = B = \{*\}$, then $C_{AB}(K, L)$ is the natural function space in **Simp**.

(d) Quasi-topological spaces :

Let **QTop** be the collection of all quasi-topological spaces and quasi-continuous maps. P. I. Booth [3] studied quasi-topological spaces to obtain a convenient category of topological spaces for homotopy theory. In fact, he obtained fibrewise exponential laws in **QTop** using categorical arguments. However a pathological problem is raised. H. Herrlich and M. Rajagopalan [11] showed that unfortunately **QTop** fails to be a category, since its objects are proper classes rather than sets. We note that, the quasi-category **QTop** is large enough to contain all topological spaces of interest to algebraic topologists. Indeed the category $\mathbf{C_g}$ of compactly generated spaces allows an obvious embedding as a full bireflective, concerete subcategory of **QTop**. L.D. Nel [18] introduced a method to form a topological universe (= quasitopos) using a certain completion procedure applying to a preuniverse (= a "start-up" category) structured with suitably specified coverings. As an example, he suggested topological unverses which are very similar to **QTop** and to the cartesian closed category $\mathbf{C}_g$. The category $\mathbf{Comp}_\alpha$ of compact Hausdorff spaces with cardinality bounded by a given α form a small pre-universe when all finite coverings are selected. Let $\mathbf{QTop}_\alpha$ be the topological universe completion of $\mathbf{Comp}_\alpha$. It is easy to check that the quasi-category **QTop** is a kind of transfinite colimit of $\mathbf{QTop}_\alpha$. We also note that the category $\mathbf{C}_{\mathbf{g}_\alpha}$ of compactly generated spaces with cardinality bounded by α is a bireflective subcategory of $\mathbf{QTop}_\alpha$ and a bicoreflective subcategory of $\mathbf{C_g}$. Therefore we suggest the topological universe $\mathbf{QTop}_\alpha$ as a good replacement of **QTop**.

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