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HOMOTOPY THEORY FOR (BRAIDED) CAT-GROUPS

by Antonio R. GARZON and Jesus G. MIRANDA

RESUME. Les catégories de Gr-catégories (tressées ou symétriques) ou les catégories équivalentes de modules croisés (2-modules croisés réduits ou modules stables) représentent des modèles algébriques pour les types d'espaces connexes dont les groupes d'homotopie sont nuls pour les ordres autres que 1,2 (ou 2,3, ou n, n+1 pour $n > 2$).
Dans cet article, on munit ces catégories d'une structure de catégorie à modèle fermée, et on étudie la théorie d'homotopie associée. On construit l'espace des chemins, le cylindre, l'espace des lacets et la suspension. On identifie les relations d'homotopie déduites de ces structures. Ceci conduirait à une classification algébrique des classes d'homotopie d'applications continues entre espaces topologiques connexes dont les groupes d'homotopie sont nuls sauf ceux d'ordre n, n+1 pour un $n > 1$.

INTRODUCTION

In his lecture at the International Congress of Mathematicians (1950), [33], J.H.C. Whitehead outlined the general aims of "algebraic homotopy" which, in particular, included the following basic homotopy classification problems: classify the homotopy types of polyhedra X, Y,..., by algebraic data; compute the set of homotopy classes of maps $[X, Y]$ in terms of the classifying data for X, Y. Both problems, including other types of spaces, have been studied in the last 40 years by many authors.

Algebraic models of homotopy types have been often obtained by giving functors which carry the spaces under consideration to algebraic objects like chain complexes, crossed complexes, chain Lie algebras,..., and then showing an equivalence between the respective homotopy categories.

In this way, recall that groups are algebraic models of 1-types, that is, there is a classifying space functor

$$B : Groups \to CW\text{–}complexes$$

such that for any group G the space BG is connected and satisfies $\pi_1 BG \cong G$ and $\pi_j BG = 0$ for $j > 1$, and further any pointed, connected CW-complex X with $\pi_j = 0$ for $j > 1$ is of the homotopy type of $B\pi_1 X$.

Crossed modules of groups, introduced by J.H.C. Whitehead, [32] (see also [3]), are algebraic models of 2-types. There is a classifying space functor

$$B : Crossed\ modules \to CW\text{–}complexes$$

such that if $\rho : L \to M$ is a crossed module then $B(L \to M)$ is connected and it has $\pi_1 B(L \to M) \cong coker\rho$, $\pi_2 B(M \to L) \cong ker\rho$ and $\pi_j B(L \to M) = 0$ for $j > 2$. Further, any connected CW-complex X with $\pi_j X = 0$ for $j > 2$ is of the homotopy type of $B(L \to M)$ for some crossed module $L \to M$; in fact the crossed module describing the 2-type of a CW-complex X is $\pi_2(X, X^1) \xrightarrow{\partial_2} \pi_1(X^1)$ where X^1 denotes the 1-skeleta of X (see [26], [25]). Note that the category of crossed modules of groups is equivalent to the category $\mathbf{Cat(Gp)}$ of internal categories (groupoids) in groups and so this last category also provides algebraic models for 2-types.

Algebraic models of 3-types were given by Conduché, [18], by means of the category of 2-crossed modules. This category is equivalent to the category of simplicial groups with Moore complex of length 2 and also it is equivalent to the category of braided regular crossed modules introduced by Brown and Gilbert in [5]. On the other hand, the category of cat^2-groups, [25], is equivalent to the category of crossed squares, [21], and they also provide algebraic models of 3-types. In [19], Ellis has used homotopical methods of crossed squares for computation with homotopy 3-types. Note that a crossed square has associated a non-abelian group complex which is in fact a 2-crossed module (see [5]).

All these results have found higher-dimensional versions (see [25], [15]) which give algebraic models for homotopy n-types, $n \geq 1$.

The study of connected spaces with only two non-zero consecutive homotopy groups is particularly interesting. Crossed modules provide algebraic models for such spaces in the lowest dimensions. The full subcategory of the category of 2-crossed modules whose objects have trivial righthand side groups, called reduced 2-crossed modules, is just the category of braided crossed modules of groups in the sense of Brown-Gilbert, [5], and it provides algebraic models for spaces X with $\pi_j X = 0$ for $j \neq 2, 3$. Note that this category is equivalent to the category of strict braided categorical groups (see [24], [13]). Further, reduced 2-crossed modules with an extra condition of symmetry (usually called stable crossed modules, [18]) are the same as strict symmetric categorical groups and both categories provide algebraic models for spaces X with $\pi_j X = 0$ for $j \neq n, n+1$ and $n \geq 3$ (see [18], [13]).

With regard to the classification of the set of homotopy classes of continous maps $[X, Y]$, this problem has found, in several particular cases, different solutions which involve the use of suitable cohomology sets. We are interested in the homotopy classification of maps into a space with two consecutive non-trivial homotopy groups. Note that this problem is somewhat equivalent to the dual problem studied in [1] which describes, in terms of cohomology groups and cohomology operations, the homotopy classification of maps $X \to Y$ when X has two non zero homology groups in exactly two consecutive dimensions.

Eilenberg and Mac Lane gave a homotopy classification for maps $X \to Y$ when X is a CW-complex and Y is a space with a unique non trivial homotopy group π in dimension $n \geq 2$, in terms of the cohomology group $H^n(X, \pi)$. There have been generalisations of this result to local coefficients. This classification theorem was generalized in [10] by showing that, if X is a CW-complex with skeletal filtration $\mathbf{X}$ and C is any crossed complex, there is a natural bijection of homotopy classes $[X, BC] \cong [\pi\mathbf{X}, C]$ where $\pi\mathbf{X}$ is the fundamental crossed complex of X and BC is the image of C under the classifying space functor $B : Crossed\ complexes \to CW\text{–}complexes$. A similar result for 3-types is given recently in [19]. The special case when C is a reduced crossed module and with a different definition of BC was proved in [12]. Also, in [12], Eilenberg-MacLane's classification theorem was generalised to the case when Y has the homotopy type of a reduced 2-crossed module, and then in [13] to the case when Y has the homotopy type of a stable crossed module. These classifications have found generalizations, when X and Y are spaces with $\pi_j = 0$ for all $j \neq n, n+1$, $n = 0, 1, 2$, in terms of isomorphism classes of (braided) monoidal functors for $(n = 2)$ $n = 1$ (see [16]).

In a quite different way, Baues, [2], has extended Whitehead's results on the algebraic classification of 3-dimensional connected CW-spaces. He replaces crossed modules and crossed complexes by quadratic modules and quadratic complexes and he obtains a homotopy classification of maps from a 3-dimensional connected CW-complex X to some other connected CW-space Y and also the homotopy classification of 4-dimensional connected CW-spaces.

In both problems, to find algebraic models for spaces and to classify homotopy classes of maps, it is to be hoped, as is pointed out in [29], that the "algebra" reflects the "geometry" in the spaces or, in other words, that the homotopy structure of the spaces can be described in an algebraic way. To do that requires to be able to do homotopy theory in the algebraic setting.

A well known and quite powerful context in which an abstract homotopy theory can be developed is supplied by a category with a closed model structure in the sense of Quillen, [30]. The object of this paper is to show that each of the categories $\mathbf{Cat(Gp)}$ (internal categories in groups), $\mathcal{B}\mathbf{Cat(Gp)}$ (strict braided categorical groups) and $\mathcal{S}\mathbf{Cat(Gp)}$ (strict symmetric categorical groups) (and so the equivalent categories of crossed modules, $\chi\mathbf{M(Gp)}$, of reduced 2-crossed modules, $\mathbf{2}-\chi\mathbf{M_{red}(Gp)}$, and of stable crossed modules, $\chi\mathbf{M_{st}(Gp)}$) may be given the structure of a Quillen closed model category. Moreover, we study the corresponding associated homotopy theories by identifying cylinder and path constructions in each case; we then characterize the homotopy relations deduced from these constructions. This then allows the determination of homotopy classes of maps between spaces X and Y such that $\pi_j X = \pi_j Y = 0$ for all $j \neq n, n+1$, $n \geq 1$, in terms of homotopy classes $[\delta(X), \delta(Y)]$ of morphisms between the corresponding algebraic models $\delta(X)$ and $\delta(Y)$.

In §1 we recall the categories $\mathbf{Cat(Gp)}$, $\mathcal{B}\mathbf{Cat(Gp)}$, $\mathcal{S}\mathbf{Cat(Gp)}$ and their equivalence with $\chi\mathbf{M(Gp)}$, $\mathbf{2}-\chi\mathbf{M_{red}(Gp)}$ and $\chi\mathbf{M_{st}(Gp)}$ respectively. Also we recall some general facts about Quillen's model categories. In §2 we define (co)-fibrations and weak equivalences in each one of the categories $\mathbf{Cat(Gp)}$, $\mathcal{B}\mathbf{Cat(Gp)}$ and $\mathcal{S}\mathbf{Cat(Gp)}$, we characterize the cofibrations in an adequate form and then we prove that, with respect to these classes of morphisms, each of these categories is a closed model category. In §3 we identify path and cylinder constructions, loop and suspension functors and we characterize the homotopy relations deduced from these constructions. Finally, in §4, we translate definitions and results of §2 and §3 to the categories $\chi\mathbf{M(Gp)}$, $\mathbf{2}-\chi\mathbf{M_{red}(Gp)}$ and $\chi\mathbf{M_{st}(Gp)}$.

In this paper we use additive notation for groups.

1 Preliminaries

Let $\mathbf{Simp(Gp)}$ be the category of simplicial groups, i.e., $\mathbf{Simp(Gp)}$ is the functor category $\mathbf{Gp}^{\Delta^{op}}$ where $\mathbf{Gp}$ is the category of groups and Δ is the category whose objects are the ordered sets $[0]=\{0\}$, $[1]=\{0,1\}$,..., and the arrows are the order preserving functions between them.

An 1-truncated simplicial group is a diagram of groups and group morphisms

$$\begin{tikzcd} G_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & G_0 \arrow[l, bend right=40, "s_0"'] \end{tikzcd}$$

such that $d_0 s_0 = d_1 s_0 = Id_{G_0}$. A morphism between 1-truncated simplicial groups is a diagram

$$\begin{tikzcd} G_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "f_1"] & G_0 \arrow[l, bend right=40, "s_0"'] \arrow[d, "f_0"] \\ H_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & H_0 \arrow[l, bend right=40, "s_0"'] \end{tikzcd}$$

where f_1 and f_0 are group morphisms such that $f_0 d_0 = d_0 f_1$, $f_0 d_1 = d_1 f_0$ and $f_1 s_0 = s_0 f_0$. We will denote $\mathbf{Tr^1(Simp(Gp))}$ the category of 1-truncated simplicial groups.

In what follows $\mathbf{Cat(Gp)}$ will denote the category of internal categories in the category of groups. An object of $\mathbf{Cat(Gp)}$, called a cat-group, will be represented by a diagram of groups and group morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O \arrow[l, bend right, "I"'] \end{tikzcd}$$

such that $sI = tI = Id_O$, and the composition of two morphisms $x, y \in A$ with $s(x) = t(y)$ will be denoted by $x \circ y$. Note that the composition of two such morphisms in A is determined by $x \circ y = x - I_{s(x)} + y = y - I_{t(y)} + x$.

A cat-group can be given equivalently (see [25]) as a 1-truncated simplicial group $\begin{tikzcd} G_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & G_0 \arrow[l, bend right, "s_0"'] \end{tikzcd}$ satisfying $[Kerd_0, Kerd_1] = 0$. Then one can consider $\mathbf{Cat(Gp)}$ as a subcategory of $\mathbf{Tr^1(Simp(Gp))}$ and the inclusion functor $J : \mathbf{Cat(Gp)} \to \mathbf{Tr^1(Simp(Gp))}$ has a left adjoint $\mathcal{P} : \mathbf{Tr^1(Simp(Gp))} \to \mathbf{Cat(Gp)}$ given by

$$\mathcal{P}\left(\begin{tikzcd} G_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & G_0 \arrow[l, bend right, "s_0"'] \end{tikzcd} \right) = \begin{tikzcd} \frac{G_1}{[Kerd_0, Kerd_1]} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & G_0 \arrow[l, bend right, "s_0"'] \end{tikzcd} .$$

Recall that limits in $\mathbf{Cat(Gp)}$ are calculated by computing them on objects and arrows. Colimits in $\mathbf{Tr^1(Simp(Gp))}$ are computed dimensionwise and then colimits in $\mathbf{Cat(Gp)}$ are computed by applying $\mathcal{P}$ to those constructed in $\mathbf{Tr^1(Simp(Gp))}$.

Given a cat-group $\mathcal{G}$: $\begin{tikzcd} A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O \arrow[l, bend right, "I"'] \end{tikzcd}$ recall that $\pi_0(\mathcal{G}) = coequ(s, t) = O/t(Ker(s))$ and $\pi_1(\mathcal{G}) = Ker(s) \cap Ker(t)$.

A braiding for a cat-group $\mathcal{G}$ (see [24], [13]) is a map

$$\begin{array}{ccc} O \times O & \xrightarrow{\tau} & A \\ (p, q) & \longmapsto & \tau_{p,q} \end{array}$$

which satisfies

a) $s\tau_{p,q} = p + q; \quad t\tau_{p,q} = q + p$:

b) Naturality:

Given $x, y \in A$; $x : p \to p'$, $y : q \to q'$, the following square is commutative

$$\begin{tikzcd}
p+q \arrow[r, "x+y"] \arrow[d, "\tau_{p,q}"'] & p'+q' \arrow[d, "\tau_{p',q'}"] \\
q+p \arrow[r, "y+x"] & q'+p'
\end{tikzcd}$$

c) hexagon axiom:

For any $p, q, n \in O$ the following diagrams are commutative

$$\begin{tikzcd}
 & p+(q+n) & \\
(p+q)+n \arrow[ur, equal] & & p+(n+q) \arrow[ul, "I_p+\tau_{n,q}"'] \arrow[d, equal] \\
n+(p+q) \arrow[u, "\tau_{(p+q),n}"] \arrow[dr, equal] & & (p+n)+q \\
 & (n+p)+q \arrow[ur, "\tau_{n,p}+I_q"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (p+q)+n & \\
p+(q+n) \arrow[ur, equal] & & (q+p)+n \arrow[ul, "\tau_{q,p}+I_n"'] \arrow[d, equal] \\
(q+n)+p \arrow[u, "\tau_{(q+n),p}"] \arrow[dr, equal] & & q+(p+n) \\
 & q+(n+p) \arrow[ur, "I_q+\tau_{n,p}"'] &
\end{tikzcd}$$

d) $\tau_{0,p} = \tau_{p,0} = I_p$

A cat-group together with a braiding is usually called a braided cat-group.

Given braided cat-groups $(\mathcal{G}, \tau)$, $(\mathcal{G}', \tau')$, a morphism between them is a morphism of cat-groups which is compatible with τ in the sense that the following square is commutative

$$\begin{tikzcd}
O \times O \arrow[r, "\tau"] \arrow[d, "f_0 \times f_0"'] & A \arrow[d, "f_1"] \\
O' \times O' \arrow[r, "\tau'"] & A'
\end{tikzcd}$$

$\mathcal{B}\mathbf{Cat}(\mathbf{Gp})$ will denote the category of braided cat-groups. There exists then a forgetful functor $U : \mathcal{B}\mathbf{Cat}(\mathbf{Gp}) \to \mathbf{Cat}(\mathbf{Gp})$.

A braided cat-group $(\mathcal{G}, \tau)$ is called a symmetric cat-group if $\tau_{p,q}^{-1} = \tau_{q,p}$, $\forall p, q \in O$. In such a case, τ is usually called a symmetry.

We will denote $\mathcal{S}\mathbf{Cat}(\mathbf{Gp})$ the full subcategory of $\mathcal{B}\mathbf{Cat}(\mathbf{Gp})$ whose objects are the symmetric cat-groups and $In : \mathcal{S}\mathbf{Cat}(\mathbf{Gp}) \to \mathcal{B}\mathbf{Cat}(\mathbf{Gp})$ will denote the inclusion functor.

A detailed list of examples of symmetric cat-groups is given in [13].

Recall now that the category $\mathbf{Cat}(\mathbf{Gp})$ is equivalent to the category $\chi\mathbf{M}(\mathbf{Gp})$ of crossed modules of groups (see for example [18]). Given a crossed module $\mathcal{L} : (L \xrightarrow{\rho} M)$, the equivalence is given by the functor $\Phi : \chi\mathbf{M}(\mathbf{Gp}) \to \mathbf{Cat}(\mathbf{Gp})$ where $\Phi(\mathcal{L})$ is the following cat-group

$$\begin{tikzcd} L \rtimes M \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M \arrow[l, bend right=60, "I"'] \end{tikzcd}$$

with $s(l,m) = m$, $t(l,m) = \rho(l) + m$, $I(m) = (0,m)$. It is easy to see that the composition of two morphisms is $(l',m') \circ (l,m) = (l', \rho(l) + m) \circ (l,m) = (l' + l, m)$.

The quasi inverse Φ^{-1} associates to any cat-group $\begin{tikzcd} A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O \arrow[l, bend right=60, "I"'] \end{tikzcd}$,

the crossed module $Ker\ s \xrightarrow{t} O$ where the action is given by ${}^{x}l = I(x) + l - I(x)$, $x \in O$, $l \in Ker\ s$.

A crossed module $\mathcal{L} : (L \xrightarrow{\rho} M)$ together with a map $\{-,-\} : M \times M \to L$ satisfying the following identities is called a reduced 2-crossed module (see [18]):

1. $\rho\{m,m'\} = m + m' - m - m'$.
2. $\{\rho(l), m\} = l -{}^{m}l$.
3. $\{m, \rho(l)\} = {}^{m}l - l$.
4. $\{m, m' + m''\} = \{m,m'\} + {}^{m'}\{m, m''\}$.
5. $\{m + m', m''\} = {}^{m}\{m', m''\} + \{m, m''\}$.

A morphism between reduced 2-crossed modules is a morphism of crossed modules $\phi : \mathcal{L} \to \mathcal{L}'$ which satisfies $\phi_1\{m, m'\} = \{\phi_0(m), \phi_0(m')\}$ for all $m, m' \in M$. We will write $\mathbf{2} - \chi\mathbf{M_{red}}(\mathbf{Gp})$ for the category of reduced 2-crossed modules. Obviously there is a forgetful functor $U : \mathbf{2} - \chi\mathbf{M_{red}}(\mathbf{Gp}) \to \chi\mathbf{M}(\mathbf{Gp})$.

The category $\mathbf{2} - \chi\mathbf{M_{red}}(\mathbf{Gp})$ is just the category of braided crossed modules of groups in the sense of [5] and it is equivalent to the full subcategory of $\mathbf{Simp}(\mathbf{Gp})$ whose objects have trivial Moore complex at dimensions other than one and two and, also, it is equivalent to the full subcategory of the category of 2-crossed modules, [18], with trivial righthand side groups. Moreover, $\mathbf{2} - \chi\mathbf{M_{red}}(\mathbf{Gp})$ is equivalent to the category $\mathcal{B}\mathbf{Cat}(\mathbf{Gp})$ and the equivalence is given by the functor $\Phi : \mathbf{2} - \chi\mathbf{M_{red}}(\mathbf{Gp}) \to \mathcal{B}\mathbf{Cat}(\mathbf{Gp})$, which associates to any reduced 2-crossed module $(\ \mathcal{L} : (L \xrightarrow{\rho} M)\ , \{-,-\})$ the braided cat-group $\Phi(\mathcal{L}) : (\ \begin{tikzcd} L \rtimes M \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M \arrow[l, bend right, "I"'] \end{tikzcd}\ , \tau)$ where $\begin{tikzcd} L \rtimes M \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M \arrow[l, bend right, "I"'] \end{tikzcd}$ is the cat-group associated to the crossed module $L \xrightarrow{\rho} M$, and $\tau : M \times M \to L \rtimes M$ is given by $\tau_{m,m'} = (\{m', m\}, m + m')$.

The quasi inverse functor associates to any braided cat-group $\mathcal{G} : \begin{tikzcd} A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O \arrow[l, bend right, "I"'] \end{tikzcd}$ the crossed module associated to the cat-group $\mathcal{G}$ together with the map

$$\begin{array}{rcl} O \times O & \xrightarrow{\{-,-\}} & Ker\ s \\ (p, q) & \longmapsto & \tau_{q,p} - I_p - I_q \end{array}$$

Note that there is a full and faithful functor from $\mathbf{2} - \chi\mathbf{M_{red}}(\mathbf{Gp})$ to the category of crossed squares (see [18]); it is given by associating to any $(\ \mathcal{L} : (L \xrightarrow{\rho} M)\ , \{-,-\})$ the following crossed square

$$\begin{tikzcd} L \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \arrow[d, equal] \\ M \arrow[r, equal] & M \end{tikzcd} \qquad \text{with function } h = \{-,-\} : M \times M \to L$$

In [19], Ellis defined a functor

$\delta : 3 - \textit{dimensional reduced CW} - \textit{spaces} \longrightarrow \textit{Crossed squares}$

whose image lies in $\mathbf{2} - \chi\mathbf{M_{red}}(\mathbf{Gp})$ when the CW-space is, for example, the 2-sphere S^2 or a wedge $\bigvee_{1\leq i\leq k} S^2$ of k 2-spheres. In fact, in the first case,

$$\delta(S^2) = \begin{array}{ccc} C_\infty & \xrightarrow{0} & C_\infty \\ {\scriptstyle 0}\downarrow & & \parallel \\ C_\infty & = & C_\infty \end{array}$$

where C_∞ is the infinite cyclic group generated by the 2-cell t of S^2 and $h : C_\infty \times C_\infty \to C_\infty$ is given by $h(t^i, t^j) = t^{ij}$, that is, $\delta(S^2)$ is the reduced 2-crossed module $(C_\infty \xrightarrow{0} C_\infty , \{-,-\} = h)$. In the case $X = \bigvee_{1\leq i\leq k} S^2$

$$\delta(X) = \begin{array}{ccc} P \otimes P & \xrightarrow{[\ ,\ ]} & P \\ {\scriptstyle [\ ,\ ]}\downarrow & & \parallel \\ P & = & P \end{array}$$

where P is the free group of rank k, $P \otimes P$ is the non-abelian tensor product of groups defined in [11], the homomorphism $[\ ,\ ]$ sends a generator $p \otimes p'$ to $[p,p'] = pp'p^{-1}p'^{-1}$, with $h(p,p') = p \otimes p'$ for $p, p' \in P$ and with P acting on itself by conjugation. Thus, $\delta(X)$ is the reduced 2-crossed module $(P \otimes P \xrightarrow{[\ ,\ ]} P, \{-,-\} = h)$.

We will write $\chi\mathbf{M_{st}}(\mathbf{Gp})$ for the full subcategory of $\mathbf{2} - \chi\mathbf{M_{red}}(\mathbf{Gp})$ whose objects (called stable crossed modules) are those reduced 2-crossed modules $(\mathcal{L} : (L \xrightarrow{\rho} M) , \{-,-\})$ such that $\{m,m'\} + \{m',m\} = 0$ (see [18]). Then the equivalence $\Phi : \mathbf{2} - \chi\mathbf{M_{red}}(\mathbf{Gp}) \to \mathcal{B}\mathbf{Cat}(\mathbf{Gp})$ restricts to an equivalence $\chi\mathbf{M_{st}}(\mathbf{Gp}) \simeq \mathcal{S}\mathbf{Cat}(\mathbf{Gp})$.

Next we will recall some facts about Quillen's model categories (see [30]).

A closed model category is a category $\underline{\mathbf{C}}$ endowed with three distinguished families of morphisms called fibrations, cofibrations and weak equivalences satisfying the axioms CM1-CM5 as stated in [31].

Given a model category $\underline{\mathbf{C}}$, the category obtained by formal inversion of the weak equivalences is denoted $Ho(\underline{\mathbf{C}})$.

If X is an object of $\underline{\mathbf{C}}$, a cylinder for X is a factorization of the codiagonal morphism

$$X \coprod X \xrightarrow{i_0+i_1} X \otimes I \xrightarrow{\sigma} X$$

where $i_0 + i_1$ is a cofibration and σ is a weak equivalence.

A path object for an object Y of $\underline{\mathbf{C}}$ is a factorization of the diagonal morphism

$$Y \xrightarrow{\sigma} Y^I \xrightarrow{(\partial_0,\partial_1)} Y \times Y$$

where (∂_0, ∂_1) is a fibration and σ is a weak equivalence.

If $f, g \in Hom_{\underline{\mathbf{C}}}(X, Y)$, a left (resp. right) homotopy from f to g is defined to be a morphism $k : X \otimes I \to Y$ (resp. $h : X \to Y^I$) such that $ki_0 = f$ and $ki_1 = g$ (resp. $\partial_0 h = g$ and $\partial_1 h = f$). The morphism f is said to be left (resp. right) homotopic to g if such a left (right) homotopy exists. When X is cofibrant (resp. Y is fibrant) "is left homotopic to" (resp. "is right homotopic to") is an equivalence relation in $Hom_{\underline{\mathbf{C}}}(X, Y)$. Moreover, if X is cofibrant and Y is fibrant, then the left and right homotopy relations on $Hom_{\underline{\mathbf{C}}}(X, Y)$ coincide.

Finally, it is illustrative for the development of this paper, to recall the known Quillen's model structures in some categories which provide algebraic models of homotopy types in low dimensions.

The category of groupoids is a closed model category where the fibrations are the fibrations of groupoids, [4], the weak equivalences are the equivalences of categories and the cofibrations are the morphisms which are injective on objects. In this model category every object is fibrant and cofibrant. If I denotes the groupoid with two objects 0 and 1, their identities and two morphisms $i : 0 \to 1$ and $i^{-1} : 1 \to 0$, then I has properties analogous to the unit interval in the homotopy theory of spaces. Homotopy of groupoids can be described by a cylinder object $\mathcal{G} \otimes I (= \mathcal{G} \times I)$ or by a path object $\mathcal{H}^I$ (which is just the category (groupoid) of functors from I to $\mathcal{H}$).

Algebraic models for non-necessarily connected 2-types are provided by any of the following equivalents categories: that of crossed modules over groupoids [9], the category of 2-groupoids, [28], and the category of simplicial groupoids with trivial Moore complex in dimensions greater than 1, [20]. These categories come supplied with monoidal closed structures and a unit interval object from which the homotopy theory derives (for example, in the category of 2-groupoids, the unit interval object is the interval groupoid I as above). Also, these categories support a Quillen's model structure (see [28], [20]) where, for example, in the category of crossed modules over groupoids, a morphism $f : \mathcal{L} \to \mathcal{L}'$ represented by the diagram

$$\begin{tikzcd} \mathcal{L}: \arrow[d, "f"'] & L \arrow[r, "\rho"] \arrow[d, "f_2"'] & M \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_1"] & O \arrow[d, "f_0"] \\ \mathcal{L}': & L' \arrow[r, "\rho'"] & M' \arrow[r, shift left] \arrow[r, shift right] & O' \end{tikzcd}$$

is a fibration if (f_1, f_0) is a fibration of groupoids and for any $p \in O$,

$$\begin{tikzcd} L(p) \arrow[r, "\rho"] \arrow[d, "f_2"'] & M(p) \arrow[d, "f_1"] \\ L'(f_0(p)) \arrow[r, "\rho'"] & M'(f_0(p)) \end{tikzcd}$$

is a fibration of crossed modules of groups, i.e., $f_2 : L(p) \to L'(f_0(p))$ is surjective (see §4); the morphism f is a weak equivalence if

$$\pi_0\left(\begin{tikzcd} M \arrow[r, shift left] \arrow[r, shift right] & O \arrow[l, bend right] \end{tikzcd} \right) \cong \pi_0\left(\begin{tikzcd} M' \arrow[r, shift left] \arrow[r, shift right] & O' \arrow[l, bend right] \end{tikzcd} \right)$$

and, for any $p \in O$, f induces group isomorphisms

$$Coker\left(L(p) \xrightarrow{\rho} M(p)\right) \cong Coker\left(L'(f_0(p)) \xrightarrow{\rho'} M'(f_0(p))\right)$$

and

$$Ker\left(L(p) \xrightarrow{\rho} M(p)\right) \cong Ker\left(L'(f_0(p)) \xrightarrow{\rho'} M'(f_0(p))\right)$$

(see §4).

Recall also that the category of crossed complexes ([9],[10]) was shown in [6] to carry the structure of a closed model category by using the notion of fibration introduced by Howie, [22], and of weak equivalence defined in [8]. Crossed modules over groupoids are crossed complexes of dimensions less than or equal to 2, and for these, fibrations and weak equivalences are just the same classes of morphisms we have given explicitly above for crossed modules.

2 Cat(Gp) ($\mathcal{B}$Cat(Gp) or $\mathcal{S}$Cat(Gp)) as a closed model category

Let

$$\begin{tikzcd}
\mathcal{G}: \arrow[d, "f"] & G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "f_1"] & G_0 \arrow[l, bend right, "I"'] \arrow[d, "f_0"] \\
\mathcal{H}: & H_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H_0 \arrow[l, bend right, "I"']
\end{tikzcd}$$

be a morphism of cat-groups.

Definition 2.1. *i) f is said to be a fibration if it is a fibration of groupoids (i.e., if the canonical morphism $G_1 \to H_{1\,s}\times_{f_0}G_0$ is surjective).*

ii) f is said to be a weak equivalence if it is an equivalence of categories.

iii) f is said to be a cofibration if it has the LLP with respect to the trivial fibrations.

Note that the category **Cat(Gp)** is equivalent to the category of 1-hypergroupoids of groups and that the above structure coincides, through the equivalence, with the model structure in this last category shown in [14]. In §4 we will compare this model structure with that on crossed complexes, [6].

Considering the functors $\mathcal{S}\mathbf{Cat}(\mathbf{Gp})\overset{In}{\hookrightarrow}\mathcal{B}\mathbf{Cat}(\mathbf{Gp})\overset{U}{\to}\mathbf{Cat}(\mathbf{Gp})$ we also give the following:

Definition 2.2. *A morphism $f : \mathcal{G} \to \mathcal{H}$ in $\mathcal{B}$***Cat(Gp)** *(resp. in $\mathcal{S}$***Cat(Gp)***) is a fibration or weak equivalence if $U(f)$ (resp. $UIn(f)$) is a fibration or weak equivalence in* **Cat(Gp)***. The morphism f is a cofibration in $\mathcal{B}$***Cat(Gp)** *(resp. in $\mathcal{S}$***Cat(Gp)***) if it has the LLP with respect to the trivial fibrations in $\mathcal{B}$***Cat(Gp)** *(resp. in $\mathcal{S}$***Cat(Gp)***).*

In the following, $\underline{\mathbf{C}}$ will denote any of the categories **Cat(Gp)**, $\mathcal{B}$**Cat(Gp)** or $\mathcal{S}$**Cat(Gp)**.

Note that every object in $\underline{\mathbf{C}}$ is fibrant.

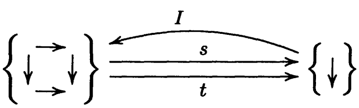
The weak equivalences can be characterized easily as follows:

Proposition 2.3. *A morphism $f : \mathcal{G} \to \mathcal{H}$ in $\underline{\mathbf{C}}$ is a weak equivalence if and only if $\pi_0(f)$ and $\pi_1(f)$ are isomorphisms.*

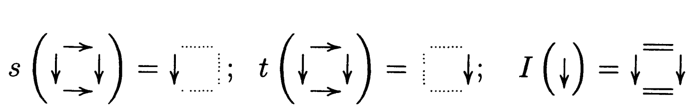
Definition 2.4. *Given two morphisms in $\underline{\mathbf{C}}$, $f, g : \mathcal{G} \to \mathcal{H}$, a homotopy from f to g is a natural transformation $\alpha : f \Rightarrow g$ which is a group morphism.*

This homotopy relation is an equivalence relation on the set of morphisms in $\underline{\mathbf{C}}$ from $\mathcal{G}$ to $\mathcal{H}$ and the set of equivalence classes will be denoted by $[\mathcal{G}, \mathcal{H}]$. Note that if f and g are homotopic then $\pi_0(f) = \pi_0(g)$ and $\pi_1(f) = \pi_1(g)$.

Given $\mathcal{H} \in \underline{\mathbf{C}}$ let us consider the object of $\underline{\mathbf{C}}$, $\mathcal{H}^I$, which is the cat-group whose objects are H_1 and whose morphisms are the commutative squares of elements of H_1 with multiplication defined pointwise, i.e.,



where



and, if τ is the braiding (symmetry) in $\mathcal{H}$, the braiding (symmetry) in $\mathcal{H}^I$ is given as follows:

For any $x, y \in (\mathcal{H}^I)_0 = H_1$ we define $\tau'_{x,y} : x + y \to y + x$ as the commutative square

$$\begin{tikzcd} sx + sy \arrow[r, "\tau_{sx,sy}"] \arrow[d, "x+y"'] & sy + sx \arrow[d, "y+x"] \\ tx + ty \arrow[r, "\tau_{tx,ty}"] & ty + tx \end{tikzcd}$$

In this way τ' is a braiding (symmetry) in $\mathcal{H}^I$ and so $\mathcal{H}^I \in \underline{\mathrm{C}}$. We will verify only the naturality of τ'. This means that, given $f : x \to x'$ and $g : y \to y'$, the following diagram must be commutative

$$\begin{tikzcd} & x' + y' \arrow[dr, "\tau'_{x',y'}"] & \\ x + y \arrow[ur, "f+g"] \arrow[dr, "\tau'_{x,y}"'] & & y' + x' \\ & y + x \arrow[ur, "g+f"'] & \end{tikzcd}$$

Now, $f : x \to x'$ and $g : y \to y'$ are commutative squares

$$\begin{tikzcd} sx \arrow[r, "f_0"] \arrow[d, "x"'] & sx' \arrow[d, "x'"] \\ tx \arrow[r, "f_1"] & tx' \end{tikzcd} \qquad \begin{tikzcd} sy \arrow[r, "g_0"] \arrow[d, "y"'] & sy' \arrow[d, "y'"] \\ ty \arrow[r, "g_1"] & ty' \end{tikzcd}$$

and so, we have to prove that the two rhombuses in the following diagram are commutative

$$\begin{tikzcd}
 & & \cdot \arrow[d, "x'+y'"] \arrow[drr, "\tau_{sx',sy'}"] & & \\
\cdot \arrow[urr, "f_0+g_0"] \arrow[d, "x+y"'] \arrow[drr, "\tau_{sx,sy}"'] & & \cdot \arrow[drr, "\tau_{tx',ty'}"] & & \cdot \arrow[d, "y'+x'"] \\
\cdot \arrow[urr, "f_1+g_1"] \arrow[drr, "\tau_{tx,ty}"'] & & \cdot \arrow[urr, "g_0+f_0"'] \arrow[d, "y+x"] & & \cdot \\
 & & \cdot \arrow[urr, "g_1+f_1"'] & &
\end{tikzcd}$$

that is, $\tau_{sx',sy'}(f_0+g_0) = (g_0+f_0)\tau_{sx,sy}$ and $\tau_{tx',ty'}(f_1+g_1) = (g_1+f_1)\tau_{tx,ty}$. But $sx = sf_0$, $sx' = tf_0$, $sy = sg_0$, $sy' = tg_0$, $tx = sf_1$, $tx' = tf_1$, $ty = sg_1$, $ty' = tg_1$ and then it is easy to see that the required identities are just the naturality conditions of τ with respect to the morphisms f_0, g_0 and f_1, g_1.

Note that this construction determines a functor $(-)^I : \underline{\mathbf{C}} \to \underline{\mathbf{C}}$ which clearly preserves fibrations and weak equivalences.

Proposition 2.5. *Given $\mathcal{H} \in \underline{\mathbf{C}}$, there is a factorization of the diagonal morphism*

$$\mathcal{H} \xrightarrow{\sigma} \mathcal{H}^I \xrightarrow{(\partial_0,\partial_1)} \mathcal{H} \times \mathcal{H}$$

where σ is a weak equivalence and (∂_0, ∂_1) is a fibration.

Proof. Consider σ and (∂_0, ∂_1) defined by:

$$\sigma_0(\cdot) = (\,\cdot = \cdot\,), \quad \sigma_1(x) = \left(\begin{array}{ccc} \cdot & \xrightarrow{x} & \cdot \\ \| & & \| \\ \cdot & \xrightarrow{x} & \cdot \end{array}\right)$$

$$(\partial_0, \partial_1)_0(x) = (t(x), s(x)) \in H_0 \times H_0$$

$$(\partial_0,\partial_1)_1\left(\begin{tikzcd} \cdot \arrow[r, "x"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "y"] & \cdot \end{tikzcd}\right) = (y,x) \in H_1 \times H_1$$

Then it is straightforward to see that (∂_0, ∂_1) is a morphism in $\underline{\mathrm{C}}$ which is a fibration.

On the other hand, it is also clear that σ is a morphism in $\underline{\mathrm{C}}$ which is a weak equivalence since $\partial_0\sigma = Id_{\mathcal{H}}$ and $Id_{\mathcal{H}^I}$ is homotopic to $\sigma\partial_0$ where the homotopy γ is defined, for any object $x : p \to q$ in $\mathcal{H}^I$, by the commutative diagram

$$\gamma_x = \left(\begin{tikzcd} p \arrow[r, "x"] \arrow[d, "x"'] & q \arrow[d, equal] \\ q \arrow[r, equal] & q \end{tikzcd}\right)$$

$\square$

Proposition 2.6. *Let $f, g : \mathcal{G} \to \mathcal{H}$ two morphisms in $\underline{\mathrm{C}}$. To give a homotopy from f to g is equivalent to give a morphism $h : \mathcal{G} \to \mathcal{H}^I$ such that $\partial_0 h = g$ and $\partial_1 h = f$.*

Proof. Let us suppose that $h : \mathcal{G} \to \mathcal{H}^I$ is a morphism in $\underline{\mathrm{C}}$ such that $\partial_0 h = g$ and $\partial_1 h = f$. Let $p \in G_0$ and define $\alpha_p = h_0(p)$. It is clear that α is a group morphism and α_p is an arrow in $\mathcal{H}$ such that

$$s(\alpha_p) = sh_0(p) = \partial_1 h_0(p) = f_0(p); \quad t(\alpha_p) = th_0(p) = \partial_0 h_0(p) = g_0(p)$$

The naturality of α follows that for any $x \in H_1$, $x : p \to q$, $h_1(x)$ is a commutative square and

$$h_1(x) = \left(\begin{tikzcd} f_0(p) \arrow[r, "f_1(x)"] \arrow[d, "\alpha_p"'] & f_0(q) \arrow[d, "\alpha_q"] \\ g_0(p) \arrow[r, "g_1(x)"'] & g_0(q) \end{tikzcd}\right)$$

Conversely, if α is a natural transformation from f to g which is a group morphism, we define $h_0(p) = \alpha_p$ and for any $x : p \to q$

$$h_1(x) = \left(\begin{array}{ccc} f_0(p) & \xrightarrow{f_1(x)} & f_0(q) \\ {\scriptstyle\alpha_p}\downarrow & & \downarrow{\scriptstyle\alpha_q} \\ g_0(p) & \xrightarrow[g_1(x)]{} & g_0(q) \end{array} \right)$$

It is clear that $h : \mathcal{G} \to \mathcal{H}^I$ is a morphism in $\mathbf{Cat(Gp)}$. If f and g are morphisms in $\mathcal{B}\mathbf{Cat(Gp)}$ (resp. $\mathcal{S}\mathbf{Cat(Gp)}$), then, for any $p, q \in G_0$

$$\tau'_{h_0p,h_0q} = \left(\begin{array}{ccc} f_0(p) + f_0(q) & \xrightarrow{\tau_{f_0p,f_0q}} & f_0(q) + f_0(p) \\ {\scriptstyle h_0(p)+h_0(q)}\downarrow & & \downarrow{\scriptstyle h_0(q)+h_0(p)} \\ g_0(p) + g_0(q) & \xrightarrow{\tau_{g_0p,g_0q}} & g_0(q) + g_0(p) \end{array} \right)$$

and

$$h_1(\tau_{p,q}) = \left(\begin{array}{ccc} f_0(p) + f_0(q) & \xrightarrow{f_1(\tau_{p,q})} & f_0(q) + f_0(p) \\ {\scriptstyle\alpha_{p+q}}\downarrow & & \downarrow{\scriptstyle\alpha_{q+p}} \\ g_0(p) + g_0(q) & \xrightarrow{g_1(\tau_{p,q})} & g_0(q) + g_0(p) \end{array} \right)$$

and so $\tau'_{h_0p,h_0q} = h_1(\tau_{p,q})$; thus, h is a morphism in $\mathcal{B}\mathbf{Cat(Gp)}$ (resp. $\mathcal{S}\mathbf{Cat(Gp)}$).

Finally, it is clear that $\partial_0 h = g$ and $\partial_1 h = f$. $\square$

Corollary 2.7. *Given $\mathcal{G}, \mathcal{H} \in \underline{\mathbf{C}}$ and $h : \mathcal{G} \to \mathcal{H}^I$ a morphism in* $\mathbf{Cat(Gp)}$, *then h is a morphism in $\underline{\mathbf{C}}$ if and only if $\partial_0 h$ and $\partial_1 h$ are morphisms in $\underline{\mathbf{C}}$.*

Proposition 2.8. *Any morphism $f : \mathcal{G} \to \mathcal{H}$ in $\underline{\mathbf{C}}$ can be factored*

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "f"] \arrow[dr, "\eta"'] & & \mathcal{H} \\ & \mathcal{G} \times_{\mathcal{H}} \mathcal{H}^I \arrow[ur, "\kappa"'] & \end{tikzcd}$$

where η is a weak equivalence and κ is a fibration.

Proof. It is clear that $\partial_0\sigma f = f$ and so there exists an unique morphism η making the following diagram commutative

$$\begin{tikzcd}
\mathcal{G} \arrow[dr, dotted, "\eta"] \arrow[drr, "\sigma f"] \arrow[ddr, "Id"'] & & \\
& \mathcal{G} \times_{\mathcal{H}} \mathcal{H}^I \arrow[r, "\gamma"] \arrow[d, "\epsilon"] & \mathcal{H}^I \arrow[d, "\partial_0"] \\
& \mathcal{G} \arrow[r, "f"'] & \mathcal{H}
\end{tikzcd}$$

The morphism κ is the composition $\partial_1\gamma$.

Note that the objects of $\mathcal{G} \times_{\mathcal{H}} \mathcal{H}^I$ are pairs $(p, x) \in G_0 \times H_1$ such that $t(x) = f_0(p)$, and a morphism from (p, x) to (q, y) is given by a morphism $z : p \to q$ or, equivalently, by a commutative diagram

$$\begin{tikzcd}
s(x) \arrow[r, "x"] \arrow[d, "y^{-1}f_1(z)x"'] & f_0(p) \arrow[d, "f_1(z)"] \\
s(y) \arrow[r, "y"'] & f_0(q)
\end{tikzcd}$$

The group of morphisms of $\mathcal{G} \times_{\mathcal{H}} \mathcal{H}^I$ is then the group of triplets (x, z, y) where $x, y \in H_1,\ z \in H_0$ and $t(x) = sf_1(z),\ t(y) = tf_1(z)$.

The functors η and γ are defined as follows: $\eta_0(p) = (p, I_{f_0 p})$ and, for any $z \in G_1,\ \eta_1(z)$ is given by the morphism z; with regard to γ we have

$$\gamma_0(p, x) = x \text{ and } \gamma_1(x, z, y) = \left(\begin{tikzcd}
s(x) \arrow[r, "x"] \arrow[d, "y^{-1}f_1(z)x"'] & f_0(p) \arrow[d, "f_1(z)"] \\
s(y) \arrow[r, "y"'] & f_0(q)
\end{tikzcd}\right)$$

It is clear that $Hom_{\mathcal{G}}(p, q) \cong Hom_{\mathcal{G}\times_{\mathcal{H}}\mathcal{H}^I}(\eta_0(p), \eta_0(q))$ and for any object (p, x) in $\mathcal{G} \times_{\mathcal{H}} \mathcal{H}^I$, the identity in p determines an (iso)morphism from $\eta_0(p)$ to (p,x). Consequently, the functor η is full, faithful and dense and so, it is a weak equivalence.

On the other hand, given $(p,x) \in \left(\mathcal{G} \times_{\mathcal{H}} \mathcal{H}^I\right)_0$ and $x' \in H_1$ such that $s(x') = \kappa_0(p,x)$, the morphism $(x, I_p, x(x')^{-1})$ satisfies $\kappa_1(x, I_p, x(x')^{-1}) = x'$ and $s(x, I_p, x(x')^{-1}) = (p,x)$, and so, κ is a fibration.

□

Let us consider now the following objects in $\mathbf{Cat(Gp)}$:

$$0 = 0 \Longrightarrow 0\ ;\ \mathcal{I}_0 = \mathbb{Z} \underset{0}{\overset{0}{\rightrightarrows}} 0\ ;\ \mathcal{I}_1 = \mathbb{Z} \Longrightarrow \mathbb{Z}$$

and $\mathcal{I} = \mathcal{P}\left(\mathbb{Z} * \mathbb{Z} * \mathbb{Z} \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{Z} * \mathbb{Z}\right)$ where $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{Z} * \mathbb{Z}$ is the 1-truncated simplicial group in which the morphisms s, t, I are determined by the following relations:

$$Iu_0 = v_0;\ Iu_1 = v_2$$

$$sv_0 = u_0;\ sv_1 = u_0;\ sv_2 = u_1;\quad tv_0 = u_0;\ tv_1 = u_1;\ tv_2 = u_1$$

where $u_i : \mathbb{Z} \to \mathbb{Z} * \mathbb{Z},\ i = 0,1$ and $v_j : \mathbb{Z} \to \mathbb{Z} * \mathbb{Z} * \mathbb{Z},\ j = 0,1,2$ are the canonical injections. We also denote $v_j,\ j = 0,1,2$ the induced morphisms $\mathbb{Z} \to \frac{\mathbb{Z}*\mathbb{Z}*\mathbb{Z}}{[Ker\ s, Ker\ t]}$.

Note that these cat-groups are objects in $\underline{\mathbf{C}}$; the only non trivial case is $\mathcal{I}$ and now we define a symmetry on it.

To do that, we put $\tau_{u_0(1),u_1(1)} = v_1(1) + v_0(1) - v_1(1) + v_2(1)$ and $\tau_{u_0(-1),u_1(1)} = v_2(1) - v_0(1) + v_1(1) - v_2(1) - v_1(1) + v_2(1)$. In the general case, we use these definitions and the fact that τ must satisfy the hexagon axiom. For example, $\tau_{u_0(-1),u_1(1)+u_0(1)+u_1(1)} = (I_{u_1(1)} + I_{u_0(1)} + \tau_{u_0(-1),u_1(1)}) \circ (I_{u_1(1)} + I_{u_1(1)}) \circ (\tau_{u_0(-1),u_1(1)} + I_{u_0(1)} + I_{u_1(1)})$. It is straightforward to check that τ is, certainly, a symmetry in $\mathcal{I}$.

Let $\mathcal{G} : G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0$ be an object in $\underline{\mathbf{C}}$. Then it is easy to see that giving a morphism from I_0 to $\mathcal{G}$ is equivalent to giving an element

$x \in G_1$ such that $s(x) = t(x) = 0$ and giving a morphism from I_1 to $\mathcal{G}$ is equivalent to giving an element $p \in G_0$. Next we analyze in more detail what it means to give a morphism from $\mathcal{I}$ to $\mathcal{G}$.

Lemma 2.9. *Let $\mathcal{G} \in \underline{\mathbf{C}}$. Then, to give a morphism $\mathcal{I} \to \mathcal{G}$ is equivalent to giving a morphism in $\mathcal{G}$.*

Proof. First we note that, for any 1-truncated simplicial group $\mathcal{G}$, the simplicial identities imply that giving a morphism in $\mathbf{Tr^1}(\mathbf{Simp}(\mathbf{Gp}))$ from
$$\begin{tikzcd} \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathbb{Z} * \mathbb{Z} \arrow[l, bend right, "I"'] \end{tikzcd}$$
to $\mathcal{G}$ is equivalent to choosing an element $x \in G_1$. Then, if $\mathcal{G} \in \mathbf{Cat}(\mathbf{Gp})$, using the adjunction $\mathcal{P} \vdash J$, to give a morphism from $\mathcal{I}$ to $\mathcal{G}$ is just to give an arrow in $\mathcal{G}$.

Now, the only thing that remains to prove is that the associated morphism in $\mathbf{Cat}(\mathbf{Gp})$ to any arrow in $\mathcal{G}$ is, in fact, a morphism in $\underline{\mathbf{C}}$.

For any $x : p \to q$, $x \in G_1$, the associated morphism $f : \mathcal{I} \to \mathcal{G}$ is determined by $f_0(u_0(1)) = p$, $f_0(u_1(1)) = q$, $f_1(v_0(1)) = I_p$, $f_1(v_1(1)) = x$ and $f_1(v_2(1)) = I_q$. Then, we have to prove that $f_1(\tau_{a,b}) = \tau_{f_0(a),f_0(b)}$ for any $a, b \in \mathbb{Z} * \mathbb{Z}$. In the case $a = u_0(1)$ and $b = u_1(1)$, $\tau_{p,q} = f_1(v_1(1) + v_0(1) - v_1(1) + v_2(1)) = x + I_p - x + I_q$ and this last relation is true because of the naturality of τ applied to the morphisms x and I_q. Other cases are shown in a similar way. □

Next we will consider the morphisms $i_0, i_1 : \mathcal{I}_1 \to \mathcal{I}$ determined by $u_0(1)$ and $u_1(1)$ respectively.

Proposition 2.10. *Let $f : \mathcal{G} \to \mathcal{H}$ be a morphism in $\underline{\mathbf{C}}$. Then:*

i) f is a fibration if and only if f has the RLP with respect to the morphism $i_0 : \mathcal{I}_1 \to \mathcal{I}$

ii) f is a trivial fibration if and only if f has the RLP with respect to the morphisms $0 \to \mathcal{I}_1$, $\mathcal{I}_0 \to 0$ and $i_0 + i_1 : \mathcal{I}_1 \coprod \mathcal{I}_1 \to \mathcal{I}$.

Proof.

i) It is clear since giving a commutative diagram in $\underline{\mathbf{C}}$

$$\begin{tikzcd} \mathcal{I}_1 \arrow[r] \arrow[d, "i_0"'] & \mathcal{G} \arrow[d, "f"] \\ \mathcal{I} \arrow[r] & \mathcal{H} \end{tikzcd}$$

is equivalent to giving an element $p \in G_0$ and an element $x \in H_1$ such that $f_0(p) = s(x)$ and to find a lifting in the diagram is equivalent to give an element $y \in G_1$ such that $f_1(y) = x$ and $s(y) = p$.

ii) It is straightforward to see that f has the RLP with respect to $0 \to I_1$ (resp. $i_0 + i_1 : \mathcal{I}_1 \coprod \mathcal{I}_1 \to \mathcal{I}$, resp. $\mathcal{I}_0 \to 0$) if and only if f_0 is surjective (resp. $\pi_1(f)$ is surjective and $\pi_0(f)$ is injective, resp. $\pi_1(f)$ is injective).

Now if f satisfies the RLP with respect to the above morphisms, we have that f is a weak equivalence (see proposition 2.3). To see that f is a fibration, take $(y,p) \in H_{1s}\times_{f_0}G_0$; since f_0 is surjective there exists $q \in G_0$ such that $f_0(q) = t(y)$ and using that f is an equivalence of categories, there exists $x \in G_1$, $x : p \to q$ such that $f_1(x) = y$.

Conversely, if f is a trivial fibration, the only thing that remains to prove is that f_0 is surjective, but this is clear because given $q \in H_0$ and using that $\pi_0(f)$ is surjective, there exists $y \in H_1$ such that $t(y) = q$ and $s(y) = f_0(p)$ for some $p \in G_0$, and so, since f is a fibration, there is $x \in G_1$ such that $f_1(x) = y$. Then, $f_0(t(x)) = q$. □

Proposition 2.11. *Any morphism $f : \mathcal{G} \to \mathcal{H}$ in $\underline{\mathbf{C}}$ can be factored as a cofibration followed by a trivial fibration.*

Proof. To get the required factorization of f use the above characterization of the trivial fibrations and the "small object argument" (see [30]). □

Proposition 2.12. *A morphism $f : \mathcal{G} \to \mathcal{H}$ in $\underline{\mathbf{C}}$ is a cofibration if and only if it is a retract of a morphism in $\underline{\mathbf{C}}$ which is of the form*

$$\begin{tikzcd} \mathcal{G}: \arrow[dd, "h"] & G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[dd, "h_1"] & G_0 \arrow[l, bend right, "I"'] \arrow[dd, "h_0"] \\ \\ \mathcal{T}: & T_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 * FV \arrow[l, bend right, "I"'] \end{tikzcd}$$

where FV is the free group on the set V.

Proof. Since the cofibrations are closed under retracts, we will do the proof only for the morphisms as above.

Given a commutative diagram in $\underline{\mathbf{C}}$

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{T} \arrow[r, "\beta"'] & \mathcal{Y} \end{tikzcd} \quad \text{where } f \text{ is a trivial fibration}$$

the lifting $\gamma : \mathcal{T} \to \mathcal{X}$ is defined as follows:

Since $f_0 : X_0 \to Y_0$ is surjective, one can define a morphism $FV \to X_0$, and this one together with α_0 determine $\gamma_0 : T_0 \to X_0$ satisfying $f_0\gamma_0 = \beta_0$.

On the other hand, given $z \in T_1$, $z : p \to q$, there exists an unique $x \in X_1, x : \gamma_0(p) \to \gamma_0(q)$, such that $f_1(x) = \beta_1(z)$ because f is an equivalence of categories. Thus, we define $\gamma_1(z) = x$, and it is straightforward to see that γ is a morphism in $\mathbf{Cat(Gp)}$.

Moreover, γ is a morphism in $\underline{\mathbf{C}}$ because for any $p, q \in T_0$ the morphisms $\gamma_1(\tau_{p,q})$ and $\tau_{\gamma_0 p,\gamma_0 q}$ have the same source and target, and also, $f_1(\gamma_1(\tau_{p,q})) = f_1(\tau_{\gamma_0 p,\gamma_0 q})$. Thus, using that f is an equivalence of categories one has that $\gamma_1(\tau_{p,q}) = \tau_{\gamma_0 p,\gamma_0 q}$.

Conversely, if f is a cofibration, we factor $f = pi$ as in proposition 2.11 and we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "i"] \arrow[d, "f"'] & \mathcal{T} \arrow[d, "p"] \\ \mathcal{H} \arrow[r, equal] & \mathcal{H} \end{tikzcd}$$

in which there exists a lifting because f has the LLP with respect to the trivial fibrations. Then, it is clear that f is a retract of i. Also, one can see that i has the required form according to the construction of $\mathcal{T}$. $\square$

Corollary 2.13. *An object* $\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \arrow[l, bend right, "I"'] \end{tikzcd}$ $\in \underline{\mathbf{C}}$ *is cofibrant if and only if* G_0 *is a free group.*

Proposition 2.14. *The trivial cofibrations in* $\underline{\mathbf{C}}$ *have the LLP with respect to the fibrations.*

Proof. Using proposition 2.12 we only will prove the required LLP when the cofibration is of the form

$$\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "h_1"] & G_0 \arrow[l, bend right, "I"'] \arrow[d, "h_0"] \\ T_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 * FV \arrow[l, bend right, "I"'] \end{tikzcd}$$

Now, given a commutative diagram in $\underline{\mathbf{C}}$

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{T} \arrow[r, "\beta"'] & \mathcal{Y} \end{tikzcd}$$ where f is a fibration

the lifting $\gamma : \mathcal{T} \to \mathcal{X}$ is defined as follows:

Since $\pi_0(h)$ is an isomorphism, given $v \in V$ there exists $g_v \in G_0$ and $k_v \in T_1$ such that $s(k_v) = g_v$ and $t(k_v) = v$. Now, because f is a fibration and $\beta_1(k_v)$ is an element in Y_1 such that $s(\beta_1(k_v)) = \beta_0(g_v) = f_0\alpha_0 g_v$, there is $x_v \in X_1$ such that $f_1(x_v) = \beta_1(k_v)$ and $s(x_v) = \alpha_0(g_v)$. Thus, we have maps $\gamma_0 : V \to X_0$ given by $\gamma_0(v) = t(x_v)$ and $k : V \to T_1$ given by $k(v) = k_v$ which induce group morphisms $\gamma_0 : FV \to X_0$ and $k : FV \to T_1$. The morphism γ_0 together with α_0 determine a morphism $\gamma_0 : G_0 * FV \to X_0$ and it is clear that $f_0\gamma_0 = \beta_0$. On the other hand, the morphisms k and $I : G_0 \to T_1$ induce a morphism $k : G_0 * FV \to T_1$ which satisfies $tk = Id$ and $Image(sk) = G_0$.

Now, for any $v \in V$ we define $\gamma_1(k_v) = x_v$, and for any $g \in G_0$ we put $\gamma_1(k_g) = I_{\alpha_0(g)}$. Then, we have defined γ_1 for any element which is in the image of $k : G_0 * FV \to T_1$.

Finally, we have $y = k_{t(y)} \circ \left((k_{t(y)})^{-1}\ y\ k_{s(y)}\right) \circ (k_{s(y)})^{-1}$ for any $y \in T_1$. Using that h is an equivalence of categories, we can identify G_1 with $h_1(G_1)$ and then we define $\gamma_1(y) = \gamma_1(k_{t(y)}) \circ \alpha_1\left((k_{t(y)})^{-1}\ y\ k_{s(y)}\right) \circ \gamma_1((k_{s(y)})^{-1})$.

It is straightforward to see that $\gamma = (\gamma_1, \gamma_0)$ is a morphism in **Cat(Gp)** which satisfies $f\gamma = \beta$ and $\gamma h = \alpha$.

Next we will see that γ is a morphism in $\underline{\mathbf{C}}$, i.e., $\gamma_1(\tau_{p,q}) = \tau_{\gamma_0 p, \gamma_0 q}$

Using the naturality of τ we obtain that $(k_{q+p})^{-1}\ \tau_{p,q}\ k_{p+q} = \tau_{g_p,g_q}$ (where $g_p = s(k_p)$), and according to the definition of γ_1 we have that $\gamma_1(\tau_{p,q})$ is the arrow making the following diagram commutative

$$\left(\begin{tikzcd} \gamma_0 p + \gamma_0 q \arrow[r, dotted] \arrow[d, "(\gamma_1(k_{p+q}))^{-1}"] & \gamma_0 q + \gamma_0 p \\ \alpha_0 g_p + \alpha_0 g_q \arrow[r, "\alpha_1(\tau_{g_p,g_q})"'] & \alpha_0 g_q + \alpha_0 g_q \arrow[u, "\gamma_1(k_{q+p})"'] \end{tikzcd}\right) = \left(\begin{tikzcd} \gamma_0 p + \gamma_0 q \arrow[r, dotted] \arrow[d, "(\gamma_1 k_p)^{-1} + (\gamma_1 kq)^{-1}"] & \gamma_0 q + \gamma_0 p \\ \gamma_0 g_p + \gamma_0 g_q \arrow[r, "\tau_{\gamma_0 g_p, \gamma_0 g_q}"] & \gamma_0 g_q + \gamma_0 g_q \arrow[u, "\gamma_1 k_q + \gamma_1 k_p"'] \end{tikzcd}\right)$$

which is exactly $\tau_{\gamma_0 p, \gamma_0 q}$. □

Theorem 2.15. *The category $\underline{\mathbf{C}}$ with the classes of morphisms given in Definition 2.1 or Definition 2.2 is a closed model category.*

Proof. CM1 is known, and CM2 and CM3 are clear. The only non trivial part of CM4 is given in proposition 2.14. The factorization of any morphism as a cofibration followed by a trivial fibration is given in proposition 2.11. Finally, to get the other factorization required in CM5 use proposition 2.8 to factor $f = \kappa\eta$ and then factor η as in proposition 2.11. □

3 Homotopy theory in Cat(Gp) ($\mathcal{B}$Cat(Gp) or $\mathcal{S}$Cat(Gp))

In this section we study the homotopy theory in $\underline{\mathbf{C}}$ associated to the closed model structure defined in §2.

Propositions 2.5 and 2.6 give directly the following:

Proposition 3.1. *Given $\mathcal{H} \in \underline{\mathbf{C}}$, $\mathcal{H}^I$ is a path object for $\mathcal{H}$ in $\underline{\mathbf{C}}$ and for any morphisms $f, g : \mathcal{G} \to \mathcal{H}$ in $\underline{\mathbf{C}}$, to give a right homotopy from f to g is equivalent to give a homotopy from f to g.*

The path construction determines the loop functor $\Omega : \underline{\mathbf{C}} \to \underline{\mathbf{C}}$ defined as $\Omega(\mathcal{H}) = Ker(\partial_0, \partial_1)$. Then $\Omega(\mathcal{H})$ can be identified with the object of $\underline{\mathbf{C}}$ associated to the abelian group $\pi_1(\mathcal{H})$, i.e., $\begin{tikzcd} \pi_1(\mathcal{H}) \arrow[r, shift left, "Id"] \arrow[r, shift right, "Id"'] & \pi_1(\mathcal{H}) \arrow[l, bend right, "Id"'] \end{tikzcd}$.

In the following we will give a cylinder construction in $\underline{\mathbf{C}}$.

Let

$$\mathcal{G} = \begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \arrow[l, bend right, "I"'] \end{tikzcd}$$

an object of $\underline{\mathbf{C}}$ and consider the following 1-truncated simplicial group

$$\begin{tikzcd} G_1 * G_1 * G_1 \arrow[r, shift left, "\overline{s}"] \arrow[r, shift right, "\overline{t}"'] & G_0 * G_0 \arrow[l, bend right, "\overline{I}"'] \end{tikzcd}$$

where $\overline{s}$,$\overline{t}$ and $\overline{I}$ are determined as follows: suppose $u_i : G_0 \to G_0 * G_0$ the i-th canonical injection, $i = 0, 1$, and $v_j : G_1 \to G_1 * G_1 * G_1$ the corresponding j-th injection, $j = 0, 1, 2$; then $\overline{s}$, $\overline{t}$ and $\overline{I}$ are determined by the relations

$$\overline{I}u_0 = v_0 I; \ \overline{I}u_1 = v_2 I$$

$$\overline{s}v_0 = u_0 s; \ \overline{s}v_1 = u_0 s; \ \overline{s}v_2 = u_1 s; \ \ \overline{t}v_0 = u_0 t; \ \overline{t}v_1 = u_1 t; \ \overline{t}v_2 = u_1 t$$

Now, if N is the congruence in $G_1 * G_1 * G_1$ generated by the relations

- $x + y = y + x$; $x \in Ker\ \overline{s}$, $y \in Ker\ \overline{t}$
- $v_1 x = v_0 x$; $x \in Ker\ t$

– $v_1 x = v_2 x;\ x \in Ker\ s$

we have a cat-group

$$\mathcal{G} \otimes I = \left(\begin{tikzcd} \frac{G_1 * G_1 * G_1}{N} \arrow[r, shift left, "\overline{s}"] \arrow[r, shift right, "\overline{t}"'] & G_0 * G_0 \arrow[l, bend right, "\overline{I}"'] \end{tikzcd} \right)$$

We also will denote by $v_j,\ j = 0, 1, 2$, the induced morphism

$$G_1 \to \frac{G_1 * G_1 * G_1}{N}.$$

Proposition 3.2. *Given $\mathcal{G} \in \underline{\mathbf{C}}$ then $\mathcal{G} \otimes I \in \underline{\mathbf{C}}$.*

Proof. We first prove that for any $x : p \to q \in G_1$ the following squares are commutative:

$$\begin{tikzcd} u_0 p \arrow[r, "v_1 I_p"] \arrow[d, "v_1 x"'] & u_1 p \arrow[d, "v_2 x"] \\ u_1 q \arrow[r, equal, "v_2 I_q"'] & u_1 q \end{tikzcd} \qquad \begin{tikzcd} u_0 p \arrow[r, equal, "v_0 I_p"] \arrow[d, "v_0 x"'] & u_0 p \arrow[d, "v_1 x"] \\ u_0 p \arrow[r, "v_1 I_q"'] & u_1 q \end{tikzcd} \qquad \begin{tikzcd} u_0 p \arrow[r, "v_1 I_p"] \arrow[d, "v_0 x"'] & u_1 p \arrow[d, "v_2 x"] \\ u_0 p \arrow[r, "v_1 I_q"'] & u_1 q \end{tikzcd}$$

The commutativity of the first one means that $v_2 x \circ v_1 I_p = v_1 x$, i.e., $v_2 x - \overline{I}_{\overline{s} v_2 x} + v_1 I_p = v_1 x$ but, $\overline{I}_{\overline{s} v_2 x} = v_2 I_p$, and so, we need to prove that $v_2(x - I_p) = v_1(x - I_p)$ which is true because $x - I_p \in Ker\ s$. The commutativity of the others diagrams is proved in a similar way.

Now, given $\alpha \in \{0, 1, 2\}$ we consider β and γ as follows: if $\alpha = 0$ then $\beta = \gamma = 1$; if $\alpha = 1$ then $\beta = 1$ and $\gamma = 2$ and if $\alpha = 2$, $\beta = \gamma = 2$.

Then we have that the following squares are commutative

$$\begin{tikzcd} u_{\beta-1} p \arrow[r, "v_\beta I_p"] \arrow[d, "v_\alpha x"'] & u_1 p \arrow[d, "v_2 x"] \\ u_{\gamma-1} q \arrow[r, "v_\gamma I_q"'] & u_1 q \end{tikzcd}$$

which allows us to show that for any $x_i : p_i \to q_i$, $1 \leq i \leq n$, $x_i \in G_1$, the following square is commutative

$$\begin{tikzcd}
u_{\beta_1 - 1}p_1 + \cdots + u_{\beta_n - 1}p_n \arrow[r, "v_{\beta_1}I_{p_1} + \cdots + v_{\beta_n}I_{p_n}"] \arrow[d, "v_{\alpha_1}x_1 + \cdots + v_{\alpha_n}x_n"'] & u_1(p_1 + \cdots + p_n) \arrow[d, "v_2(x_1 + \cdots + x_n)"] \\
u_{\gamma_1 - 1}q_1 + \cdots + u_{\gamma_n - 1}q_n \arrow[r, "v_{\gamma_1}I_{q_1} + \cdots + v_{\gamma_n}I_{q_n}"'] & u_1(q_1 + \cdots + q_n)
\end{tikzcd}$$

Now, let $p, q \in G_0 * G_0$. Then, $p = \sum_{i=1}^{n} u_{\delta_i}p_i$ and $q = \sum_{j=1}^{m} u_{\delta'_j}q_j$ where $\delta_i, \delta'_j = 0, 1$ and we define $\tau'_{p,q}$ as the following composition

$$\begin{tikzcd}
p + q \arrow[r, "\tau'_{p,q}"] \arrow[d, "\sum_{i=1}^{n} v_{\delta_i+1}I_{p_i} + \sum_{j=1}^{m} v_{\delta'_j+1}I_{q_i}"'] & q + p \\
\sum_{i=1}^{n} u_1 p_i + \sum_{j=1}^{m} u_1 q_j \arrow[r, "v_2(\tau_{(\sum p_i, \sum q_j)})"'] & \sum_{j=1}^{m} u_1 q_j + \sum_{i=1}^{n} u_1 p_i \arrow[u, "\left(\sum_{j=1}^{m} v_{\delta'_j+1}I_{q_i} + \sum_{i=1}^{n} v_{\delta_i+1}I_{p_i}\right)^{-1}"']
\end{tikzcd}$$

It is straightforward to see that $\tau'_{p,q}$ is well defined and that τ' satisfies the same properties than τ.

$\square$

Note that the object $\mathcal{I}$ considered in §2 is just $\mathcal{I}_1 \otimes I$.

Lemma 3.3. *Given $\mathcal{G} \in \underline{\mathrm{C}}$ there is a factorization of the codiagonal morphism*

$$\mathcal{G} \coprod \mathcal{G} \xrightarrow{i_0 + i_1} \mathcal{G} \otimes I \xrightarrow{\sigma} \mathcal{G}$$

Proof. Let $i_0 = (v_0, u_0)$ and $i_1 = (v_2, u_1)$. It is clear, according to the definition of the braiding in $\mathcal{G} \otimes I$ (see proposition 3.2), that i_1 is a morphism in $\underline{\mathbf{C}}$. Also, i_0 is a morphism in $\underline{\mathbf{C}}$ since the following square is commutative

$$\begin{tikzcd} u_0(p+q) \arrow[r, "v_0\tau_{p,q}"] \arrow[d, "v_1 I_{p+q}"] & u_0(q+p) \\ u_1(p+q) \arrow[r, "v_2\tau_{p,q}"'] & u_1(q+p) \arrow[u, "(v_1 I_{q+p})^{-1}"'] \end{tikzcd}$$

Both morphisms i_0 and i_1 induce the morphism in $\underline{\mathbf{C}}$, $i_0 + i_1 : \mathcal{G} \coprod \mathcal{G} \to \mathcal{G} \otimes I$.

On the other hand, the identities $\sigma_0 u_i = Id_{G_0}$, $i = 0, 1$ and $\sigma_1 v_j = Id_{G_1}$, $j = 0, 1, 2$, determine a morphism $\sigma : \mathcal{G} \otimes I \to \mathcal{G}$ which is given explicitly by $\sigma_0 \left(\sum_{i=1}^{n} u_{\delta_i} p_i \right) = \sum_{i=1}^{n} p_i$, for any $\sum_{i=1}^{n} u_{\delta_i} p_i \in G_0 * G_0$, and $\sigma_1 \left(\sum_{j=1}^{m} v_{\epsilon_j} x_j \right) = \sum_{j=1}^{m} x_j$ for any $\sum_{j=1}^{m} v_{\epsilon_j} x_j \in \frac{G_1 * G_1 * G_1}{N}$, $\epsilon_j = 0, 1, 2$.

It is clear finally that $\sigma i_0 = \sigma i_1 = Id_{\mathcal{G}}$. □

Lemma 3.4. *Given $\mathcal{G}, \mathcal{H} \in \underline{\mathbf{C}}$ and $k : \mathcal{G} \otimes I \to \mathcal{H}$ a morphism in* **Cat(Gp)**, *then k is a morphism in $\underline{\mathbf{C}}$ if and only if ki_0 and ki_1 are morphisms in $\underline{\mathbf{C}}$.*

Proof. The only non trivial thing is to prove that, if ki_0 and ki_1 are morphisms in $\underline{\mathbf{C}}$ then k is also a morphism in $\underline{\mathbf{C}}$.

Let us denote $f = ki_0$ and $g = ki_1$; thus $f_0 = k_0 u_0$, $g_0 = k_0 u_1$, $f_1 = k_1 v_0$ and $g_1 = k_1 v_2$.

We want to prove that, for any $p, q \in G_0 * G_0$, $k_1(\tau'_{p,q}) = \tau_{k_0 p, k_0 q}$. Now, it is straightforward to check that, if $p, q \in G_0$, then $k_1(\tau'_{u_1 p, u_1 q}) = \tau_{g_0 p, g_0 q}$, $k_1(\tau'_{u_0 p, u_1 q}) = \tau_{f_0 p, g_0 q}$, $k_1(\tau'_{u_1 p, u_0 q}) = \tau_{g_0 p, f_0 q}$ and $k_1(\tau'_{u_0 p, u_0 q}) = \tau_{f_0 p, f_0 q}$.

For example, the above second relation is equivalent to the commutativity of the diagram

$$\begin{tikzcd} f_0 p + g_0 q \arrow[r, "\tau_{f_0 p, g_0 q}"] \arrow[d, "k_1 v_1 I_p + Id"] & g_0 q + f_0 p \arrow[d, "Id + k_1 v_1 I_p"'] \\ g_0 p + g_0 q \arrow[r, "\tau_{g_0 p, g_0 q}"] & g_0 q + g_0 p \end{tikzcd}$$

which is true because this diagram is just $\tau'_{k_1 v_1 I_p, Id}$ (see the definition of τ' in $\mathcal{H}^I$).

For general $p = \sum_{i=1}^{n} u_{\delta_i} p_i$ and $q = \sum_{j=1}^{m} u_{\delta'_j} q_j$ in $G_0 * G_0$, we use induction on n and m as the following example suggests. Suppose $p = u_0 p_0 + u_1 p_1$ and $q = u_0 q_0$; then one has $\tau'_{p,q} = (\tau'_{u_0p_0,u_0q_0} + Id_{u_1p_1}) \circ (Id_{u_0p_0} + \tau'_{u_1p_1,u_0q_0})$ from where

$$k_1(\tau'_{p,q}) = k_1(\tau'_{u_0p_0,u_0q_0} + Id_{u_1p_1}) \circ k_1(Id_{u_0p_0} + \tau'_{u_1p_1,u_0q_0}) =$$
$$[k_1(\tau'_{u_0p_0,u_0q_0}) + Id_{k_0(u_1p_1)}] \circ [(Id_{k_0(u_0p_0)} + k_1(\tau'_{u_1p_1,u_0q_0})] =$$
$$= [\tau_{(k_0u_0p_0,k_0u_0q_0)} + Id_{k_0(u_1p_1)}] \circ [(Id_{k_0(U_0p_0)} + \tau(k_0u_1p_1, k_0u_0q_0)] =$$
$$= \tau_{(k_0u_0p_0 + k_0u_1p_1, k_0u_0q_0)} = \tau_{k_0p,k_0q}$$

$\square$

Proposition 3.5. *The functor* $- \otimes I : \underline{\mathbf{C}} \to \underline{\mathbf{C}}$ *is left adjoint to the functor* $(-)^I : \underline{\mathbf{C}} \to \underline{\mathbf{C}}$.

Proof. Consider $\varphi : Hom_{\mathbf{Cat(Gp)}}(\mathcal{G}, \mathcal{H}^I) \to Hom_{\mathbf{Cat(Gp)}}(\mathcal{G} \otimes I, \mathcal{H})$ the map given by $\varphi(h) = k$ where, if $f = \partial_1 h$ and $g = \partial_0 h$, (k_1, k_0) is given as follows: Let k_0 be determined by $k_0 u_0 = f_0$ and $k_0 u_1 = g_0$ and consider the morphism in $\mathbf{Tr^1(Simp(Gp))}$

$$\begin{tikzcd}
G_1 * G_1 * G_1 \arrow[r, shift left, "\overline{s}"] \arrow[r, shift right, "\overline{t}"'] \arrow[d, "\overline{k}_1"] & G_0 * G_0 \arrow[l, bend right, "\overline{I}"'] \arrow[d, "k_0"] \\
H_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H_0 \arrow[l, bend right, "I"']
\end{tikzcd}$$

determined by $\overline{k}_1 v_0 = f_1$, $\overline{k}_1 v_2 = g_1$ and $\overline{k}_1 v_1$ given, for any $x : p \to q$, by

$$\overline{k}_1(v_1 x) = \left(\begin{tikzcd}
f_0(p) \arrow[r, "f_1(x)"] \arrow[d, dotted, no head] & f_0(q) \arrow[d, "h_0(q)"] \\
\cdot \arrow[r, dotted, no head] & g_0(q)
\end{tikzcd}\right) = \left(\begin{tikzcd}
f_0(p) \arrow[r, dotted, no head] \arrow[d, "h_0(p)"] & \cdot \arrow[d, dotted, no head] \\
g_0(p) \arrow[r, "g_1(x)"'] & g_0(q)
\end{tikzcd}\right)$$

where

$$\left(\begin{tikzcd} f_0(p) \arrow[r, "f_1(x)"] \arrow[d, "h_0(p)"] & f_0(q) \arrow[d, "h_0(q)"'] \\ g_0(p) \arrow[r, "g_1(x)"'] & g_0(q) \end{tikzcd}\right) = h_1(x)$$

This morphism $(\overline{k}_1, k_0)$ determines, using the adjunction $\mathcal{P} \vdash J$, a morphism in $\mathbf{Cat(Gp)}$, $(\overline{k}_1, k_0) : \mathcal{P}\left(\begin{tikzcd} G_1 * G_1 * G_1 \arrow[r, shift left, "\overline{s}"] \arrow[r, shift right, "\overline{t}"'] & G_0 * G_0 \arrow[l, bend right, "\overline{I}"'] \end{tikzcd}\right) \to \mathcal{H}$

and this morphism induces another one $(k_1, k_0) : \mathcal{G} \otimes I \to \mathcal{H}$ because, if $x \in Ker(t)$, $\overline{k}_1(v_1x - v_0x) = 0$, and, if $x \in Ker(s)$, $\overline{k}_1(v_2x - v_1x) = 0$ as can be easily checked out.

On the other hand, given a morphism in $\mathbf{Cat(Gp)}$, $k : \mathcal{G}\otimes I \to \mathcal{H}$, let $\phi : Hom_{\mathbf{Cat(Gp)}}(\mathcal{G} \otimes I, \mathcal{H}) \to Hom_{\mathbf{Cat(Gp)}}(\mathcal{G}, \mathcal{H}^I)$ be given by $\phi(k) = h$ defined by

$$h_0(p) = \left(f_0p \xrightarrow{k_1v_1I_p} g_0p \right); \quad h_1(x : p \to q) = \left(\begin{tikzcd} f_0p \arrow[r, "f_1x"] \arrow[d, "h_0(p)"] & f_0q \arrow[d, "h_0(q)"'] \\ g_0p \arrow[r, "g_1x"'] & g_0q \end{tikzcd}\right)$$

where $f = ki_0$ and $g = ki_1$.

Note that the last square is commutative because in $\mathcal{G}\otimes I$ one has, for any $x : p \to q \in G_1$, the relation $v_1I_q \circ v_0x = v_2x \circ v_1I_p$ (see proposition 3.2)

It is clear that $\partial_1 h = ki_0$ and $\partial_0 h = ki_1$ and then φ and ϕ give a bijection

$$Hom_{\underline{\mathbf{C}}}(\mathcal{G} \otimes I, \mathcal{H}) \cong Hom_{\underline{\mathbf{C}}}(\mathcal{G}, \mathcal{H}^I)$$

because, by corollary 2.7 and lemma 3.4, one has that $k \in Hom_{\underline{\mathbf{C}}}(\mathcal{G} \otimes I, \mathcal{H})$ if and only if $ki_0, ki_1 \in Hom_{\underline{\mathbf{C}}}(\mathcal{G}, \mathcal{H})$ if and only if $\partial_1 h, \partial_0 h \in Hom_{\underline{\mathbf{C}}}(\mathcal{G}, \mathcal{H})$ if and only if $h \in Hom_{\underline{\mathbf{C}}}(\mathcal{G}, \mathcal{H}^I)$.

□

Proposition 3.6. *If $\mathcal{G}$ is a cofibrant object in $\underline{\mathbf{C}}$ then $\mathcal{G}\otimes I$ is a cylinder object for $\mathcal{G}$ in $\underline{\mathbf{C}}$.*

Proof. We have according to lemma 3.3, a factorization of the codiagonal morphism

$$\mathcal{G}\coprod\mathcal{G} \xrightarrow{i_0+i_1} \mathcal{G}\otimes I \xrightarrow{\sigma} \mathcal{G}$$

and it is clear that, if $\mathcal{G}$ is cofibrant, i_0+i_1 is a cofibration according to the characterization of the cofibrations given in proposition 2.12

On the other hand, σ is a weak equivalence because $\sigma i_1 = Id_{\mathcal{G}}$ and $Id_{\mathcal{G}\otimes I}$ is homotopic to $i_1\sigma$, the homotopy $\alpha : Id_{\mathcal{G}} \Rightarrow i_1\sigma$ being given, for any $p=\sum_{i=1}^{n} u_{\delta_i}p_i \in G_0 * G_0$, by $\alpha_p = v_{\delta_1+1}I_{p_1} + \dots + v_{\delta_n+1}I_{p_n} : p \to u_1(p_1+\dots+p_n)$. The naturality is deduced from the relations proved for $\mathcal{G}\otimes I$ in proposition 3.2.

$\square$

Note that the functor $-\otimes I : \underline{\mathbf{C}} \to \underline{\mathbf{C}}$ preserves cofibrations and weak equivalences.

Corollary 3.7. *If $\mathcal{G}$ is a cofibrant object in $\underline{\mathbf{C}}$ and $f,g : \mathcal{G} \to \mathcal{H}$ are two morphisms in $\underline{\mathbf{C}}$, then f and g are right homotopic if and only if they are left homotopic if and only if they are homotopic.*

Corollary 3.8. *If $\mathcal{G}$ is a cofibrant object in $\underline{\mathbf{C}}$ and $\mathcal{H}$ is any object of $\underline{\mathbf{C}}$ then*

$$Hom_{Ho(\underline{\mathbf{C}})}(\mathcal{G},\mathcal{H}) = [\mathcal{G},\mathcal{H}]$$

The cylinder construction determines a suspension functor $\Sigma : \underline{\mathbf{C}} \to \underline{\mathbf{C}}$ defined by $\Sigma(\mathcal{G})$=Coker(i_0+i_1); this can be described using the following:

Proposition 3.9. *Given $\mathcal{G} \in \underline{\mathbf{C}}$, Coker $(i_0+i_1 : \mathcal{G}\coprod\mathcal{G} \to \mathcal{G}\otimes I)$ can be identified with the object of $\underline{\mathbf{C}}$*

$$\begin{tikzcd}
(\pi_0(G))_{ab} \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] & 0 \arrow[l, bend right=60, "0"']
\end{tikzcd}$$

Proof. Let us consider the 1-truncated simplicial group

$$\begin{tikzcd} \frac{G_1 * G_1 * G_1}{N'} \arrow[r, shift left, "\bar{s}"] \arrow[r, shift right, "\bar{t}"'] & G_0 * G_0 \arrow[l, bend right, "\bar{I}"'] \end{tikzcd}$$

where N' is the congruence generated by the relations $v_1 x = v_0 x$, $x \in Ker\ t$; $v_1 x = v_2 x$, $x \in Ker\ s$ and also consider the following morphism f in $\mathbf{Tr^1}(\mathbf{Simp}(\mathbf{Gp}))$

$$\begin{tikzcd} G_1 * G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "v_0+v_2"] & G_0 * G_0 \arrow[l, bend right, "I"'] \arrow[d, "u_0+u_1"] \\ \frac{G_1 * G_1 * G_1}{N'} \arrow[r, shift left, "\bar{s}"] \arrow[r, shift right, "\bar{t}"'] & G_0 * G_0 \arrow[l, bend right, "\bar{I}"'] \end{tikzcd}$$

Note that, if $\mathcal{G} \in \mathbf{Cat}(\mathbf{Gp})$ then $\mathcal{P}(f) = i_0 + i_1$ and therefore $\mathcal{P}(Coker(f)) = \Sigma(\mathcal{G})$.

Now, $Coker(f)$ is the 1-truncated simplicial group $\begin{tikzcd} H \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] & 0 \arrow[l, bend right, "0"'] \end{tikzcd}$ where $H = Coker(v_0 + v_2)$ is the quotient group of $G_1 * G_1 * G_1$ under the congruence generated by the relations $v_0 x = v_2 x = 0$, $\forall x \in G_1$; $v_1 x = 0$, $x \in Ker(s)$ *or* $x \in Ker(t)$. Since $[Ker\ s, Ker\ t] = 0$ we have that $H \cong \frac{G_1}{Ker\ s + Ker\ t} \cong \pi_0(\mathcal{G})$. Then, by applying $\mathcal{P}$ we have that $(\Sigma(\mathcal{G}))_1 = (\pi_0(\mathcal{G}))_{ab}$.

For any object $\mathcal{G} \in \underline{\mathbf{C}}$ it is easy to see that $\Sigma(U(\mathcal{G})) = Coker(i_0 + i_1)$.

□

Note that $\Sigma(\mathcal{I}_1) = \mathcal{I}_0$.

Corollary 3.10. *The functor $\Sigma : \underline{\mathbf{C}} \to \underline{\mathbf{C}}$ is left adjoint to the functor Ω.*

4 Model structure and homotopy theory in the categories $\chi\mathbf{M}(\mathbf{Gp})$, $\mathbf{2}-\chi\mathbf{M}_{\mathbf{red}}(\mathbf{Gp})$ and $\chi\mathbf{M}_{\mathbf{st}}(\mathbf{Gp})$.

In this section we use the commutative diagram of categories and functors

$$\begin{tikzcd}
\chi\mathbf{M}_{\mathbf{st}}(\mathbf{Gp}) \arrow[r, "\Phi"] \arrow[d, hook, "In"'] & \mathcal{S}\mathbf{Cat}(\mathbf{Gp}) \arrow[d, hook, "In"'] \\
\mathbf{2}-\chi\mathbf{M}_{\mathbf{red}}(\mathbf{Gp}) \arrow[r, "\Phi"] \arrow[d, "U"'] & \mathcal{B}\mathbf{Cat}(\mathbf{Gp}) \arrow[d, "U"'] \\
\chi\mathbf{M}(\mathbf{Gp}) \arrow[r, "\Phi"] & \mathbf{Cat}(\mathbf{Gp})
\end{tikzcd}$$

to define model structures in the categories $\chi\mathbf{M}(\mathbf{Gp})$, $\mathbf{2}-\chi\mathbf{M}_{\mathbf{red}}(\mathbf{Gp})$ and $\chi\mathbf{M}_{\mathbf{st}}(\mathbf{Gp})$, and to study the associated homotopy theories.

The equivalence $\Phi : \chi\mathbf{M}(\mathbf{Gp}) \to \mathbf{Cat}(\mathbf{Gp})$ and the model structure in $\mathbf{Cat}(\mathbf{Gp})$ (see definition 2.1) together with propositions 2.3 and 2.12 allow us to consider the following model structure in $\chi\mathbf{M}(\mathbf{Gp})$ (cf. [14]): the fibrations are those morphisms $f = (f_1, f_0) : (\mathcal{L} : (L \xrightarrow{\rho} M)) \to (\mathcal{L}' : (L' \xrightarrow{\rho'} M'))$ such that f_1 is surjective, the weak equivalences are those $f = (f_1, f_0)$ such that the induced morphisms $Coker(\rho) \to Coker(\rho')$ and $Ker(\rho) \to Ker(\rho')$ are isomorphisms and the cofibrations are the retracts of those morphisms $f = (f_1, f_0)$ where $M' = M * FV$. Note that the inclusion functor from $\chi\mathbf{M}(\mathbf{Gp})$ to the category of crossed modules over groupoids clearly preserves weak equivalences but it does not preserve fibrations because, in this last category, fibrations (f_1, f_0) between crossed modules of groups also require that f_0 be surjective (see §1).

In the same way as above, the following equivalences of categories (see §1)

$$\mathbf{2}-\chi\mathbf{M}_{\mathbf{red}}(\mathbf{Gp}) \simeq \mathcal{B}\mathbf{Cat}(\mathbf{Gp}), \ \chi\mathbf{M}_{\mathbf{st}}(\mathbf{Gp}) \simeq \mathcal{S}\mathbf{Cat}(\mathbf{Gp})$$

allow us to consider model structures in $\mathbf{2}-\chi\mathbf{M}_{\mathbf{red}}(\mathbf{Gp})$ and $\chi\mathbf{M}_{\mathbf{st}}(\mathbf{Gp})$.

In these closed model categories fibrations, cofibrations or weak equivalences are just those morphisms f such that $U(f)$ (resp. $UIn(f)$) is a fibration, a cofibration or a weak equivalence in $\chi\mathbf{M}(\mathbf{Gp})$.

We will denote by $\underline{\mathbf{D}}$ any of the categories $\chi\mathbf{M}(\mathbf{Gp})$, $\mathbf{2} - \chi\mathbf{M}_{\mathbf{red}}(\mathbf{Gp})$ or $\chi\mathbf{M}_{\mathbf{st}}(\mathbf{Gp})$.

Next we will make explicit some homotopy constructions in $\underline{\mathbf{D}}$ which are the ones corresponding to those given in $\underline{\mathbf{C}}$ in §3.

Definition 4.1. *Given two morphisms in $\underline{\mathbf{D}}$, $f, g : \mathcal{L} \to \mathcal{L}'$, a homotopy from f to g is a f_0-derivation $\eta : M \to L'$ such that $\eta\rho = g_1 - f_1$ and $\rho'\eta = g_0 - f_0$.*

This homotopy relation is an equivalence relation on the set of morphisms in $\underline{\mathbf{D}}$ from $\mathcal{L}$ to $\mathcal{L}'$ and the set of equivalence classes will be denoted by $[\mathcal{L}, \mathcal{L}']$. It is clear that $[\mathcal{L}, \mathcal{L}'] \cong [\Phi(\mathcal{L}), \Phi(\mathcal{L}')]$.

It should be noted that if one eliminates the freeness assumption, Whitehead's homotopy systems of dimension 2, [32], are just crossed modules of groups and the homotopy relation defined by Whitehead for morphisms between two such objects is that considered in the above definition. More in general, without the assumption of only one vertex, homotopies for crossed complexes have been extensively studied (see [11], [6], [9]) and, for reduced crossed complexes of dimension 2, one has again the definition of homotopy given in 4.1. Homotopies between morphisms of crossed complexes have been used for example by Huebschmann, [23], to interpret the cohomology $H^n(G, A)$ of a group G with coefficients in a G-module A.

Proposition 4.2. *Given $\mathcal{L} \in \underline{\mathbf{D}}$ there is a factorization of the diagonal morphism*

$$\mathcal{L} \xrightarrow{\sigma} \mathcal{L}^I \xrightarrow{(\partial_0, \partial_1)} \mathcal{L} \times \mathcal{L}$$

where σ is a weak equivalence and (∂_0, ∂_1) is a fibration.

Proof. Given $\mathcal{L} \in \underline{\mathbf{D}}$ and considering $\Phi(\mathcal{L}) \in \underline{\mathbf{C}}$, there exists (see proposition 2.5) a factorization in $\underline{\mathbf{C}}$ of the diagonal morphism

$$\Phi(\mathcal{L}) \xrightarrow{\sigma} \Phi(\mathcal{L})^I \xrightarrow{(\partial_0, \partial_1)} \Phi(\mathcal{L}) \times \Phi(\mathcal{L})$$

and then applying Φ^{-1}

$$\mathcal{L} \xrightarrow{\sigma} \Phi^{-1}(\Phi(\mathcal{L})^I) \xrightarrow{(\partial_0,\partial_1)} \mathcal{L} \times \mathcal{L}$$

is the required factorization.

Next we make explicit the object of $\underline{\mathbf{D}}$, $\Phi^{-1}(\Phi(\mathcal{L})^I)$, and also the morphisms σ and (∂_0, ∂_1).

Considering $\mathcal{L} : (L \xrightarrow{\rho} M))$ the underlying crossed module of $\mathcal{L} \in \underline{\mathbf{D}}$, it is straightforward to see that the underlying crossed module of $\mathcal{L}^I = \Phi^{-1}(\Phi(\mathcal{L})^I)$ is

$$\mathcal{L}^I : \left(\begin{array}{ccc} L \rtimes L & \xrightarrow{\bar{\rho}} & L \rtimes M \\ (l', l) & \longmapsto & (l', \rho(l)) \end{array} \right)$$

where $L \rtimes L$ is the semidirect product with action of L on itself by conjugation and the action of $L \rtimes M$ on $L \rtimes L$ is given by

$${}^{(l_1,m)}(l', l) = (l_1 + {}^m l' - {}^{m+\rho(l)-m} l_1, {}^m l)$$

The factorization of the diagonal morphism is

$$\begin{array}{ccccc} L & \xrightarrow{\sigma_1} & L \rtimes L & \xrightarrow{(\partial_0,\partial_1)_1} & L \times L \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \bar{\rho}} & & \downarrow{\scriptstyle \rho\times\rho} \\ M & \xrightarrow[\sigma_0]{} & L \rtimes M & \xrightarrow[(\partial_0,\partial_1)_0]{} & M \times M \end{array}$$

where

$$\sigma_0(m) = (0, m); \;\; \sigma_1(l) = (0, l)$$

$$(\partial_0, \partial_1)_0(l, m) = (\rho(l) + m, m); \;\; (\partial_0, \partial_1)_1(l', l) = (l' + l, l)$$

The braiding (symmetry) τ' in $\Phi(\mathcal{L})^I$ determines a map $\{-,-\} : (L \rtimes M) \times (L \rtimes M) \to L \rtimes L$ which is defined by

$$\{(l, m), (l', m')\} = \tau'_{(l',m'),(l,m)} - I_{(l,m)} - I_{(l',m')}$$

and it is straightforward to check that

$$\{(l,m),(l',m')\} = \left(l + {}^{m}l' + \{m,m'\} - {}^{m'}l - l' - \rho\{m,m'\}, \{m,m'\}\right).$$

□

Proposition 4.3. *Let $f,g : \mathcal{L} \to \mathcal{L}'$ two morphisms in $\underline{\mathbf{D}}$. To give a homotopy from f to g is equivalent to give a morphism $h : \mathcal{L} \to \mathcal{L}'^I$ such that $\partial_0 h = g$ and $\partial_1 h = f$, i.e., a right homotopy from f to g.*

Proof. Let $\eta : M \to L'$ be a f_0-derivation

$$\begin{tikzcd} L \arrow[r, "\rho"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & M \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] \arrow[dl, "\eta"'] \\ L' \arrow[r, "\rho'"] & M' \end{tikzcd}$$

such that $\eta\rho = g_1 - f_1$ and $\rho'\eta = g_0 - f_0$. Then we define $h_0 : M \to L' \rtimes M'$ and $h_1 : L \to L' \rtimes L'$ as follows:

$$\begin{array}{rcl} M & \longrightarrow & L' \rtimes M' \\ m & \longmapsto & (\eta(m), f_0(m)) \end{array} \qquad \begin{array}{rcl} L & \longrightarrow & L' \rtimes L' \\ l & \longmapsto & (\eta\rho(l), f_1(l)) \end{array}$$

and it is straightforward to see that $h = (h_1, h_0)$ is a morphism in $\underline{\mathbf{D}}$ satisfying $(\partial_0, \partial_1)h = (g, f)$.

Conversely, if h is a homotopy from f to g, we have a commutative diagram

$$\begin{tikzcd} L \arrow[r, "h_1"'] \arrow[rr, bend left, "(g_1,f_1)"] \arrow[d, "\rho"] & L' \rtimes L' \arrow[r, "(\partial_0,\partial_1)_1"'] \arrow[d, "\overline{\rho}'"] & L' \times L' \arrow[d, "\rho' \times \rho'"] \\ M \arrow[r, "h_0"] \arrow[rr, bend right, "(g_0,f_0)"'] & L' \rtimes M' \arrow[r, "(\partial_0,\partial_1)_0"] & M' \times M' \end{tikzcd}$$

Then, if we denote $h_0(m) = (h_{0,0}(m), h_{0,1}(m))$, it is straightforward to see that the map $\eta : M \to L'$ defined by $\eta(m) = h_{0,0}(m)$ is a f_0-derivation satisfying $\eta\rho(l) = g_1(l) - f_1(l)$, $\forall l \in L$, and $\rho'\eta(m) = g_0(m) - f_0(m)$, $\forall m \in M$. $\square$

Corollary 4.4. *Given $\mathcal{L}, \mathcal{L}' \in \underline{\mathbf{D}}$ and morphisms $f, g : \mathcal{L} \to \mathcal{L}'$, then f and g are right homotopic if and only if they are homotopic.*

Corollary 4.5. *If $\mathcal{L}$ is a cofibrant object in $\underline{\mathbf{D}}$ and $\mathcal{L}'$ is any object in $\underline{\mathbf{D}}$, then*

$$Hom_{Ho(\underline{\mathbf{D}})}(\mathcal{L}, \mathcal{L}') = [\mathcal{L}, \mathcal{L}'].$$

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References

[1] M.G. BARRATT, Track groups II, *Proc. London Math. Soc.* (3) 5 (1955), 258-329.

[2] H.J. BAUES, Combinatorial homotopy and 4-dimensional complexes, *de Gruyter Expos. Math.* vol. 2, 1991.

[3] A.L. BLAKERS, On the relations between homotopy and homology groups, *Ann. of Math.* vol. 49 (1948), 428-461.

[4] R.BROWN, Fibrations of groupoids, *J. of Algebra* 15 (1970), 103-132.

[5] R.BROWN and N.D. GILBERT, Algebraic models of 3-types and automorphism structures for crossed modules, *Proc. London Math. Soc.* (3) 59 (1989), 51-73.

[6] R. BROWN and M. GOLASINSKI, A model structure for the homotopy category of crossed complexes, *Cahiers Topologie Géom. Diff. Catégoriques* 30 (1989), 61-82.

[7] R. BROWN and P.J. HIGGINS, Tensor products and homotopies for ω-groupoids and crossed complexes, *J. Pure Appl. Algebra* 47 (1987), 1-33.

[8] R.BROWN and P.J. HIGGINS, The algebra of cubes, *J. Pure Appl. Algebra* 21 (1981), 233-260.

[9] R. BROWN and P.J. HIGGINS, Crossed complexes and chain complexes with operators, *Math Proc. Camb. Phil. Soc.* 107 (1990), 33-57.

[10] R. BROWN and P.J. HIGGINS, The classifying space of a crossed complex, *Math Proc. Camb. Phil. Soc.* 110 (1991), 95-120.

[11] R. BROWN and J.L. LODAY, Van Kampen theorems for diagram of spaces, *Topology* 26 (1987), 311-335.

[12] M. BULLEJOS and A. M. CEGARRA, A 3-dimensional non-abelian cohomology with applications to homotopy classification of continuous maps, *Canadian Journal of Mathematics*, 43 (2) (1991), 265-296.

[13] M. BULLEJOS, P. CARRASCO and A.M. CEGARRA, Cohomology with coefficients in symmetric cat-groups. An extension of Eilenberg-MacLane's classification theorem, *Math. Proc. Camb. Phil. Soc.* 114 (1993), 163-189.

[14] J.G. CABELLO and A.R. GARZON, Closed model structures for algebraic models of n-types, *J. Pure Appl. Algebra* 103 (1995), 287-302.

[15] P. CARRASCO and A.M CEGARRA, Group theoretic algebraic models for homotopy types, *J. Pure Appl. Algebra* 75 (1991), 195-235.

[16] P. CARRASCO and A.M. CEGARRA, Applications de la notion du Nerf d'une Gr-catégorie (tressée), *C. R. Acad. Sci. Paris* t. 321, Série I (1995), 395-398.

[17] P. CARRASCO and A.M. CEGARRA, Extensions centrales de Gr-catégories par des Gr-catégories tressées, *C. R. Acad. Sci. Paris* t. 321, Série I (1995), 527-530.

[18] D. CONDUCHE, Modules croisés généralisés de longeur 2, *J. Pure Appl. Algebra* 34 (1984), 155-178.

[19] G. ELLIS, Crossed squares and combinatorial homotopy, *Math. Z.* 214 (1993), 93-110.

[20] A.R. GARZON and J.G. MIRANDA, Models for homotopy n-types in diagram categories, *Applied Categorical Structures* 4, (1996), 1-14.

[21] D. GUIN-WALERY and J.L. LODAY, Obstructions a l'excision en K-theorie algébrique, *L. N. in Math* 854 *Springer* (1981), 179-216.

[22] J. HOWIE, Pullback functors and crossed complexes, *Cah. Top. Géom. Diff.* XX-3 (1979), 281-296.

[23] J. HUEBSCHMANN, Crossed n-fold extensions and cohomology, *Comm. Math. Helv.* 55 (1980), 302-314.

[24] A. JOYAL and R. STREET, Braided tensor categories, *Advances in Math.* (1) 82 (1991), 20-78.

[25] J.L. LODAY, Spaces with finitely many non-trivial homotopy groups, *J. Pure Appl. Algebra* 24 (1982), 179-202.

[26] S. MaC LANE and J.H.C. WHITEHEAD, On the 3-type of a complex, *Proc. Nat. Acac. Sci. USA* 30 (1956), 41-48.

[27] J.G. MIRANDA , Estructuras de modelos y teoría de homotopía en categorías de grupos y grupoides simpliciales, Ph.D. thesis, University of Granada (1995).

[28] I. MOERDIJK and J. SVENSSON , Algebraic classification of equivariant homotopy 2-types, I, *J. Pure Appl. Algebra* 89, (1993), 187-216.

[29] T. PORTER, Abstract Homotopy theory. The interaction of category theory and homotopy theory, *U.C.N.W. Maths Preprint* 92.04 (1992).

[30] D. QUILLEN, Homotopical Algebra, *L.N. in Math.* 43, *Springer* (1967).

[31] D. QUILLEN, Rational homotopy theory, *Annals of Math.* 90 (1969), 205-295.

[32] J.H.C. WHITEHEAD, Combinatorial homotopy II, *Bull. A.M.S.* 55 (1949), 453-496.

[33] J.H.C. WHITEHEAD, Algebraic Homotopy Theory, *Proc. Int. Congress of Mathematicians, Harvard* 2 (1950), 354-357.

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