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TOPOLOGICAL PROPERTIES OF HIGH-DIMENSIONAL HANDLES

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Résumé. Dans cet article on donne une démonstration plus brève d'un rèsultat de [3] concernant le groupe des auto-équivalences d'homotopie, qui gardent l'orientation, de la somme connexe $X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$ de $p \geq 1$ copies de $\mathbb{S}^1 \times \mathbb{S}^n$, modulo les homotopes à l'application identique. Ce résultat se rapporte à un article précédent de Hosokawa et Kawauchi [7] sur les surfaces non nouées dans des espaces à quatre dimensions. En effet, on étend leur résultat principal (en généralité plus grande) pour les plongements $f: X \to \mathbb{R}^{n+3}$ de X dans l'espace euclidien $(n+3)$-dimensionnel. Par conséquent on classifie le type d'homotopie du complémentaire de $f(X)$ dans $\mathbb{R}^{n+3}$ en donnant des exemples de variétés qui ont le même type d'homotopie qu'un bouquet de sphères et qui ne peuvent être fibrées sur un cercle.

1. Introduction and results.

Through the paper we work in the piecewise–linear (PL) category in the sense of [17], and we shall omit the prefix PL. Therefore the terms *homeomorphism* and *homotopy equivalence* mean PL homeomorphism and PL homotopy equivalence, respectively.

In the following X always denotes the connected sum of p copies of $\mathbb{S}^1 \times \mathbb{S}^n$, i.e. we set $X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$, $p \geq 1$, $n \geq 2$.

In [3] we gave a proof of the high-dimensional version of a classical theorem of Montesinos on handle presentations of closed orientable PL 4-manifolds [15]. For this we studied the group of orientation -preserving auto-homeomorphisms resp. homotopy self-equivalences of $X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$, $p \geq 1$, modulo those pseudo-isotopic resp. homotopic to the identity. Recall that two homeomorphisms $f, g: X \to X$ are said to be *pseudo-isotopic* if there is a homeomorphism $F: X \times I \to X \times I$ ($I = [0,1]$) such that $F(x,0) = f(x)$ and

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$F(x,1) = g(x)$ for all $x \in X$ (see for example [2], [10], and [19]). Let $\mathcal{D}_0(X)$ be the group of pseudo-isotopy classes of orientation-preserving homeomorphisms of X, and $\mathcal{E}_0(X)$ the group of homotopy classes of orientation-preserving homotopy self-equivalences of X.

Essential for the proof is the following

Theorem 1. *Given* $X = \#_p\left(\mathbb{S}^1 \times \mathbb{S}^n\right)$, $n \geq 2$, $p \geq 1$, *there exists a short exact sequence*

$$1 \to \bigoplus_{p+1} \mathbf{Z}_2 \to \mathcal{D}_0(X) \cong \mathcal{E}_0(X) \to \operatorname{Out}(\Pi_1) \to 1,$$

where $\Pi_1 = \Pi_1(X) \simeq \underset{p}{*}\mathbf{Z}$ *is the free group with* p *generators, and* $\operatorname{Out}(\Pi_1)$ *is the outer automorphism group of* Π_1, *i.e. automorphisms modulo inner automorphisms.*

Observe that the group $\mathcal{D}_0(X)$ is not a direct sum of the other two terms of the sequence for $p > 1$. Indeed, diffeomorphisms of X, which permute the p summands $\mathbb{S}^1 \times \mathbb{S}^n$, also permute the p rotations along n-spheres.

In [3] we gave a geometric proof of Theorem 1, and also indicated an algebraic one. One purpose of the present paper is to give a shorter algebraic proof of Theorem 1 using the group $\mathcal{E}_0(X)$. This is based on algebraic lemmas which are interesting by themselves (see Lemmas 6 and 9). In fact, with these we can simplify the proof of the main result of Hosokawa and Kawauchi [7] on unknotted surfaces in four-spaces, extending it (in greater generality) for embeddings $f\colon X \to \mathbb{R}^{n+3}$ of X into the Euclidean $(n+3)$-space. We also classify the homotopy type of the complement of $f(X)$ in $\mathbb{R}^{n+3}$, giving examples of manifolds homotopy equivalent to a bouquet of spheres which cannot be fibered over a circle.

More precisely, we say that a locally tame subspace K of $\mathbb{R}^{n+3}$ is a *knotted* (orientable) *∂-handle of genus p* if K is homeomorphic to X. Two knotted ∂-handles K and K' are *equivalent* if there is an orientation-preserving auto-homeomorphism of $\mathbb{R}^{n+3}$ sending K onto K'. The equivalence class of knotted ∂-handle is called its *knot type*. We say that K is *unknotted* in $\mathbb{R}^{n+3}$ if there exists an $(n+2)$-dimensional (solid) handlebody $Y = \#_p(\mathbb{S}^1 \times D^{n+1})$, standardly embedded in $\mathbb{R}^{n+3}$, such that $\partial Y = K$. For $n = 1$, this coincides with the concept of unknotted (orientable) surface in the Euclidean 4-space, first introduced and studied by Hosokawa and Kawauchi in [7].

The following theorems extend some results of [7] to dimension n (for more details on definitions see Section 4).

Theorem 2. *Let K^{n+1} be a knotted ∂-handle of genus p in $\mathbb{R}^{n+3}$. Then the fundamental group of $\mathbb{R}^{n+3}\backslash K$ is isomorphic to $\mathbb{Z}$ if and only if K is stably unknotted in $\mathbb{R}^{n+3}$, i.e. an unknotted ∂-handle can be obtained from K by hyperboloidal transformations along trivial 1-handles.*

Theorem 3. *Suppose that K^{n+1} is a knotted ∂-handle of genus p in $\mathbb{R}^{n+3}$, $n \geq 2$, with $\Pi_1(\mathbb{R}^{n+3}\backslash K) \cong \mathbb{Z}$. Then the complement $\mathbb{R}^{n+3}\backslash K$ is (simple) homotopy equivalent to the wedge $\vee_p\mathbb{S}^{n+1} \vee \mathbb{S}^1 \vee \mathbb{S}^{n+2}$.*

We remark that manifolds homotopy equivalent to a wedge of spheres of various dimension were also treated in [6]. There the classification of thickenings of a wedge of spheres is reduced to the classification of concordance classes of embeddings of a number of solid tori in the boundary of a solid high-dimensional handlebody. For other related results concerning manifolds with free fundamental group we refer to [1] and [4]. Concepts and notations from homotopy theory are standard, and can be found for example in [5], [8], and [14]. For a reference on homological algebra see [16].

2. Homotopy self-equivalences of $X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$.

Throughout the section we shall assume $n \geq 3$. In this case we have

$$\Pi_{n+1}(\mathbb{S}^n) \cong \mathbb{Z}_2.$$

The arguments for $n = 2$ (in which case $\Pi_3(\mathbb{S}^2) \cong \mathbb{Z}$) are slightly different.

In this section we are going to prove the following

Proposition 4. *Let $\theta_0\colon \mathcal{E}_0(X) \to \mathrm{Out}(\Pi_1)$ be the canonical homomorphism. Then there is an exact sequence*

$$\oplus_p\mathbb{Z}_2 \longrightarrow \mathcal{E}_0(X) \xrightarrow{\theta_0} \mathrm{Out}(\Pi_1) \longrightarrow 1.$$

In the next section it will be proved that $\oplus_p\mathbb{Z}_2$ injects into $\mathcal{E}_0(X)$. The proof of Proposition 4 is based on obstruction theory (see for example [5] and [18]). We first need some algebraic lemmas which are interesting by themselves.

Lemma 5. *Let $f, g\colon X \to X$ be two degree one maps. If $f_* = g_*\colon \Pi_1 \to \Pi_1$, then*

$$f_* = g_*\colon \Pi_q \to \Pi_q$$

for all $q \leq n$.

Proof. We observe that $\Pi_i(X) = 0$ for $1 < i < n$, hence $f_* = g_*\colon \Pi_q \to \Pi_q$ for all $q < n$. By [12], p. 341, Poincaré duality and $\deg(f) = \deg(g) = 1$, we have the following commutative diagrams

$$\begin{tikzcd}
H_n(\tilde{X};\mathbf{Z}) \arrow[r, equal] \arrow[d, "\tilde{f}_*"'] & H_n(\tilde{X};\mathbf{Z}) \arrow[r, "\simeq", "PD"'] \arrow[d, "\tilde{g}_*"] & H^1_c(\tilde{X};\mathbf{Z}) \arrow[r, equal] \arrow[d, "\tilde{f}^*"'] & H^1_c(\tilde{X};\mathbf{Z}) \arrow[d, "\tilde{g}^*"] \\
H_n(\tilde{X};\mathbf{Z}) \arrow[r, equal] & H_n(\tilde{X};\mathbf{Z}) \arrow[r, "\simeq", "PD"'] & H^1_c(\tilde{X};\mathbf{Z}) \arrow[r, equal] & H^1_c(\tilde{X};\mathbf{Z})
\end{tikzcd}$$

$$\begin{tikzcd}
H^1_c(\tilde{X};\mathbf{Z}) \arrow[r, equal] \arrow[d, "\tilde{f}^*"'] & H^1_c(\tilde{X};\mathbf{Z}) \arrow[r, "\simeq"] \arrow[d, "\tilde{g}^*"] & H^1(\Pi_1;\mathbf{Z}[\Pi_1]) \arrow[r, equal] \arrow[d, "f^*_*"'] & H^1(\Pi_1;\mathbf{Z}[\Pi_1]) \arrow[d, "g^*_*"] \\
H^1_c(\tilde{X};\mathbf{Z}) \arrow[r, equal] & H^1_c(\tilde{X};\mathbf{Z}) \arrow[r, "\simeq"] & H^1(\Pi_1;\mathbf{Z}[\Pi_1]) \arrow[r, equal] & H^1(\Pi_1;\mathbf{Z}[\Pi_1])
\end{tikzcd}$$

where $\tilde{f}, \tilde{g}\colon \tilde{X} \to \tilde{X}$ are the liftings of f, g respectively to the universal covering space $\tilde{X}$ of X.

Since the hypothesis $f_* = g_*\colon \Pi_1 \to \Pi_1$ directly implies $f^*_* = g^*_*$, it follows that $\tilde{f}_* = \tilde{g}_*\colon H_n(\tilde{X};\mathbf{Z}) \to H_n(\tilde{X};\mathbf{Z})$. Then the Hurewicz isomorphism

$$H_n(\tilde{X};\mathbf{Z}) \simeq \Pi_n(\tilde{X}) \simeq \Pi_n(X)$$

implies that $f_* = g_*\colon \Pi_n \to \Pi_n$ as required. □

Lemma 6. *Let $\Lambda = \mathbf{Z}[\Pi_1]$ be the group ring of $\Pi_1(X)$, where $X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$. Let $g_1, g_2, \ldots, g_p \in \Pi_1(X)$ be canonical generators and let*

$$\sigma = (g_1 - 1, g_2 - 1, \ldots, g_p - 1) \in \oplus_p\Lambda.$$

Then the Λ-module $\Pi_n(X)$ is Λ-isomorphic to $(\oplus_p\Lambda)/\sigma\Lambda$.

Proof. Observe that

$$\Pi_n(X) \cong \Pi_n(\tilde{X}) \cong H_n(\tilde{X};\mathbf{Z}) \cong H_n(X;\Lambda) \cong H^1(X;\Lambda)$$

by the Hurewicz isomorphism theorem and by Poincaré duality.

Let $X^{(q)}$ be the q-skeleton of the standard cellular decomposition

$$e^0 \cup pe^1 \cup pe^n \cup e^{n+1}$$

of X. Then the chain complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(\tilde{X}^{(1)}, \tilde{X}^{(0)}) & \longrightarrow & H_0(\tilde{X}^{(0)}) & \longrightarrow & H_0(\tilde{X}^{(1)}) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & I(\Lambda) & \xrightarrow{i} & \Lambda & \xrightarrow{\epsilon} & \mathbf{Z} & \longrightarrow & 0,
\end{array}$$

gives

$$H^1(X;\Lambda) \cong \frac{\mathrm{Hom}_\Lambda(I(\Lambda),\Lambda)}{\mathrm{Im}\, i^\#}.$$

Here ϵ is the augmentation map, $I(\Lambda) = \mathrm{Ker}(\epsilon)$ is the augmentation ideal, and $i^\#\colon \mathrm{Hom}_\Lambda(\Lambda,\Lambda) \to \mathrm{Hom}_\Lambda(I(\Lambda),\Lambda)$ is the homomorphism induced by $i\colon I(\Lambda) \to \Lambda$.

The augmentation ideal $I(\Lambda) \cong \oplus_p\Lambda$ has generators $g_1-1, g_2-1, \ldots, g_p-1$. Given $\varphi \in \mathrm{Hom}_\Lambda(\Lambda,\Lambda)$, then $i^\#(\varphi)$ corresponds to

$$\sigma\varphi(1) \in \oplus_p\Lambda \cong \mathrm{Hom}_\Lambda(I(\Lambda),\Lambda),$$

proving the lemma. □

Lemma 7. *The canonical map*

$$H^n(X;\Pi_n(X)) \to \mathrm{Hom}_\Lambda(H_n(\widetilde{X}),\Pi_n(X))$$

is an isomorphism.

Proof. This is a consequence of the universal coefficient spectral sequence

$$\mathrm{Ext}^p_\Lambda(H_q(\widetilde{X}),\Pi_n(X)) \Rightarrow H^{p+q}(X,\Pi_n(X)).$$

The only contributions for $H^n(X;\Pi_n(X))$ are

$$\mathrm{Ext}^0_\Lambda(H_n(\widetilde{X}),\Pi_n(X)) \cong \mathrm{Hom}_\Lambda(H_n(\widetilde{X}),\Pi_n(X))$$

and

$$\mathrm{Ext}^n_\Lambda(H_0(\widetilde{X}),\Pi_n(X)) \cong \mathrm{Ext}^n_\Lambda(\mathbf{Z},\Pi_n(X)) \cong H^n(*_p\mathbf{Z};\Pi_n(X)) \cong 0.$$

For the latter isomorphism observe that the wedge $\vee_p \mathbb{S}^1$ is the classifying space of the group $*_p\mathbb{Z}$. $\square$

Remark. Using $H^n(X;\Pi_n(X)) \cong H_1(X;\Pi_n(X))$ with

$$\Pi_n(X) \cong (\oplus_p\Lambda)/\sigma\Lambda,$$

we obtain the following exact sequence:

$$H_1(X;\Lambda) \to H_1(X;\oplus_p\Lambda) \to H_1(X;(\oplus_p\Lambda)/\sigma\Lambda) \to \mathbb{Z}\otimes_\Lambda \Lambda \to \mathbb{Z}\otimes_\Lambda (\oplus_p\Lambda).$$

Now $H_1(X;\Lambda) \cong H_1(X;\oplus_p\Lambda) \cong 0$ and $\mathbb{Z}\otimes_\Lambda \Lambda \to \mathbb{Z}\otimes_\Lambda (\oplus_p\Lambda)$ is the null homomorphism because σ goes to zero. Hence we obtain more precisely

$$H^n(X;\Pi_n(X)) \cong \mathbb{Z}.$$

However we do not need this specific result in our context.

Corollary 8. *Let $f,g\colon X \to X$ be orientation-preserving homotopy equivalences with $f_* = g_*\colon \Pi_1(X) \to \Pi_1(X)$. Then the only obstruction for the existence of a homotopy between f and g lies in $H^{n+1}(X;\Pi_{n+1}(X))$.*

Proof. Obviously there is no obstruction on the 1-skeleton. By standard obstruction theory f and g are then homotopic over $X^{(n-1)}$. The n-dimensional obstruction lies in

$$H^n(X;\Pi_n(X)) \cong \mathrm{Hom}_\Lambda(H_n(\widetilde{X}),\Pi_n(X))$$

and it is equal to

$$H_n(\widetilde{X}) \xrightarrow{\cong} \Pi_n(X) \xrightarrow{f_*-g_*} \Pi_n(X),$$

hence vanishes by Lemma 5. $\square$

The proof of Proposition 4 will follow from Corollary 8 and from the isomorphism given below.

Lemma 9.

$$H^{n+1}(X;\Pi_{n+1}(X)) \cong \oplus_p \mathbb{Z}_2.$$

Proof. Let $\Sigma_1^n, \Sigma_2^n, \ldots, \Sigma_{p-1}^n \subset X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$ be n-spheres which are determined by the connected sums. Let X^* denote the space obtained from X by adjoining $(n+1)$-discs along Σ_i^n for $i = 1, 2, \ldots, p-1$. In particular X^* is homotopy equivalent to $\vee_p(\mathbb{S}^1 \times \mathbb{S}^n)$. Moreover let us denote $X_0 = X \backslash \{(n+1)\text{- open disc}\}$. It is not difficult to see that the following sequence of homotopy equivalences holds:

$$X_0 \simeq \vee_p((\mathbb{S}^1 \times \mathbb{S}^n) \backslash \overset{\circ}{D}{}^{n+1}) \simeq \vee_p(\mathbb{S}^1 \vee \mathbb{S}^n) \simeq (\vee_p \mathbb{S}^1) \vee (\vee_p \mathbb{S}^n).$$

Hence we have

$$\Pi_n(X_0) \cong \Lambda \otimes (\oplus_p \mathbb{Z}) \cong \oplus_p \Lambda$$

and

$$\Pi_{n+1}(X_0) \cong \Lambda \otimes (\oplus_p \mathbb{Z}_2) \cong \oplus_p(\Lambda/2\Lambda).$$

Let us consider the homotopy sequence of the pair (X, X_0):

$$\cdots \to \Pi_{n+1}(X_0) \to \Pi_{n+1}(X) \to \Pi_{n+1}(X, X_0) \to \Pi_n(X_0) \to \Pi_n(X) \to 0.$$

Since $\Pi_{n+1}(X, X_0) \cong \Lambda$, $\Pi_n(X_0) \cong \oplus_p \Lambda$ and $\Pi_n(X) \cong (\oplus_p \Lambda)/\sigma\Lambda$ (by Lemma 6), it follows that the short sequence

$$0 \to \Pi_{n+1}(X, X_0) \to \Pi_n(X_0) \to \Pi_n(X) \to 0$$

is exact. This implies that the homomorphism $\Pi_{n+1}(X_0) \to \Pi_{n+1}(X)$ is surjective.

Now we have

$$H^{n+1}(X;\Pi_{n+1}(X)) \cong H_0(X;\Pi_{n+1}(X)) \cong \mathbb{Z} \otimes_\Lambda \Pi_{n+1}(X),$$

and therefore the homomorphism

$$\mathbb{Z} \otimes_\Lambda \Pi_{n+1}(X_0) \to \mathbb{Z} \otimes_\Lambda \Pi_{n+1}(X) \cong H^{n+1}(X;\Pi_{n+1}(X))$$

is also surjective, i.e.

$$\mathbb{Z} \otimes_\Lambda \Pi_{n+1}(X) \cong H^{n+1}(X;\Pi_{n+1}(X))$$

is a direct sum of copies of $\mathbb{Z}_2$.

We consider now the exact homotopy sequence of the pair (X^*, X). Since $\Pi_{n+1}(X^*, X) \cong \oplus_{p-1}\Lambda$ and $\Pi_{n+1}(X^*) \cong \Pi_{n+1}(\widetilde{X}^*)$ is a direct sum of copies of $\mathbf{Z}_2$, we obtain

$$\cdots \to \Pi_{n+1}(X) \to \Pi_{n+1}(X^*) \to 0,$$

hence $\mathbf{Z}\otimes_\Lambda \Pi_{n+1}(X) \to \mathbf{Z}\otimes_\Lambda \Pi_{n+1}(X^*)$ is surjective. So we have the following composition of surjective homomorphisms

$$\mathbf{Z} \otimes_\Lambda \Pi_{n+1}(X_0) \to \mathbf{Z} \otimes_\Lambda \Pi_{n+1}(X) \to \mathbf{Z} \otimes_\Lambda \Pi_{n+1}(X^*).$$

Since $\Pi_{n+1}(\tilde{X}^*)$ has at least p copies of $\mathbf{Z}_2$ we obtain

$$\mathbf{Z} \otimes_\Lambda \Pi_{n+1}(X_0) \cong \oplus_p \mathbf{Z}_2 \cong \mathbf{Z} \otimes_\Lambda \Pi_{n+1}(X^*),$$

hence $\mathbf{Z} \otimes_\Lambda \Pi_{n+1}(X) \cong \oplus_p \mathbf{Z}_2$. This completes the proof. $\square$

In order to finish the proof of Proposition 4 we have to prove the surjectivity of $\mathcal{E}_0(X) \to \mathrm{Out}(\Pi_1(X))$. For this we refer to [12]. Any $\xi \in \mathrm{Out}(\Pi_1(X))$ can be realized by a homeomorphism $f\colon X \to X$. If f has degree 1, then $[f] \in \mathcal{E}_0(X)$. If f has degree -1, we compose f with the homeomorphism

$$r' = \#_p(\mathrm{Id}_{\mathbf{S}^1} \times r)\colon \#_p(\mathbf{S}^1 \times \mathbf{S}^n) \to \#_p(\mathbf{S}^1 \times \mathbf{S}^n),$$

where r is the reflection on the first coordinate.

Remark: Theorem 1 would follow from an equivariant version of Corollary 16.4 of the textbook of Hu (see [8]). However we shall explicitely construct all the elements of $\oplus_p \mathbf{Z}_2$ in the next section.

3. The injection $\oplus_p \mathbf{Z}_2 \to \mathcal{E}_0(X)$.

Now we are going to realize the obstructions involved in Theorem 1.

Let $\{g_i\}$, $i = 1, \ldots, p$, be a free basis of $\Pi_1(X) \cong *_p\mathbf{Z}$, where $X = \#_p(\mathbf{S}^1 \times \mathbf{S}^n)$, $p \geq 1$, $n \geq 3$. Obviously g_i is the homotopy class of the i-th $\mathbf{S}^1$-factor $\mathbf{S}^1_i$ of X. Let $\mathrm{Aut}(\Pi_1)$ be the group of automorphisms of the fundamental group $\Pi_1 = \Pi_1(X)$ of X. As proved in [11] and [12], the group $\mathrm{Aut}(\Pi_1)$ is generated by sliding 1-handles, twisting 1-handles and permuting 1-handles. More precisely, for $i = 2, \ldots, p$ $(p > 1)$ define $\phi_i \in \mathrm{Aut}(\Pi_1)$ by setting $\phi_i(g_1) = g_i$, $\phi_i(g_i) = g_1$ and $\phi_i(g_j) = g_j$ for each $j \neq i$, $j \neq 1$. Permuting the 1-handles g_i and g_j corresponds to the automorphism $\phi_i \circ \phi_j \circ \phi_i^{-1}$. It follows that $\phi_i^2 = 1$ and by [11], [12] there exist homeomorphisms $f_i \colon X \to X$ (*permuting 1-handles*) such that $f_{i*} = \phi_i$. Then define $\sigma \in \mathrm{Aut}(\Pi_1)$ by setting $\sigma(g_1) = g_1^{-1}$ and $\sigma(g_i) = g_i$ for $i \neq 1$. Twisting the 1-handle g_i corresponds to the automorphism $\phi_i \circ \sigma \circ \phi_i^{-1}$. Obviously $\sigma^2 = 1$. Furthermore there exist homeomorphisms of X (*twisting 1-handles*) which realize σ and $\phi_i \circ \sigma \circ \phi_i^{-1}$ for $i \geq 2$. Finally we define $\psi \in \mathrm{Aut}(\Pi_1)$, $p > 1$, by setting $\psi(g_1) = g_1 g_2$ and $\psi(g_i) = g_i$ for $i \geq 2$ (*sliding 1-handles*).

Let $\Sigma_i = \mathbb{S}_i^n$ be the i-th $\mathbb{S}^n$-factor of $X = \#_p \left(\mathbb{S}^1 \times \mathbb{S}^n\right)$, $p \geq 1$, $n \geq 3$. Following [11], we show that rotations of X parallel to Σ_i generate the obstruction subgroup

$$\mathrm{Ker}\, \theta_0 \cong \oplus_p \Pi_1 \left(\mathrm{SO}(n+1)\right) \cong \oplus_p \mathbb{Z}_2 .$$

Let

$$\alpha \colon (\mathbb{S}^1, 1) \to (\mathrm{SO}(n+1), \mathrm{id})$$

be a loop representing a homotopy class of $\Pi_1 \left(\mathrm{SO}(n+1)\right) \cong \mathbb{Z}_2$ ($n \geq 3$).

Then α induces a homeomorphism

$$h_\alpha \colon \mathbb{S}^n \times I \to \mathbb{S}^n \times I$$

defined by

$$h_\alpha(x,t) = (\alpha(t)x, t)$$

for all $x \in \mathbb{S}^n$ and $t \in I = [0,1]$. Obviously h_α is the identity on the boundary $\partial(\mathbb{S}^n \times I) = \mathbb{S}^n \times 0 \cup \mathbb{S}^n \times 1$.

Now let M^{n+1} be a closed oriented $(n+1)$-manifold and let Σ^n be an oriented n-sphere embedded in M. Suppose $\varphi \colon \mathbb{S}^n \times I \to M$ is an orientation-preserving embedding such that $\varphi(\mathbb{S}^n \times 0) = \Sigma$. Because $h_\alpha =$ identity on $\partial(\mathbb{S}^n \times I)$, one obtains a homeomorphism

$$h_\alpha^\Sigma \colon M \to M$$

defined by

$$h_\alpha^\Sigma(x) = \begin{cases} x & if \quad x \in M \setminus \mathrm{Im}\, \varphi \\ \varphi \circ h \circ \varphi^{-1}(x) & if \quad x \in \mathrm{Im}\, \varphi. \end{cases}$$

We call the homeomorphism h_α^Σ the *α-rotation of M parallel to Σ* (briefly, a *rotation*). Obviously the pseudo-isotopy class of h_α^Σ depends only on the homotopy (resp. isotopy) class of α (resp. Σ).

If $M^{n+1} = X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$, $p \geq 1$, $n \geq 3$, let $\Sigma_i = \mathbb{S}^n_i$ be the i-th $\mathbb{S}^n$-factor of X. We set

$$h_{i,\alpha} = h_\alpha^{\Sigma_i}$$

for $i = 1, \dots, p$ and $[\alpha] \in \Pi_1(\mathrm{SO}(n+1)) \cong \mathbb{Z}_2$. One can choose $h_{i,\alpha}$ to be the identity on the union $\cup_{i=1}^p \Sigma_i$. Because $(h_{i,\alpha})_* =$ identity on $\Pi_q(X)$ for all $q \leq n$, we have that $h_{i,\alpha} \in \operatorname{Ker}\theta_0$, $i = 1, \dots, p$ (Here θ_0 is the canonical homomorphism $\mathcal{E}_0(X) \to \mathrm{Out}(\Pi_1)$ considered in Section 2). Moreover $h_{i,\alpha} \circ h_{j,\beta} = h_{j,\beta} \circ h_{i,\alpha}$ $(i \neq j)$, each $h_{i,\alpha}$ commutes with the generators of $\mathrm{Aut}(\Pi_1)$ and $h_{i,\alpha}$ is pseudo-isotopic to the identity if and only if $[\alpha] = 0$. Thus we have shown that the rotations $h_i = h_{i,\alpha}$ of X parallel to the n-spheres Σ_i generate $\operatorname{Ker}\theta_0$ if $[\alpha]$ is the generator of $\Pi_1(\mathrm{SO}(n+1)) \cong \mathbb{Z}_2$. In particular, this shows that the term $\oplus_p \mathbb{Z}_2$ injects into $\mathcal{D}_0(X) \cong \mathcal{E}_0(X)$.

More precisely, we can interpret our results in the following way (which is related to Lemma 5.4 of [11]):

Proposition 10. *Let $X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$, $p \geq 1$, $n \geq 3$, and let $f\colon X \to X$ be an orientation-preserving homeomorphism such that $\theta_0(f) = 1$, i.e. $f_* =$ identity on $\Pi_1(X)$. Then there exist loops (obstructions)*

$$\alpha_i\colon (\mathbb{S}^1, 1) \to (\mathrm{SO}(n+1), \mathrm{Id})$$

$(i = 1, 2, \dots, p)$ such that f is pseudo-isotopic to the product

$$h_{1,\alpha_1} \circ h_{2,\alpha_2} \circ \cdots \circ h_{p,\alpha_p}.$$

Moreover, the pseudo-isotopy can be chosen keeping the union $\cup_{i=1}^p \Sigma_i$ fixed.

In other words, the rotations $h_i = h_{i,\alpha}$ $(i = 1, \dots, p)$ constitute a free basis of

$$\operatorname{Ker}\theta_0 \cong \oplus_p \Pi_1(\mathrm{SO}(n+1)) \cong \oplus_p \mathbb{Z}_2,$$

where $[\alpha]$ is the generator of $\Pi_1(\mathrm{SO}(n+1)) \cong \mathbb{Z}_2$.

4. Unknotted handles in Euclidean spaces.

In this section we are going to prove Theorems 2 and 3.

Let K^{n+1} be a knotted ∂-handle of genus p in $\mathbb{R}^{n+3}$, i.e. K is a locally tame subspace homeomorphic to $X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$. An oriented $(n+2)$-cell B in $\mathbb{R}^{n+3}$ is said to *span* K *as a 1-handle* if $B \cap K = (\partial B) \cap K$ and this intersection is the union of disjoint two $(n+1)$-cells and $(K \cup \partial B) \setminus \operatorname{int}(\partial B \cap K)$ has an orientation compatible with both the orientations of $K \setminus \operatorname{int}(\partial B \cap K)$

(induced from K) and $\partial B \backslash \operatorname{int}(\partial B \cap K)$ (induced from B). If $B_1, \ldots, B_q$ are mutually disjoint oriented $(n+2)$-cells in $\mathbb{R}^{n+3}$ which span K as 1-handles, then the knotted ∂-handle of genus $p+q$

$$h^1(K; B_1, \ldots, B_q) = (K \cup \bigcup_{i=1}^{q} \partial B_i) \backslash \operatorname{int}(K \cap \bigcup_{i=1}^{q} \partial B_i)$$

with orientation induced from $K \backslash \operatorname{int}(K \cap \bigcup_{i=1}^{q} \partial B_i)$ is said to be *obtained from* K *by hyperboloidal transformations along* 1-*handles* (see [7] for $n = 1$). A 1-handle B on K in $\mathbb{R}^{n+3}$ is *trivial* if there exists an $(n+3)$-cell N in $\mathbb{R}^{n+3}$ containing B such that $N \cap K = \partial N \cap K$ and this intersection is an $(n+1)$-cell. Note that the attaching two $(n+1)$-cells of B to K are contained in the $(n+1)$-cell $\partial N \cap K$ since we have

$$(\partial B) \cap K = B \cap K \subset N \cap K = (\partial N) \cap K.$$

Hyperboloidal transformations along trivial 1-handles do not alter the fundamental groups of the complements in $\mathbb{R}^{n+3}$. In particular, if K is unknotted, then $\Pi_1(\mathbb{R}^{n+3} \backslash K) \cong \Pi_1(\mathbb{R}^{n+3} \backslash \mathbb{S}^{n+1}) \cong \mathbb{Z}$, since $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+3}$ is the standardly embedded $(n+1)$-sphere in $\mathbb{R}^{n+3}$.

Proof of Theorem 2.

Sufficient condition. If $h^1 = h^1(K; B_1, \ldots, B_q)$ is unknotted for some q, then we have

$$\mathbb{Z} \cong \Pi_1(\mathbb{R}^{n+3} \backslash h^1) \cong \Pi_1(\mathbb{R}^{n+3} \backslash K)$$

because B_i are trivial 1-handles.

Necessary condition. Clearly, there are 1-handles $B_1, \ldots, B_q$ on K such that $h^1(K; B_1, \ldots, B_q)$ is unknotted in $\mathbb{R}^{n+3}$. The assertion now follows from the lemma below.

Lemma 11. *If* $\Pi_1(\mathbb{R}^{n+3} \backslash K) \cong \mathbb{Z}$, *then an arbitrary* 1-*handle* B *on* K *is trivial.*

Proof. Let α be a simple proper arc in B such that the union $K \cup \alpha$ is a spine of the union $K \cup B$. By sliding α along K and by deforming α itself we can assume that α is attached to $K \backslash B$ as follows. There is a generating 1-sphere C of K which intersects α into two endpoints x^+ and x^-. Let $\delta^\pm$ be a regular neighborhood of $x^\pm$ in α and define $\alpha' = \operatorname{cl}(\alpha \backslash (\delta^+ \cup \delta^-))$. We now join the endpoints of α' with a simple arc γ such that the loop $\gamma \cup \alpha' \subset \mathbb{R}^{n+3} \backslash K$. Obviously, $\gamma \cup \alpha'$ is in general not homologous to zero in $\mathbb{R}^{n+3}\backslash K$. However, by twisting γ along C we can assume that the simple loop $\gamma \cup \alpha'$ is homologous to zero in $\mathbb{R}^{n+3}\backslash K$ (use $H_1(\mathbb{R}^{n+3}\backslash K) \cong \mathbb{Z}$ and the fact that the generator of $H_1(\mathbb{R}^{n+3}\backslash K)$ is given by a loop transversal to C). Since $H_1(\mathbb{R}^{n+3}\backslash K) \cong \Pi_1(\mathbb{R}^{n+3}\backslash K)$, $\gamma \cup \alpha'$ is null-homotopic in $\mathbb{R}^{n+3}\backslash K$. Hence it bounds a singular 2-disk. By general position and the embedded disk theorem, $\gamma \cup \alpha'$ bounds an embedded 2-disk in $\mathbb{R}^{n+3}\backslash K$ as $n+3 \geq 5$ (the case $n = 1$ was treated in [7] so we can assume $n > 1$). Then $K \cup \alpha$ is ambient isotopic to K, so we can find an $(n+3)$-cell N containing B such that $N \cap K = (\partial N) \cap K$ and this intersection is an $(n+1)$-cell, i.e. B is a trivial 1-handle on K. □

Proof of Theorem 3.

Suppose that $\Pi_1(\mathbb{R}^{n+3}\backslash K) \cong \mathbb{Z}$, $n > 1$. It is convenient to consider K in the $(n+3)$-sphere $\mathbb{R}^{n+3} \cup \{\infty\} = \mathbb{S}^{n+3}$. We shall identify $\Pi_1(\mathbb{S}^{n+3}\backslash K) = \mathbb{Z}$. Then for every i $(1 \leq i \leq n)$ we have

$$\begin{aligned} H_{i+1}(\widetilde{\mathbb{S}^{n+3}\backslash K}; \mathbb{Z}) &\cong H_{i+1}(\mathbb{S}^{n+3}\backslash K; \mathbb{Z}[\mathbb{Z}]) \\ &\cong H_{i+2}(\mathbb{S}^{n+3}, \mathbb{S}^{n+3}\backslash K; \mathbb{Z}[\mathbb{Z}]) \\ &\cong H^{n-i+1}(K; \mathbb{Z}[\mathbb{Z}]) \\ &\cong H_i(X; \mathbb{Z}[\mathbb{Z}]), \end{aligned}$$

where $X = \#_p(\mathbb{S}^1 \times \mathbb{S}^n)$ as usual, and $\mathbb{Z}[\mathbb{Z}]$ is the integral group ring of $\mathbb{Z}$.

Now $H_i(X; \mathbb{Z}[\mathbb{Z}])$ injects into $H_i(X; \Lambda)$, where $\Lambda = \mathbb{Z}[*_p\mathbb{Z}]$.

Thus we obtain

$$H_i(X; \mathbb{Z}[\mathbb{Z}]) \cong 0$$

for $1 \leq i < n$, and (use Lemma 6)

$$H_n(X; \mathbb{Z}[\mathbb{Z}]) \cong (\oplus_p \mathbb{Z}[\mathbb{Z}])/\sigma\mathbb{Z}[\mathbb{Z}] \cong \oplus_p \mathbb{Z}[\mathbb{Z}],$$

since σ is zero on $\mathbb{Z}[\mathbb{Z}]$.

Hence $H_i(\widetilde{\mathbb{S}^{n+3}\backslash K}; \mathbb{Z}) \cong 0$ for $1 \leq i \leq n$, and $H_{n+1}(\widetilde{\mathbb{S}^{n+3}\backslash K}; \mathbb{Z})$ is $\mathbb{Z}[\mathbb{Z}]$-free of rank p. Next, we shall show that $H_{n+2}(\widetilde{\mathbb{S}^{n+3}\backslash K}; \mathbb{Z}) \cong 0$.

Let M be the $(n+3)$-manifold obtained from $\mathbb{S}^{n+3}$ by removing the interior of a regular neighborhood of K in $\mathbb{S}^{n+3}$. The exact homology sequence

$$H_{n+3}(M; \mathbb{Q}) \cong 0 \to H_{n+3}(\mathbb{S}^{n+3}; \mathbb{Q}) \to H_{n+3}(\mathbb{S}^{n+3}, M; \mathbb{Q}) \to H_{n+2}(M; \mathbb{Q}) \to 0$$

splits, hence

$$\begin{aligned}\mathbb{Q} \oplus H_{n+2}(M;\mathbb{Q}) &\cong H_{n+3}(\mathbb{S}^{n+3}, M;\mathbb{Q}) \\ &\cong H_{n+3}(\mathbb{S}^{n+3}, \mathbb{S}^{n+3}\backslash K;\mathbb{Q}) \cong H^0(K;\mathbb{Q}) \cong \mathbb{Q},\end{aligned}$$

i.e. $H_{n+2}(M;\mathbb{Q}) \cong 0$. This implies that $H_{n+2}(\widetilde{M};\mathbb{Q})$ is finitely generated over $\mathbb{Q}$. Using $H_{n+3}(\widetilde{M};\mathbb{Z}) \cong 0$, from the partial Poincaré duality theorem for infinite cyclic coverings (see [9]) we obtain $H^{n+2}(\widetilde{M};\mathbb{Q}) \cong H_0(\widetilde{M}, \partial\widetilde{M};\mathbb{Q}) \cong 0$ as $\partial\widetilde{M}$ is connected. In fact, the homomorphism $H_1(\partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is onto since we have

$$\begin{aligned}H_1(M, \partial M;\mathbb{Z}) &\cong H^{n+2}(M;\mathbb{Z}) \cong FH_{n+2}(M;\mathbb{Z}) \oplus TH_{n+1}(M;\mathbb{Z}) \\ &\cong H_{n+2}(M;\mathbb{Q}) \oplus TH_{n+1}(M;\mathbb{Z}) \cong TH_{n+1}(M;\mathbb{Z}).\end{aligned}$$

Now $H_{n+1}(M;\mathbb{Z})$ is torsion free as follows from the Mayer-Vietoris exact sequence

$$H_{n+2}(\mathbb{S}^{n+3}) \cong 0 \to H_{n+1}(\partial M) \to H_{n+1}(M) \oplus H_{n+1}(K) \to H_{n+1}(\mathbb{S}^{n+3}) \cong 0,$$

where

$$H_{n+1}(K) \cong H_{n+1}(\#_p(\mathbb{S}^1 \times \mathbb{S}^n)) \cong \mathbb{Z}$$

and

$$H_{n+1}(\partial M) \cong H^1(\partial M) \cong FH_1(\partial M)$$

is $\mathbb{Z}$-free. However $H_{n+2}(\widetilde{M};\mathbb{Z}) \cong H_{n+2}(M;\mathbb{Z}[\mathbb{Z}])$ is a torsion free abelian group.

Therefore

$$H_{n+2}(\widetilde{\mathbb{S}^{n+3}\backslash K};\mathbb{Z}) \cong H_{n+2}(\widetilde{M};\mathbb{Z}) \cong FH_{n+2}(\widetilde{M};\mathbb{Z}) \cong H_{n+2}(\widetilde{M};\mathbb{Q}) \cong 0.$$

Summarizing we have obtained

$$H_i(\widetilde{\mathbb{S}^{n+3}\backslash K};\mathbb{Z}) \cong \begin{cases} 0 & 1 \le i \le n \\ \oplus_p \mathbb{Z}[\mathbb{Z}] & i = n+1 \\ 0 & i \ge n+2. \end{cases}$$

Let $f_1, \ldots, f_p \colon (\mathbb{S}^{n+1}, *) \to (\mathbb{S}^{n+3}\backslash K, x_0)$ be maps representing a $\mathbb{Z}[\mathbb{Z}]$-basis for

$$\Pi_{n+1}(\mathbb{S}^{n+3}\backslash K, x_0) \cong H_{n+1}(\widetilde{\mathbb{S}^{n+3}\backslash K};\mathbb{Z}) \cong \oplus_p \mathbb{Z}[\mathbb{Z}]$$

and let

$$f\colon (\mathbb{S}^1, *) \to (\mathbb{S}^{n+3}\backslash K, x_0)$$

be a map representing a generator of $\Pi_1(\mathbb{S}^{n+3}\backslash K, x_0) \cong \mathbb{Z}$.

The wedge of maps

$$\vee_p f_i \vee f\colon (\vee_p \mathbb{S}^{n+1} \vee \mathbb{S}^1, *) \to (\mathbb{S}^{n+3}\backslash K, x_0)$$

clearly gives a homotopy equivalence (and hence a simple homotopy equivalence since the Whitehead group of $\mathbb{Z}$ is trivial).

Therefore $\mathbb{R}^{n+3}\backslash K = (\mathbb{S}^{n+3}\backslash K)\backslash\{\infty\}$ is homotopy equivalent to the wedge $\vee_p \mathbb{S}^{n+1} \vee \mathbb{S}^1 \vee \mathbb{S}^{n+2}$ as requested. □

Using Theorem 3, and following in part [7], we also prove the non-fibered property of the exterior of a knotted ∂-handle in $\mathbb{S}^{n+3}$. More precisely, we have

Proposition 12. *For any knotted ∂-handle K^{n+1} of genus p ($p \geq 1$) in $\mathbb{S}^{n+3}$, the complement $\mathbb{S}^{n+3}\backslash K$ cannot be fibered over a circle.*

Proof. Let M be the $(n+3)$-manifold obtained from $\mathbb{S}^{n+3}$ by removing the interior of a regular neighbourhood of K in $\mathbb{S}^{n+3}$. Suppose that $n > 1$ (for $n = 1$ see [7]). If $\mathbb{S}^{n+3}\backslash K$ and hence M is fibered over a circle, then the infinite cyclic connected covering $\widetilde{M}$ of M can be written as the topological product of a compact connected $(n+2)$-manifold N and the real line $\mathbb{R}$. In particular, we have that

$$H_*(\widetilde{M}; \mathbb{Q}) \cong H_*(N \times \mathbb{R}; \mathbb{Q})$$

is finitely generated over $\mathbb{Q}$. However, we are going to show that $H_{n+1}(\widetilde{M}; \mathbb{Q})$ is of rank p as $\mathbb{Q} < t >$-module, where $\mathbb{Q} < t >$ is the rational group ring of the covering translation group $< t >$ of $\widetilde{M}$. Note that $\mathbb{Q} < t >$ is a principal ideal domain. Thus $H_{n+1}(\widetilde{M}; \mathbb{Q})$ would be infinitely generated over $\mathbb{Q}$, giving a contradiction. Therefore, for $p \geq 1$, M and hence $\mathbb{S}^{n+3}\backslash K$ cannot be fibered over a circle.

First we observe that

$$\begin{aligned} H_{i+1}(M; \mathbb{Q}) &\cong H_{i+1}(\mathbb{S}^{n+3}\backslash K; \mathbb{Q}) \\ &\cong H_{i+2}(\mathbb{S}^{n+3}, \mathbb{S}^{n+3}\backslash K; \mathbb{Q}) \\ &\cong H^{n-i+1}(K; \mathbb{Q}) \\ &\cong H_i(X; \mathbb{Q}) \cong \begin{cases} 0 & 1 < i < n \\ \mathbb{Q} & i = 0 \\ \oplus_p \mathbb{Q} & i = 1 \vee i = n \end{cases} \end{aligned}$$

In order to show that

$$\mathrm{rk}_{\mathbb{Q}<t>} H_{n+1}(\widetilde{M};\mathbb{Q}) = p,$$

let us consider the following part of the Wang exact sequence

$$H_{n+1}(\widetilde{M};\mathbb{Q}) \xrightarrow{t-1} H_{n+1}(\widetilde{M};\mathbb{Q}) \xrightarrow{\pi_*} H_{n+1}(M;\mathbb{Q}) \cong \oplus_p \mathbb{Q},$$

where $\pi\colon \widetilde{M} \to M$ is the covering projection. Since $H_1(M;\mathbb{Q}) \cong \mathbb{Q}$, it follows that

$$t-1\colon H_1(\widetilde{M};\mathbb{Q}) \xrightarrow[\cong]{} H_1(\widetilde{M};\mathbb{Q}),$$

and hence

$$\pi_*\colon H_{n+1}(\widetilde{M};\mathbb{Q}) \longrightarrow H_{n+1}(M;\mathbb{Q}) \cong \oplus_p \mathbb{Q}$$

is surjective. We set

$$H_{n+1}(\widetilde{M};\mathbb{Q}) \cong \oplus_r \mathbb{Q} < t > \oplus T,$$

where T denotes the $\mathbb{Q} < t >$-torsion part of $H_{n+1}(\widetilde{M};\mathbb{Q})$.

Since $H_1(M,\partial M;\mathbb{Q}) \cong 0$ (because $H_1(M,\partial M;\mathbb{Z}) \cong TH_{n+1}(M;\mathbb{Z})$ as shown in the proof of Theorem 3), it follows that $H_1(\widetilde{M},\partial\widetilde{M};\mathbb{Q})$ is a finitely generated $\mathbb{Q} < t >$-torsion module, and that

$$t-1\colon H_1(\widetilde{M},\partial\widetilde{M};\mathbb{Q}) \xrightarrow[\cong]{} H_1(\widetilde{M},\partial\widetilde{M};\mathbb{Q}).$$

Let us consider a decomposition

$$\frac{\mathbb{Q}<t>}{(f_1(t))} \oplus \cdots \oplus \frac{\mathbb{Q}<t>}{(f_s(t))}$$

of $H_1(\widetilde{M},\partial\widetilde{M};\mathbb{Q})$. According to the partial Poincaré duality theorem for infinite cyclic coverings (see [9], Theorem 2.3 (II)), T is $\mathbb{Q} < t >$-isomorphic to

$$\frac{\mathbb{Q}<t>}{(f_1(t^{-1}))} \oplus \cdots \oplus \frac{\mathbb{Q}<t>}{(f_s(t^{-1}))},$$

and hence $t-1\colon T \to T$ is a $\mathbb{Q}<t>$-isomorphism. Therefore, we have the exact sequence

$$\begin{array}{c} H_{n+1}(\widetilde{M};\mathbb{Q})/T \cong \oplus_r \mathbb{Q}<t> \\ \Big\downarrow{\scriptstyle t-1} \\ H_{n+1}(\widetilde{M};\mathbb{Q})/T \cong \oplus_r \mathbb{Q}<t> \\ \Big\downarrow{\scriptstyle \pi_*} \\ H_{n+1}(M;\mathbb{Q}) \cong \oplus_p \mathbb{Q} \\ \Big\downarrow \\ 0 \end{array}$$

since

$$H^1(\widetilde{M},\partial\widetilde{M};\mathbb{Q}) \cong H_1(\widetilde{M},\partial\widetilde{M};\mathbb{Q}) \cong H_{n+1}(\widetilde{M};\mathbb{Q})$$

by the partial Poincaré duality theorem [9]. From this it follows that $r=p$, i. e.

$$\mathrm{rk}_{\mathbb{Q}<t>}\, H_{n+1}(\widetilde{M};\mathbb{Q}) = p$$

as requested. Thus the proof is complete. $\square$

Example. In [20] Zeeman defined the process of k-twist-spinning a smooth n-knot in $\mathbb{S}^{n+2}$. The result is a smooth $(n+1)$-knot in $\mathbb{S}^{n+3}$, uniquely determined by the original n-knot and the integer k. The complement of such a k-twist-spun knot in $\mathbb{S}^{n+3}$ is a bundle over a circle with covering group $\mathbb{Z}_k$, and typical fiber homeomorphic with the k-fold covering of $\mathbb{S}^{n+2}$ branched over the original n-knot. Moreover, $\mathbb{S}^1$ acts on $\mathbb{S}^{n+3}$ so as to leave the k-twist spun knot setwise invariant, and map the complement fiberwise. In particular, if $k=\pm 1$, then the result is unknotted. More recently, Litherland [13] showed that combining certain rollings with twists yields $(n+1)$-knots which are again fibered over $\mathbb{S}^1$. Now any bundle over $\mathbb{S}^1$ with typical fiber F is equivalent to a fibration of type

$$F \times_g \mathbb{S}^1 \to \mathbb{S}^1,$$

where $g\colon F \to F$ is a homeomorphism, and $F \times_g \mathbb{S}^1$ denotes the quotient space obtained from $F \times [0,1]$ identifying $x \times 0$ with $g(x) \times 1$ for any point $x \in F$.

Let us consider a knotted ∂-handle K^{n+1} obtained from a k-twist spun knot Σ^{n+1} in $\mathbb{S}^{n+3}$ ($k \neq \pm 1$) by a hyperboloidal transformation along a trivial 1-handle. Then K is knotted since

$$\begin{aligned} \Pi_1(\mathbb{S}^{n+3}\backslash K) &\cong \Pi_1(\mathbb{S}^{n+3}\backslash \Sigma) \\ &\cong \Pi_1(F \times_g \mathbb{S}^1) \\ &\cong \Pi_1(F) \times_{g_*} \mathbb{Z} \end{aligned}$$

is not isomorphic to $\mathbb{Z}$ (use $k \neq \pm 1$). Here $g_*\colon \Pi_1(F) \to \Pi_1(F)$ is the induced automorphism on Π_1. Furthermore, if G is a group and $h\colon G \to G$ an automorphism, then $G \times_h \mathbb{Z} < t >$ denotes the extension of G by the infinite cyclic group $\mathbb{Z} < t >$ generated by t in which conjugation by t induces h on G. Finally the complement $\mathbb{S}^{n+3}\backslash K$ is not fibered over $\mathbb{S}^1$ by Proposition 12.

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