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A REMARK ON TOPOLOGICAL SPACES, GRIDS, AND TOPOLOGICAL SYSTEMS

by J. ADAMEK* and M.-C. PEDICCHIO

RESUME. La variété des grilles introduite par Barr & Pedicchio contient la duale de TOP comme sous-quasi-variété; on prouve ici qu'elle est équivalente à la comma-catégorie de toutes les algèbres de Boole atomiques complètes dans la catégorie des cadres. Autrement dit, les grilles sont dualement équivalentes aux systèmes topologiques de Vickers.

INTRODUCTION. The category TOP of topological spaces is dually equivalent to a quasivariety of algebras. This has been proved in [BP_1], where a variety of algebras, called grids, was introduced together with a single implication specifying a subquasivariety equivalent to TOP^{op}. A similar result, using 2-sorted algebras, can be obtained as follows: every topology on a set X is nothing else than a subframe F of the CABA (complete atomic Boolean algebra) B of all subsets of X. Thus, topological spaces can be identified with injective frame homomorphisms $\varphi\colon F \to B$ from a frame, F, to a CABA, B. Now drop the injectivity and consider *all* frame homomorphisms $\varphi\colon F \to B$. More precisely, consider the comma-category FRM $\downarrow$ CABA of the (non-full) subcategory of CABAs and CABA-homomorphisms in FRM, the category of frames and frame-homomorphisms. This category can be, in a very natural sense, considered as a variety of 2-sorted algebras: we have sorts *frame* and *boole*, the operations are

(i) joins and finite meets in the sort *frame*,

(ii) joins and negation in the sort *boole* and

(iii) a unary operation φ in the sort *frame* $\to$ *boole*.

The equations are (i) those presenting frames in the sort *frame*, (ii) those presenting CABAs in the sort *boole*, and (iii) those presenting φ as a frame homomorphism. Within this variety, then, the single implication

$$\varphi(x) = \varphi(y) \Rightarrow x = y$$

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specifies topological spaces as a dual subquasivariety.

The obvious advantage of grids in comparison to the above variety is that grids are one-sorted algebras. We are going to show, following a suggestion of A. Joyal, that the two approaches are in fact equivalent: the category of grids is equivalent to FRM $\downarrow$ CABA. Moreover, we also observe that the category of topological systems of S. Vickers [V] is dually equivalent to FRM $\downarrow$ CABA, thus, the main result of our paper can be interpreted to say that topological systems are dually equivalent to a variety of (single sorted) algebras.

We are grateful to A. Carboni for fruitful discussions on the subject and for a suggestion of a more direct proof that FRM $\downarrow$ CABA is a variety (see Remark 12 below).

I. Topological Systems and Frame-Homomorphisms

Recall from [V] that a *topological system* is a triple (X, F, R) where X is a set, F is a frame, and $R \subseteq X \times F$ is a relation satisfying

(1) $$xR \bigvee_{i \in I} u_i \iff (\exists_i) xRu_i \qquad \text{(for } I \text{ arbitrary)}$$

and

(2) $$xR \bigwedge_{j \in J} u_j \iff (\forall_j) xRu_j \qquad \text{(for } J \text{ finite).}$$

A *continuous map* from one topological system (X, F, R) to another one (X', F', R') is a pair of functions $f\colon X \to X'$ and $h\colon F' \to F$ such that h is a frame homomorphism satisfying

(3) $$xRh(y) \iff f(x)R'y.$$

We denote by TOPSYS the resulting category of topological systems.

Notation. We denote by FRM $\downarrow$ CABA the category whose objects are triples (F, B, φ) where F is a frame, B a CABA, and $\varphi\colon F \longrightarrow B$ a frame-homomorphism. Morphisms from (F, B, φ) to (F', B', φ') are commutative squares

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & B \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ F' & \xrightarrow[\varphi']{} & B' \end{array}$$

where h_1 is a frame-homomorphism and h_2 a CABA-homomorphism.

Proposition 1. *The categories* TOPSYS *and* FRM $\downarrow$ CABA *are dually equivalent.*

Proof. For each topological system (X, F, R) we denote by $H(X, F, R)$ the map

$$\varphi\colon F \longrightarrow \mathcal{P}X, \quad \varphi(u) = \{x \in X; xRu\}$$

which, due to (1) and (2), is a frame-homomorphism. Any continuous map $(f,h) : (X,F,R) \longrightarrow (X',F',R')$ yields a morphism $(h,\mathcal{P}f)$: $H(X',F',R') \to H(X,F,R)$ in FRM $\downarrow$ CABA, by (3), and it is obvious that H: TOPSYS$^{\mathrm{op}}$ $\to$ FRM $\downarrow$ CABA is a functor, which is full and faithful, since (3) is equivalent to $\mathcal{P}f \cdot \varphi = \varphi' \cdot h$. It remains to show that H is isomorphism-dense: every object of FRM $\downarrow$ CABA is isomorphic to one of the form $\varphi\colon F \to \mathcal{P}X$, and the latter is $H(X, F, R)$ for R defined by xRu iff $x \in \varphi(u)$. □

Proposition 2. TOP *is dually equivalent to the full, regularly epireflective subcategory of* FRM $\downarrow$ CABA *formed by all monomorphisms* $\varphi\colon F \hookrightarrow B$.

Proof. I. The category M of all monomorphisms in FRM $\downarrow$ CABA is equivalent to TOP$^{\mathrm{op}}$. In fact, let $\mathcal{P}$:TOP$^{\mathrm{op}}$ $\to$ CABA be the functor assigning to each space the CABA of all subsets, and to each continuous map the preimage-map, and let Ω: TOP$^{\mathrm{op}}$ $\to$ FRM be the usual subfunctor of $\mathcal{P}$ of all open sets. The following functor H: TOP$^{\mathrm{op}}$ $\to M$:

$$HX = (\Omega X \hookrightarrow \mathcal{P}X)$$

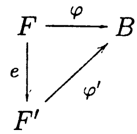
and

$$Hf = (\Omega f, \mathcal{P}f)$$

is an equivalence of categories: it is, obviously, full and faithful, and it is isomorphism dense since each object of M is isomorphic to one of the form $\varphi\colon F \to \mathcal{P}X$, φ an inclusion map, and then F is a topology on X, yielding a space with $HX = (\varphi\colon F \to \mathcal{P}X)$.

II. M is regularly epireflective in FRM $\downarrow$ CABA.

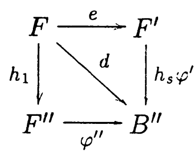
In fact, for each object φ of FRM $\downarrow$ CABA consider a (regular epi, mono)-factorization in FRM



(see [BGO]). Then

$$(e, \mathrm{id}_B) : (F \xrightarrow{\varphi} B) \longrightarrow (F' \xrightarrow{\varphi'} B)$$

is a regular epimorphism in FRM $\downarrow$ CABA, and this is a reflection of φ in M. In fact, given a monomorphism $\varphi''\colon F'' \to B''$ and a morphism $(h_1, h_2)\colon \varphi \longrightarrow \varphi''$ in FRM $\downarrow$ CABA, we use the diagonal fill-in



to obtain a morphism $(d, h_2)\colon \varphi' \to \varphi''$ with $(h_1, h_2) = (d, h_2) \cdot (e, id)$. □

Corollary 3. TOP$^{\mathrm{op}}$ *is equivalent to a quasivariety of 2-sorted algebras.*

Remark 4. The category FRM $\downarrow$ CABA also contains FRM as a full, coreflective subcategory if each frame F is identified with the homomorphism of F into $\mathcal{P}(ptF)$ (where ptF denotes the set of all points of F), assigning to each y the set of all points p with $p(y) = 1$. This, as well as the above Proposition 2, is a dualization of the corresponding statement on TOPSYS in [V].

II. Grids and Frame-Homomorphisms

Definition 5 (see [BP$_1$]). A *grid* is a frame G together with a unary operation $'$ satisfying the following axioms, where $u^\uparrow = u \vee u'$ and $u_\downarrow = u \wedge u'$:

(G1) $u'' = u$

(G2) $(-)^\uparrow$ and $(-)_\downarrow$ are $\bigvee$-homomorphisms

(G3) $(-)^\uparrow$ is a $\wedge$-homomorphism

(G4) $(u \wedge v)_\downarrow = u \wedge v_\downarrow$

(G5) for each u the interval $[u_\downarrow, u^\uparrow]$ is a CABA with $'$ and $\bigvee$ the CABA-operations.

Lemma 6. *For every element u of a grid we have*

(i) $u_\downarrow = u \wedge 1' = u' \wedge 1'$;

thus

(ii) $u \leq 1'$ *iff* $u \leq u'$ (*thus*. $u^\uparrow = u'$);

and

(iii) $'$ *is a frame-homomorphism from* $[0, 1']$ *to* G.

Proof. (i) follows from (G4) applied to $v = 1$ (for u and u'). (ii) follows from (i). Using (ii) and (G2), we see that for $u_i \in [0, 1']$

$$(\bigvee u_i)' = \bigvee u_i^\uparrow = \bigvee u_i'$$

and using (ii) and (G3),

$$(\wedge u_i)' = \wedge u_i^\uparrow = \wedge u_i'.$$

□

Lemma 7. *For each element u of a grid put*

$$u_1 = u \wedge 1' \qquad \textit{and} \qquad u_2 = u \vee 1'.$$

Then

$$u = u_1' \wedge u_2 \qquad \textit{and} \qquad u_2 \leq u_1' \vee 1'.$$

This representation of u is unique: if $x \in [0, 1']$ and $y \in [1', 1]$ fulfil

$$u = x' \wedge y \qquad \textit{and} \qquad y \leq x' \vee 1',$$

then $x = u_1$ and $y = u_2$.

Proof. The equality $u = u_1' \wedge u_2$ is derived as follows:

$$\begin{aligned} u_1' \wedge u_2 &= (u \wedge 1')' \wedge u_2 \\ &= (u_\downarrow)' \wedge u_2 \quad \text{by Lemma 6 (i)} \\ &= u^\uparrow \wedge u_2 \quad \text{by (G5)} \\ &= (u \vee u') \wedge (u \vee 1') \\ &= u \vee (u \wedge 1') \vee (u' \wedge u) \vee (u' \wedge 1') \quad \text{by distributivity.} \\ &= u \vee u_\downarrow \quad \text{by Lemma 6 (i)} \\ &= u \end{aligned}$$

Next

$$\begin{aligned} u_2 &= u \vee 1' \\ &\leq u^\uparrow \vee 1' \\ &= (u_\downarrow)' \vee 1' \quad \text{by (G5)} \\ &= u_1' \vee 1' \quad \text{by Lemma 6 (i).} \end{aligned}$$

Given x, y as above, we compute

$$\begin{aligned} u_1 &= (x' \wedge y) \wedge 1' \\ &= (x' \wedge 1') \wedge y \\ &= (x \wedge 1') \wedge y \quad \text{by Lemma 6 (i)} \\ &= x \wedge (1' \wedge y) \qquad x \leq 1' \leq y \\ &= x \end{aligned}$$

and due to $1' \leq y \leq x' \vee 1'$ also

$$u_2 = (x' \wedge y) \vee 1' = (x' \vee 1') \wedge (y \vee 1') = y.$$

□

Denote by GRID the category of grids and grid homomorphisms, i.e., frame homomorphisms preserving the given unary operation. We will prove that this category is equivalent to FRM $\downarrow$ CABA. Define a functor

$$K\colon \text{GRID} \longrightarrow \text{FRM} \downarrow \text{CABA}$$

on objects $(G,')$ as follows: the interval $[0, 1']$ in G is a frame and the interval $[1', 1]$ is a CABA by (G5); put

$$K(G,') = [0, 1'] \xrightarrow{\varphi} [1', 1]$$

where

$$\varphi(u) = u' \vee 1' \qquad \text{for all } u \leq 1'.$$

(Due to Lemma 6 (iii), φ is a frame homomorphism). The definition of K on homomorphisms $h\colon (G,') \to (\bar{G},')$ is by means of the domain - codomain restriction of h: since $h(1') = h(1)' = 1'$ we have restrictions $h_1\colon [0,1']_G \longrightarrow [0,1']_{\bar{G}}$ and $h_2\colon [1',1]_G \longrightarrow [1',1]_{\bar{G}}$. and we put

$$K(h) = (h_1, h_2).$$

The square

$$\begin{array}{ccc} [0,1']_G & \xrightarrow{\varphi_G} & [1',1]_G \\ {\scriptstyle h_1}\big\downarrow & & \big\downarrow{\scriptstyle h_2} \\ [0,1']_{\bar{G}} & \xrightarrow[\varphi_{\bar{G}}]{} & [1',1]_{\bar{G}} \end{array}$$

commutes because

$$\begin{aligned} h_2 \cdot \varphi_G(u) &= h(u' \vee 1') \\ &= h(u)' \vee 1' \\ &= h_1(u)' \vee 1' \\ &= \varphi_{\bar{G}} \cdot h_1(u). \end{aligned}$$

It is easy to verify that K is a well-defined functor. We are going to verify that K is an equivalence of categories.

Lemma 8. *For each object* $\varphi\colon F \to B$ *of* FRM $\downarrow$ CABA *the following subframe*

$$H(F, B, \varphi) = \{(u, b) \in F \times B;\ b \leq \varphi(u)\}$$

of $F \times B$ *together with the unary operation*

$$(u, b)' = (u, \varphi(u) \wedge \neg b)$$

forms a grid.

Proof. (G1): From $b \leq \varphi(u)$ we have $\varphi(u) \wedge b = b$, thus

$$\begin{aligned} (u, b)'' &= (u, \varphi(u) \wedge \neg b)' \\ &= (u, \varphi(u) \wedge \neg[\varphi(u) \wedge \neg b]) \\ &= (u, \varphi(u) \wedge (\neg\varphi(u) \vee b)) \\ &= (u, 0_{\mathrm{B}} \vee (\varphi(u) \wedge b)) \\ &= (u, b) \end{aligned}$$

(G2) and (G3): In fact, since $b \leq \varphi(u)$, we have

$$(u,b)^{\uparrow} = (u, b \vee (\varphi(u) \wedge \neg b)) = (u, \varphi(u) \wedge 1_B) = (u, \varphi(u))$$

and

$$(u,b)_{\downarrow} = (u, b \wedge \varphi(u) \wedge \neg b) = (u, 0_{\mathrm{B}}).$$

Thus (G2) and (G3) easily follow from the fact that φ is a frame homomorphism.

(G4):

$$((u,b) \wedge (v,c))_{\downarrow} = (u \wedge v, 0_{\mathrm{B}}) = (u,b) \wedge (v,c)_{\downarrow}.$$

(G5): The interval $[(u, 0_{\mathrm{B}}), (u, \varphi(u))]$ is isomorphic to the interval $[0_{\mathrm{B}}, \varphi(u)]$ of B, which is a CABA. □

Now we can define a functor H: FRM $\downarrow$ CABA $\rightarrow$GRID, which assigns to each homomorphism $F \xrightarrow{\varphi} B$ the grid $H(F, B, \varphi)$ of Lemma 8, and to each morphism $(h_1, h_2)\colon (F, B, \varphi) \longrightarrow (\bar{F}, \bar{B}, \bar{\varphi})$ the homomorphism $H(h_1, h_2) : H(F, B, \varphi) \longrightarrow H(\bar{F}, \bar{B}, \bar{\varphi})$ given by

$$(u,b) \mapsto (h_1(u), h_2(b)).$$

This is well-defined, since $b \leq \varphi(u)$ implies $h_2(b) \leq \bar{\varphi}(h_1(u))$ (due to $\bar{\varphi} h_1 = h_2 \varphi$), and since h_1, h_2 are frame homomorphisms, so is $H(h_1, h_2)$. Let us check that $H(h_1, h_2)$ preserves $'$:

$$\begin{aligned} H(h_1,h_2)(u,b)' &= (h_1(u), h_2(\varphi(u) \wedge \neg b)) \\ &= (h_1(u), h_2\varphi(u) \wedge \neg h_2(b)) \\ &= (h_1(u), \bar{\varphi} h_1(u) \wedge \neg h_2(b)) \\ &= (h_1(u), h_2(b))' \\ &= (H(h_1,h_2)(u,b))'. \end{aligned}$$

Thus, H is a well-defined functor.

Theorem 9. *The categories* GRID *and* FRM $\downarrow$ CABA *are equivalent. In fact, both HK and KH are naturally isomorphic to identity functors.*

Proof. I. We define a natural transformation

$$\psi\colon \mathrm{Id}_{GRID} \longrightarrow HK$$

by

$$\psi_{(G,')}(u) = (u \wedge 1', u \vee 1') \qquad \text{for all } u \in G.$$

It is easy to see that this mapping is a frame homomorphism, let us verify that it preserves $'$, i.e., that $(u \wedge 1', u \vee 1')' = (u' \wedge 1', u' \vee 1')$. The first coordinates agree by Lemma 6 (i), since $u \wedge 1' = u' \wedge 1'$. The second coordinate of the left-hand side is, by definition of K and H.

$$\begin{aligned}((u \wedge 1')' \vee 1') \wedge (u \vee 1')' &= (u \vee u' \vee 1') \wedge (u \vee 1')' \quad \text{by Lemma 6 (i)} \\ &= [(u \vee 1') \wedge (u \vee 1')'] \vee [u' \wedge (u \vee 1')'] \\ &\qquad\qquad\qquad\qquad \text{by distributivity} \\ &= 1' \vee [u' \wedge (u \vee 1')'] \quad \text{by (G5)} \\ &= (u' \vee 1') \wedge (u \vee 1')' \quad \text{by distributivity and (G5).}\end{aligned}$$

To prove that the last expression is equal to the second coordinate of the right hand side, i.e., to $u' \vee 1'$, it is sufficient, by (G5) to verify that $(u' \vee 1') \wedge (u \vee 1') = 1'$ (since this is equivalent to $u' \vee 1' \leq (u \vee 1')')$:

$$\begin{aligned}(u' \vee 1') \wedge (u \vee 1') &= (u \wedge u') \vee (u' \wedge 1') \vee (1' \wedge u) \vee 1' \quad \text{by distributivity} \\ &= (u' \wedge 1') \vee 1' \quad \text{by Lemma 6 (i)} \\ &= 1'.\end{aligned}$$

Thus, $\psi_{(G,')}$ is a grid homomorphism. It is bijective, which follows from Lemma 7. The naturality of ψ follows immediately from the fact that every grid homomorphism preserves $1'$.

II. KH is naturally isomorphic to the identity functor of FRM $\downarrow$ CABA. In fact, given an object $\varphi \colon F \longrightarrow B$ put $G = H(F, B, \varphi)$, then $1'_G = (1_F, 0_B)$, thus

$$[0, 1']_G = F \times \{0_B\} \quad \text{and} \quad [1', 1]_G = \{1_F\} \times B.$$

We see that KH assigns to $\varphi \colon F \longrightarrow B$ the homomorphism from $F \times \{0_B\}$ to $\{1_F\} \times B$ given by $(u, 0_B) \mapsto (1_F, \varphi(u))$. The canonical isomorphism from the latter object to $\varphi \colon F \longrightarrow B$ (given by dropping 0_B on the first sort and 1_F on the second one) is obviously natural. □

Corollary 10. *The category of topological systems is dually equivalent to the category of grids.*

Remark 11. $\mathrm{TOP}^{\mathrm{op}}$ is presented in the variety FRM $\downarrow$ CABA by the single implication $\varphi(x) = \varphi(y) \Longrightarrow x = y$ (see Introduction). This translates, under the above equivalence K, to the single implication $u^{\dagger} \vee 1' = v^{\dagger} \vee 1' \Longrightarrow u^{\dagger} = v^{\dagger}$ for grids, which is precisely the implication used in [BP_1], [BP_2].

Remark 12. (A. Carboni) The fact that FRM $\downarrow$ CABA is equivalent to a 1-sorted variety can be seen directly as follows: if a many sorted variety $\mathscr{V}$ has the property that the terminal object in $\mathscr{V}$ has no proper subobjects, then $\mathscr{V}$ is equivalent to a 1-sorted variety. (In fact, $\mathscr{V}$ has then a regular projective regular generator, viz., the free $\mathscr{V}$-algebra generated by a single variable in each sort.) Since FRM $\downarrow$ CABA has the terminal object $\mathrm{id}\colon 1 \to 1$ which has no proper subobjects, the result follows.

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