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# EXPONENTIAL OBJECTS IN THE CONSTRUCT *PRAP*

*by E. LOWEN, R. LOWEN & C. VERBEECK*

**Résumé.** Dans cet article, on étudie l'exponentialité de la catégorie *PRAP* des espaces de pré-approximation (pre-approach spaces) et contractions. Un espace de pré-approximation peut être considéré comme un espace avec des pré-distances entre des points et des ensembles. On présente une caractérisation interne des objets exponentiels dans *PRAP*: ce sont les espaces de pré-approximation pour lesquels les pré-distances sont déterminées par les pré-distances entre des points et des ensembles singleton. On montre que la catégorie des objets exponentiels est l'enveloppe bi-coréflective des espaces de pré-approximation finis. Ce résultat peut être appliqué à la situation des espaces prétopologiques et alors on trouve la caractérisation connue des objets exponentiels dans *PRTOP*: ce sont les espaces prétopologiques générés de manière finie.

# 1 Introduction

It is the purpose of this paper to give an internal description of the exponential objects in *PRAP*, the category of pre-approach spaces. This category was introduced by E. and R. Lowen in [4] as an extensional supercategory of *AP*, the category of approach spaces [7]. With respect to *AP*, *PRAP* can be viewed as the counterpart of *PRTOP*, the category of pretopological spaces as introduced by G. Choquet in [1], with respect to *TOP*.

In [6] E. Lowen and G. Sonck were able to describe the exponential objects in *PRTOP* making use of the initially dense object **3**, the space with 3 points and one non-trivial neighborhood. Their technique not only allowed to give an internal description of the exponential objects in *PRTOP*, but moreover allowed for an extensive investigation of exponentiality related to coreflective subcategories of *PRTOP*. In [5] it was shown that in *PRAP* there exists a canonical counterpart of **3**, namely $\mathbb{P}^*$, i.e., the set $[0,\infty] \cup \{p\}$, where $p \notin [0,\infty]$, equipped with a particular pre-approach structure which we do not describe here. $\mathbb{P}^*$, as **3** in *PRTOP*, is initially dense in *PRAP*, but, as it e.g. follows from the results of this paper, and unlike **3**, it is not finitely generated. The complexity of $\mathbb{P}^*$ when compared to the simplicity of **3** made the transposition of the technique for *PRTOP* to *PRAP* rather awkward and elaborate.

Recently G. Richter presented an alternative technique [12], based on the fact that exponential objects allow for a particular interchange between the construction of final structures and of products. His technique avoids the intervenience of **3**, and complementary to the results of [6] allowed for an investigation of exponentiality in relation with certain epi-reflective subcategories of *PRTOP*.

Using Richter's method we are able in this paper to give an internal description of the exponential objects in *PRAP*. Whereas Richter's technique in *PRTOP* is relatively simple, for the case of *PRAP* it too becomes more elaborate. First we needed to devise a "functional" version of it in the setting of *PRAP*, second we had to apply the technique a continuous number of times and finally we had to "glue" everything together. Finally we are also able to show that the terminology "finitely generated" for the exponential objects justly applies also to the situation in *PRAP*.

# 2 Preliminaries

In 1988, E. Lowen and R. Lowen [4] introduced the category *PRAP* of pre-approach spaces. Given a set $X$, we denote its power set by $2^X$ and the set of its finite subsets by $2^{(X)}$. A map $\delta : X \times 2^X \to [0,\infty]$ is called a *pre-distance* if it fulfils

**(D1)** $\forall A \in 2^X, \forall x \in X : x \in A \Rightarrow \delta(x,A) = 0.$

**(D2)** $\forall x \in X : \delta(x,\emptyset) = \infty.$

**(D3)** $\forall A,B \in 2^X, \forall x \in X : \delta(x,A \cup B) = \delta(x,A) \wedge \delta(x,B).$

There are several equivalent ways of defining a pre-approach space. In this paper we will work mainly with local pre-distances and with limit functions. If $X$ is a set, a collection $(\mathcal{A}(x))_{x \in X}$ of ideals in $[0,\infty]^X$ is called a *pre-approach system* if it fulfils

**(A1)** $\forall x \in X, \forall \varphi \in \mathcal{A}(x) : \varphi(x) = 0.$

**(A2)** $\forall x \in X, \forall \varphi \in [0,\infty]^X :$

$$\left(\forall \varepsilon, N \in ]0,\infty[, \exists \varphi_N^\varepsilon \in \mathcal{A}(x) \text{ s.t. } \varphi \wedge N \leq \varphi_N^\varepsilon + \varepsilon\right) \Rightarrow \varphi \in \mathcal{A}(x).$$

The members of $\mathcal{A}(x)$ are called *local pre-distances* in $x$. For ease in notation we shall, whenever convenient denote a pre-approach system $(\mathcal{A}(x))_{x\in X}$ also simply by $\mathcal{A}$. $(X,\mathcal{A})$ is called a *pre-approach space*. If $(X,\mathcal{A})$ and $(X',\mathcal{A}')$ are pre-approach spaces and if $f : (X,\mathcal{A}) \to (X',\mathcal{A}')$ is a function, then $f$ is a *contraction* if

$$\forall x \in X, \forall \varphi' \in \mathcal{A}'(f(x)) : \varphi' \circ f \in \mathcal{A}(x).$$

The construct with as objects "pre-approach spaces" and as morphisms "contractions" is denoted by *PRAP*. This construct is related to the construct *AP* of approach spaces which has been extensively studied in [8]. One of the motivations for introducing *PRAP* is that it provides a unifying theory for both pretopological spaces (= Čech closure spaces) and pre-metric spaces.

If $X$ is endowed with a pretopological structure $q := (\mathcal{V}(x))_{x\in X}$, where $\mathcal{V}(x)$ is the neighborhood filter in $x$ (which need not satisfy the open kernel condition) then there is a related pre-approach system $(\mathcal{A}_q(x))_{x\in X}$ where

$$\mathcal{A}_q(x) = \left\{\varphi \in [0,\infty]^X \mid \varphi(x) = 0 \text{ and } \{\varphi < \varepsilon\} \in \mathcal{V}(x) \text{ for every } \varepsilon > 0\right\}.$$

Through this embedding continuous maps between pretopological spaces correspond exactly with contractions.

If $X$ is endowed with a pre-metric $d$, i.e., a function $d : X \times X \to [0,\infty]$ which is zero on the diagonal, then a natural pre-approach system $(\mathcal{A}_d(x))_{x\in X}$ is given by

$$\mathcal{A}_d(x) = \left\{\varphi \in [0,\infty]^X \mid \varphi \leq d(x,\cdot)\right\}.$$

Through this embedding non-expansive maps correspond exactly with contractions. So both the constructs *PRTOP* of pretopological spaces and continuous maps and *PRMET* of pre-metric spaces and non-expansive maps are fully embedded in *PRAP*.

*PRTOP* is both coreflectively and bireflectively embedded whereas *PRMET* is a coreflective and finitely productive subconstruct.

If $(X,\mathcal{A})$ is a pre-approach space then $(\Lambda(x))_{x\in X}$ is a *basis* for $\mathcal{A}$ if it fulfils $\mathcal{A}(x)=\hat{\Lambda}(x)$ for every $x\in X$, where

$$\hat{\Lambda}(x)=\left\{\varphi\in[0,\infty]^X \mid \forall\varepsilon,N\in\,]0,\infty[,\exists\psi\in\Lambda(x):\varphi\wedge N\leq\psi+\varepsilon\right\}.$$

*PRAP* is a well fibred topological construct. Initial and final structures are described as follows:
If $f:X\to Y$ is a map and $\varphi\in[0,\infty]^X$ then we denote $f(\varphi)$ the function defined as $f(\varphi):Y\to[0,\infty]:y\mapsto\inf\{\varphi(x)\mid f(x)=y\}$. Let $((X_i,\mathcal{A}_i))_{i\in I}$ be a class of *PRAP*-objects. If $(f_i:X\to(X_i,\mathcal{A}_i))_{i\in I}$ is a source then the collection $(\Lambda(x))_{x\in X}$ is a basis for the initial pre-approach system, where for all $x\in X$

$$\Lambda(x)=\left\{\sup_{j\in J}\varphi_j\circ f_j \mid J\in 2^{(I)},\varphi_j\in\mathcal{A}_j(f_j(x)),\forall j\in J\right\}.$$

If $(f_i:(X_i,\mathcal{A}_i)\to X)_{i\in I}$ is a sink then the collection $(\Lambda(x))_{x\in X}$ is a basis for the final pre-approach system, where for all $x\in X$

$$\Lambda(x)=\begin{cases}\bigcap_{\substack{i\in I\text{ s.t.}\\ f_i^{-1}(x)\neq\emptyset}}\bigcap_{z\in f_i^{-1}(x)}\left\{f_i(\varphi)\mid\varphi\in\mathcal{A}_i(z)\right\} & \text{if }\bigcup_{i\in I}f_i^{-1}(x)\neq\emptyset\\ \left\{\varphi\in[0,\infty]^X\mid\varphi(x)=0\right\} & \text{if }\bigcup_{i\in I}f_i^{-1}(x)=\emptyset.\end{cases}$$

Just like pretopological spaces, pre-approach spaces can also be characterized by convergence (see [4]). The difference with pretopological spaces however is that with each filter and each point we can associate a distance the point "is away from being a limit point" of the filter. First some notations. Given a set $X$, $F(X)$ stands for the set of all filters on $X$; if $\mathcal{F}\in F(X)$, then $U(\mathcal{F})$ stands for the set of all ultrafilters finer than $\mathcal{F}$. If $\mathcal{G}\subset 2^X$ then

$$\text{stack}_X\,\mathcal{G}:=\left\{B\subset X\mid\exists G\in\mathcal{G}:G\subset B\right\},$$

if $\mathcal{G}$ consists of a single set $G$ we write $\text{stack}_X G$ and if moreover $G$ consists of a single point $a$, we write $\text{stack}_X a$ for short. If no confusion can occur, we drop the subscript $X$.

**Definition 2.1** *A map* $\lambda:F(X)\to[0,\infty]^X$ *is called a* pre-approach limit *if it fulfils*

*(CAL1)* $\forall x \in X : \lambda(\text{stack}x)(x) = 0.$

*(CAL2)* $\mathcal{F} \subset \mathcal{G} \Rightarrow \lambda(\mathcal{G}) \leq \lambda(\mathcal{F}).$

*(PRAL) For any family* $(\mathcal{F}_j)_{j\in J}$ *of filters on* $X$:

$$\lambda\Big(\bigcap_{j\in J} \mathcal{F}_j\Big) = \sup_{j\in J} \lambda(\mathcal{F}_j).$$

It follows from the results in [8] that all the above structures are equivalent. We shall now state those transitions between the above structures which are needed in the sequel.

**Proposition 2.2** *1. If* $\mathcal{A}$ *is a pre-approach system on* $X$ *then the map* $\lambda_{\mathcal{A}} : F(X) \to [0,\infty]^X$ *defined by*

$$\lambda_{\mathcal{A}}\mathcal{F}(x) := \sup_{\varphi\in\mathcal{A}(x)} \inf_{F\in\mathcal{F}} \sup_{y\in F} \varphi(y)$$

*is a pre-approach limit on* $X$.

2. *If* $\lambda$ *is a pre-approach limit on* $X$ *then the system* $\mathcal{A}_\lambda$ *where for all* $x \in X$ :

$$\mathcal{A}_\lambda(x) := \Big\{\varphi \in [0,\infty]^X \mid \forall \mathcal{U} \in U(X) : \sup_{U\in\mathcal{U}} \inf_{y\in U} \varphi(y) \leq \lambda\mathcal{U}(x)\Big\}$$

*is a pre-approach system on* $X$.

■

**Proposition 2.3** *If* $(X,\lambda)$ *and* $(X',\lambda')$ *are pre-approach spaces and* $f : X \to X'$ *is a map then* $f$ *is a contraction if and only if for every filter* $\mathcal{F}$ *on* $X$, $\lambda'(\text{stack} f(\mathcal{F})) \circ f \leq \lambda(\mathcal{F})$. ■

**Proposition 2.4** *1. If* $\delta$ *is a pre-distance on* $X$ *then the system* $\mathcal{A}_\delta$ *where for all* $x \in X$ :

$$\mathcal{A}_\delta(x) := \Big\{\varphi \in [0,\infty]^X \mid \forall A \subset X : \inf_{a\in A} \varphi(a) \leq \delta(x,A)\Big\}$$

*is a pre-approach system on $X$ and the map $\lambda_\delta : F(X) \to [0,\infty]^X$ defined by*

$$\lambda_\delta \mathcal{F}(x) := \sup_{A \in \sec \mathcal{F}} \delta(x,A)$$

*is a pre-approach limit on $X$.*

2. *If $d$ is a pre-metric on $X$ then the map $\delta_d : X \times 2^X \to [0,\infty]$ defined by*

$$\delta_d(x,A) = \inf_{a \in A} d(x,a)$$

*is a pre-distance on $X$, the map $\lambda_d : F(X) \to [0,\infty]^X$ defined by*

$$\lambda_d \mathcal{F}(x) = \inf_{F \in \mathcal{F}} \sup_{y \in F} d(x,y)$$

*is a pre-approach limit on $X$ and the system $\mathcal{A}_d$ where for all $x \in X$,*

$$\mathcal{A}_d(x) = \{\varphi \in [0,\infty]^X \mid \varphi \leq d(x,\cdot)\}$$

*is a pre-approach system on $X$.*

■

A set $X$ equipped with a pre-approach limit (or, equivalently, a pre-distance or pre-approach system) is called a *pre-approach space* and is usually denoted $(X,\lambda)$. The associated pre-distance or pre-approach system are usually denoted simply $\delta$, $\mathcal{A}$ instead of $\delta_\lambda$, $\mathcal{A}_\lambda$ unless confusion might occur.

**Definition 2.5** *An object $X$ in a category $C$ with products is called an* exponential object *in $C$ provided that the functor $X \times - : C \to C$ has a right adjoint.*

In a topological construct, exponential objects are characterized by the existence of canonical function spaces: $X$ is exponential in a topological construct $C$ if and only if for each $C$-object $Y$ the set $C(X,Y)$ of all $C$-morphisms from $X$ to $Y$ can be endowed with a $C$-structure $\xi$ such that

1. The evaluation map $ev_{X,Y} : X \times (C(X,Y),\xi) \to Y : (x,f) \mapsto f(x)$ is a $C$-morphism.

2. For each $C$-object $Z$ and each $C$-morphism $h : X \times Z \to Y$ the map $h^* : Z \to (C(X,Y),\xi)$ defined by $h^*(z)(x) := h(x,z)$ is a $C$-morphism.

A category $C$ is called *cartesian closed* if every $C$-object is exponential. For more information on cartesian closedness and exponential objects we refer to [2], [3], [9], [10]. *PRAP* is not cartesian closed [4]. However, there exists a cartesian closed supercategory of *PRAP*. It is defined as follows:

**Definition 2.6** *A map* $\lambda : F(X) \to [0,\infty]^X$ *is called a* convergence-approach limit *if it fulfils the properties (CAL1) and (CAL2) of Definition 2.1, and the following weakening of (PRAL):*

*(CAL3)* $\forall \mathcal{F}, \mathcal{G} \in F(X) : \lambda(\mathcal{F} \cap \mathcal{G}) = \lambda(\mathcal{F}) \vee \lambda(\mathcal{G})$.

*The pair* $(X,\lambda)$ *is called a* convergence-approach space.

A map $f : (X,\lambda) \to (X',\lambda')$ between two convergence-approach spaces is a contraction if and only if $\forall \mathcal{F} \in F(X) : \lambda'(\text{stack} f(\mathcal{F})) \circ f \leq \lambda(\mathcal{F})$. In [4], it was proved that the category *CAP* of convergence-approach spaces and contractions is a cartesian closed topological supercategory of *PRAP*. If $(X,\lambda_X)$ and $(Y,\lambda_Y)$ are convergence-approach spaces then the canonical convergence-approach limit $\lambda_c$ on the set $C(X,Y)$ of all contractions from $X$ to $Y$ is defined by

$$\lambda_c\Psi(f) := \inf\left\{\alpha \in [0,\infty] \mid \forall \mathcal{F} \in F(X) : \lambda_Y(\text{stack}\Psi(\mathcal{F})) \circ f \leq \lambda_X(\mathcal{F}) \vee \alpha\right\}$$

for all $\Psi \in F(C(X,Y))$ and $f \in C(X,Y)$. Note that the infimum is actually a minimum, and that the set of numbers $\alpha$ satisfying the above condition is $[\lambda_c\Psi(f),\infty]$.

In [5], E. and R. Lowen proved that *PRAP* is finally dense in *CAP*. Application of Theorems 3.1 and 3.3 in Schwarz [13] gives the following useful characterization of exponential objects in *PRAP*.

**Proposition 2.7** *For a pre-approach space $X$ the following are equivalent:*

1. *$X$ is exponential in PRAP.*
2. *$\lambda_c$ is a pre-approach limit on $C(X,Y)$ for all pre-approach spaces $Y$.*
3. *$X \times -$ preserves coproducts and quotient maps.* ■

Although this characterization is useful at times, it does not give an internal description of the exponential objects in *PRAP*. Making use of this result, this however is what we shall obtain in the following section.

# 3 Exponentiality in *PRAP*

First of all we begin by observing that every pre-metric pre-approach space is exponential in *PRAP*. For a pre-metric pre-approach space $(X,\lambda_X) = (X,d)$ we will denote the open ball with center $x$ and radius $\varepsilon$ in $(X,d)$ by $B(x,\varepsilon)$. If $\Psi$ is a filter on $C(X,Y)$ and $F \subset X$, we denote $\Psi(\text{stack}F)$ by $\Psi(F)$. We require the following lemma.

**Lemma 3.1** *Let $(X,\lambda_X)$ be a pre-metric pre-approach space.*

1. *If $\lambda_X \mathcal{F}(x) < \varepsilon$ then stack$B(x,\varepsilon) \subset \mathcal{F}$.*

2. *If furthermore $(Y,\lambda_Y)$ is a pre-approach space, $f \in C(X,Y)$, $\alpha \in [0,\infty]$, and $\Psi$ is a filter on $C(X,Y)$, then the following are equivalent:*

   *(a)* $\forall \mathcal{F} \in F(X), \forall x \in X : \lambda_Y(\text{stack}\Psi(\mathcal{F}))(f(x)) \leq \lambda_X \mathcal{F}(x) \vee \alpha$,

   *(b)* $\forall \varepsilon > 0, \forall x \in X : \lambda_Y(\text{stack}\Psi(B(x,\varepsilon)))(f(x)) \leq \varepsilon \vee \alpha$.

**Proof.**

1. This follows from the observation that

$$\begin{aligned}\lambda_X \mathcal{F}(x) < \varepsilon &\Rightarrow \inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) < \varepsilon \\ &\Rightarrow \exists F \in \mathcal{F}, \forall y \in F : d(x,y) < \varepsilon \\ &\Rightarrow \text{stack}B(x,\varepsilon) \subset \mathcal{F}.\end{aligned}$$

2. To show that (a) $\Rightarrow$ (b), let $\varepsilon > 0$ and $x \in X$, choose $\mathcal{F} :=$stack$B(x,\varepsilon)$ and note that

$$\lambda_X(\text{stack}B(x,\varepsilon))(x) = \inf_{F\supset B(x,\varepsilon)} \sup_{y\in F} d(x,y) \leq \varepsilon.$$

To show that (b) $\Rightarrow$ (a) let $\mathcal{F}$ be an arbitrary filter on $X$. Then, according to the first part of the lemma

$$\lambda_Y(\text{stack}\,\Psi(\mathcal{F}))(f(x)) \leq \lambda_Y(\text{stack}\,\Psi(B(x,\varepsilon)))(f(x)) \leq \varepsilon \vee \alpha,$$

for every $x \in X$ and for every $\varepsilon > 0$ satisfying $\lambda_X \mathcal{F}(x) < \varepsilon$. This completes the proof.

$\blacksquare$

**Theorem 3.2** *Every pre-metric pre-approach space is exponential in PRAP.*

**Proof.** Let $(X, \lambda_X) = (X, d)$ be a pre-metric pre-approach space and let $(Y, \lambda_Y)$ be a pre-approach space. Let $(\Psi_i)_{i \in I}$ be a family of filters on $C(X,Y)$, $f \in C(X,Y)$, let $x \in X$ and $\varepsilon > 0$. It is easily verified that

$$\text{stack}\left(\bigcap_{i \in I} \Psi_i\right)(B(x,\varepsilon)) = \bigcap_{i \in I} \text{stack}\,\Psi_i(B(x,\varepsilon)).$$

Define $\alpha := \sup_{i \in I} \lambda_c \Psi_i(f)$. Note that

$$\begin{aligned}
\lambda_Y\left(\text{stack}\left(\bigcap_{i \in I} \Psi_i\right)(B(x,\varepsilon))\right)(f(x)) &= \lambda_Y\left(\bigcap_{i \in I} \text{stack}\,\Psi_i(B(x,\varepsilon))\right)(f(x)) \\
&= \sup_{i \in I} \lambda_Y(\text{stack}\,\Psi_i(B(x,\varepsilon)))(f(x)).
\end{aligned}$$

Now, for any $i \in I$

$$\lambda_Y(\text{stack}\,\Psi_i(B(x,\varepsilon)))(f(x)) \leq \lambda_X(\text{stack}\,B(x,\varepsilon))(x) \vee \lambda_c \Psi_i(f)$$

due to the definition of $\lambda_c \Psi_i(f)$. Noting that $\lambda_X(\text{stack}\,B(x,\varepsilon))(x) \leq \varepsilon$, we get

$$\lambda_Y\left(\text{stack}\left(\bigcap_{i \in I} \Psi_i\right)(B(x,\varepsilon))\right)(f(x)) \leq \varepsilon \vee \sup_{i \in I} \lambda_c \Psi_i(f) = \varepsilon \vee \alpha.$$

According to Lemma 3.1.2, it then follows that

$$\lambda_Y\left(\text{stack}\left(\bigcap_{i \in I} \Psi_i\right)(\mathcal{F})\right)(f(x)) \leq \lambda_X \mathcal{F}(x) \vee \alpha$$

for every $x \in X$ and every filter $\mathcal{F}$ on $X$. Applying the definition of $\lambda_c$ we can thus infer

$$\lambda_c\Big(\bigcap_{i\in I}\Psi_i\Big)(f) \leq \alpha = \sup_{i\in I}\lambda_c\Psi_i(f),$$

which, the other inequality being trivial, entails

$$\lambda_c\Big(\bigcap_{i\in I}\Psi_i\Big) = \sup_{i\in I}\lambda_c\Psi_i.$$

This means that $(C(X,Y),\lambda_c)$ is a pre-approach space, and consequently it follows from 2.7 that $(X,\lambda_X)$ is exponential in *PRAP*. ■

**Proposition 3.3** *Let $(X,\mathcal{A})$ be a pre-approach space. Then the following are equivalent:*

1. *$\mathcal{A}(x)$ has a largest element for every $x \in X$.*

2. $\forall x \in X, \forall A \subset X : \delta(x,A) = \inf_{a\in A}\delta(x,\{a\})$.

3. *$(X,\mathcal{A})$ is a pre-metric pre-approach space.*

■

The third condition actually is a restatement of the second, saying that $\delta(x,A)$ is just the pre-distance between $x$ and $A$ as defined in the pre-metric space $(X,d_\delta)$ where $d_\delta(x,a) := \delta(x,\{a\})$.

Second we shall now show that every exponential object in *PRAP* is a pre-metric space. Hereto we need the following preliminary results. Given a pre-approach space $(X,\mathcal{A})$ with limit function $\lambda$ and given $\varepsilon \in [0,\infty]$ we define a pretopological structure $q_\varepsilon$ on $X$ in the following way: a filter $\mathcal{F}$ converges to $x$ in $q_\varepsilon$ if and only if $\lambda\mathcal{F}(x) \leq \varepsilon$. Further we put, for any $x \in X$, $\varepsilon_x := \sup\{\varepsilon \in [0,\infty[ \mid q_\varepsilon \text{ is discrete in } x\}$. Furthermore for every $x \in X$ and $\varepsilon \in [0,\infty]$ let $\theta_x^\varepsilon$ be the function defined by $\theta_x^\varepsilon(y) := \varepsilon$ if $y \neq x$ and $\theta_x^\varepsilon(x) := 0$.

**Lemma 3.4** 1. *For every $\varphi \in \mathcal{A}(x)$:*

$$\inf_{y\neq x}\varphi(y) \leq \varepsilon_x.$$

2. *For every $\varepsilon < \varepsilon_x : \theta_x^\varepsilon \in \mathcal{A}(x)$.*

**Proof.**

1. Suppose there exists a $\varphi \in \mathcal{A}(x)$ with $\inf_{y \neq x} \varphi(y) > \varepsilon_x$. Fix positive real numbers $a$ and $\delta$ such that $\varepsilon_x < a + \delta < \inf_{y \neq x} \varphi(y)$. Now let $\mathcal{F}$ be a filter on $X$, different from stack$x$. Then

$$\lambda \mathcal{F}(x) = \sup_{\psi \in \mathcal{A}(x)} \inf_{F \in \mathcal{F}} \sup_{y \in F} \psi(y) \geq \inf_{F \in \mathcal{F}} \sup_{y \in F} \varphi(y).$$

   Now $\sup_{y \in F} \varphi(y) > a + \delta$ for every $F \in \mathcal{F}$, so $\lambda \mathcal{F}(x) \geq a + \delta > a$, hence $\mathcal{F} \overset{q_a}{\not\to} x$. By the arbitrariness of $\mathcal{F}$ this means $q_a$ is discrete in $x$, but since $a > \varepsilon_x$, this is a contradiction.

2. Let $\varepsilon < \varepsilon_x$ and suppose $\theta_x^\varepsilon$ does not belong to $\mathcal{A}(x)$, then, by Proposition 2.2, there exists an ultrafilter $\mathcal{U}$ on $X$ such that

$$\sup_{U \in \mathcal{U}} \inf_{y \in U} \theta_x^\varepsilon(y) > \lambda \mathcal{U}(x),$$

   whence there exists some $U \in \mathcal{U}$ not containing $x$ and $\lambda \mathcal{U}(x) < \varepsilon$, i.e., $\mathcal{U}$ differs from stack$x$ and $\mathcal{U} \overset{q_\varepsilon}{\to} x$. This however implies that $q_\varepsilon$ is not discrete in $x$, which contradicts the definition of $\varepsilon_x$.

■

**Lemma 3.5** *For every $x \in X$, $\theta_x^{\varepsilon_x}$ belongs to $\mathcal{A}(x)$.*

**Proof.** For $\varepsilon_x = 0$ there is nothing to show. Suppose $\varepsilon_x > 0$ and for every $\gamma \in \,]0, \varepsilon_x[$, define $\psi^\gamma := \theta_x^{\varepsilon_x - \gamma}$. Then $\psi^\gamma$ belongs to $\mathcal{A}(x)$ for all $\gamma \in \,]0, \varepsilon_x[$ because of the previous lemma. Since moreover for every $\gamma \in \,]0, \varepsilon_x[$:

$$\theta_x^{\varepsilon_x} = \inf_{\beta \in \,]0, \varepsilon_x[} \psi^\beta + \beta \leq \psi^\gamma + \gamma,$$

it follows from (A2) that $\theta_x^{\varepsilon_x} \in \mathcal{A}(x)$. ■

**Theorem 3.6** *If $(X, \lambda)$ is exponential in PRAP, then $\mathcal{A}(x)$ has a largest element for every $x \in X$.*

**Proof.** Fix $x \in X$. Define

$$Z := \bigcup_{\varphi \in \mathcal{A}(x)} X \times \{\varphi\}$$

equipped with the coproduct structure in *PRAP*. Let $Y$ be the quotient of $Z$ in *PRAP* by identification of the points $(x, \varphi)$, $\varphi \in \mathcal{A}(x)$, i.e., the quotient with respect to the map

$$f : Z \to Y : (z, \mu) \mapsto \begin{cases} (z, \mu) & \text{if } z \neq x \\ a & \text{if } z = x. \end{cases}$$

Now for every $\varphi \in \mathcal{A}(x)$ and for every $\nu_\varphi \in \mathcal{A}(x)$ define

$$\begin{cases} \psi_\varphi^{\nu_\varphi}(z, \mu) := \infty & \text{if } \mu \neq \varphi \\ \psi_\varphi^{\nu_\varphi}(z, \varphi) := \nu_\varphi(z). \end{cases}$$

A straightforward verification shows that every $\psi_\varphi^{\nu_\varphi}$ belongs to the pre-approach system $\mathcal{A}_Z(x, \varphi)$. This implies that

$$\xi \circ pr_X^{X \times Z} \vee \psi_\varphi^{\nu_\varphi} \circ pr_Z^{X \times Z} \in \mathcal{A}_{X \times Z}(x, (x, \varphi))$$

for every $\xi \in \mathcal{A}(x)$ and every $\nu_\varphi \in \mathcal{A}(x)$, where $pr_X^{X \times Z}$ and $pr_Z^{X \times Z}$ are the projections of $X \times Z$ onto $X$ respectively $Z$. Let us now consider the final *PRAP*-structure on $X \times Y$ with respect to the map

$$1_X \times f : X \times Z \to X \times Y.$$

It is easy to see that for every $x' \in X$, every $y \in X \setminus \{x\}$, and every $\mu \in \mathcal{A}(x)$,

$$\inf_{\varphi \in \mathcal{A}(x)} \left( \varphi \circ pr_X^{X \times Z} \vee \psi_\varphi^{\nu_\varphi} \circ pr_Z^{X \times Z} \right)(x', (y, \mu)) = \inf_{\varphi \in \mathcal{A}(x)} \left( \varphi(x') \vee \psi_\varphi^{\nu_\varphi}(y, \mu) \right) \tag{1}$$

is the value in $(x', (y, \mu))$ of a local pre-distance in $(x, a)$ with respect to this final structure on $X \times Y$ for every collection $(\nu_\varphi)_{\varphi \in \mathcal{A}(x)} \subset \mathcal{A}(x)$. Since $(X, \lambda)$ is exponential in *PRAP* and $f : Z \to Y$ is final, $1_X \times f : X \times Z \to X \times Y$ is also final (see Proposition 2.7), so (1) is also a local pre-distance in $(x, a)$ with respect to the product structure on $X \times Y$. Let us first investigate this product structure. The final local distances in $a$ on $Y$ are characterized as follows: $\chi \in \mathcal{A}_Y(a)$ if and only if there exists a family $(\nu'_\varphi)_{\varphi\in\mathcal{A}(x)}$ in $\mathcal{A}(x)$ such that:

$$\forall (y,\mu) \in Y\setminus\{a\} : \begin{cases} \chi(y,\mu) = \inf_{\varphi\in\mathcal{A}(x)} \psi_\varphi^{\nu'_\varphi}(y,\mu), \\ \chi(a) = 0. \end{cases}$$

We further know that

$$\Lambda_{X\times Y}(x,a) := \left\{\xi\circ pr_X^{X\times Y} \vee \chi\circ pr_Y^{X\times Y} \mid \xi\in\mathcal{A}(x), \chi\in\mathcal{A}_Y(a)\right\}$$

is a basis for the product pre-approach system on $X\times Y$, where $pr_X^{X\times Y}$ and $pr_Y^{X\times Y}$ are the projections of $X\times Y$ onto $X$ respectively $Y$. From these observations we infer that for every $\varepsilon, N \in \,]0,\infty[$ there exist a function $\xi_N^\varepsilon \in \mathcal{A}(x)$ and a collection $\left(\nu'^{\varepsilon}_{N,\varphi}\right)_{\varphi\in\mathcal{A}(x)} \subset \mathcal{A}(x)$ such that for every $x'\in X$, every $y\in X\setminus\{x\}$, and every $\mu\in\mathcal{A}(x)$,

$$\inf_{\varphi\in\mathcal{A}(x)} \left(\varphi(x')\vee\psi_\varphi^{\nu_\varphi}(y,\mu)\right)\wedge N \le \xi_N^\varepsilon(x') \vee \inf_{\varphi\in\mathcal{A}(x)} \psi_\varphi^{\nu'^{\varepsilon}_{N,\varphi}}(y,\mu)+\varepsilon.$$

If we choose $\nu_\varphi := 0$ for every $\varphi\in\mathcal{A}(x)$, this reduces to

$$\mu(x')\wedge N \le \xi_N^\varepsilon(x')\vee\nu'^{\varepsilon}_{N,\mu}(y)+\varepsilon.$$

Now take $N\in\,]0,\infty[$ fixed and choose $\mu\in\mathcal{A}_N(x) := \{\varphi\in\mathcal{A}(x) \mid \varphi\le N\}$. Then for every $x'\in X$, for every $y\in X\setminus\{x\}$, and for every $\varepsilon\in\,]0,\infty[$:

$$\mu(x') \le \left(\xi_N^\varepsilon\vee\theta_x^{\varepsilon_x}\right)(x')\vee\nu'^{\varepsilon}_{N,\mu}(y)+\varepsilon.$$

Hence for every $x'\in X$ and every $\varepsilon\in\,]0,\infty[$:

$$\begin{aligned} \mu(x') &\le \inf_{y\in X\setminus\{x\}} \left(\left(\xi_N^\varepsilon\vee\theta_x^{\varepsilon_x}\right)(x')\vee\nu'^{\varepsilon}_{N,\mu}(y)+\varepsilon\right) \\ &= \left(\xi_N^\varepsilon\vee\theta_x^{\varepsilon_x}\right)(x')\vee\inf_{y\in X\setminus\{x\}}\nu'^{\varepsilon}_{N,\mu}(y)+\varepsilon. \end{aligned}$$

Since $\nu'^{\varepsilon}_{N,\mu}\in\mathcal{A}(x)$, we know however from Lemma 3.4 that

$$\inf_{y\in X\setminus\{x\}}\nu'^{\varepsilon}_{N,\mu}(y)\le\varepsilon_x,$$

whence

$$\mu(x') \leq \left(\xi_N^\varepsilon \vee \theta_x^{\varepsilon_x}\right)(x') + \varepsilon. \tag{2}$$

If we now define

$$\xi_N := \inf_{\varepsilon \in ]0,\infty[} \left(\left(\xi_N^\varepsilon \vee \theta_x^{\varepsilon_x}\right) + \varepsilon\right) \wedge N,$$

then $\xi_N \in \mathcal{A}_N(x)$. Moreover, $\xi_N$ is the largest element of $\mathcal{A}_N(x)$, as it follows from (2) that every $\mu \in \mathcal{A}_N(x)$ is smaller than $\xi_N$.

Finally we now define

$$\xi := \sup_{N \in ]0,\infty[} \xi_N,$$

then for every $N \in ]0,\infty[$: $\xi \wedge N \geq \xi_N$. On the other hand,

$$\begin{aligned} \xi \wedge N &= \left(\sup_{M \in ]0,\infty[} \xi_M\right) \wedge N \\ &= \sup_{M \in ]0,\infty[} (\xi_M \wedge N) \end{aligned}$$

and since $\xi_M \wedge N$ belongs to $\mathcal{A}_N(x)$ for every $M \in ]0,\infty[$, it follows that $\xi \wedge N \leq \xi_N$. By (A2) this implies that $\xi$ belongs to $\mathcal{A}(x)$. Furthermore $\xi$ is the largest element of $\mathcal{A}(x)$. Indeed, suppose there exist a function $\psi \in \mathcal{A}(x)$ and an element $z$ in $X$ such that $\psi(z) > \xi(z)$. Take $N > \psi(z)$, then we have $\psi(z) \wedge N > \xi(z) \wedge N = \xi_N(z)$ and $\psi \wedge N \in \mathcal{A}_N(x)$, which contradicts $\xi_N$ being the largest element of $\mathcal{A}_N(x)$. This completes the proof. ■

Combining Theorem 3.2, Proposition 3.3 and Theorem 3.6, we can now give the following internal description of the exponential objects in *PRAP*:

**Theorem 3.7** *For a pre-approach space $(X,\lambda)$ the following are equivalent:*

1. *$(X,\lambda)$ is exponential in PRAP.*

2. *$(X,\lambda)$ is pre-metric.*

3. *$\mathcal{A}(x)$ contains a largest element for every $x \in X$.*

■

As already stated, *PRTOP* is a bireflective and bicoreflective subcategory of *PRAP*. Moreover, the class of exponential objects in *PRTOP* shows much resemblance with the class of exponential objects in *PRAP*, as well categorically as objectwise.

**Definition 3.8** *A pretopological space* $(X,q)$ *is* finitely generated *if every point* $x$ *in* $X$ *has a smallest neighborhood* $V_x$.

Of course, this means $V_x = \bigcap \mathcal{V}_q(x) \in \mathcal{V}_q(x)$ for every $x \in X$. In [6], E. Lowen and G. Sonck proved the following result:

**Theorem 3.9** *A pretopological space is exponential in PRTOP if and only if it is finitely generated.* ■

Note the similarity with the situation in *PRAP*: by Theorem 3.7 a pre-approach space $(X,\lambda)$ is exponential in *PRAP* if and only if $\mathcal{A}(x)$ possesses a largest element for every $x \in X$. Since the concept of neighborhood filter in a pretopological space is generalized by the concept of pre-approach system in a pre-approach space, we see that such a "finitely generated" pre-approach space could be a logical generalization of a finitely generated pretopological space. This is justified by the following fact.

**Proposition 3.10** *A pretopological space* $(X,q)$ *is finitely generated if and only if* $\mathcal{A}_q(x)$ *has a largest element for every* $x \in X$.

**Proof.** Note that $\mathcal{A}_q(x)$ has $\{\theta_V \mid V \text{ neighborhood of } x \text{ for } q\}$ as a basis where $\theta_V(y) := 0$ if $y \in V$ and $\theta_V(y) := \infty$ if $y \notin V$. Suppose for every $x \in X$ there exists a smallest element $V_x$ of the neighborhood filter $\mathcal{V}_q(x)$. Then

$$\begin{aligned}
\varphi \in \mathcal{A}_q(x) &\iff \forall \varepsilon, N \in ]0,\infty[, \exists V_N^\varepsilon \in \mathcal{V}_q(x) : \varphi \wedge N \leq \theta_{V_N^\varepsilon} + \varepsilon \\
&\iff \forall \varepsilon, N \in ]0,\infty[, \varphi \wedge N \leq \theta_{V_x} + \varepsilon \\
&\iff \varphi \leq \theta_{V_x}.
\end{aligned}$$

Conversely, suppose $\mathcal{A}_q(x)$ has a largest element $\varphi_x$ for every $x \in X$. Then for every neighborhood $V$ of $x$, $\theta_V \leq \varphi_x$. If we define $V_x := \bigcap \mathcal{V}_q(x)$, then we see that $\theta_{V_x} = \sup_{V \in \mathcal{V}_q(x)} \theta_V \leq \varphi_x$, so $\theta_{V_x}$ belongs to $\mathcal{A}_q(x)$. This implies $V_x = \{\theta_{V_x} < 1\} \in \mathcal{V}_{\mathcal{A}}(x)$ where $\mathcal{V}_{\mathcal{A}}(x)$ is the neighborhood filter of $x$ in the *PRTOP*-bicoreflection of $(X, \mathcal{A}_q)$, i.e., in $(X,q)$. ■

So we can conclude that the exponential objects in *PRAP* are characterized by a generalization of the property which characterizes the exponential objects in *PRTOP*. There are also categorical similarities between the categories of exponential objects in *PRTOP* and *PRAP*. It is a well known fact that the full subcategory *FING* of *PRTOP* whose objects are the finitely generated pretopological spaces is the bicoreflective hull of the finite spaces in *PRTOP*. We will prove a similar result for *PRMET* in *PRAP*.

**Proposition 3.11** *PRMET is the bicoreflective hull of the finite spaces in PRAP.*

**Proof.** Let $(X,\delta)$ be a finite pre-approach space, then for the subsets $A$ of $X$ and for all elements $x$ of $X$,

$$\delta(x,A) = \inf_{a\in A} \delta(x,\{a\}),$$

so by Proposition 3.3 $(X,\delta)$ is a pre-metric pre-approach space. As *PRMET* is bicoreflective in *PRAP*, the bicoreflective hull of the finite pre-approach spaces is a subcategory of *PRMET*.

Conversely, let $(X,d)$ be a pre-metric space. We have to show that $(X,\mathcal{A}_d)$ belongs to the bicoreflective hull of the finite pre-approach spaces. For every finite subset $A \subset X$ define $d_A(x,y) := d(x,y)$ for every $x,y \in A$ and define

$$f:\sum_{A\ \text{finite}\subset X} A \longrightarrow X : (x,A) \mapsto x.$$

Then

$$(A,d_A) \overset{j_A}{\hookrightarrow} \sum_{A\ \text{finite}\subset X} (A,d_A) \overset{f}{\rightarrow} (X,\mathcal{A}_d)$$

is final in *PRAP* for every $A \subset X$ ($j_A$ is the canonical injection):

Let $\mathcal{A}(x)$ be the final pre-approach system in $x$. We are to show that $\mathcal{A} = \mathcal{A}_d$. A basis for $\mathcal{A}(x)$ is given by

$$\begin{aligned}\Lambda(x) &= \bigcap_{A\ \text{finite}\ni x} \left\{f(\varphi) \mid \varphi \in \mathcal{A}_{\Sigma}(x,A)\right\}\\ &= \bigcap_{A\ \text{finite}\ni x} \left\{f(\varphi) \mid \exists\psi \in \mathcal{A}_{d_A}(x), \forall y \in A : \varphi(y,A) = \psi(y)\right\}\end{aligned}$$

$$
\begin{aligned}
&= \bigcap_{A\,\text{finite}\ni x} \left\{ f(\varphi) \mid \exists \psi \le d_A(x,.), \forall y \in A : \varphi(y,A) = \psi(y) \right\} \\
&= \bigcap_{A\,\text{finite}\ni x} \left\{ f(\varphi) \mid \forall y \in A : \varphi(y,A) \le d_A(x,y) \right\}.
\end{aligned}
$$

So

$$
\begin{aligned}
\varphi' \in \Lambda(x) &\iff \forall A \text{ finite } \ni x, \exists \varphi \text{ s.t. } \forall y \in A : \varphi(y,A) \le d(x,y) \\
&\qquad \text{and } \varphi' = f(\varphi) \\
&\iff \forall A \text{ finite } \ni x, \exists \varphi \text{ s.t. } \forall y \in A : \varphi(y,A) \le d(x,y) \\
&\qquad \text{and } \forall z \in X : \varphi'(z) = \inf_{B\,\text{finite}\ni z} \varphi(z,B) \\
&\iff \forall A \text{ finite } \ni x, \forall y \in A : \varphi'(y) \le d(x,y) \\
&\iff \forall z \in X : \varphi'(z) \le d(x,z) \\
&\iff \varphi' \in \mathcal{A}_d(x),
\end{aligned}
$$

so $\mathcal{A} = \mathcal{A}_d$. ■

**Remark 3.12** *PRMET is cartesian closed since it is the bicoreflective hull of the finite pre-approach spaces and it contains only exponential objects ([11]).*

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