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*CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES*
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# CENTRAL EXTENSIONS AND RECIPROCITY LAWS

*by Jean-Luc BRYLINSKI*

**Résumé.** Cet article développe la théorie d'un groupe agissant sur un groupoïde connexe. Une telle action donne lieu à une extension canonique du groupe. On démontre que toute extension de groupe est obtenue par ce procédé. Lorsque le groupe a un objet fixe dans le groupoïde, on prouve que l'extension de groupe est scindée. On donne diverses applications de ce théorème de point fixes. La première est une preuve catégorique de la loi de réciprocité quadratique sur un corps global quelconque; c'est en fait une distillation de l'essence catégorique de la preuve due à A. Weil. La seconde application est la démonstration de la nature purement symplectique du théorème des point fixes de Atiyah-Bott pour un fibré en droites holomorphe sur une variété kaehlérienne.

## Introduction

In this paper we develop a geometric description of group extensions based on the concept of a group acting on a groupoid. We prove a new type of fixed point theorem: if the group fixes some object of the category, then the extension splits. The splitting then has numerical consequences. As an application, we derive the quadratic reciprocity law over any global field $E$ of characteristic not equal to 2. The relevant groupoid has as objects the lagrangian subgroups of the group $\mathbb{A}^2$, where $\mathbb{A}$ denotes the additive group of adeles of $E$. The morphisms are given by intertwining operators between two models of the Stone-von Neumann representation of the Heisenberg-Weil group associated to $\mathbb{A}^2$. The group $SL(2,E)$ acts on the groupoid in a natural way, and the fixed object is the lagrangian subgroup $E^2$ of $\mathbb{A}^2$.

When a group $G$ acts on some manifold, one can make $G$ act on various categories of bundles over the manifold, where the bundles are equipped with extra structures such as a connection. Then we show that the extension of $G$ splits if and only if there exists an equivariant bundle in our category. In the case of a group $G$ of symplectomorphisms of a quantizable symplectic manifold, the category consists of the quantum line bundles and the central extension of $G$ is the well-known Kostant central extension. In general, one can often interpret a fixed object as corresponding to some equivariant bundle, or to some equivariant fiber functor. If $G$ has a fixed point, then the central extension splits. Given two fixed points, the corresponding splittings differ by an explicit character of $G$. This can be used to show that in the case of a torus acting by automorphisms of a Kaehler manifold, the right hand side of the Atiyah-Bott fixed point formula can be given a purely symplectic interpretation, up to a character of $G$. This recovers a result of Jeffrey [4].

I first started thinking about the results of this paper in the Spring of 1992, and I talked about some aspects of them at colloquiums at Cornell and Princeton Universities at that time. I am grateful to Corrado De Concini, Pierre Deligne, Lisa Jeffrey, Victor Kac and Dennis McLaughlin for useful discussions. This research was supported in part by NSF grant DMS-9203517.

## 1. ACTION OF A GROUP ON A CATEGORY

It hardly needs to be pointed out that the notion of a group acting on a set is one of the most prevalent notions in geometry. In fact, if one takes F. Klein to his word, this is the same as geometry. The notion of a group acting on a category $C$ is more complicated, although it is implicitly encountered every time a group acts on a space.

**Definition 1.1.** *An action of a group $G$ on a category $C$ consists of a functor $T_g : C \to C$ for any $g \in G$, together with invertible natural transformations $\phi_{g,h} : T_{gh} \to T_g \circ T_h$, defined for any $g, h \in G$, satisfying the following conditions:*

*(A) For any $g_1, g_2, g_3 \in G$, the following diagram is commutative:*

$$\begin{tikzcd}
T_{g_1g_2g_3} \arrow[r, "\phi_{g_1,g_2g_3}"] \arrow[d, "\phi_{g_1g_2,g_3}"] & T_{g_1}T_{g_2g_3} \arrow[d, "T_{g_1}(\phi_{g_2,g_3})"] \\
T_{g_1g_2}T_{g_3} \arrow[r, "\phi_{g_1,g_2}T_{g_3}"] & T_{g_1}T_{g_2}T_{g_3}
\end{tikzcd}$$

*(B) We have $T_1 = Id$ and $\phi_{1,g} = Id$, $\phi_{g,1} = Id$.*

I recently learned from Sasha Beilinson that this definition appears already in Verdier's thesis. it was also recently proposed by Kapranov [5] in connection with his conjectural framework for higher dimensional Langlands philosophy.

The basic example is the following. Let $G$ act by homeomorphisms on a space $X$, 'and let $C$ be the category such that an object of $C$ is a vector bundle $E \to X$ over $X$, and a morphism $E \to F$ is a bundle isomorphism. Then for any $g \in G$, define the functor $T_g$ as the pull-back by $g^{-1}$ so that

$$T_g = g_* = (g^{-1})^*.$$

For any vector bundle $E$, and for $g, h \in G$, we have a natural isomorphism between $(gh)_*(E)$ and $g_*(h_*(E))$. This gives the natural transformation $\phi_{g,h}$. The coherency condition (A) is obvious (so obvious that it takes some time to convince oneself that there is something to prove).

To obtain a group extension of $G$ from an action of $G$ on a category $C$, one needs some assumptions on the category, which we now give:

(C) $C$ is a groupoid, i.e. every arrow in $C$ is invertible.

(D) $C$ is connected, i.e. for two objects $P_1$ and $P_2$ of $C$, there exists an arrow from $P_1$ to $P_2$.

Then for any object $P$ of $C$, we have the fundamental group $\pi_1(C, P)$ of $C$, which is the group $Aut(P)$ of arrows from $P$ to itself. Given an arrow $f : P_1 \to P_2$, we have an induced isomorphism $f_* : \pi_1(C, P_1) \to \pi_1(C, P_2)$. The isomorphism depends on the choice of the arrow $f$, but only modulo inner conjugations. Hence if we introduce the category $Out\ Gr$, whose objects are groups, and whose arrows are classes of group isomorphisms modulo inner automorphisms, we have a well-defined object $\pi_1(C)$ of $Out\ Gr$. Note that the automorphism group of $\pi \in Ob(Out\ Gr)$ is the group $Out(\pi) := Aut(\pi)/Inn(\pi)$ of outer automorphisms of $\pi$.

The action of $G$ on $C$ induces an action of $G$ on the object $\pi_1(C)$ of $Out\ Gr$, i.e. a group homomorphism $f : G \to Out(\pi_1(C))$.

**Proposition 1.2.** *(I) For an object $P$ of $C$, there is an associated group extension*

$$1 \to \pi_1(C, P) \to \tilde{G}_P \to G \to 1 \tag{1}$$

*for which the induced group homomorphism $G \to Out(\pi_1(C))$ coincides with $f$. We have: $\tilde{G}_P = \{(g, \alpha) : g \in G, \alpha \in Hom_C(T_g(P), P)\}$.*

*(II) Any arrow $P_1 \to P_2$ in $C$ induces an isomorphism of extensions from $\tilde{G}_{P_1}$ to $\tilde{G}_{P_2}$. This isomorphism depends on the arrow only up to inner conjugation by an element of the subgroup $\pi_1(C)$.*

**Proof.** We define $\tilde{G}_P = \{(g, \alpha) : g \in G, \alpha \in Hom_C(T_g(P), P)\}$ as in (1), with product law

$$(g_1, \alpha_1) \cdot (g_2, \alpha_2) = (g_1 g_2, \alpha_1 \circ T_{g_1}(\alpha_2)). \tag{2}$$

The associativity follows from the commutative diagram (A). The inverse of $(g, \alpha)$ is $(g^{-1}, T_{g^{-1}}(\alpha)^{-1} \circ \phi_{g^{-1},g})$, where we use $T_1(P) = P$. So $\tilde{G}_P$ is a group, we have a group homomorphism $p : \tilde{G}_P \to G$, $p(g, \alpha) = g$; $p$ is surjective because of condition (D). The kernel of $p$ identifies with $Aut(P) = \pi_1(C, P)$, and the group homomorphism $G \to Out(\pi_1(C, P))$ with $f$.

Let $h : P_1 \to P_2$ be an arrow in $C$. Then we have a homomorphism of extensions $h_* : \tilde{G}_{P_1} \to \tilde{G}_{P_2}$ such that $h_*(g, \alpha) = (g, h\alpha T_g(h)^{-1})$. The composition of arrows induces the composite of the corresponding homomorphisms of group extensions. For $P_1 = P_2$, an automorphism $h$ of $P_1$ induces the corresponding inner automorphism of $\tilde{G}_{P_1}$. This proves (2). ∎

We will now explain the significance of this group extension in a geometric context. Consider some category $C$ of geometric objects over a space $X$. By geometric object we mean some type of bundle over $X$, maybe equipped with some structure like a connection, possibly satisfying some condition on the curvature. We assume that this category satisfies conditions (C) and (D). Assume that a discrete group $G$ acts on $X$ by homeomorphisms. Assume that for $P$ in the category, the pull-back object $g^*P$ also belongs to it. Then we have an action of $G$ on the category, so that $T_g$ is the pull-back under $g^{-1}$. We have the notion of a $G$-equivariant object of $C$. This is an object to which the action of $G$ lifts. The following result explains the geometric meaning of the extension of $G$ obtained in Proposition 1.2.

**Proposition 1.3.** *Assume that the group $G$ acts by homeomorphisms on a space $X$, and let $C$ some category of geometric objects on $X$, which satisfies (C) and (D). Assume that for $g \in G$ and $P \in ob(C)$, the pull-back $g^*P$ belongs to $C$. Then the following conditions are equivalent:*

*(I) The group extension $1 \to \pi_1(C, P) \to \tilde{G}_P \to G \to 1$ of Proposition 1.2 splits.*

*(II) There exists a $G$-equivariant object of $C$.*

**Proof.** A splitting $s : G \to \tilde{G}$ of the extension is a map $g \in G \mapsto \phi_g : g_*P \tilde{\to}$ such that $\phi_{gh} = h_*(\phi_g) \circ \phi_h$ for all $g, h \in G$. This amounts exactly to an action of $G$ on the bundle $P$ which lifts the action on $X$. This shows that (I) implies (II). The same reasoning, joined with the fact that all objects of $P$ are isomorphic, show that (II) implies (I). ∎

We are mostly interested in the case where $\pi_1(C)$ is abelian, in which case $Out(\pi_1(C)) = Aut(\pi_1(C))$. if the action of $G$ on $\pi_1(C)$ is trivial, then Proposition 1.2 gives a central extension of $G$ by $\pi_1(C)$, which is defined uniquely, up to a unique isomorphism.

We illustrate Proposition 1.3 with the following example. Let $M$ be an oriented Riemannian manifold such that $H^1(M, \mathbb{Z}/2) = 0$. Assume that $G$ acts on $M$ by orientation-preserving isometries. Let $P \to M$ be the oriented orthonormal bundle, which is a principal $SO(n)$-bundle. Assume that the structue group of this bundle can be lifted to $Spin(n)$, in other words $M$ is a Spin manifold. We construct a category as follows. The objects of $C$ are $Spin(n)$-bundles $Q \to M$, together with an isomorphism $\phi : Q/\mathbb{Z}/2 \tilde{\to} P$ of $SO(n)$-bundles over $M$. A morphism is an isomorphism of $Spin(n)$-bundles, which is induces the identity on $P$. The group $C$ acts on $C$ by pull-backs. The assumptions (A)-(D) are satisfied, therefore we have a central extension $\tilde{G}$ of $G$ by $\pi_1(C) = \mathbb{Z}/2\mathbb{Z}$. The central extension splits if and only there exists a $G$-equivariant $Spin(n)$-structure on $M$.

If we work with differentiable categories and differentiable group actions, we get extensions of Lie groups.

**Proposition 1.4.** *Any group extension $1 \to K \to \tilde{G} \xrightarrow{p} G \to 1$ arises from an action of $G$ on some groupoid $C$ satisfying conditions (A)-(D).*

**Proof.** We define a groupoid $C$ as follows. The set $Ob(C)$ of objects of $C$ is equal to $G$. To $g \in G$ associate the set $p^{-1}(g) \subset \tilde{G}$ equipped with the action of $K$ by right translations. Given $g_1, g_2 \in G^2$, an arrow from $g_1$ to $g_2$ is a bijection $\phi : p^{-1}(g_1) \simeq p^{-1}(g_2)$ which is $K$-equivariant. Thus $\phi$ is given by right multiplication by some element $h$ of $\tilde{G}$ such that $p(h) = g_1^{-1}g_2$. Given $g \in G$, let $T_g : C \to C$ be the following functor. $T_g(g_1) = gg_1$ on objects, and $T_g(h) = h$ on arrows. More precisely, if we view right multiplication by $h \in \tilde{G}$ as an arrow from $g_1$ to $g_2$, then $T_g(h)$ is the arrow $h$ from $gg_1$ to $gg_2$. It is clear that axioms (A)-(D) are verified. We then describe the extension $1 \to Aut(1) \to E \to G \to 1$. The group of automorphisms of $1 \in G = Ob(C)$ identifies with $K$. The group $E$ consists of pairs $(g, h)$, where $g \in G$, $h \in \tilde{G}$ and $p(h) = g$, with product $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$. But this identifies with $\tilde{G}$. ∎

So we have a sort of universal construction of group extensions in terms of groups acting on groupoids. As an illustration, we study a type of induction operation for group extensions. Let $1 \to K \to \tilde{G} \xrightarrow{p} G \to 1$ be a group extension, and let $f : G \to Out(K)$ the associated outer action of $G$ on $K$. Let $H$ be a group which contains $G$ as a subgroup. Then we construct a group extension $1 \to A \to \tilde{H} \to H \to 1$; when $K$ is abelian, the group $A$ is the group $Map_G(H, K)$ of $G$-equivariant maps $H \to K$, equipped with pointwise product, and $H$ acts on this group in the natural way. If $K$ is not abelian, the description of $A$ is less simple. To construct the new group extension, we introduce a groupoid $C$. The set $Ob(C)$ is equal to the set of maps $m : H \to G$ such that $m(gh) = g \cdot m(h)$ for any $g \in G$, $h \in H$. Note that such a map $m$ is completely by $m^{-1}(1) \subset H$, which is a set of representatives for the left $G$-cosets in $H$. Given $m_1, m_2 : H \to G$ as above, an arrow $m_1 \to m_2$ is a family $\phi_h$ of $K$-equivariant bijections $\phi_h : p^{-1}(m_1(h)) \tilde{\to} p^{-1}(m_2(h))$ such that for $g \in G$, $h \in H$, $\phi_h$ and $\phi_{gh}$ are given by the same element of $\tilde{G}$. The group $H$ acts naturally on $C$. For $h' \in H$, the functor $T_{h'}$ transforms the object $(m_h)$ into the family $(m_{h(h')^{-1}})$, and the arrow $(\phi_h)$ into the arrow $(\phi_{h(h')^{-1}})$.

Given a set $S \subset H$ of representatives for the left $G$-cosets, there is an an associated object of $C$. its automorphism group $A$ is simply the cartesian power $K^S$. There results a homomorphism $\lambda : H \to Out(K^S)$, which may be described as follows. First for any $g \in G$, one needs to choose an automorphism $\tilde{f}(g)$ of $K$ representing $f(g) \in Out(K)$. Let $h \in H$; then for any $s \in S$, one can write $sh^{-1} = g_s s'$ for a unique pair $(g_s, s') \in G \times S$. Then $\lambda(h)$ is the class of the automorphism which maps $(h_s)_{s \in S}$ to the element of $K^S$ whose $s'$ component is $\tilde{f}(g_s) \cdot h_s$.

We summarize this construction in

**Proposition 1.5.** *Let $G \subset H$ and let $1 \to K \to \tilde{G} \to G \to 1$ be a group extension. Then there exists an induced extension $1 \to A \to \tilde{H} \to H \to 1$, where $A$ is isomorphic to the cartesian power $K^{G \backslash H}$. We have:*

*(I) For $K$ abelian, $A$ is isomorphic to $Map_G(H, K)$ as an $H$-module, and the above construction represents the Shapiro isomorphism*
*$H^2(G, K) \tilde{\to} H^2(H, Map_G(H, K))$ in group cohomology.*

*(II) Assume that the homomorphism $f : G \to Out(K)$ lifts to a homomorphism $\tilde{f} : G \to Aut(K)$. Then the extension of $G$ splits as a semi-direct product if and only if the induced extension of $H$ splits.*

## 2. CONSTRUCTIONS OF CENTRAL EXTENSIONS

We will give several examples of our construction of central extensions, which are related to equivariant bundles and to reciprocity laws.

First let $(M, \omega)$ be a symplectic manifold, and assume that the cohomology class of $\omega$ is integral, so that there exists some line bundle with curvature $K = 2\pi\sqrt{-1} \cdot \omega$. Let $G$ be some group of symplectomorphisms of $(M, \omega)$. Let $C$ be the following category. The objects of $C$ are pairs $(L, \nabla)$, where $L$ is a line bundle on $M$, and $\nabla$ a connection on $L$ with curvature $\omega$. A morphism $h : (L_1, \nabla_1) \to (L_2, \nabla_2)$ is an isomorphism $h : L_1 \xrightarrow{\sim} L_2$ of line bundles which is compatible with the connections. Then $G$ acts on $C$ by $T_g(L, \nabla) = (g^{-1})^*(L, \nabla) = g_*(L, \nabla)$. The natural transformation $\phi_{g_1,g_2}$ is the obvious one. Axioms (A), (B) and (C) are easily verified. Axiom (D) will hold true if all pairs $(L, \nabla)$ are isomorphic, and this happens when any flat line bundle on $M$ is trivial, i.e. $H^1(M, \mathbb{C}^*)$. The group $\pi_1(C)$ is equal to $\mathbb{C}^*$, and the action of $G$ on $\mathbb{C}^*$ is trivial. So in this case we get a central extension of the group $G$ of symplectomorphisms by $\mathbb{C}^*$. This is isomorphic to Kostant's central extension ([6], see also [3]).

There are several variants of this construction. If one works instead with a category of circle bundles with connection, one gets a central extension by the circle group $\mathbb{T}$ instead of $\mathbb{C}^*$. If $\omega$ is not integral, but has period group $\Lambda \subset \mathbb{R}$, one can work with a category of $\mathbb{C}/\Lambda$-torsors, and one recovers Weinstein's generalization of the Kostant central extension [15].

We now introduce a category inspired by the Stone-von Neumann theorem. This will be crucial in our proof of the quadratic reciprocity law. Let $B$ be a locally compact abelian topological group which is second countable. Let $\hat{B}$ be the Pontryagin dual of $B$. The there is an associated <u>Heisenberg group</u> $H = H(B)$ introduced by A. Weil [14]. We will give a description of it slightly different from Weil's; we will always assume that the map $x \mapsto x^2$ from $H$ to itself is invertible; the inverse map will be denoted $x \mapsto x^{1/2}$. The Heisenberg group $H = H(B)$ is the group $H = \mathbb{T} \times B \times \hat{B}$ with the product

$$(z_1, b_1, \chi_1) \cdot (z_2, b_2, \chi_2) = (z_1 z_2 \chi_2(b_1^{1/2}) \chi_1(b_2^{1/2})^{-1}, b_1 \cdot b_2, \chi_1 \cdot \chi_2) \quad (3)$$

The commutator factors through the non-degenerate skew-symmetric pairing
$S((b_1, \chi_1), (b_2, \chi_2)) = \chi_2(b_1) \cdot \chi_1(b_2)$. Note that $H(B)$ is a central extension of $B \times \hat{B}$ by $\mathbb{T}$, with center equal to $\mathbb{T}$. Our description of $H$ has the advantage that the symplectic group $Sp(B \times \hat{B})$, which consists

of all automorphisms of $H \times \hat{B}$ preserving the skew-symmetric pairing, acts naturally on $H$, the action on the factor $\mathbb{T}$ of $H = \mathbb{T} \times B \times \hat{B}$ being trivial.

Recall now the generalized Stone-von Neumann theorem.

**Theorem 2.1.** *(see [14] There is exactly one isomorphism class of irreducible unitary representation $\rho : H(B) \to Aut(\mathcal{H})$ such that $\rho(z) = z \cdot Id$ for $z \in \mathbb{T}$.*

A representation satisfying the assumption of Theorem 2.1 will be called a Stone-von Neumann representation of $H$. A construction of such a representation is obtained from any lagrangian subgroup $\Lambda$ of $B \times \hat{B}$. This means that $\Lambda$ is maximal isotropic with respect to $S$. Then $\mathbb{T} \times \Lambda \subset H$ is a commutative subgroup, and we have the character $\eta(z,x) = z$ of $\mathbb{T} \times \Lambda$. Pick a Haar measure $\mu$ on $B \times \hat{B}$.

**Lemma 2.2.** *Let $\mathcal{H}(\Lambda)$ be the induced representation*

$$\mathcal{H}(\Lambda) = \{f : H \to \mathbb{C}; f(hg) = \eta(g)^{-1} \cdot f(h), \ \forall h \in \mathbb{T} \times \Lambda$$
$$\text{and} \int_{B \times \hat{B}/\Lambda} |f|^2 d\mu < \infty\}.$$

*Let $\rho(g) \cdot f(h) = f(g^{-1}h)$. Then $(\mathcal{H}(\Lambda), \rho)$ is a Stone-von Neumann representation of $H$.*

Note that any $g \in Sp(B \times \hat{B})$ induces an intertwining isomorphism $g_* : \mathcal{H}(\Lambda) \tilde{\to} H(g \cdot \Lambda)$, such that $(g_* f)(h) = f(g^{-1}h)$.

We then come to the category $C$. An object of $C$ is a lagrangian subgroup $\Lambda$ of $B \times \hat{B}$. An arrow $f : \Lambda_1 \to \Lambda_2$ is a unitary intertwining operator $f : \mathcal{H}(\Lambda_1) \tilde{\to} \mathcal{H}(\Lambda_2)$. The group $G$ will be some subgroup of the group $Sp(B \times \hat{B})$ of symplectic automorphisms of $B \times \hat{B}$. The action is given by the functor $T_g$ which sends $\Lambda$ to $g(\Lambda)$ and $f : \mathcal{H}(\Lambda_1) \tilde{\to} \mathcal{H}(\Lambda_2)$ to the intertwining operator $T_g(f)$ defined so as to give a commutative diagram

$$\begin{array}{ccc} \mathcal{H}(\Lambda_1) & \xrightarrow{f} & \mathcal{H}(\Lambda_2) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ \mathcal{H}(g \cdot \Lambda_1) & \xrightarrow{T_g(f)} & \mathcal{H}(g \cdot \Lambda_2) \end{array} .$$

The natural transformation $\phi_{g_1,g_2}$ is the obvious one.

The group of automorphisms of an object $\Lambda$ is equal to $\mathbb{T}$, and $Sp(B \times \hat{B})$ acts trivially on $\mathbb{T}$. So from Proposition 1.2 one derives a central extension

$$1 \to \mathbb{T} \to Mp(B \times \hat{B}) \to Sp(B \times \hat{B}) \to 1 \tag{4}$$

where $Mp(B \times \hat{B})$ is the so-called metaplectic group (Weil actually gives this name to a subgroup of $Mp(B \times \hat{B})$, which is a central extension of $Sp(B \times \hat{B})$ by $\mathbb{Z}/2$).

Of particular interest is the case $B = K^n$, for $K$ some local field, i.e. $K = \mathbb{R}$ or $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$, or a power series field $\mathbb{F}_q((x))$ over a finite field of characteristic not equal to 2. Then one can identify $\hat{K}$ with $K$ in the following way. One chooses a non-trivial character $\psi : K \to \mathbb{T}$; then the continuous homomorphism $x \mapsto (y \mapsto \psi(x \cdot y))$ from $K$ to $\hat{K}$ is an isomorphism. Thus for $B = K^n$ we have $\hat{B} \tilde{\to} K^n$, and $Sp(B \times \hat{B})$ is the symplectic group $Sp(2n, K)$.

The last example is taken from work of Arbarello, De Concini and Kač [1], and is an analog for buildings of the Kostant central extension. Let $k$ be a field, and let $E$ be a vector space over $k$ (not of finite dimension). Two subspaces $F_1$ and $F_2$ of $E$ are called commensurable if $F_1 \cap F_2$ is of finite codimension in both $F_1$ and $F_2$. We fix a commensurability class $S$ of subspaces of $E$. We introduce a category $C$, whose objects are subspaces $F \subset E$ in the given commensurability class. To define the arrows, we need the following construction.

**Proposition 2.3.** *There exists a unique way to assign a* 1*-dimensional vector space* $(F_1|F_2)$ *to a pair of subspaces in the class* $C$*, together with isomorphisms*

$$\omega(F_1, F_2, F_3) : (F_1|F_2) \otimes_k (F_2|F_3) \tilde{\to} (F_1|F_3) \tag{5}$$

*such that:*

*(I) For* $F_1 \subseteq F_2$*, we have:*

$$(F_1|F_2) = \wedge^{\max}(F_2|F_1).$$

*For* $F_1 \subseteq F_2 \subseteq F_3$*, the isomorphism* $\omega(F_1, F_2, F_3)$ *is the canonical one.*

*(II) Given $F_1, F_2, F_3, F_4 \in S$, we have a commutative diagram*

$$\begin{array}{ccc}
(F_1|F_2)\otimes(F_2|F_3)\otimes(F_3|F_4) & \xrightarrow{\sim} & (F_1|F_3)\otimes(F_3|F_4) \\
\downarrow \simeq & & \downarrow \simeq \\
(F_1|F_2)\otimes(F_2|F_4) & \xrightarrow{\sim} & (F_1|F_4)
\end{array}$$

Then we define $Hom_C(F_1, F_2)$ to be $(F_1|F_2)\setminus\{0\}$, the set of non-zero elements in the line $(F_1|F_2)$. Composition of arrows is induced by the isomorphism $\omega(F_1, F_2, F_3)$, and (2) guarantees that composition of arrows is associative. The category $C$ is a connected groupoid, and $Aut(F) = k^*$ for any object $F$, so that $\pi_1(C) = k^*$.

Let $G = GL(E, S)$ be the subgroup of $GL(E)$ which preserves the commensurability class $S$. Then $G$ operates on the category $C$. For $g \in G$, define the functor $T_g : C \to C$ by $T_g(F) = g \cdot F$, and $T_g : (F_1|F_2)\xrightarrow{\sim}(g\cdot F_1|g\cdot F_2)$ is the canonical isomorphism, which exists by the uniqueness part of Proposition 2.3. Then we actually have $T_{gh} = T_g T_h$ for $g, h \in G$, so we take $\phi_{g,h} = Id$. Thus we have an action of the group $G$ on the category $C$. Consequently we have a central extension

$$1 \to k^* \to \widetilde{GL(E,S)} \to GL(E,S) \to 1. \qquad (6)$$

In particular let $E = k((x))^n$, and let $S$ be the commensurability class of the "lattice" $k[[x]]^n \subset k((x))^n$. Then the linear group $GL(n, k((x)))$ preserves this commensurability class, hence we have a central extension $\widetilde{GL(n,k((x)))}$ of $GL(n, k((x)))$. For $n = 1$ we have the following computation

**Proposition 2.4.** *Let $f, g \in k((x))^*$. Let $\tilde{f}$, $\tilde{g}$ be elements of $\widetilde{k((x))^*}$ which map to $f$, resp. $g$. Then the commutator $[\tilde{f}, \tilde{g}]$ is equal to*

$$\left[\frac{f^{v(g)}}{g^{v(f)}}\right](0), \qquad (7)$$

*where $v$ denotes the* <u>*valuation*</u> *of a formal power series (order of zero at the origin).*

**Proof.** As both the commutator $[\tilde{f}, \tilde{g}]$ and the expression (7) are bilinear in $f$ and $g$ and skew-symmetric, we may reduce the proof to two cases:

(a) $v(f) = 0$, $g = x$;

(b) $v(f) = v(g) = 0$.

In the first case, we pick the object $\mathcal{O} = k[[x]]$ of $C$, and we take $\tilde{f} = (f, 1)$, where 1 is the canonical element of $(f \cdot \mathcal{O}|\mathcal{O}) = (\mathcal{O}|\mathcal{O})$. And we pick $\tilde{x} = (x, [1])$, where $[1] \in (x \cdot \mathcal{O}|\mathcal{O}) = k \cdot 1$ is the element 1. The product $\tilde{f}\tilde{x}$ is equal to $(fx, \omega(1 \otimes f([1])))$. Now $f([1]) \in (x \cdot \mathcal{O}|\mathcal{O})$ is the transform of [1] under $f$, so it is the class $[f(0)]$ of $f$ in $\mathcal{O}/x \cdot \mathcal{O}$. On the other hand, we have: $\tilde{x}\tilde{f} = (fx, \omega([1] \otimes x \cdot 1)$, and $x \cdot 1$ is again the canonical element of $(x\mathcal{O}|x\mathcal{O})$. So we find $[\tilde{f}, \tilde{x}] = f(0)$ as claimed.

In the second case, it is easy that the two lifts $\tilde{f}$ and $\tilde{g}$ commute. ∎

## 3. THE FIXED POINT THEOREM

Given a central extension $1 \to A \to \tilde{G} \to G \to 1$ of $G$, we have a natural homomorphism $f : H_2(G, \mathbb{Z}) \to A$. Recall its concrete definition. First we compute the multiplier group $H_2(G, \mathbb{Z})$ of Schur using a free presentation $1 \to R \to F \xrightarrow{p} G \to 1$. Then $H_2(G, \mathbb{Z}) = F' \cap R/[F, R]$, where $F'$ denotes the commutator subgroup of $F$. Let $[x_1, y_1] \cdots [x_g, y_g]$ be an element of $F' \cap R$. Then setting $a_i = p(x_i)$, $b_i = p(y_i)$, we have the relation $[a_1, b_1] \cdots [a_g, b_g] = 1$ in $G$. Then chose lifts $\tilde{a}_i$ of $a_i$ and $\tilde{b}_i$ of $b_i$ in $\tilde{G}$. The expression $[\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g]$ belongs to $A$, and is independent of the choice of $\tilde{a}_1, \cdots, \tilde{b}_g$. We then have:

$$f([x_1, y_1] \cdots [x_g, y_g]) = [\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g] \in A. \tag{8}$$

This may be interpreted in a way which does not involve the given free presentation of $G$: any family of elements $(a_1, \cdots, a_g, b_1, \cdots, b_g)$ such that $[a_1, b_1] \cdots [a_g, b_g] = 1$ gives rise to an element of $H_2(G, \mathbb{Z})$.

A splitting of the central extension $1 \to A \to \tilde{G} \xrightarrow{q} G \to 1$ is a group homomorphism $s : G \to \tilde{G}$ such that $qs = Id_G$. The existence of a splitting implies that the central extension is isomorphic to the trivial one $1 \to A \to A \times G \to G \to 1$, hence that the corresponding homomorphism $f : H_2(G, \mathbb{Z}) \to A$ is trivial.

**Theorem 3.1.** *Let $C$ be a category satisfying the assumptions (C) and (D) of §1, and let the group $G$ act on $C$. Assume that $A = \pi_1(C)$ is abelian, and that the action of $G$ is trivial. Assume that there exists an object $P$ of $C$ such that $T_g(P) = P$ for all $g \in G$. Then the central extension $1 \to A \to \tilde{G} \to G \to 1$ of §1 has a natural splitting.*

**Proof.** Use the notations of the proof of Proposition 1.2. We use the object $P$ to describe the central extension. We set $s(g) = (g, Id_P)$. This gives a splitting of the central extension. ∎

## 4. THE QUADRATIC RECIPROCITY LAW

We will use the Fixed Point Theorem 3.1 to obtain a proof of the ordinary reciprocity law.

First of all we consider the metaplectic central extension $1 \to \mathbb{T} \to Mp(2,K) \to SL(2,K) = Sp(2,K) \to 1$ of §2, for $F$ a local field. There is an induced homomorphism $f : H_2(SL(2,K),\mathbb{Z}) \to \mathbb{T}$. We recall that for elements $a, b \in K^*$ we have a corresponding element of $H_2(SL(2,K),\mathbb{Z})$, denoted by $\{a,b\}$; this construction of Steinberg [12] works over any field. In the language of §4, this element $\{a,b\}$ can be constructed as follows. One introduces the free group $F$ on generators $X(u), Y(u)$ (for $u \in K$). There is a surjective homomorphism $p : F \to SL(2,K)$ such that $p(X(u)) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $p(Y(u)) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$. Choose an element $k$ of $K^*$ such that $c = k^2 \neq 1$. Then introduce for $u \in K^*$ the element $w(u) = X(u)Y(-u^{-1})X(u)$ of $F$, which maps to $\begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}$ of $SL(2,K)$. Then $h(u) = w(u) \cdot w(1)^{-1}$ maps to $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. In $SL(2,K)$ we have the expression $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = [\begin{pmatrix} t & 0 \\ o & t^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \frac{t}{c-1} \\ 0 & 1 \end{pmatrix}]$, and a similar expression for $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Now write a product of commutators in $F$ as follows. Write down $h(a)h(b)h(ab)^{-1}$ as a product of elements of the type $X(u)$ or $Y(v)$, and replace each of these by a commutator in $F$ which has the same image in $SL(2,K)$. Then one gets a product of commutators which maps to $1 \in SL(2,K)$; this gives the element $\{a,b\}$ of $H_2(SL(2,K),\mathbb{Z})$.

**Lemma 4.1.** *[8] For $K$ a local field and for the metaplectic central extension of $SL(2,K)$, we have:*

$$f(\{a,b\}) = (a,b) \tag{9}$$

*where $(a,b)$ is the* <u>*Hilbert norm residue symbol*</u>

$$(a,b) = \begin{cases} +1 & \text{if } ax^2+by^2=1 \text{ has a solution } (x,y)\in K^2 \\ -1 & \text{otherwise} \end{cases}. \tag{10}$$

**Proof.** This well-known fact follows easily by comparing Matsumoto's formula [9] for a cocycle representing the central extension of $SL(2,K)$ by $K_2(K)$ with Kubota's formula [7] for a cocycle representing the metaplectic central extension. ∎

Now assume that $K$ is a non-archimedean local field with ring of integers $O$. The additive character $\psi : K \to \mathbb{T}$ is called *unramified* if its kernel is equal to $O$.

**Lemma 4.2.** *Assume the character $\psi : K \to \mathbb{T}$ is unramified and that the residue field of $O$ has characteristic different from 2. Then the central extension $1 \to \mathbb{T} \to Mp(2n,K) \to Sp(2n,K) \to 1$ splits canonically over the subgroup $Sp(2n,O)$.*

**Proof.** It is easy to see that $O^{2n}$ is a lagrangian subgroup of $K^{2n}$, which is fixed under $Sp(2n,O)$. This gives an object $O^{2n}$ of the category $C$, such that $T_g \cdot O^{2n} = O^{2n}$ for any $g \in Sp(2n,O)$. By the fixed point theorem, the central extension of $Sp(2n,O)$ splits. ∎

Using this splitting, we may view $Sp(2n, O)$ as a subgroup of $Mp(2n, K)$.

Now let $E$ be a global field of characteristic not equal to 2. Let $S$ be the set of non-trivial places (not-trivial absolute values) of $E$. For each place $\mathcal{P}$ denote by $E_{\mathcal{P}}$ the completion of $E$ at $\mathcal{P}$, which is a local field. There exists a choice a choice of non-trivial characters $\psi_{\mathcal{P}} : E_{\mathcal{P}} \to \mathbb{T}$ such that

(1) for almost all non-archimedean places $\mathcal{P}$, the character $\psi_{\mathcal{P}}$ is unramified.

(2) for all $a \in E$, we have the product formula:

$$\prod_{\mathcal{P} \in S} \psi_{\mathcal{P}}(a) = 1.$$

Note that Lemma 1 and condition (1) imply that all but finitely many $\psi_{\mathcal{P}}(a)$ are non-trivial.

For a number field, the construction of such a family $(\psi_{\mathcal{P}})$ is given in Tate's thesis [13]. For the field of meromorphic functions on a curve over $\mathbb{F}_q$, it involves the choice of a meromorphic differential on the curve.

In order to obtain the quadratic reciprocity law in the form given to it by Hilbert, we introduce the ring of adeles $\mathbb{A}$, which is the restricted product $\mathbb{A} = \prod^{r}_{\mathcal{P} \in S} E_{\mathcal{P}}$. The ring $\mathbb{A}$ consists of families $(x_{\mathcal{P}} \in E_{\mathcal{P}})$, such that for almost all $\mathcal{P}$, $x_{\mathcal{P}}$ belongs to $O_{\mathcal{P}}$. We have the diagonal embedding $E \hookrightarrow \mathbb{A}$, which identifies $E$ with a discrete subring of $\mathbb{A}$, with compact quotient. We have the character $\psi$ of $\mathbb{A}$ such that $\psi((x_{\mathcal{P}})) = \prod_{\mathcal{P} \in S} \psi_{\mathcal{P}}(x_{\mathcal{P}})$. This character has trivial restriction to $E$. We use this character to identify $\mathbb{A}$ with its dual [13]. We consider the central extension

$$1 \to \mathbb{T} \to Mp(2n, \mathbb{A}) \to Sp(2n, \mathbb{A}) \to 1 \tag{11}$$

of the symplectic subgroup with coefficients in $\mathbb{A}$. We are concerned mostly with the case $n = 1$.

The group $Sp(2n, \mathbb{A})$ is the restricted product of the groups $Sp(2n, E_{\mathcal{P}})$, with respect to the subgroups $Sp(2n, O_{\mathcal{P}})$. One can take the restricted product

$\prod_{\mathcal{P}\in S}^r Mp(2n, E_{\mathcal{P}})$ of the groups $Mp(2n, E_{\mathcal{P}})$, with respect to the subgroups $Sp(2n, O_{\mathcal{P}})$. This is a central extension of $\prod_{\mathcal{P}\in S}^r Sp(2n, E_{\mathcal{P}})$ by $\oplus_{\mathcal{P}\in S} \mathbb{T}$. We have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & \oplus_{\mathcal{P}\in S} \mathbb{T} & \to & \prod_{\mathcal{P}\in S}^r Mp(2n, E_{\mathcal{P}}) & \to & \prod_{\mathcal{P}\in S}^r Sp(2n, E_{\mathcal{P}}) & \to & 1 \\
 & & \downarrow \Sigma & & \downarrow & & \downarrow || & & \\
1 & \to & \mathbb{T} & \to & Mp(2n, \mathbb{A}) & \to & Sp(2n, \mathbb{A}) & \to & 1
\end{array} \tag{12}$$

Consider the case $n = 1$. Let $a = (a_{\mathcal{P}})_{\mathcal{P}\in S}$, $b = (b_{\mathcal{P}})_{\mathcal{P}\in S}$ be invertible adeles (i.e. ideles). For each $\mathcal{P} \in S$, we have the element $\{a_{\mathcal{P}}, b_{\mathcal{P}}\}$ of $H_2(SL(2, E_{\mathcal{P}}), \mathbb{Z})$. Let $f : H_2(SL(2, \mathbb{A}), \mathbb{Z}) \to \mathbb{T}$ be the homomorphism associated to the central extension (1-11) of $SL(2, \mathbb{A})$, and let $f_{\mathcal{P}} : H_2(SL(2, E_{\mathcal{P}}), \mathbb{Z}) \to \mathbb{T}$ be the homomorphism associated to the local central extension. Then we have:

$$f(\{a, b\}) = \prod_{\mathcal{P}\in S} f_{\mathcal{P}}(\{a_{\mathcal{P}}, b_{\mathcal{P}}\}) = \prod_{\mathcal{P}\in S} (a_{\mathcal{P}}, b_{\mathcal{P}}). \tag{13}$$

For almost every place $\mathcal{P} \in S$, we have: $a_{\mathcal{P}}, b_{\mathcal{P}} \in O_{\mathcal{P}}^*$, hence $f_{\mathcal{P}}(\{a_{\mathcal{P}}, b_{\mathcal{P}}\}) = 1$. So the product in (13) is indeed finite.

Now we note:

**Lemma 4.3.** *The subgroup $E^2$ of $\mathbb{A}^2$ is lagrangian.*

**Proof.** The character $\psi : \mathbb{A} \to \mathbb{T}$ which we use to identify $\mathbb{A}$ with the dual group has the property that $E$ is its own orthogonal with respect to the pairing $\mathbb{A} \times \mathbb{A} \to \mathbb{T}$ given by $\psi(x \cdot y)$. Hence $E^2$ is its own orthogonal with respect to the pairing

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \psi(x_1 y_2 - x_2 y_1).$$

∎

We see that the subgroup $SL(2, E)$ of $SL(2, \mathbb{A})$ fixes the lagrangean subgroup $E^2$ of $\mathbb{A}^2$. We view $E^2$ as an object of the category $C$, such that $T_g(E^2) = E^2$ for all $g \in SL(2, E)$. So we have:

**Theorem 4.4.** *The restriction to $SL(2, E)$ of the central extension (11) has a canonical splitting.*

This is a group-theoretic version of the quadratic reciprocity law. Indeed, the expression (13) must be equal to 1 when $a$ and $b$ belong to $E^*$. Therefore we obtain:

**Corollary 4.5.** *(Hilbert's form of the quadratic reciprocity law) For $a, b \in E^*$, we have:*

$$\prod_{\mathcal{P}\in S} (a, b)_{\mathcal{P}} = 1.$$

We refer to [11] for a discussion of the equivalence between this product formula of Hilbert and the usual reciprocity law in the case $E = \mathbb{Q}$; this equivalence is of course an elementary fact.

The proof given here is somewhat similar spirit to Weil's proof [14], which itself is akin to one of Gauss's proofs, namely the one based on theta-functions. The role of theta-functions in Weil's proof is to give concrete vectors in the model $\mathcal{H}(E^2)$ of the (adelic) Stone-von Neumann representation. The point of the present categorical approach is to get rid of all computations, save those over local fields which were recalled in §2.

## 5. THE ATIYAH-BOTT FIXED POINT THEOREM

Theorem 3.1 is not the only case when the existence of a fixed point implies a "reciprocity law". For a group of symplectomorphisms we have an analogous theorem.

**Theorem 5.1.** *(see [3, Theorem 2.4.12]) Let $(M, \omega)$ be a connected symplectic manifold such that $H^1(M, \mathbb{C}^*) = 0$. Assume that the cohomology class of $\omega$ is integral. Let $G$ be a group of symplectomorphisms of $(M, \omega)$. Assume that there exists a point $x$ of $M$ which is fixed by $G$. Then the Kostant central extension $1 \to \mathbb{T} \to \tilde{G} \to G \to 1$ splits.*

**Proof.** Let $(L, \nabla)$ be a line bundle with curvature $2\pi i\omega$. From §2 we know that $\tilde{G}$ is the group of pairs $(g, \phi)$, where $g \in G$, and $\phi : g_*(L, \nabla) \tilde{\rightarrow} (L, \nabla)$. We define a section $s : G \rightarrow \tilde{G}$ by $s(g) = (g, \phi)$ where $\phi$ is the unique such isomorphism which induces the identity on the fibers at $x$. This makes sense because the fiber at $x$ of $g_*(L)$ is the fiber $L_x$, as $g \cdot x = x$. Then $s$ is the required section of $\tilde{G} \rightarrow G$. ∎

We note a Lie algebra analog of Theorem 5.1. For $(M, \omega)$ a symplectic manifold, we have the Lie algebra $H_M$ of <u>hamiltonian</u> vector fields on $M$, and the central extension

$$0 \rightarrow \mathbb{R} \rightarrow C^\infty(M) \xrightarrow{p} H_M \rightarrow 0 \tag{14}$$

Here for $f \in C^\infty(M)$, $p(f)$ is the hamiltonian vector field $X_f$. This central extension may be described, for any $x \in M$, by the Lie algebra 2-cocycle

$$c(\xi, \eta) = \omega_x(\xi_x, \eta_x). \tag{15}$$

(see [10], [3]).

**Proposition 5.2.** *Let $\mathfrak{h}_x \subset H_M$ be the Lie subalgebra of $H_M$ consisting of those hamiltonian vector fields which vanish at $x$. Then the central extension $\pi^{-1}(\mathfrak{h}_x) \rightarrow \mathfrak{h}_x$ induced by (14) is split.*

**Proof.** This is obvious from the formula (15) for the 2-cocycle of this extension. ∎

Returning to a group $G$ of symplectomorphisms which admits a fixed point, Theorem 5.1 has a numerical consequence which we state in the simplest case $G = \mathbb{Z}^2$.

**Corollary 5.3.** *(see [3, Corollary 2.4.13]) Let $(M, \omega)$ be a symplectic manifold, and let $f, g$ be commuting symplectomorphisms. Assume that there exists a point $x \in M$ which is fixed by both $f$ and $g$. Then given any smooth mapping $\sigma : [0,1] \rightarrow [0,1] \rightarrow M$ such that $\sigma(t, 1) = f \cdot \sigma(t, 0)$ and $\sigma(1, u) = g \cdot \sigma(0, u)$, we have:*

$$\int_{[0,1]\times[0,1]} \sigma^*\omega \in \mathbb{Z}.$$

We will show how the splitting theorem can be deduced from the abstract fixed point theorem (Theorem 3.1). Given a connected groupoid in which all automorphism groups are identified with $\mathbb{C}^*$, a fiber functor is a functor $F$ from $C$ to the category $C_0$ of $\mathbb{C}^*$-sets on which the $\mathbb{C}^*$-action is simply transitive. There is a groupoid $C^*$ whose objects are the fiber functors $F : C \to C_0$, and whose arrows are natural transformations between fiber functors. Given an action of the group $G$ on $C$ in the sense of §1, one obtains in a natural way an action of $G$ on $C^*$; indeed to any $g \in G$ we associate the functor $T_g^* : C^* \to C^*$ which sends a fiber functor $F$ to $F \circ T_{g^{-1}}$. The central extension associated to the $G$-groupoid $C^*$ is the opposite of the one associated to the $G$-groupoid $C$. In the situation of Theorem 5.1, the category C, constructed in §2. has as objects the line bundles over $M$, equipped with a connection $\nabla$ whose curvature is equal to $2\pi\sqrt{-1}\cdot\omega$. Every point $x$ of $M$ gives a fiber functor $F_x : C \to C_0$, such that $F_x(L, \nabla) = L_x \setminus \{0\}$. If $x$ is a fixed point of $G$, then the fiber functor $F_x$ is fixed by $T_g^*$ for all $g \in G$, since $T_g^*(F_x) \cdot L = F_x(g^{-1}L) = [(g^{-1})^* L]_x = L_{g^{-1}x} = L_x$. By Theorem 4.1 the central extension of $G$ associated to $C^*$ must split.

It may happen that the group $G$ of symplectomorphisms has no fixed point but that there is a fiber functor which is fixed under the $G$-action. This is illustrated by the following Theorem.

**Theorem 5.4.** *Let $(M, \omega)$ be a simply-connected quantizable symplectic manifold, and let $G$ be a Lie group which acts on $M$ by symplectomorphisms. Assume there is a connected lagrangian submanifold $\Lambda \subset M$ which is invariant under $G$. Then the Kostant central extension of $G$ splits if either*

*(a) $\Lambda$ is simply-connected*

*or (b) $G$ is 1-connected.*

**Proof.** As usual, let $C$ be the category of pairs $(L, \nabla)$ consisting of a line bundle $L$ and a connection $\nabla$ whose curvature is $2\pi\sqrt{-1}\cdot\omega$. In cases (a) and (b) we describe a fiber functor $F : C \to C_0$ from $C$ to the category of $\mathbb{C}^*$-torsors, which is invariant under the group action. Since $\Lambda$ is lagrangian, the restriction of $L$ to $\Lambda$ is a flat line bundle. In case (a), we let $F(L,\nabla)$ be the $\mathbb{C}^*$-torsor $\Gamma^*_{hor}(\Lambda,L)$ consisting of the non-zero flat sections of $L_{/\Lambda}$. Then for $g\in G$ we have:

$$(T_g^*F)(L,\nabla)=\Gamma^*_{hor}(\Lambda,g_*L)=\Gamma^*_{hor}(g\cdot\Lambda,L)=\Gamma^*_{hor}(\Gamma,L)=F(L,\nabla).$$

In case (b), let $\tilde{\Lambda}\to\Lambda$ be a universal covering space. Since $G$ is 1-connected, it is easy to see that the action of $G$ on $\Lambda$ lifts to a an action on $\tilde{\Lambda}$. Then we let $F(L,\nabla)$ be the $\mathbb{C}^*$-torsor consisting of the non-zero flat sections of the pull-back of $L_{/\Lambda}$ to $\tilde{\Lambda}$. So if $f:\tilde{\Lambda}\to M$ is the composite map, then $F(L,\nabla)=\Gamma^*_{hor}(\tilde{\Lambda},f^*L)$. It is then easy to show that $T_g^*F=F$ for any $g\in G$. ∎

We will study what happens when the group $G$ of symplectomorphisms admits two fixed points $x$ and $y$. Then we have two splittings $s_x,s_y:G\to\tilde{G}$ of the central extension $\tilde{G}$ of $G$ by $\mathbb{C}^*$. We thus have $s_y(g)=s_x(g)\chi(g)$, where $\chi:G\to\mathbb{C}^*$ is a character. We have two descriptions of $\chi$. The first description involves choosing a $G$-equivariant line bundle $L$ with connection $\nabla$, whose curvature is equal to $2\pi\sqrt{-1}\cdot\omega$. Such an equivariant line bundle exists because the central extension is split. Then let $\lambda_x:G\to\mathbb{C}^*$ be the character giving the action of $G$ on the fiber $L_x$, and define $\lambda_y$ similarly. Then we have:

$$\chi=\lambda_y\cdot\lambda_y^{-1}.$$

On the other hand, any path $\gamma$ from $x$ to $y$ determines an isomorphism of fiber functors $F_x\xrightarrow{H_\gamma}F_y$. The character $\chi(g)$ may be interpreted as the composition $H_{g\cdot\gamma}H_\gamma^{-1}$, which in turn is the holonomy of the line bundle $L$ around that loop $(g\gamma)*\gamma^{-1}$. Comparing these two descriptions of $\chi(g)$, we obtain

**Theorem 5.5.** *Let $G$ be a group of symplectomorphisms of $(M,\omega)$ and let $L$ be a $G$-equivariant line bundle, equipped with a connection $\nabla$ whose curvature is equal to $2\pi\sqrt{-1}\cdot\omega$. Let $x,y$ be two fixed points of $G$, let $\gamma$ be any path from $x$ to $y$, and let $g\cdot\gamma$ be its transform under $g$. Then we have*

$$\lambda_x=\lambda_y\cdot H_{(g\cdot\gamma)*\gamma^{-1}}(L,\nabla)$$

*where $H_{g\cdot\gamma*\gamma^{-1}}(L,\nabla)$ denotes the holonomy of $(L,\nabla)$ around the loop $(g\cdot\gamma)*\gamma^{-1}$. If $\sigma$ is a surface in $M$ which bounds $(g\cdot\gamma)*\gamma^{-1}$, then*

$$\lambda_x = \lambda_y \cdot exp(-2\pi\sqrt{-1}\cdot \int_\sigma \omega).$$

This result was proved by Jeffrey in her thesis [4]. We will explain its significance in relation with the Atiyah-Bott fixed point theorem and geometric quantization. First recall that in the absence of a group, if $X$ is a compact Kaehler manifold with symplectic form $\omega$ and $L$ is a holomorphic line bundle with curvature $2\pi\sqrt{-1}\omega$, the holomorphic Euler characteristic $\chi(X,L)=\sum_i(-1)^i dim\ H^i(X,L)$ is computed by the Riemann-Roch theorem $\chi(X,L)=\langle ch(L)\cdot Td(X),[X]\rangle$. The right hand side of this formula only depends on the symplectic manifold $M$, as the Chern character $ch(L)$ depends only on $\omega$ and the Todd class $Td(X)$ only depends on the symplectic structure of $X$. In many cases of interest we have the vanishing $H^i(X,L)=0$ for $i>0$, and then the quantization space $H^0(X,L)$ has a dimension which is an intrinsic invariant of the symplectic manifold.

Now assume that $T$ is a compact torus which acts on $X$ as a group of Kaehler automorphisms, and that the line bundle $L$ is (holomorphically) equivariant under $T$. The Atiyah-Bott fixed point formula gives an expression for the virtual character $\sum_i(-1)^i Tr(t,H^i(X,L))$ in terms of the components $F$ of the fixed point set $X^T$. For each component $F$, the character $\lambda_x$ of $T$ is independent of $x\in F$, and will be denoted by $\lambda_F$. Let $D_F(t)=\sum_j(-1)^j Tr(t,N_F^*)$, where $N_F^*$ is the conormal bundle to $F\hookrightarrow X$. The Atiyah-Bott fixed point formula then says:

$$\sum_i(-1)^i Tr(t,H^i(X,L)) = \sum_F \frac{\lambda_F(t)\cdot ch(F,L_{/F})}{D_F(t)}.$$

Assume that $X$ is simply-connected. We wish to show that up to a character of $T$, the right hand side is a purely symplectic invariant of $T$ acting by on $X$ by symplectomorphisms. Indeed, we cannot compute each character $\lambda_F$ by purely symplectic methods, but from

Theorem 5.5 we see that for two components $F, F'$ of $X^T$, the ratio $\lambda_F \lambda_{F'}^{-1}$ only depends on the symplectic structure of $X$ and on the action of $T$. It is well-known that $D_F(t)$ only depends on the symplectic structure on the normal space to $F$, not on its complex structure. And we have seen that that the same is true of the Euler characteristic $\sum_i (-1)^i Tr(t, H^i(X, L))$. Hence we obtain

**Corollary 5.6.** *Let $(X, \omega)$ be a simply-connected compact symplectic manifold, let $T$ be a compact torus acting on $X$ by symplectomorphisms, and let $L$ be a $T$-equivariant line bundle equipped with a $T$-invariant connection whose curvature is equal to $2\pi\sqrt{-1} \cdot \omega$. Then there exists a virtual representation $V$ of $T$ with the following properties. Assume that $X$ admits a complex structure for which $\omega$ is a Kaehler form. Then $L$ is a holomorphic line bundle and the action of $T$ on $L$ preserves the holomorphic structure. Furthermore, there exists a character $\chi$ of $T$ such that*

$$\sum_i (-1)^i Tr(t, H^i(X, L)) = \chi(t) \cdot Tr(t, V) \text{ for all } t \in T.$$

We note that this ambiguity of a character of $T$ cannot be avoided, because a given action of $T$ on $L$ can always be twisted by such a character, which symplectic geometry does not see.

The significance of Corollary 5.6 for geometric quantization is that the virtual representation of $T$ produced by holomorphic cohomology with respect to a Kaehler polarization is intrinsically associated to the $T$-symplectic manifold.

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