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PANAGIS KARAZERIS

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COMPACT TOPOLOGIES ON LOCALLY PRESENTABLE CATEGORIES

by Panagis KARAZERIS

RESUME. Les topologies sur les catégories localement présentables généralisent les notions familières suivantes, d'une part les topologies de Grothendieck sur des petites catégories, d'autre part les topologies de Gabriel sur des catégories abéliennes à générateurs. Dans cet article on introduit une condition, qui peut être vérifiée pour les topologies précédentes, appelée "compacité".

Dans le cas des topologies de Grothendieck, cette condition signifie qu'un recouvrement quelconque a un sous-recouvrement fini. Les topologies compactes correspondantes ont des localisations fermées dans la catégorie donnée pour des colimites filtrantes monomorphiques.

On examine aussi la fermeture des objets séparés et des faisceaux pour les colimites filtrantes. Les topologies compactes sur une catégorie localement de présentation finie forment un locale. Si cette catégorie est un topos cohérent, alors le locale est compact et localement compact. Si le topos est celui des faisceaux sur un espace cohérent, alors le locale en question est celui des ouverts pour la topologie recollée ("path topology") sur cet espace.

1 Introduction

By a topology on a locally presentable category we mean a universal closure operator on such a category:

Basic Definition *A* **topology** *on a category with pullbacks is a process, which associates to each subobject* $S \rightarrowtail A$ *of an object* A*, another subobject* $j_A(S) \rightarrowtail A$

$j_A(S) \leq A$,

$j_A(j_A(S)) = j_A(S)$,

$S \leq T$ implies $j_A(S) \leq j_A(T)$,

$f^{-1}(j_A(S)) = j_B(f^{-1}(S))$, *for all arrows* $f\colon B \to A$ *in the category and* f^{-1} *denoting pulling back along* f.

This notion has become a standard category theoretic tool since the 1970's, but some examplification of it may still be enlightening: To start with a somewhat trivial case, when the category in question is a lattice A, a topology on it is nothing else but a nucleus (as in [10], p. 48). Indeed, there is a one to one correspondence between the two, sending a nucleus j on A to the closure operator which, to $a \leq b$ assigns $j_b(a) = j(a) \wedge b$ and, on the other hand, sending a universal closure operator to the nucleus that has the closure of a in the top element of A as value on $a \in A$.

On the category of abelian groups we may consider the topology, which assigns to a subgroup H of G the closure $j(H) = \{g \in G \mid \exists n \in \mathbf{N} \text{ s.t } n.g \in H\}$, i.e the elements of G that are torsion with respect to H. This example may be held responsible for the term "torsion theory" used in ring theory to describe topologies on module, or more generally, abelian categories. It also provides for one possible justification for the use of the word topology in this setting: Such a piece of data renders the ring over which we consider the modules linearly topological. The other and probably more burdensome justification is the use of the word by Grothendieck in generalizing the notion of covering family from topological spaces to small categories, in view of the correspondence between universal closure operators on presheaf categories and such covering families on the indexing category, leading to the developement of topos theory. [1]

In a manner similar to that of topos theory, universal closure operators on a locally α-presentable $(l.\alpha - p)$ category correspond to endomorphisms, satisfying the Lawvere- Tierney axioms, of an object Ω that serves as a subobject classifier for the category ([6]). We know, by the general theory of these categories given in [9], that such an $\mathcal{E}$ is equivalent to a category of those $\underline{Set}$-valued functors from a small category $\mathbf{C}$ with colimits of size less than α (α a regular cardinal) that preserve the colimits in question. The object Ω, then, lives in the full presheaf category $\underline{Set}^{\mathbf{C}^{op}}$. When the category happens to be a topos, the object Ω is the subobject classifier for the topos.

A universal closure operator on a locally presentable category corresponds also to a family of subobjects of the generators, satisfying properties analogous to those of a Gabriel topology on an abelian category with generators ([17]), or of a Grothendieck topology on a small category (or better on the corresponding presheaf category) ([14]). An extra property is required to ensure that, when viewing the objects in a locally presentable category as left exact functors, the closure of a subobject is left exact ([5]). (We refer to any of the above variants of a universal closure operator as a topology.)

Contrary to the classical cases of abelian categories or toposes, a topology, in the above sense, does not always give rise to a localization, although a localization uniquely determines a topology. A topology on a locally presentable category determines a full subcategory of it, the one consisting of the objects having the unique extension property with respect to dense monos. Such a subcategory is reflective ([8]) but the reflection now need not be left exact.

In the first section we introduce a condition on a topology on a locally finitely presentable $(l.f.p)$ category, that we call compactness, since for

[1] We should also mention here the work by Castellini, Dikranjan, Giuli and others (cf. for example Dikranjan and Giuli, Closure Operators I, Topology and Applications 27 (1987), 129-143) which places the study of closure operators in categories in a more general framework and succeeds, among other things, in accounting for the ordinary closure operators of General Topology as well. We do not pursue these ideas here though.

Grothendieck topologies it simply means that every covering has a finite subcovering. Such kind of finiteness conditions on a topology have been considered in the past, either for Grothendieck topologies ([21]), or in the abelian context ([17], [18]). Here we elaborate on that notion of compactness. We start from what compactness should intuitively mean: when the supremum of a family of subobjects is dense (with respect to the topology), then so is the supremum of some finite subfamily. We do not necessarily require that property for subobjects of arbitrary objects, but for subobjects of, say, the representables or of the free object on one generator, in other words for subobjects of objects in a set of regular presentable generators. We arrive at an invariant equivalent formulation: A topology is compact if, viewed as a closure operator, it commutes with directed suprema of subobjects of any object. What we said here holds for $l.\alpha - p$ categories and notions of β-compactness (in the obvious generalized sense), when β is larger than α.

The condition was considered in connection with the problem of finding when the category of sheaves is closed under filtered colimits (similar considerations led to it in [21], [17], [18]). This problem is taken up in the second section. First, we show that the compactness of a topology implies that the subcategory of separated objects is closed under filtered colimits. We believe that this result has not been noticed even for the case of Grothendieck topologies. On that we base the rest of the results in that section. Concerning the full subcategory of sheaves, we have the following: It is closed under monomorphic filtered colimits when the topology is compact and, conversely, the topology induced by a localization is compact when the localization is closed under monomorphic filtered colimits. When the ambient category is locally coherent, meaning that finitely generated subobjects of finitely presentable ones are finitely presentable ([7]), the sheaves are closed under filtered colimits. A condition, stronger than compactness, sufficient for the preservation of filtered colimits by the inclusion of sheaves is also discussed. That condition generalizes one, which is known to be equivalent to the preservation of filtered colimits in the abelian case ([18]). Still, the results of the second section are valid for $l.\alpha - p$ categories and β-compact topologies, β larger than α.

The set of localizations of a topos or of a category of modules over a ring, ordered under inclusion is the dual of a frame. Or, equivalently, the set of topologies, with the order induced by the order of subobjects, is a frame. Similar results hold for the localizations of a locally presentable category in which finite limits commute with filtered colimits. Despite the ramification of concepts related to localization, the above results hold for each of the two branches: In a locally presentable category satisfying the exactness property mentioned above, localizations form the dual of a frame ([3]), while topologies form a frame ([6]).The compact topologies on an *l.f.p* category constitute a subframe of that of all topologies. When the *l.f.p* category is a coherent topos then the frame in question, which in this connection can be thought off as the frame of coherent extensions of the coherent theory classified by the topos ([13]), is a locally compact, compact one. More specifically when the topos is that of sheaves on a coherent space, the frame can be identified as the frame of opens for the patch topology on the given coherent space.

We should point out here that the aforementioned frame can be identified with other well known constructions in special cases, notably, when the *l.f.p* category is R-modules over a commutative ring R, with the dual Zariski spectrum of the ring ([11]). This fact though can be expressed in a more comprehensive way using the formalism of Gabriel topologies on algebraic quantales. For that we intend to present this connection elsewhere.

The work presented here is part of the author's Ph.D thesis ([11]). I wish to express my sincere gratidude to my teacher Anders Kock for his skillful teaching and his support. I would also like to thank Professors F. Borceux and F. W. Lawvere for comments that contributed to the shaping of this work, as well as M. Adelmann and J. Schmidt for useful discussions.

2 Compact Topologies

When we are dealing with a Grothendieck topology or a topology on a category of modules we want to be able to verify whether a topology has a cerain property by looking at a presentation of it in terms of dense sieves or dense (topologizing, in the ring-theoreric terminology) ideals.

In particular the intuitive idea of compactness - a family of subobjects has dense supremum if a finite subfamily of it does - should be referring to subobjects of objects of a certain type (free, representable, the ground ring etc.), without on the other hand depending on the choice of it. This is the aim of the next Proposition.

2.1 Proposition. *Let $\mathcal{E}$ be a locally α -presentable ($l.\alpha - p$) category, j a topology on it and $\mathcal{M}$ a set of regular generators, which are α-presentable. The following are equivalent:*

(i) *Whenever a β-directed family of subobjects of an object in $\mathcal{M}$ has j-dense supremum, there is a member of the family which is j-dense.*

(ii) *Whenever a β-directed family of subobjects of an object in $\mathcal{E}$ has j-dense supremum, there is a member of the family which is j-dense.*

(iii) *Any j-dense subobject of an α-presentable P contains a β- generated subobject that is j-dense in P.*

(iv) *The closure operator commutes with β-directed suprema of subobjects of α-presentable objects.*

(v) *The closure operator commutes with β-directed suprema of subobjects of all objects.*

Proof: The implications $(v) \Rightarrow (iv)$ and $(ii) \Rightarrow (i)$ are trivial, while $(iv) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$ follow immediately from the fact that in a $l.\alpha - p$ category every object can be written as the β-directed supremum of the β-generated subobjects of it. Also the implication $(iii) \Rightarrow (ii)$ follows immediately if we note that a β-directed supremum of subobjects is the monomorphic β-filtered colimit of those subobjects.

$(i) \Rightarrow (ii)$: Recall from [9] (Satz 7.6) that in a $l.\alpha - p$ category every α-presentable object can be written as a retract of a colimit of size less than α of objects in $\mathcal{M}$, where $\mathcal{M}$ is as in the statement above. Let then P be an α-presentable object, $m: P \to colim_J G_k$ a split monic with section e, with $G_k \in \mathcal{M}$, $card J < \alpha$. Let $\{A_i \rightarrowtail P | i \in I\}$ a

β-directed family of subobjects of P, the supremum of which is dense in P. Consider the following pullback:

$$\begin{array}{ccccc} j(\bigvee_I A_i) \times_P G_k & \cong j(\bigvee_I A_i \times_P G_k) & \longrightarrow & G_k & \\ & \downarrow & & & \searrow \ colim_J G_k \\ & j(\bigvee_I A_i) & \rightarrowtail & P & \swarrow \end{array}$$

By the universality of the closure operator and the commutation of β-directed suprema with pullbacks in $\mathcal{E}$ we have

$$j(\bigvee_I A_i) \times_P G_k \cong j((\bigvee_I A_i) \times_P G_k) \cong j(\bigvee_I (A_i \times_P G_k))$$

The iso in the lower row gives an iso in the corresponding upper row and then the compactness hypothesis gives an iso $j(A_i \times_P G_k) \cong G_k$, for some $i \in I$. For each $k \in J$, now, we have a morphism from G_k to a $j(A_i)$ (the inverse for the iso we found followed by the projection of the pullback

$$j(A_i \times_P G_k) \cong j(A_i) \times_P G_k$$

to $j(A_i)$) and, since there are less than α many k's and I is β-directed, we can fix $i \in I$ so that for all the G_k's in the colimit there is $G_k \to j(A_i)$, hence there is a factorization $g\colon colim_J G_k \to j(A_i)$. In that way $g.m$ becomes a right inverse for the mono $j(A_i) \rightarrowtail P$, so $A_i \rightarrowtail P$ is dense.

$(iii) \Rightarrow (iv)$. Here we have to appeal to the representation of $\mathcal{E}$ as the category of α-lex Set-valued functors, $\mathcal{E} = \alpha - Lex(\mathbf{C}^{op}, \underline{Set})$. In the representation just mentioned there is a canonical choice for $\mathbf{C}$, namely $\mathbf{C} = \mathcal{E}_{\alpha-p}$. Then the α-presentable objects of $\mathcal{E}$ are exactly the representables. Subsequently we deliberately avoid to distinguish between the two. Let us then be given a β-directed family of subobjects $\{A_i \rightarrowtail P | i \in I\}$ of an α-presentable object P. Notice then that the inequality $\bigvee_I j_P(A_i) \leq j_P(\bigvee_I A_i)$ always holds. For the other inequality recall that the closure of a subobject $A \rightarrowtail P$ of a representable is

$$j_P(A)(Q) = \{x: Q \to P | x^{-1}(A) \in D_j(Q)\},$$

where by $D_j(Q)$ we mean the set of j-dense subobjects of Q (cf. [5]). Then $x \in j_P(\bigvee_I A_i)(Q)$ iff $x^{-1}(A_i) = x^{-1}(A_i) \in D_j(Q)$, in which case (ii) tells us that there is $i \in I$ with $x^{-1}(A_i) \in D_j(Q)$, so that $x \in j_P(A_i)(Q)$. That way we have shown that $j_P(\bigvee_I A_i) \leq \bigvee_I j_P(A_i)$ in the full presheaf category $\underline{Set}^{\mathbf{C}^{op}}$. Since the inclusion $\mathcal{E} \hookrightarrow \underline{Set}^{\mathbf{C}^{op}}$ preserves β-filtered colimits the same inequality holds in $\mathcal{E}$.

$(iv) \Rightarrow (v)$: We consider an object A of $\mathcal{E}$ and a family $\{A_i \rightarrowtail A \mid i \in I\}$ of subobjects of it, which is β-directed. Write A as the α-filtered colimit of α-presentable objects above it. Consider then the pullback of $\bigvee_I j(A_i) \rightarrowtail j(\bigvee_I A_i) \rightarrowtail A$ along an injection

$$P \to colim_{\mathcal{E}_{\alpha-p}/A} P \cong A$$

$$\begin{array}{ccccc}
\bigvee_I j(A_i) \times_A P & \cong & j(\bigvee_I A_i) \times_A P & \rightarrowtail & P \\
\downarrow & & \downarrow & & \downarrow \\
\bigvee_I j(A_i) & \rightarrowtail & j(\bigvee_I A_i) & \rightarrowtail & A
\end{array}$$

The following hold:

$$j(\bigvee_I A_i) \times_A P \cong j((\bigvee_I A_i) \times_A P \cong j(\bigvee_I (A_i \times_A P))$$
$$\cong \bigvee_I j(A_i \times_A P) \cong \bigvee_I (j(A_i) \times_A P) \cong \bigvee_I j(A_i) \times_A P$$

using appropriately the universality of the closure operator, the commutation of the closure with β-directed suprema of subobjects of α-presentable objects and the commutation in $\mathcal{E}$ of pullbacks with β-directed suprema. Moreover, the commutation in $\mathcal{E}$ of pullbacks with α-filtered colimits gives:

$$colim_{\mathcal{E}_{\alpha-p}/A}(\bigvee_I j(A_i) \times_A P) \cong j(\bigvee_I A_i) \times_A colim_{\mathcal{E}_{\alpha-p}/A} P$$

$$\cong j(\bigvee_I A_i) \times_A A_i \cong j(\bigvee_I A_i)$$

So having an arrow $j(\bigvee_I A_i) \times_A P \cong \bigvee_I j(A_i) \times_A P \to \bigvee_I A_i$ for each $P \in \mathcal{E}_{\alpha-p}/A$ we get a factorization through the colimit $j(\bigvee_I A_i)$, $g : j(\bigvee_I \longrightarrow A_i) \bigvee_I j(A_i)$. Chasing the diagram we show for the inclusions $i_1 : \bigvee_I j(A_i) \rightarrowtail A$, $i_2 : j(\bigvee_I A_i) \rightarrowtail A$ and the projection $s_P j(\bigvee_I A_i) \times_A P \longrightarrow j(\bigvee_I A_i)$ that $i_1.g.s_P = i_2.s_P$ for all P, giving $i_1.g = i_2$, so g is an iso as required. ∎

The previous equivalences allow us now to say what a compact topology is.

Definition *Let $\mathcal{E}$ be $l.a-p$ category, j a topology on it, β a cardinal, $\alpha \leq \beta$. We say that j is β-***compact** *if it satisfies any of the equivalent conditions given in Proposition 1.1 above. We reserve the word* **compact** *for the case $\alpha = \beta = \aleph_0$.*

Examples: a. Any Grothendieck topology on a small **C** given in terms of coverings with the property that every covering family contains a finite subfamily which covers, is a compact topology on the $l.f.p$ category $\underline{Set}^{\mathbf{C}^{op}}$. See also the discussion following Proposition 3.4.

b. Any locale with the property that any supremum can be attained by using less than κ elements (κ a regular cardinal) gives rise to a κ-compact topology when viewed as a site (with the canonical topology). If the locale happens to be a boolean algebra, then the condition just stated is the κ-chain condition.

c. The classical closure operator (torsion theory) on the $l.f.p.$ category of abelian groups, assigning to a subgroup $H \rightarrowtail G$ the closure $j(H) = \{g \in G \mid \exists n \in \mathbf{N} \text{ such that } n.g \in H\}$, is a compact one.

d. It was shown in [4] that when $\mathcal{E}$ is a $l.\alpha - p.$ category with universal colimits, then the α-lex closure operator is a universal one (the α-lex closure assigns to a subfunctor G of an α-lex functor $F: \mathcal{E}_{\alpha-p}^{op} \to \underline{Set}$ the smallest α-lex subfunctor of F that contains G). In the inductive construction of the α-lex closure each step commutes with α-directed suprema. Hence, the α-lex closure commutes with α-directed suprema giving rise to an α-compact topology in this case.

e. Since a locally presentable category $\mathcal{E}$ is well-powered any topology j on it will be β-compact for some β sufficiently large (in partcular it will be β-compact for a $\beta \geq \alpha$, where α is the presentability rank of $\mathcal{E}$).

3 Preservation of filtered colimits

We consider the following proposition to be the basic one of this section, since it is the main technical means for arriving at a characterization of compactness in terms of preservation of monomorphic filtered colimits.

3.1 Theorem. *Let $\mathcal{E}$ be a $l.\alpha - p$ category, j a β-compact topology on it with $\alpha \leq \beta$. Then the inclusion of j-separated objects into $\mathcal{E}$, $sep_j\mathcal{E} \hookrightarrow \mathcal{E}$, preserves β-filtered colimits.*

Proof: The following things, well known from topos theory (see e.g [12]), are general "closure-theoretic" and hold in our case. First, if the diagonal of an object is j-closed then the object is j-separated and secondly, if a map $P \longrightarrow X \times X$ factors through the closure of the diagonal of X, then the equalizer

$$E \rightarrowtail P \rightrightarrows X$$

is j-dense in P. Let $I \longrightarrow sep_j\mathcal{E}$ be a β-filtered diagram of j-separated objects.

Let $X = colimX_i$, where the colimit is calculated in $\mathcal{E}$. We want to show that X is j-separated. For that it suffices to show that the diagonal of X is j-closed. Since $\mathcal{E}$ is locally presentable we check elements defined over the (α-presentable) generators. Let $(x,y)\colon P \to X \times X$ be such an element belonging to the closure of the diagonal (so that we know that the equalizer E of x and y is j-dense). We want to show that $x = y$. Since I is β-filtered and P α-presentable, x and y are defined at some (common, as it can be chosen) stage $k \in I$ and represented by elements x_k and y_k, respectively. It suffices to exhibit a later stage of definition where x_k and y_k are identified. Consider the category of indices below k, k/I. It is still β-filtered. By a theorem of Grothendieck and Verdier (proved in detail in [20]) there is a β-directed poset K cofinal in k/I. Consider now the equalizer diagrams

$$E_i = Eq(x_i, y_i) \rightarrowtail P \rightrightarrows X_k \to X_i$$

of the arrows $x_i = t_{ik}.x_k, y_i = t_{ik}.y_k$, where t_{ik} the transition maps. The commutation of β-filtered colimits with equalizers in $\mathcal{E}$ gives

$$\begin{aligned} \bigvee_I E_i &= colim_K(Eq(P \rightrightarrows X_i)) \\ &\cong Eq(P \rightrightarrows colim_K X_i) \\ &\cong Eq(P \rightrightarrows colim_I X_i) \\ &\cong E, \end{aligned}$$

the third equality provided by cofinality of K in k/I. But now E, being the equalizer of an element in the closure of the diagonal of $colim_I X_i$, is j-dense. By the β-compactness assumption one of the $E_i \longrightarrow P$ is j-dense and since the X_i's are j-separated the two elements x_i, y_i have to be equal. This concludes the proof. ∎

Remark: A special case of the theorem above is known. In [16] a notion of B-valued set was introduced, for B a complete boolean algebra, which is essentially the same as a separated presheaf on B (seen as a site with the canonical topology). It is shown there that the category $ModB$ of B-valued sets is $l.\kappa - p$ if B has the κ-chain condition. This amounts to showing that the inclusion of separated presheaves preserves κ-filtered colimits when the canonical topology on B is κ-compact (after Example (b.) in section 1). The proof there is given by a direct, long calculation involving the description of colimits in $ModB$. Except for that special case above we believe that the result of Theorem 1 is new even for the case of a Grothendieck topology. For that reason we state it separately.

3.2 Corollary. *If J is a Grothendieck topology on a small category* $\mathbf{C}$, *which is α-compact, then the inclusion $sep(\mathbf{C}, J) \hookrightarrow \underline{Set}^{\mathbf{C}^{op}}$ preserves α-filtered colimits and hence $sep(\mathbf{C}, J)$ is $l.\alpha - p$.*

We know of no direct proof of the fact that for a topology j on a locally presentable category $\mathcal{E}$, $sep_j\mathcal{E}$ is reflective in $\mathcal{E}$. Combining though the result of Theorem 2.1 with one of Adamek and Rosicky ([1], see also [15] for a special case), that a subcategory of a locally presentable one $\mathcal{E}$, closed under arbitrary limits and β-filtered colimits, is reflective in $\mathcal{E}$, we obtain:

3.3 Corollary. *Let $\mathcal{E}$ be a locally presentable category, j a topology on it. Then $sep_j\mathcal{E}$ is reflective in $\mathcal{E}$ and hence it is itself locally presentable.*

Proof: As we mentioned in Example (e.) above, the topology j will be β-compact for some β sufficiently large thus $sep_j\mathcal{E}$ will be closed in $\mathcal{E}$ under β-filtered colimits. On the other hand $sep_j\mathcal{E}$ is trivially closed in $\mathcal{E}$ under arbitrary limits, so we can conclude by the aforementioned result. ∎

As far as sheaves are concerned the following holds:

3.4 Proposition. *Let $\mathcal{E}$ be a $l.\alpha - p$ category, j a β-compact topology on it. Then the inclusion $sh_j\mathcal{E} \hookrightarrow \mathcal{E}$ preserves monomorphic β-filtered colimits.*

Proof: Let $I \longrightarrow sh_j\mathcal{E}$ be a monomorphic β-filtered diagram of j-sheaves and $X = colim_I X_i$. We wish to show that X is a j-sheaf. We already know that it is j-separated. We also know that in order to be a sheaf it suffices to be orthogonal to dense monics with codomain α-presentable. So consider such a monic $A \rightarrowtail P$, and an arbitrary $A \rightarrowtail X$. We want to produce an extension of the latter (which will then be unique). But the subobject $A \rightarrowtail P$ can be refined to a j-dense $F \rightarrowtail P$ one, with F being β-generated:

$$\begin{array}{ccccc} F & \rightarrowtail & A & \rightarrowtail & P \\ & \searrow & \downarrow & \swarrow & \\ & & X = colim_I X_i & & \end{array}$$

We can produce an extension of $F \rightarrowtail A \longrightarrow X$ along $F \rightarrowtail P$, as the following calculation shows :

$$hom_{\mathcal{E}}(F, X) \cong colim_I hom_{\mathcal{E}}(F, X_i) \cong colim_I hom_{\mathcal{E}}(P, X_i) \cong hom_{\mathcal{E}}(P, X)$$

(where the middle bijection is due to the fact that the X_i's are sheaves). Then, by the separatedness of X, it turns out that this is also an extension of $A \to X$ along $A \rightarrowtail P$, finishing the proof. ∎

If the ambient category $\mathcal{E}$ has suitable properties, compactness can have stronger implications concerning the preservation of filtered colimits by the inclusion of sheaves. Namely, in some $l.\alpha - p$ categories β-generated may coincide with β-presentable ($\alpha \leq \beta$). Categories with this property are called locally β-noetherian. Examples of such categories are these of boolean algebras and R-algebras over a noetherian ring R (for $\alpha = \beta = \aleph_0$), small categories (for $\alpha = \aleph_0$, $\beta = \aleph_1$) and commutative C^*-algebras (for $\alpha = \beta = \aleph_1$) (see e.g [7], Ch. 5). In other $l.\alpha - p$ categories α-generated subobjects of α-presentable ones are α-presentable. Such categories are called locally α-coherent and have been characterised by S. Fakir ([7], chapter 9). A typical example is that of R-modules over a coherent ring R. Finally, for in a category of presheaves on a small category with pullbacks, finitely generated subobjects of the representables are finitely generated (See [21], Remark 15, p. 301). This is so because, if a sieve $R \rightarrowtail h_C$ is finitely generated (meaning that there are finitely many $C_i \to C$ generating R), then there is an epi

$$\coprod_{i \in I} h_{C_i} \twoheadrightarrow R.$$

If the site has pullbacks that epi is the coequalizer of the diagram

$$\coprod_{i,j} h_{C_i \times_C C_j} \rightrightarrows \coprod_{i=1}^{n} h_{C_i} \twoheadrightarrow R$$

(the diagram being the kernel pair of $\coprod_{i=1}^{n} h_{C_i} \twoheadrightarrow R$), hence R is finitely presentable as an object of $\underline{Set}^{\mathbf{C}^{op}}$ By a similar proof we can show that in a coherent topos finitely generated subobjects of representables are finitely presentable. In all the above cases the compactness of the topology implies the preservation of filtered colimits by the inclusion of sheaves.

Remark. In the case of sheaves for a site (with pullbacks) the result of Proposition 2.4 is known (cf. [21], Theorem 11). It follows from an easy calculation involving the "double plus" construction of the associated sheaf functor. We do not know which full subcategories of a locally presentable one $\mathcal{E}$, closed under limits and β-filtered colimits, for some β, arise as categories of sheaves for a topology. We know, though, that localizations of such an $\mathcal{E}$ arise as categories of sheaves. In connection with that we have the following:

3.5 Proposition. *Let $\mathcal{E}$ be a $l.\alpha-p$ category and $\mathcal{L} \hookrightarrow \mathcal{E}$ a localization of it such that the inclusion preserves monomorphic β-filtered colimits (we assume that $\alpha \leq \beta$). Then the topology on $\mathcal{E}$ inducing $\mathcal{L}$ is β-compact.*

Proof: Recall from [6] (Proposition 1.5) that the topology inducing the localization $i : \mathcal{L} \hookrightarrow \mathcal{E}$, with left exact reflection r, assigns to a subobject $S \rightarrowtail A$ the pullback

$$\begin{array}{ccc} j(S) & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \eta_A} \\ ir(S) & \longrightarrow & ir(A) \end{array}$$

of the monic (by the left exactness of r) $irS \rightarrowtail irA$ along the unit of the adjunction $\eta_A \colon A \to irA$. Given now a β-directed family of subobjects $\{S_k \rightarrowtail A \mid k \in I\}$ the closure of

$$j(\bigvee_{k\in I} S_k) \cong ir(j(\bigvee_{k\in I} S_k) \times_{irA} A) \cong (\bigvee_{k\in I} irS_k \times_{irA} A)) \cong \bigvee_{k\in I} j(S_k),$$

(since i preserves monomorphic filtered colimits and the latter commute with pullbacks), as required. ∎

Summarizing the facts we have seen so far, we obtain:

3.6 Theorem. *When $\mathcal{E}$ is a $l.\alpha - p$ category such that topologies on it agree with localizations, then β-compact topologies ($\alpha \leq \beta$) correspond to localizations preserving monomorphic β-filtered colimits.*

We are presented naturally with the problem of characterizing those topologies, which, in the above context, correspond to localizations closed under β-filtered colimits. The problem has been considered for abelian locally finitely presentable categories by M. Prest in [18]. The non-abelian formulation of the condition given there turns out to be sufficient for the preservation of filtered colimits by the inclusion of sheaves, but not necessary. Yet, as it is weaker than, for example, requiring the local coherence of the ambient category, so we give a proof for its sufficiency.

3.7 Proposition. *Let $\mathcal{E}$ be a $l.\alpha - p$ category and j a topology on it satisfying the following two conditions*

(i) j is β-compact ($\alpha \leq \beta$)

(ii) If F is a β-generated subobject of an α-presentable Q, then it admits a presentation by a coequalizer

$$K \rightrightarrows P \twoheadrightarrow F,$$

where P is β-presentable and K contains a β-generated dense subobject.

Then the inclusion $sh_j\mathcal{E} \hookrightarrow \mathcal{E}$ preserves β-filtered colimits.

Proof: Let $I \to sh_j\mathcal{E}$ be a β-filtered diagram of j-sheaves and $X = colim_I X_i$. We wish to show that X is a j-sheaf. We already know that it is j-separated. It suffices to check orthogonality with respect to a dense subobject of an α-presentable one $A \rightarrowtail Q$. As in Prop. 2.4, since X is separated it suffices to take $A = F$, a β-generated subobject of Q. There is a presentation of it

$$K \rightrightarrows P \twoheadrightarrow F,$$

with K having a β-generated dense subobject. Consider a map $A \to X$. In the following diagram

$$\begin{array}{ccccccc} G \rightarrowtail & K & \rightrightarrows & P & \twoheadrightarrow & F \rightarrowtail & Q \\ & & & \downarrow & & \downarrow & \\ & & & X_i & \longrightarrow & colim_I X_i & \end{array}$$

we get a factorization of $P \to F \to X$ through an X_i, since P is β-presentable. The two composites

$$G \rightarrowtail K \rightrightarrows P \to X_i$$

are equal (upon replacing X_i with a further X_k in the diagram) since they agree when preceeded by an injection into the colimit. But $G \rightarrowtail K$ is dense and X_i is a sheaf, so the two composites

$$K \rightrightarrows P \longrightarrow X_i$$

are equal, since they agree when they are restricted along a dense mono. That way we obtain a factorization of $P \to X_i$ through $F \to X_i$. There is a unique extension of $F \to X_i$ along $F \rightarrowtail Q$ and the composite $Q \to X_i \to X$ is the required extension to the given map $F \to X$. ∎

The necessity of the condition in the abelian case seems to hinge on the good combination of exactness and finiteness properties available for an abelian locally finitely presentable category. Namely, the fact that the kernel of an epi between finitely presentable objects is finitely generated (cf. [19] Prop. I.3.2), as the following proof shows.

3.8 Theorem. *(M. Prest, [18], Th. 2.3) A localization of a locally finitely presentable abelian category is closed under filtered colimits iff the corresponding Gabriel topology is compact and has the further property that, if a map between finitely presentable objects has dense image, then its kernel contains a finitely generated dense subobject.*

Proof: The sufficiency is clearly implied by the preceeding proposition. For the necessesity, let $P \to Q$ be a morphism between finitely presentable objects with dense image, and let K be its kernel. Then, applying the localization functor to the sequence

$$K \rightarrowtail P \twoheadrightarrow Q$$

we get a sequence

$$rK \rightarrowtail rP \twoheadrightarrow rQ,$$

where rK remains the kernel of $rP \twoheadrightarrow rQ$, rP and rQ remain finitely presentable and $rP \twoheadrightarrow rQ$ is an epi, since the image of $P \longrightarrow Q$ is dense. So, rK is finitely generated, which means that there is a finitely generated subobject of K dense in K. ∎

4 The frame of compact topologies

The fact that, under the assumption that filtered colimits commute with finite limits, the set of all topologies on a $l.\alpha - p$ is a frame was shown in [6]. It is the analogue of the important result given in [3] that, under the same assumptions, the localizations of a locally presentable category form a co-frame.

4.1 Theorem. *Let $\mathcal{E}$ be a l.f.p category. Then the set of compact topologies is a subframe of the frame of all topologies.*

Proof: As mentioned in the introduction to this section, topologies are given pointwise by acting on subobjects of α-presentable objects. The binary infimum of topologies is then given pointwise by intersection of subobjects. The supremum, however, of a set J of topologies is given by the following construction (cf. [6], Theorem 3, p. 313): For a subobject $A \rightarrowtail P$ of an α-presentable object define $j'_P(A) = \bigvee_K j_{k_1} \circ \ldots \circ j_{k_n}(A)$, where the supremum ranges over the set K of all the finite sequences of J (hence it is a directed one). Then we repeat the process over $j'_P(A)$ and so on. The process terminates because locally presentable categories are well powered. We first show that the set of compact topologies is closed in the frame of all topologies under arbitrary suprema. Let the

J be a set of compact topologies and $\{A_i \rightarrowtail P \mid i \in I\}$ a directed set of subobjects of a finitely presentable object P. Since each topology $j \in J$ commutes with directed suprema, and suprema commute with each other we have:

$$\begin{aligned} j'_P(\bigvee_{i\in I} A_i) &= Kj_{k_1} \circ \ldots \circ j_{k_n}(\bigvee_{i\in I} A_i) \\ &= \bigvee_{k\in K}\bigvee_{i\in I} j_{k_1} \circ \ldots \circ j_{k_n}(A_i) \\ &= \bigvee_{i\in I}\bigvee_{k\in K} j_{k_1} \circ \ldots \circ j_{k_n}(A_i) \\ &= \bigvee_{i\in I} j'_P(A_i) \end{aligned}$$

Iterating the application of j' we will always be getting the supremum outside the argument of it, so eventually

$$(\bigvee J)_P(\bigvee_{i\in I} A_i) = \bigvee_{i\in I}(\bigvee J)_P(A_i),$$

as required. It is also closed under binary infima because, if j and k are compact topologies and the A_i's are as above, we have:

$$\begin{aligned} (j \wedge k)_P(\bigvee_{i\in I} A_i) &= j_P(\bigvee_{i\in I} A_i) \wedge k_P(\bigvee_{i\in I} A_i) \\ &= \bigvee_{i\in I} j_P(A_i) \wedge \bigvee_{i\in I} k_P(A_i) \\ &= \bigvee_{i\in I}\bigvee_{i'\in I} (j_P(A_i) \wedge k_P(A_{i'})), \end{aligned}$$

since in $\mathcal{E}$ directed suprema commute with intersections. The latter double supremum, though, can be substituted by a single one, again by directedness. So finally

$$(j \wedge k)_P(\bigvee_{i\in I} A_i) = \bigvee_{i\in I}(j \wedge k)_P(A_i)$$

i.e $j \wedge k$ is compact, which concludes the proof. ∎

Remark. The above proof shows actually that in a locally presentable category, where filtered colimits commute with finite limits (so that there is available a frame of topologies), the topologies commuting with directed suprema of subobjects form a subframe of that of all topologies. We might also consider topologies on a special kind of $l.f.p$ categories, namely algebraic lattices. As we noted in the introduction, topologies on such a categories are nothing else but nucleui. Notice also that under this identification, orthogonal elements for the closure operator correspond to closed elements for the nucleus. A compact topology on an algebraic lattice is, then, a nucleus commuting with directed suprema. A frame on the other hand is a locally presentable category, where filtered colimits commute with finite limits, so we have:

4.2 Corollary. *The set of directed-suprema-preserving nuclei $N_\omega(A)$ on a frame or an algebraic lattice A is a subframe of the frame $N(A)$ of all nuclei.*

We continue with a calculation concerning directed suprema of compact topologies, which we are going to use in the sequel in studying the frame of compact topologies on a coherent topos.

4.3 Proposition. *If J is a directed family of compact topologies on a locally finitely presentable category, and if $A \rightarrowtail P$ is any subobject of a finitely presentable object, then*

$$(\bigvee J)_P(A) = \{j_P(A) \mid j \in J\}$$

. *Consequently, the set of j-dense subobjects of a finitely presentable object is the union of the sets of j-dense subobjects, $j \in J$.*

Proof: In the computation of $j'_P(A)$ as above, we can, since J is directed, dominate any finite sequence by a single j and, using the idempotency of j, get

$$j'(A) = \bigvee_{k \in K} j_{k_1} \circ \ldots \circ j_{k_n}(A) \leq \bigvee j \circ \ldots \circ j(A) = \bigvee \{j(A) \mid j \in J\},$$

while the other inequality holds trivially. In iterating j', because of the compactness of the topologies involved, we get:

$$\begin{aligned} j'(j'(A)) &= \bigvee_{k \in K} j_{k_1} \circ \ldots \circ j_{k_n}(j'(A)) \\ &= \bigvee_{k \in K} j_{k_1} \circ \ldots \circ j_{k_n}(\bigvee\{j(A) \mid j \in J\}) \\ &= \bigvee_{k \in K} \bigvee\{j_{k_1} \circ \ldots \circ j_{k_n}(j(A)) \mid j \in J\}) \\ &= \bigvee_{k \in K} \bigvee_{j \in K} k(j(A)), \end{aligned}$$

where the first supremum in the last expression is taken over all topologies k, such that each of them dominates a finite sequence in K. By the same argument as above

$$j'_P(j'_P(A)) = \bigvee\{j_P(A) \mid j \in J\},$$

i.e, the process stabilizes already after two steps, yielding the desired result. ∎

Recall from [5] that topologies on a locally α-presentable category $\mathcal{E}$, according to our Basic Definition in the Introduction, are in a one-to-one correspondence with families $D(P)$ of subobjects (the dense subobjects) of the α-presentable objects P in $\mathcal{E}$, so that the following are satisfied:

(D1) $P \in D(P)$

(D2) If $S \in D(P)$ and $x : Q \to P$ any map with α-presentable domain, then $x^{-1}(S) \in D(Q)$.

(D3) If $P = colim P_i$ is a colimit of size less than α of α-presentable objects and $S \rightarrowtail P$ is such that, for all $s_i : P_i \to P$, $s_i^{-1}(S) \in D(P_i)$, then $S \in D(P)$.

(D4) If $S \rightarrowtail P$ and there is $T \in D(P)$ such that, for all $x : Q \to P$ factoring through T, $x^{-1}(S) \in D(Q)$, then $S \in D(P)$.

As we now aim at showing the local compactness of the frame of compact topologies on a coherent topos we are going to need sufficiently many finite elements of that frame. These come about as the topologies with the property of being the smallest such that contain a given finitely generated object. This corresponds to the usual description of Grothendieck topologies in terms of generating families covering a single object. What we get though in that way is just a pretopology, i.e a collection of upper-closed families of sieves that satisfy the axioms (D1), (D2) above. We want to ensure that the topology generated by such a collection (which makes sence to talk about since the intersection of topologies is a topology) is compact if the generating families contain finitely many arrows. This is the content of the Lemma that follows, which is though stated in a greater generality. In the more general setting in which we are discussing topologies, let $\mathcal{P}$ be a set of α-presentable projective generators and D be a pretopology given in terms of dense subobjects of objects in P. Suppose that, for all $P \in \mathcal{P}$, all α-directed families $\{T_i \rightarrowtail P \mid i \in I\}$ of subobjects of P, $\bigvee_{i\in I} T_i \in D(P)$ implies that there is $i \in I$, such that $T_i \in D(P)$. Then we say that the pretopology satisfies the α-compactness property.

4.4 Lemma. *Let $\mathcal{E}$ be a regular lo- cally α-presentable category with a set P of α-presentable projective generators and D be a pretopology given by subobjects of objects in $\mathcal{P}$. Assume that applying transfinitely many times the process*

$$D^{+}(P) = \{T \rightarrowtail P \mid \exists S \rightarrowtail P, S \in D(P)\ s.t,$$
$$\forall s\colon Q \to S \rightarrowtail P, s^{-1}(S) \in D(Q)\},$$

taking a directed union at the limit ordinal step, gives a topology on $\mathcal{E}$ (as it is the case with presheaf categories, with $\mathcal{P}$ taken to be the representables, or with algebraic categories, with $\mathcal{P} = \{F1, F2, ...\}$, the set of free objects on finitely many generators, cf. [5]). If the pretopology D satisfies the α-compactness property, then the topology $D = \bigcup D_\beta$ generated by D is α -compact.

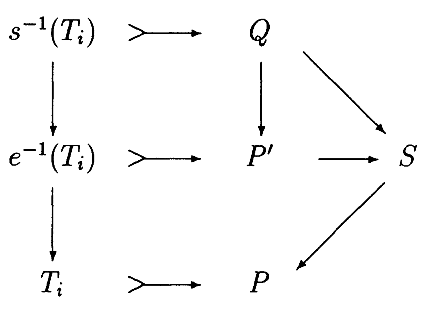
Remark: The extra assumption that iterating sufficiently many times the process described above yields a topology is needed because, although the process will necessarily cease at some point, what we get at the end may not satisfy (D3) even if D does. This is a complication that does not occur in the more familiar situations, e.g in that of Grothendieck topologies.

Proof: It suffices to show that, given an α-directed family $\{T_i \rightarrowtail P \mid i \in I\}$ of subobjects of a $P \in \mathcal{P}$ with $T_i \in D(P)$ then, for some $i \in I$, $T_i \in D(P)$. Specifically, we show that for all ordinals, all $P \in \mathcal{P}$ and all α-directed families $\{T_i \rightarrowtail P | i \in I\}$, if $\bigvee_{i\in I} T_i \in D_\gamma$ then, for some $i \in I$, $T_i \in D_\gamma$. Inspecting the inductive construction of $D(P)$ we realize that the limit ordinal step is immediate. If $T_i \in D_{\gamma+1}(P)$ then there is an $S \in D_\gamma(P)$ such that, for all $s : Q \to S \rightarrowtail P$, $Q \in P$, $s^{-1}(\bigvee_{i\in I} T_i) = \bigvee_{i\in I} s^{-1}(T_i) \in D_\gamma(Q)$.

Write S as the α-directed supremum of the α-generated subobjects F below it. Then, by the inductive hypothesis, one of those F's is in $D_\gamma(P)$. To avoid complicated notation we assume that S itself is α-generated. So, it admits a regular epi $e : P' \to S$ from an α-presentable P'. The inverse image of T_i along

$$P' \twoheadrightarrow S \rightarrowtail P$$

is in $D_\gamma(P')$. By the hypothesis, we can find $i \in I$, such that $e^{-1}(T_i)$ is in $D_\gamma(P')$.



Then, for all

$$x : P' \twoheadrightarrow S \rightarrowtail P,$$

the inverse image $x^{-1}(T_i)$ is obtained as a pullback of $e^{-1}(T_i)$ along the factorization $Q \to P'$ of x through e. Thus $x^{-1}(T_i) \in D_\gamma(P)$ and we are done. ∎

Now let $\mathbf{C}$ be a small category with pullbacks. Call a sieve $F \rightarrowtail h_C$ finitely generated if there are $C_k \to C$, $k = 1, ..., n$, so that $D \to C$ is in F iff it factors through some , for some k. For each $D \in \mathbf{C}$ define a family of subobjects of h_D by

$$u^F(D) = \{Y \rightarrowtail h_D \mid \exists t: D \to C, t^{-1}(F) \leq Y\} \vee \{h_D\},$$

where $F \rightarrowtail h_C$ is a given finitely generated sieve. Then the collection $\{u^F(D) | D \in \mathbf{C}\}$ is a pretopology on $\underline{Set}^{\mathbf{C}^{op}}$: Obviously the $u^F(D)$'s are upper closed, contain h_D and if $Y \in u^F(D)$, for any $a: E \to D$ in $\mathbf{C}$, $(ta)^{-1}(F) = a^{-1}t^{-1}(F) \leq a^{-1}(Y)$, so $a^{-1}(Y) \in u^F(E)$. As a matter of fact it is the smallest pretopology containing F. This pretopology satisfies tha compactness property because, for all $t: D \to C$, $t^{-1}(F)$ is finitely generated. The set $\{C_i \times_C D \to D \mid C_i \to C\}$ is a set of generators for $t^{-1}(F)$. As a corollary to the above Lemma we obtain that the smallest Grothendieck topology containing a finitely generated sieve $F \rightarrowtail h_C$ is compact. We denote that topology by u^F.

4.5 Theorem. *The frame of compact topologies on a coherent topos is a locally compact, compact one.*

Proof: A coherent topos is the category of sheaves for a compact Grothendieck topology on a small category with finite limits (see e.g [14], Ch.IX). Then compact topologies on that topos correspond to compact topologies on the full presheaf category above the one corresponding to the given topos (in the pointwise order of topologies). For that it suffices to prove the claim for the frame of compact topologies on a presheaf category $\underline{Set}^{\mathbf{C}^{op}}$, where $\mathbf{C}$ a small category with finite limits. Notice first that the topologies u^F discussed above are finite elements of the frame of compact topologies: If, for $F \rightarrowtail h_C$, $u^F \leq \bigvee K$, where K is a directed set of compact topologies, then, by Prop.3.3,

$$u^F(C) \subseteq (\bigvee K)(C) = \bigcup\{k(C) | k \in K\},$$

where $k(C)$ is the set of k-dense subobjects of h_C. So there is $k \in K$ with $F \in k(C)$, implying that $u^F \leq k$. In particular the largest topology, being the smallest one containing the empty sieve $\emptyset \rightarrowtail h_1$ (1 denotes the terminal object of $\mathbf{C}$) is a finite element of the frame, showing the compactness of the frame in question.

As for local compactness we show that any compact topology j is the supremum of the topologies u^F below it. Or, rephrasing, that if k is another compact topology such that, for all u^F, $u^F \leq k$ implies $u^F \leq k$, then $j \leq k$. Suppose that $X \in j(D)$, for some $D \in \mathbf{C}$. j being compact means that there is $F \rightarrowtail X \rightarrowtail h_D$, with F finitely generated and $F \in j(D)$. The latter means that $u^F \leq j$, so by assumption $u^F \leq k$, or $F \in k(D)$, giving that $X \in k(D)$, as required. ∎

By a coherent space we mean one that is sober, compact and has a basis of compact opens which is closed under finite intersection. They are also known as spectral spaces since, by a fundamental result of Hochster (see for example [10], p. 205), they are exactly the ones that arise as Zariski spectra of commutative rings. They correspond exactly to coherent frames in the sense that the opens of a coherent space form such a frame and, conversely, every coherent space arises as the space of points for such a frame.

On any coherent space we can define a topology having as subbasic closed sets the closed ones for the original topology and the compact open ones for the original topology. The patch topology has, thus, the following frame theoretic description.

Given a space X, a closed subset corresponds to $X_{c(a)}$, the fixed points for a closed nucleus $c(a) = a \vee -$ on the frame of opens of X. Similarly, a compact open corresponds to $X_{u(f)}$, the fixed points for an open nucleus $u(f) = f \rightarrow -$, where f is a compact open (finite element of the frame of opens). Thus a basic closed set corresponds to $X_{c(a)} \cup X_{u(f)}) = X_{c(a) \wedge u(f)}$. Finally an arbitrary closed set for the patch topology is of the form

$$\bigcap_{a,f}(X_{c(a)} \cup X_{u(f)}) = X_{\bigvee\{c(a)\wedge u(f) \mid a\in I, f \text{ compact}\}}$$

In this way we obtain a contravariant equivalence between the set of closed subsets for the patch topology and the set of nuclei of the form $\bigvee\{c(a) \wedge u(f) \mid a \in I, f \text{ compact}\}$. Given, now, a coherent frame A call the set of nuclei of that form the *patch frame* of A (being equivalent to that of patch opens for A the frame of opens of a coherent space. Alternatively, one could show directly that the set of nuclei with the above description is a frame).

We identify the patch frame of a coherent frame with that of compact topologies (nuclei) on the original frame, or equivalently (elaborating on Corollary IX 5.6 of [14]) with that of compact topologies on the topos of sheaves for that frame. First we need:

4.6 Lemma. *Let A be a coherent frame and f a finite element therein. Then the open nucleus $u(f) = f \to -$ is a compact one (commutes with directed suprema).*

Proof: We have to show that, whenever I is directed

$$f \to (\bigvee_{i\in I} a_i) \leq \bigvee_{i\in I}(f \to a_i)$$

Since the finite elements generate the frame, it suffices to show that, for a finite $g \in A$, $g \leq f \to (\bigvee_{i\in I} a_i)$ implies that $g \leq \bigvee_{i\in I}(f \to a_i)$. But $g \leq f \to (\bigvee_{i\in I} a_i)$ means that $g \wedge f \leq \bigvee_{i\in I} a_i$ and, by the coherence of the frame $g \wedge f$ is a finite element, so, there is an $i \in I$ such that $g \wedge f \leq a_i$, or, equivalently, $g \leq f \to a_i$. ∎

4.7 Proposition. *The patch frame obtained from a coherent frame A is isomorphic to that of compact nuclei on A.*

Proof: By the lemma, whenever f is finite, $u(f)$ is compact; $c(a)$ is always compact, thus so is any nucleus of the form

$$\bigvee\{c(a) \wedge u(f) \mid a \in I,\ f \in J,\ \ f \text{ compact}\}$$

So the patch frame is contained in the frame of compact topologies. Now, every compact nucleus j is of the form $\{k_f \mid f \in K(A)\}$, where $K(A)$ is the set of finite elements of A and $k_f = c(j(f)) \wedge u(f)$. For that, we simply modify the argument in [10], Prop. II.2.7. First, $j(f) = k_f(f)$ and, for all $a \in A$, $k_f(a) \leq j(a)$, so that j is an upper bound of the k_f's. Then, given any other upper bound l for the k_f's, we have that, for each $f \in K(A)$, $j(f) = k_f(f) \leq l(f)$. But A is coherent so any $a \in A$ can be written as $a = \bigvee\{f \in K(A) \mid f \leq a\}$ and, j, l being compact, we get

$$\begin{aligned} j(a) &= j(\bigvee\{f \in K(A) \mid f \leq a\}) \\ &= \bigvee\{j(f) \mid f \in K(A), f \leq a\} \\ &\leq \bigvee\{l(f) \mid f \in K(A), f \leq a\} \\ &= l(\{f \in K(A) \mid f \leq a\}) \\ &= l(a) \end{aligned}$$

So finally, $j = \bigvee\{k_f | f \in K(A)\}$. ∎

We close with some remarks concerning the functorial character of our constuction. In [3], § 7, it is shown that a geometric morphism between two locally presentable categories with finite limits commuting with filtered colimits induces a pair of adjoint order-preserving maps between the corresponding co-frames of localizations. Contrary to the case of localizations the functorial behaviour of the frame of topologies has not been examined. We can show that a geometric morphism between two locally presentable categories induces an order-preserving map between the frames of topologies, coinciding with the right adjoint in the adjunction given in [3] in the case that topologies correspond exactly to localizations. More presicely if $\phi^* \dashv \phi_*: \mathcal{E} \to \mathcal{F}$ is a geometric morphism between locally presentable categories with regular images, and j a topology on $\mathcal{E}$, then the assignment

$$\phi_{\sharp}(j)(A) = \bigvee\{B \rightarrowtail X \mid \phi_*(B) \leq j\phi_*(A)\},$$

where $A \rightarrowtail X$, defines a topology $\phi_{\sharp}(j)$ on $\mathcal{F}$ (the reader may consult [11] for the straithforward but rather long calculations proving this as well as the claims that follow in this paragraph). The map we produce preserves arbitrary infima, having thus a left adjoint. When we look at the corresponding frames of compact topologies on two locally finitely presentable categories and a geometric morphism between them, which preserves monomorphic filtered colimits, it no longer preserves arbitrary infima, only the finite ones, but it further preserves directed suprema. In other words, it becomes a preframe map, according to the definition given in [2]. Under the identification asserted in the Introduction, of the frame in question with the dual Zariski spectrum, the geometric morphism between module categories induced by a commutative ring homomorphism (cf. [3], §8) further induces, in the way described above, a frame map, which coincides with the usual map induced between ring spectra.

Finally, when the geometric morphism is a map of locales then the induced map $\phi_\sharp\colon N(X) \to N(Y)$ between the frames of nuclei is again a map of locales, coinciding with the one described in [10], Prop.II 2.8.

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Panagis Karazeris
Arhiepiskopou Makariou 10A
GR-26441 Patras, Greece
email:panagisk@hol.gr