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S. MANTOVANI

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## SEMILOCALIZATIONS OF EXACT AND LEFTEXTENSIVE CATEGORIES

by *S. MANTOVANI*

**RESUME.** Dans cet article on démontre qu'une catégorie  $C$  à limites projectives finies est une semilocalisation d'une catégorie exacte ssi dans  $C$  il existe des coégalisateurs de relations d'équivalence qui soient universels. On caractérise également les catégories localement distributives comme étant les semilocalisations de catégories lextensives. Enfin on montre que les semilocalisations des catégories exactes et extensives (prétopos) sont exactement les catégories à limites projectives finies avec coégalisateurs de relations d'équivalence universelles, sommes finies universelles, et paires coégalisatrices de monomorphismes universelles, ces catégories sont appelées ici semiprétopos.

### 1. Introduction

The initial problem faced here consists in looking for “structure preserving” reflections for regular categories and for locally distributive categories, exactly like localizations do for exact categories and lextensive categories. To this purpose it turns out to be useful the notion introduced in [8] of *semi-left-exact* reflection, where, if  $\mathcal{R}$  is a reflective subcategory, the units are stable under pullback along morphisms of  $\mathcal{R}$  (such reflective subcategories are called here *semilocalizations*).

We prove that:

- 1) semilocalizations of regular categories are regular
- 2) semilocalizations of locally distributive categories are locally distributive.

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When we deal with an exact category, any of its semilocalization inherits an additional property, namely existence and universality of coequalizers of equivalence relations. In Proposition 3.5 we show that this condition characterizes those regular categories that are semilocalizations of an exact category : this is done using the exact completion of a regular category (see [10], [13]).

On the other side, in Proposition 4.5 we notice that any locally distributive category can be seen as a semilocalization of a lextensive one, namely its sum completion (see [5]).

Thus it seems to be natural to consider semilocalizations of exact *and* lextensive categories, that is pretoposes, that satisfy the elementary weakening of the Giraud axioms for Grothendieck toposes. We show in 5.7 that these categories are exactly those regular categories in which finite sums, coequalizers of equivalence relations and also cokernel pairs of monomorphisms exist and are universal. We call these categories *semipretoposes*: in some sense they represent the “elementary” version of locally cartesian, locally presentable categories. Indeed, as it has very recently been proved in [11], these last categories are exactly the semilocalizations of Grothendieck toposes.

Finally, we remark that semipretoposes do not coincide with quasipretoposes, that is with those categories appearing as separated objects for a (unique) topology on a pretopos (see [6]). In fact, in [2] it is given an example (due to Adámek and Rosický) of a semilocalization of a pretopos having strong equivalence relations that are not effective, condition which is (strictly) necessary in order to have a quasipretopos.

## 2. Semilocalizations

Let  $\mathcal{C}$  be a category with finite limits (= lex) and suppose  $\mathcal{R}$  is a full replete reflective subcategory of  $\mathcal{C}$ , the inclusion functor being  $i : \mathcal{R} \rightarrow \mathcal{C}$  and the reflector  $r : \mathcal{C} \rightarrow \mathcal{R}$  with unit denote by  $r_X$ , for any  $X$  in  $\mathcal{C}$ . We will freely write  $\mathcal{R}$  instead of  $i(\mathcal{R})$ , when no confusion arises.

We will deal with a particular class of reflections, namely those which are called *semi-left-exact* in [8] and *admissible* in [9]. The approach to this kind of reflection in [8] is related to the matter of factorization systems and to limit preservation properties of the reflector  $r$ , as the following equivalent conditions show:

**Definition 2.1.** *The reflection of  $\mathcal{C}$  onto  $\mathcal{R}$  is said to be semi-left-exact if one of the following equivalent conditions holds:*

- (1) *In every pullback diagram of this form:*

$$\begin{array}{ccc} V & \xrightarrow{v} & U \\ g \downarrow & & \downarrow f \\ X & \xrightarrow{r_X} & rX \end{array}$$

*if  $U$  is in  $\mathcal{R}$ , then  $v = r_V$  (units of the reflection are stable under pullback along morphisms of  $\mathcal{R}$ );*

- (2)  *$r$  preserves pullbacks along morphisms of  $\mathcal{R}$ ;*  
 (3)  *$r$  preserves products of the form  $X \times R$ , with  $R$  in  $\mathcal{R}$ , and  $\mathcal{R}$ -equalizers, that is equalizers of pairs with codomain in  $\mathcal{R}$ .*

**Definition 2.2.** *We will call semilocalizations those reflective subcategories whose reflections are semi-left-exact.*

Examples of semilocalizations are given by localizations, by categories of separated objects in a pretopos (see [6]) and by torsion theories (not only hereditary) in abelian categories, which are all cases of reflections where units stable under pullback along any morphism (= with *stable units* in [8]). But there are also examples of semilocalizations which have not stable units, as shown in [8], ex.4.6. Another important example is given by the reflection of Hausdorff compact spaces onto Stone spaces.

Janelidze's point of view in [9] was related to the functor  $r^X : \mathcal{C}/X \rightarrow \mathcal{R}/rX$  which is obtained from  $r$ , for any  $X$  in  $\mathcal{C}$  sending  $f$  to  $r(f)$ . This functor has a right adjoint  $r_X^*$  sending any  $g : U \rightarrow rX$  to its pullback along  $r_X$ . The admissibility of  $\mathcal{R}$  is the request of full fidelity of  $r_X^*$  for any  $X$ , which turns out to be equivalent to the semi-left-exactness. The main difference between the last and the former approach is that Janelidze's use of this concept in Galois theory involves not only a reflective subcategory, but also a pullback-stable class of morphisms  $\Theta$ . In this sense admissibility respect to  $\Theta$  requires units stable under pullback along morphisms in  $\Theta$ .

We recall from [6],1.4.4. a proposition (about universality of colimits) which will be very useful for our purposes :

**Proposition 2.3.** *Let  $\mathcal{R}$  be a semilocalization of a category  $\mathcal{C}$ . Then the reflection of any colimit universal in  $\mathcal{C}$  is a colimit universal in  $\mathcal{R}$ .*

### 3. The regular case

We want to study now semilocalizations of regular and exact categories and then we need to recall some well known definitions:

**Definition 3.1.** ( see [1])

- (1) *A lex category  $\mathcal{R}$  is said to be regular when*
  - (i) *every morphism can be factored by a regular epimorphism followed by a monomorphism*
  - (ii) *regular epimorphisms are universal, i.e. they are stable under pullback.*
- (2) *An exact category is a regular category in which every equivalence relation is effective (that is it is the kernel pair of its coequalizer).*

The first observation is that that, like localizations of exact categories are still exact, semilocalizations of regular categories are still regular, as the following proposition shows:

**Proposition 3.2.** *Let  $\mathcal{R}$  be a semilocalization of a regular category  $\mathcal{C}$ . Then  $\mathcal{R}$  is regular.*

*Proof.*  $\mathcal{R}$  has finite limits, since  $\mathcal{C}$  has and  $\mathcal{R}$  is limit-closed, being reflective. Furthermore every kernel pair has a coequalizer in  $\mathcal{C}$ , whose reflection in  $\mathcal{R}$  gives us a coequalizer in  $\mathcal{R}$ . Regular epimorphisms are universal in  $\mathcal{R}$ , since they are universal in  $\mathcal{C}$  (by 2.3).  $\square$

In particular a semilocalization of an exact category is regular, even if not exact in general, as the case of separated objects for a topology in a topos may show. We would like to characterize those regular categories  $\mathcal{R}$  which appear as semilocalizations of *exact* ones. In order to do that, we can start from the observation that, again thanks to 2.3, in this case in  $\mathcal{R}$  also *coequalizers of equivalence relations* are universal. This property is not true in any regular category, e.g. it is not true in the category of Kelley spaces, as shown in [3].

We will show that this condition is also sufficient to obtain the characterization we are looking for. But we need first some preliminaries about the construction of the exact completion of a regular category.

The construction of the category of relations on a regular category makes easier to describe various completions and we are going to describe the *exact completion* of a regular category, that is the 2-adjoint to the forgetful 2-functor from the 2-category of exact categories and exact functors to the one of regular categories and exact functors (see [10]). If  $\mathcal{R}$  is a regular category, defining a *relation*  $R$  from  $X$  to  $Y$  as a subobject  $R \hookrightarrow X \times Y$ , we obtain a category whose objects coincide with the objects of  $\mathcal{R}$  and whose maps are relations between objects of  $\mathcal{R}$ , with a suitable composition which in a regular category turns out to be associative (see [10]).

$\text{Rel}(\mathcal{R})$  has an extra structure given by an *involution*  $(\ )^\circ$ , which is the identity on objects, which gives the opposite relation  $(R)^\circ$ , composing  $R \hookrightarrow X \times Y$  with the canonical isomorphism  $\langle \pi_2, \pi_1 \rangle : X \times Y \rightarrow Y \times X$ . Furthermore we have an embedding  $\mathcal{R} \rightarrow \text{Rel}(\mathcal{R})$ , given by the construction of the graph. Working in  $\text{Rel}(\mathcal{R})$ , we will call (graphs of) arrows in  $\mathcal{R}$  “*maps*”.

The following lemma sums up some of the main properties of the calculus of relations, which we will need later:

**Lemma 3.3.** *Let  $\mathcal{R}$  be a regular category; then:*

- (1) *an arrow  $R : X \rightarrow Y$  of  $\text{Rel}(\mathcal{R})$  is the graph of an arrow of  $\mathcal{R}$  (i.e. is a map) if and only if  $R^\circ$  is a right adjoint, which simply means  $RR^\circ \leq 1$ ,  $R^\circ R \geq 1$ ;*
- (2) *an arrow  $f : X \rightarrow Y$  of  $\mathcal{R}$  is a monomorphism if and only if  $f^\circ f = 1$  and is a regular epimorphism if and only if  $ff^\circ = 1$*
- (3) *for every relation  $R : X \rightarrow Y$  there exists a pair of maps  $r_0$  and  $r_1$  such that  $R = r_1 r_0^\circ$ ,  $r_0^\circ r_0 \cap r_1^\circ r_1 = 1$ ; such a pair is essentially unique (“*tabulation*” of  $R$ );*
- (4) *a square in  $\mathcal{R}$*

$$\begin{array}{ccc} U & \xrightarrow{k} & X \\ h \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

*is commutative iff  $kh^\circ \leq f^\circ g$  and is a pullback iff  $h, k$  tabulate  $f^\circ g$ ; in particular, the kernel pair of a map  $f : X \rightarrow Y$  is a tabulation of the relation  $f^\circ f$ .*

We will define the exact completion  $\mathcal{R}_{ex}$  (see [10]) as follows:

*objects* are pairs  $(X, R)$ , where  $X$  is an object of  $\mathcal{R}$  and  $R$  is an equivalence relation on  $X$ , that is a reflexive ( $1_X \leq R$ ), symmetric ( $R = R^\circ$ ) and transitive ( $RR = R$ ) endomorphism  $R : X \rightarrow X$  of  $\text{Rel}(\mathcal{R})$ ;

*arrows*  $E : (X, R) \rightarrow (Y, S)$  are relations  $E : X \rightarrow Y$  of  $\mathcal{R}$  such that  $ER = E = SE$ ,  $R \leq E^\circ E$  and  $EE^\circ \leq S$ ;

*composition* is the composition in  $\text{Rel}(\mathcal{R})$ .

It turns out that  $\mathcal{R}_{ex}$  is an exact category. Furthermore, we can embed  $\mathcal{R}$  in  $\mathcal{R}_{ex}$ , using the identities on objects in  $\mathcal{R}$  as particular equivalence relations and we identify any object  $X$  of  $\mathcal{R}$  with the pair  $(X, 1_X)$  in  $\mathcal{R}_{ex}$ . Maps in  $\mathcal{R}$  turn out to be maps in  $\mathcal{R}_{ex}$ .

In [7] it is shown when  $\mathcal{R}$  is reflective in  $\mathcal{R}_{ex}$ , namely

**Proposition 3.4.** (see [7]) *Let  $\mathcal{R}$  be a regular category. Then  $\mathcal{R}$  is a reflective subcategory of its exact completion  $\mathcal{R}_{ex}$  if and only if coequalizers of equivalence relations exist in  $\mathcal{R}$ .*

The following Proposition shows when this reflection is semi-left-exact.

**Proposition 3.5.** *Let  $\mathcal{R}$  be a regular category with coequalizers of equivalence relations. Then  $\mathcal{R}$  is a semilocalization of  $\mathcal{R}_{ex}$  if and only if coequalizers of equivalence relations are universal in  $\mathcal{R}$ .*

*Proof.* We know that given an object  $(X, R)$  of  $\mathcal{R}_{ex}$ , the unit of the reflection is the coequalizer  $q_R : X \rightarrow X/R$  of  $R$ , where  $R$  is an equivalence relation in  $\mathcal{R}$  with tabulation  $(r_0, r_1)$ , so that  $q_R r_0 = q_R r_1$ . Let us denote by  $t$  this composition and consider in  $\mathcal{R}$  the pullback  $\bar{t}$  of  $t$  along  $f$ :

$$\begin{array}{ccccc} T & \xrightarrow{t_i} & P & \xrightarrow{s} & Y \\ \downarrow \bar{f} & & p \downarrow & & f \downarrow \\ R & \xrightarrow{r_i} & X & \xrightarrow{q_R} & X/R \end{array} .$$

Since the outer diagram is a pullback and the right one is the pullback of  $q_R$  along  $f$ , for any  $i = 0, 1$  the left diagram is a pullback. By

Lemma 3.3 (4), the following equalities hold:

$$(1) \quad \bar{t} \bar{f}^\circ = f^\circ t \quad (2) \quad s p^\circ = f q_R^\circ \quad (3) \quad t_i \bar{f}^\circ = p^\circ r_i$$

Furthermore  $T = t_1 t_0^\circ$  is an equivalence relation on  $P$  and  $s t_0 = s t_1$ . This means that (4)  $T = t_1 t_0^\circ \leq s^\circ s$  holds.

We want to prove that

$$\begin{array}{ccc} (P, T) & \xrightarrow{s} & (Y, 1_Y) \\ \downarrow R_p & & \downarrow f \\ (X, R) & \xrightarrow{q_R} & (X/R, 1_{X/R}) \end{array}$$

is a pullback diagram in  $\mathcal{R}_{ex}$ .

$s : (P, T) \rightarrow (Y, 1_Y)$  is a morphism in  $\mathcal{R}_{ex}$ , that is  $s T = s$ ,  $T \leq s^\circ s$  and  $s s^\circ \leq 1$ .

The last two inequalities follow respectively from (4) and from  $s s^\circ = 1$ , since  $s$  is a regular epimorphism. As regard to the first equality, we have that  $1 \leq T$  implies  $s \leq s T$  and (4)  $T \leq s^\circ s$  implies  $s T \leq s s^\circ s = s$ .

$R_p : (P, T) \rightarrow (X, R)$  is a morphism in  $\mathcal{R}_{ex}$ , that is

$$(a) \quad R(R_p)T = R_p \quad (b) \quad T \leq (R_p)^\circ R_p \quad (c) \quad R_p(R_p)^\circ \leq R$$

$$\begin{aligned} (a) \quad R R_p T &= R_p T = R(p t_1) t_0^\circ = R(r_1 \bar{f}) t_0^\circ = \\ & \text{(by (3)) } R r_1 r_0^\circ p = R R_p = R_p \end{aligned}$$

Since  $p$  is a morphism in  $\mathcal{R}$ ,  $1 \leq p^\circ p$  and  $p p^\circ \leq 1$ , then

$$(b) \quad T = t_1 t_0^\circ \leq p^\circ (p t_1) t_0^\circ = p^\circ r_1 \bar{f} t_0^\circ = \text{(by (3))}$$

$$= p^\circ r_1 r_0^\circ p = p^\circ R_p = p^\circ R^\circ R_p = (R_p)^\circ R_p$$

$$(c) \quad R_p(R_p)^\circ = R_p p^\circ R^\circ = R_p p^\circ R \leq R R = R.$$

In order to show that we have a pullback diagram in  $\mathcal{R}_{ex}$ , using again the calculus of relations, we only need to prove that  $s(R_p)^\circ =$

$f^\circ q_R$ . But  $s(Rp)^\circ = sp^\circ R^\circ = sp^\circ R = f^\circ q_R R$  by (3) and, since  $1 \leq R \leq q_R^\circ q_R$ , we immediatly obtain

$$f^\circ q_R \leq f^\circ q_R R = s(Rp)^\circ \leq f^\circ q_R q_R^\circ q_R = f^\circ q_R$$

Now it is clear that the semi-left-exactness of this reflection means exactly that coequalizers of equivalent relations are universal in  $\mathcal{R}$ .  $\square$

Now we can conclude about semilocalizations of exact categories as:

**Corollary 3.6.** *A lex category  $\mathcal{R}$  has universal coequalizers of equivalence relations if and only if it is a semilocalization of an exact category.*

#### 4. The locally distributive case

Now we are going to consider universality of another kind of colimit, namely sums. Also here we need to recall some preliminar definitions and remarks:

**Definition 4.1.** (see [5] )

- (1) *A category  $\mathcal{C}$  with finite sums is called extensive if for each pair of objects  $X, Y$  in  $\mathcal{C}$ , the canonical functor  $+: \mathcal{C}/X \times \mathcal{C}/Y \rightarrow \mathcal{C}/(X + Y)$  is an equivalence.*
- (2) *An extensive category  $\mathcal{C}$  with all finite limits is called lextensive.*
- (3) *A category with finite sums and finite products is called distributive if the canonical arrow  $A \times B + A \times C \rightarrow A \times (B + C)$  is an isomorphism.*

**Proposition 4.2.** (see [5], [4], [12])

- (1) *A category  $\mathcal{C}$  is locally distributive if and only if it has all finite limits and universal sums.*
- (2) *A category is lextensive if and only if it is locally distributive and it has disjoint sums.*

We want to study now the behaviour of semilocalizations in the context of locally distributive categories (that are nothing else that lex categories with universal sums, by 4.2 (1)). By Proposition 2.3 again and 4.2 (1) we immediately obtain the following

**Proposition 4.3.** *A semilocalization  $\mathcal{R}$  of a locally distributive category  $\mathcal{C}$  is locally distributive.*

As before, in order to obtain a characterization of locally distributive categories by means of semilocalizations, we need a free construction, meaning the left biadjoint to the forgetful 2-functor from the 2-category EXT of extensive categories and the 2-category of CAT of categories. This turns out to be given by the process of *sum completion*, meaning the construction of the category Fam( $\mathcal{C}$ ) (see [5]), where *objects* are finite families  $(X_i)_{i \in I}$  of objects of  $\mathcal{C}$  and *arrows* are pairs  $(f, \varphi) : (X_i)_{i \in I} \rightarrow (Y_j)_{j \in J}$  with  $\varphi$  a function  $I \rightarrow J$  and  $f$  a family  $(f_i : X_i \rightarrow Y_{\varphi(i)})_{i \in I}$  of morphisms of  $\mathcal{C}$ .

Fam( $\mathcal{C}$ ) is always *extensive* and, when  $\mathcal{C}$  is *lex*, Fam( $\mathcal{C}$ ) is *lexensive* (see e.g.[CJ]).

$\mathcal{C}$  always canonically embeds into Fam( $\mathcal{C}$ ), but when  $\mathcal{C}$  has finite sums, we obtain a left adjoint to this embedding, taking as unit of the reflection of  $(X_i)_I$  the morphism  $(m, \mu) : (X_i)_I \rightarrow \Sigma_i X_i$  given by the injections. This reflection has an additional property when  $\mathcal{C}$  is locally distributive, as we will show later. First we need some informations about Fam( $\mathcal{C}$ ).

**Lemma 4.4.** *Let  $\mathcal{C}$  be a lex category with sums. In Fam( $\mathcal{C}$ ) we have:*

- (1) *the morphism  $(m, \mu) : (X_i)_I \rightarrow \Sigma_i X_i$  given by the injections is a regular epimorphism*
- (2)  *$(f, \varphi) : (X_i)_I \rightarrow (Y_j)_J$  is a monomorphism if and only if  $\varphi : I \rightarrow J$  is injective and any  $f_i : X_i \rightarrow Y_{\varphi(i)}$  is monic in  $\mathcal{C}$*
- (3) *a morphism in  $\mathcal{C}$  is a regular epimorphism in Fam( $\mathcal{C}$ ) if and only if it is a regular epimorphism in  $\mathcal{C}$ .*
- (4)  *$(e, \varepsilon) : (X_i)_I \rightarrow (Y_j)_J$  is a regular epimorphism if and only if  $\varepsilon : I \rightarrow J$  is surjective and for any  $j \in J$ ,  $\Sigma e_i : \Sigma_{\varepsilon(i)=j} X_i \rightarrow Y_j$  is a regular epimorphism in  $\mathcal{C}$ .*
- (5) *A commutative square*

$$\begin{array}{ccc}
 (P_i)_T & \xrightarrow{(p, \pi)} & (Z_k)_K \\
 (g, \gamma) \downarrow & & \downarrow (f, \varphi) \\
 (X_i)_I & \xrightarrow{(e, \varepsilon)} & (Y_j)_J
 \end{array}$$

is a pullback in  $\text{Fam}(\mathcal{C})$  if and only if

(i)  $\pi$  is the pullback of  $\varepsilon$  along  $\varphi$

(ii)  $\forall t = (i, k) \in T$  with  $\varepsilon(i) = \varphi(k) = j$

$$\begin{array}{ccc} P_{t=(i,k)} & \xrightarrow{p_t} & Z_k \\ g_t \downarrow & & \downarrow f_k \\ X_i & \xrightarrow{e_i} & Y_j \end{array}$$

is a pullback in  $\mathcal{C}$ .

*Proof.* Just (1) maybe needs some explanation, since we are dealing with sums that may not be disjoint. So given  $(m, \mu) : (X_i)_I \rightarrow \Sigma_i X_i$ , we can take its kernel pair

$$\begin{array}{ccc} (P_{ij})_{I \times J} & \xrightarrow{(p_1, \pi_1)} & (X_i)_I \\ (p_2, \pi_2) \downarrow & & \downarrow (m, \mu) \\ (X_j)_I & \xrightarrow{(m, \mu)} & \Sigma_i X_i \end{array}$$

where any  $P_{ij}$  is given by the pullback of  $m_i$  along  $m_j$ . Now if  $(f, \varphi) : (X_i)_I \rightarrow (Y_j)_J$  is such that  $(f, \varphi)(p_1, \pi_1) = (f, \varphi)(p_2, \pi_2)$ , then  $\varphi p_1 = \varphi p_2$  and so for any  $i, j$   $\varphi(i) = \varphi(j) = *$ , which means  $\varphi$  is constant. From the universal property of sums, we have then a unique morphism  $p : \Sigma_i X_i \rightarrow Y_*$  such that  $p m_i = f_i$ , which implies that  $(p, 1_*)(m, \mu) = (f, \varphi)$ . Hence  $(m, \mu)$  is the coequalizer of its kernel pair.  $\square$

**Proposition 4.5.** *Let  $\mathcal{C}$  be locally distributive. Then  $\mathcal{C}$  is a semilocalization of  $\text{Fam}(\mathcal{C})$ . Furthermore the reflector  $: \text{Fam}(\mathcal{C}) \rightarrow \mathcal{C}$  preserves products and monomorphisms.*

*Proof.* Since the units of the reflection are given by the injections into sums, and the objects of  $\mathcal{C}$  are identified as the families with only one element, the condition of universality of sums is exactly the condition of semi-left-exactness of the reflection. The preservation of products is nothing else that distributivity of  $\mathcal{C}$ . As regard to the preservation of monomorphisms, given a monomorphism  $(f, \varphi) : (X_i)_I \rightarrow (Y_j)_J$

in  $\text{Fam}(\mathcal{C})$ , then by Lemma 4.4 (2) any  $f_i : X_i \rightarrow Y_{\varphi(i)}$  is a monomorphism in  $\mathcal{C}$ . Furthermore in a locally distributive category as  $\mathcal{C}$  injections into a sum are monic (see [5]), so that, for any  $i$ ,  $n_{\varphi(i)}f_i$  is monic, where  $n_{\varphi(i)} : Y_{\varphi(i)} \rightarrow \Sigma_j Y_j$ . Since  $n_{\varphi(i)}f_i = (\Sigma f_i)m_i$  (with  $m_i : X_i \rightarrow \Sigma_i X_i$ ), the pullback in  $\mathcal{C}$  of  $(\Sigma f_i)m_i$  along itself must give the identity on  $X_i$ :

$$\begin{array}{ccccc}
 X_i & \xrightarrow{r_i} & Z_i & \xrightarrow{t_i} & X_i \\
 r_i \downarrow & & \downarrow q_i & & m_i \downarrow \\
 Z_i & \xrightarrow{q_i} & \Sigma_i Z_i & \xrightarrow{\Sigma t_i} & \Sigma_i X_i \\
 t_i \downarrow & & \downarrow \Sigma t_i & & \Sigma f_i \downarrow \\
 X_i' & \xrightarrow{m_i} & \Sigma_i X_i & \xrightarrow{\Sigma f_i} & \Sigma_j Y_j
 \end{array}$$

In the above diagram every square is a pullback and the center is given by  $\Sigma_i Z_i$  since sums are universal in  $\mathcal{C}$ . But any  $q_i$  is a monomorphism, therefore  $r_i$  (and hence  $t_i$ ) is an isomorphism, as then  $\Sigma t_i$ . This means that  $\Sigma f_i$  is monic in  $\mathcal{C}$ .  $\square$

Note that a semilocalization of a lextensive category is not usually lextensive, since it may have not disjoint sums.

Combining 4.3 and 4.5 we obtain

**Corollary 4.6.** *A lex category  $\mathcal{C}$  has universal finite sums if and only if  $\mathcal{C}$  is a semilocalization of a lextensive category.*

## 5. Semipretoposes

Now we want to consider semilocalizations of exact *and* extensive categories, i.e. of *pretoposes*. Thanks to 2.3, we know that these semilocalizations must be lex categories  $\mathcal{S}$  in which “some” colimits exist and are *universal*, namely finite sums, coequalizers of equivalence relations and also cokernel pairs of monomorphisms (this follows from the fact that in a pretopos pushouts of monomorphisms exist and are universal cf. e.g. [6]).

**Definition 5.1.** *A lex category  $\mathcal{S}$  will be called semipretopos if it has*

- (1) *universal (finite) sums*
- (2) *universal coequalizers of equivalence relations*
- (3) *universal cokernel pairs of monomorphisms.*

An equivalent definition is given by substituting (3) with

*(3') in  $\mathcal{S}$  (epimorphisms, regular monomorphisms) is a factorization of morphisms stable under pullback.*

Note that the definition of semipretoposes differs from the one given in [6] of *quasipretoposes*, categories that characterize quasilocations (reflector preserving products and monomorphisms) of pretoposes. The difference is that here we do not request *effectiveness of strong equivalence relations*, condition which is strictly necessary in order to have a quasipretopos. In fact in [2] it is shown an example (due to Adámek and Rosický) of a semilocalization of a pretopos that has not effective strong equivalence relations. (Actually they provided this example to show that locally presentable locally cartesian closed categories are different from Grothendieck quasitoposes.)

We will characterize semipretoposes exactly as semilocalizations of pretoposes (and this would be the “elementary” analog of showing that locally presentable locally cartesian closed categories are the semilocalizations of Grothendieck toposes, cf. [11]). In order to do that, we need to state some properties of semipretoposes.

First of all, we can recall some results from [6] about the behaviour of relations in semipretoposes. In fact the factorization system given by (3') restricted to monomorphisms gives rise to a *topology (universal closure operator)* on subobjects. In particular this closure operator acts on relations and, given a relation  $R$  from  $X$  to  $Y$ , we will denote its closure by  $\overline{R}$ . (Note that any morphism of  $\mathcal{S}$  considered as a relation through its graph is a closed relation; in particular  $\overline{1_X} = 1_X$ ). Its behaviour with respect to composition of relations is given by the following :

**Proposition 5.2.** *(see [6], 2.2.3) Let  $R \mapsto X \times Y$  and  $S \mapsto Y \times Z$  be composable relations in  $\mathcal{S}$ . Then  $\overline{S R} \leq \overline{S} \overline{R}$ .*

Furthermore, in a semipretopos sums are universal, and then by Proposition 3.4 of [2] and Proposition 2.12 of [5] we have that

**Proposition 5.3.** *In a semipretopos  $\mathcal{S}$*

- (1) *the initial object is strict, that is any arrow into it is invertible.*
- (2) *Given a sum  $X_1 + X_2$  with distinct injections  $m_1 : X_1 \rightarrow X_1 + X_2$  and  $m_2 : X_2 \rightarrow X_1 + X_2$ , the pullback of  $m_1$  along  $m_2$  is the closure  $\bar{0}_{12}$  of the relation  $0 \rightarrow X_1 \times X_2$ , that is  $X_1 \cap X_2 = \bar{0}_{12}$*

It is a rather long but easy exercise on composition of relations in a regular category to prove the following lemma:

**Lemma 5.4.** *Given in a regular category a family  $(X_i)_I$  of subobjects of  $X$  and relations  $R_{ij} \rightarrow X_i \times X_j$  and  $R_{hk} \rightarrow X_h \times X_k$ , with  $i, j, h, k \in I$ ,  $R_{ij}$  and  $R_{hk}$  are composable relations if considered as relations on  $X$  and*

$$R_{ij} R_{hk} = R_{ij} X_{ki} R_{hk}$$

where  $X_{ki} = X_k \cap X_i$  as subobjects of  $X$ .

*In particular,*

- (1) *when sums are universal and  $X = \sum_i X_i$ , by 5.3. (2), if  $k \neq i$*   
 $R_{ij} R_{hk} = R_{ij} \bar{0}_{ki} R_{hk}$

- (2) *and when also epimorphisms are universal, by 5.2.*

$$\bar{0}_{ki} R_{hk} \leq \bar{0}_{hi} \quad \text{and} \quad R_{ij} \bar{0}_{ki} \leq \bar{0}_{kj}$$

Now we are going to apply the process of sum completion to a semipretopos. First we will prove that the process of sum completion preserves regularity, at least in the case of locally distributivity.

In fact we have:

**Proposition 5.5.** *Let  $\mathcal{S}$  be a locally distributive category which is regular. Then  $\text{Fam}(\mathcal{S})$  is regular.*

*Proof.* We already know that  $\text{Fam}(\mathcal{S})$  is lex, so what we need to show is :

- (1) In  $\text{Fam}(\mathcal{S})$  any morphism can be factorized by a regular epimorphism followed by a monomorphism.

- (2) Regular epimorphisms are stable under pullback.

As regard to (1), given  $(f, \varphi) : (X_i)_I \rightarrow (Y_j)_J$  in  $\text{Fam}(\mathcal{S})$ , in order to have the desired factorization, first we factorize the function  $\varphi$  as  $\phi : I \rightarrow \varphi(I) = K$  followed by the inclusion  $\eta : K \hookrightarrow J$ . Now fixed an element  $k \in K$ , consider  $\Sigma f_i : \Sigma_{\varphi(i)=k} X_i \rightarrow Y_k$  and factorize this morphism in  $\mathcal{S}$  as  $e_k : \Sigma_{\varphi(i)=k} X_i \rightarrow Z_k$  followed by  $n_k : Z_k \hookrightarrow Y_k$ :

$$\begin{array}{ccc} (X_i)_{\varphi(i)=k} & \xrightarrow{m_i} & \Sigma_{\varphi(i)=k}(X_i) \\ f_i \downarrow & & \downarrow e_k \\ Y_k & \xleftarrow{n_k} & Z_k \end{array}$$

From Lemma 4.4 (4) and (2),  $(e, \phi) : (X_i)_I \rightarrow (Z_k)_K$  is a regular epimorphism and  $(n, \eta) : (Z_k)_K \hookrightarrow (Y_j)_J$  is a monomorphism in  $\text{Fam}(\mathcal{S})$  such that  $(n, \eta)(e, \phi) = (f, \varphi)$ .

As regard to (2), we can restrict ourselves to the case  $J = \{*\}$ , thanks to Lemma 4.4 (4) and (5). So given a regular epimorphism  $(e, \varepsilon) : (X_i)_I \rightarrow Y$ , let us consider its pullback  $(p, \pi_2) : (P_{ik})_{I \times K} \rightarrow (Z_k)_K$  along a morphism  $(f, \varphi) : (Z_k)_K \rightarrow Y$ :

$$\begin{array}{ccc} (P_{ik})_{I \times K} & \xrightarrow{(p, \pi_2)} & (Z_k)_K \\ (g, \pi_1) \downarrow & & \downarrow (f, \varphi) \\ (X_i)_I & \xrightarrow{(e, \varepsilon)} & Y \end{array}$$

From Lemma 4.4 (4) and (5), it follows that  $\pi_2$  is surjective.

Since for any  $(i, k) \in I \times K$

$$\begin{array}{ccc} P_{ik} & \xrightarrow{p_{ik}} & Z_k \\ g_{ik} \downarrow & & \downarrow f_k \\ X_i & \xrightarrow{e_i} & Y \end{array}$$

is a pullback in  $\mathcal{S}$ , we can factorize it as in the following diagram, where the right square represents the pullback of  $\Sigma e_i$  along  $f_k$ :

$$\begin{array}{ccccc}
 P_{ik} & \xrightarrow{g_{ik}} & T_k & \xrightarrow{r_k} & Z_k \\
 g_{ik} \downarrow & & \downarrow h_k & & f_k \downarrow \\
 X_i & \xrightarrow{m_i} & \Sigma_i X_i & \xrightarrow{\Sigma e_i} & Y
 \end{array}$$

Since  $\mathcal{S}$  is regular and  $\Sigma e_i : \Sigma_i X_i \rightarrow Y$  is a regular epimorphism in  $\mathcal{S}$  (by Lemma 4.4 (4)), then  $r_k$  is a regular epimorphism in  $\mathcal{S}$ , for any  $k \in K$ . But  $\mathcal{S}$  is also locally distributive, so any  $T_k = \Sigma_i P_{ik}$  and  $r_k = \Sigma_i p_{ik} : \Sigma_i P_{ik} \rightarrow Z_k$ . This means that also  $(p, \pi_2)$  is a regular epimorphism in  $\text{Fam}(\mathcal{S})$ .  $\square$

So starting from a semipretopos  $\mathcal{S}$ ,  $\text{Fam}(\mathcal{S})$  is an extensive regular category, in which  $\mathcal{S}$  embeds as a semilocalization. Hence we can consider its exact completion, obtaining  $\text{Fam}(\mathcal{S})_{ex}$ , which is not only exact, but also extensive, since *the exact completion preserves universal and disjoint sums* (see [6]), i.e.  $\text{Fam}(\mathcal{S})_{ex}$  is a pretopos in which  $\text{Fam}(\mathcal{S})$  embeds.

In order to apply Proposition 3.5 we have to face the problem of the existence and universality of coequalizers of equivalence relations in  $\text{Fam}(\mathcal{S})$ .

Consider then an equivalence relation  $\mathfrak{R}$  on  $(X_i)_I$  in  $\text{Fam}(\mathcal{S})$ .  $\mathfrak{R}$  is given by a monomorphism  $(m, \mu) : (R_k)_K \hookrightarrow (X_i \times X_j)_{I \times I}$ , that composed with the two projections gives rise to  $((r^0, \rho^0), (r^1, \rho^1)) : (R_k)_K \rightrightarrows (X_i)_I$ . By Lemma 4.4 (2) we can identify  $K$  with a subset of  $I \times I$ ; for any  $(i, j) \in K$ ,  $R_{ij}$  is then a relation in  $\mathcal{S}$  from  $X_i$  to  $X_j$ .  $\mathfrak{R}$  has the properties:

*reflexivity* :  $1_{(X_i)_I} \leq \mathfrak{R}$ , that is  $(i, i) \in K$  and  $(1_{X_i}) \leq R_{ii}$ ,  $\forall i \in I$

*simmetry* :  $\forall (i, j) \in K$ ,  $(j, i) \in K$  and  $R_{ij}^\circ = R_{ji}$

*transitivity*:  $\mathfrak{R}\mathfrak{R} = \mathfrak{R}$ , that is  $\bigcup_i R_{ij} R_{hi} = R_{hj}$   $(\star)$

where the union  $\bigcup_j A_j$  of subobjects  $A_j \hookrightarrow A$  in  $\mathcal{S}$  is obtained factorizing  $\sum_j A_j \rightarrow A$  by the regular factorization. (In a locally distributive regular category unions are distributive respect to composition, see [10]).

From above it is clear that  $K$  gives rise to an equivalence relation on  $I$ . Considering its coequalizer  $\varepsilon : I \rightarrow I/K = J$  in SET, for any

$j \in J$  the subfamily given by  $(R_k)_{K_j} \rightrightarrows (X_i)_{I_j}$ , where  $I_j = \varepsilon^{-1}(j)$  and  $K_j = (\varepsilon\mu)^{-1}(j)$  is again an equivalence relation.

So from now on we will restrict ourself to the case  $J = \{*\}$ , that is  $K = I \times I$ .

At this point we can consider the reflection of  $\mathfrak{R} = (R_{ij})_{I \times I}$  in  $\mathcal{S}$ , that is  $R := \Sigma_{ij} R_{ij}$ .

Since by 4.5 monomorphisms are preserved,  $\Sigma_{ij} R_{ij} = \cup_{ij} R_{ij}$ , where the union is taken in  $X \times X$  and then  $R$  is a relation on  $\Sigma_i X_i = X$ .

We want to prove first that  $R$  is an equivalence relation on  $X$  in  $\mathcal{S}$ .

(1) *Reflexivity*: from the reflexivity of  $\mathfrak{R}$ , it follows that

$$\Sigma_i 1_{X_i} = 1_{\Sigma X_i = X} \leq \Sigma_{ij} R_{ij} = R$$

(2) *Simmetry*:  $R^\circ = (\Sigma_{ij} R_{ij})^\circ = \Sigma_{ij} R_{ij}^\circ = \Sigma_{ij} R_{ji} = R$

(3) *Transitivity*:

$$\begin{aligned} RR &= \Sigma_{ij} R_{ij} \Sigma_{hk} R_{hk} = \bigcup_{ij} R_{ij} \bigcup_{hk} R_{hk} = \bigcup_{ijhk} R_{ij} R_{hk} = (\text{by 5.4 (1)}) \\ &= \bigcup_{ijh} R_{ij} R_{hi} \cup \bigcup_{ijh} R_{ij} \left( \bigcup_{k \neq i} \bar{0}_{ki} R_{hk} \right) \leq (\text{by 5.4 (2)}) \\ &\leq \bigcup_{jh} \left( \bigcup_i R_{ij} R_{hi} \right) \cup \bigcup_{ijh} R_{ij} \bar{0}_{hi} \leq (\text{by 5.4 (2) and } (\star)) \\ &\leq \bigcup_{jh} R_{hj} \cup \bigcup_{jh} \bar{0}_{hj} \leq R \cup \left( \overline{\bigcup_{jh} 0_{hj}} = \bar{0}_X \right) \leq R \cup 1_X = R \end{aligned}$$

by the reflexivity of  $R$ .

Therefore, given an equivalence relation  $\mathfrak{R} = ((r^0, \rho^0), (r^1, \rho^1)) : (R_{ij})_{I \times I} \rightrightarrows (X_i)_I$  in  $\text{Fam}(\mathcal{S})$ , we obtain an equivalence relation  $R$  in  $\mathcal{S}$ , where equivalence relations admit coequalizers. Denoting by  $\bar{r}^0, \bar{r}^1$  the sums  $\Sigma r_{ij}^0, \Sigma r_{ij}^1$  we can consider the coequalizer  $q : X \rightarrow X/R$  of  $(\bar{r}^0, \bar{r}^1) : R \rightrightarrows X$  in  $\mathcal{S}$ . We claim that

$$(qm_i, \varepsilon) : (X_i)_I \rightarrow X \rightarrow X/R \text{ is a coequalizer for } \mathfrak{R} \text{ in } \text{Fam}(\mathcal{S}).$$

*Proof.* Let  $(f, \varphi); (X_i)_I \rightarrow (Z_h)_H$  be an arrow in  $\text{Fam}(\mathcal{S})$  such that  $(f, \varphi)(r^0, \rho^0) = (f, \varphi)(r^1, \rho^1)$ . This means first that for any  $i, j \in I$ ,

$\varphi(i) = \varphi(j)$ , that is  $\varphi$  is a constant function of value say  $\bar{h}$ . Furthermore for any  $i, j \in I$ ,  $f_i r_{ij}^0 = f_j r_{ij}^1$ . From this fact and the universal property of sums, denoting by  $\bar{f}$  the sum  $\Sigma f_i : X \rightarrow Z_{\bar{h}}$ , we obtain that  $\bar{f} r^0 = \bar{f} r^1$ .

We can now use the universal property of coequalizers in  $\mathcal{R}$  to show that there exists a unique arrow  $g : X/R \rightarrow Z_{\bar{h}}$  such that  $gq = \bar{f}$ . Hence  $g(qm_i) = \bar{f}m_i$ , and this completes the proof.  $\square$

The last step we need is to show that coequalizers of equivalence relations are universal also in  $\text{Fam}(\mathcal{S})$ .

By Lemma 4.4.(5), the pullback of the coequalizer diagram  $R_{ij} \rightrightarrows X_i \rightarrow X \rightarrow X/R$  along  $(f_k) : (Y_k)_K \rightarrow X/R$  in  $\text{Fam}(\mathcal{S})$  is given by pullback diagrams in  $\mathcal{S}$ , for any  $i, j \in I, k \in K$

$$\begin{array}{ccccccc} S_{ijk} & \rightrightarrows & P_{ik} & \xrightarrow{p_{ik}} & Z_k & \xrightarrow{q_k} & Y_k \\ \downarrow & & \downarrow & & \downarrow & & f_k \downarrow \\ R_{ij} & \rightrightarrows & X_i & \xrightarrow{m_i} & \Sigma_i X_i & \xrightarrow{q} & X/R \end{array}$$

But the outer pullback diagram is the same as in

$$\begin{array}{ccccccc} S_{ijk} & \longrightarrow & S_k \rightrightarrows & Z_k & \xrightarrow{q_k} & Y_k \\ \downarrow & & \downarrow & \downarrow & & f_k \downarrow \\ R_{ij} & \xrightarrow{n_{ij}} & \Sigma R_{ij} \rightrightarrows & \Sigma X_i & \xrightarrow{q} & X/R \end{array}$$

By the universality of sums it follows from the first diagram that  $Z_k = \Sigma_i P_{ik}$  and from the second one that  $S_k = \Sigma_{ij} S_{ijk}$ , for any  $k \in K$ . Furthermore, since in  $\mathcal{S}$  coequalizers of equivalence relations are universal,  $q_k$  is a coequalizer for  $S_k$  for any  $k$ . As a consequence, using the universal property of sums as before, we obtain that  $(q_k p_{ik}) : (P_{ik})_{I \times K} \rightarrow (Y_k)_K$  is the coequalizer in  $\text{Fam}(\mathcal{S})$  of the equivalence relation  $(S_{ijk})$ .

Summing up, we have proved that

**Proposition 5.6.** *Given a semipretopos  $\mathcal{S}$ ,  $\text{Fam}(\mathcal{S})$  is an extensive regular category with universal coequalizers of equivalence relations.*

Applying Proposition 3.5 to  $\text{Fam}(\mathcal{S})$ , we obtain then that  $\text{Fam}(\mathcal{S})$  is a semilocalization of the pretopos  $\text{Fam}(\mathcal{S})_{ex}$ . Combining this result with Proposition 4.5, we can end with :

**Corollary 5.7.** *A category  $\mathcal{S}$  is a semipretopos if and only if it is a semilocalization of a pretopos.*

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S. MANTOVANI, DIPARTIMENTO DI MATEMATICA, VIA C. SALDINI, 50 – 20133 MILANO – ITALY TEL: (39)(2)26602216 FAX: (39)(2)70630346  
*E-mail address:* mantovani@vmimat.mat.unimi.it