CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

MARIA RITA CASALI
LUCA MALAGOLI

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Cahiers de topologie et géométrie différentielle catégoriques, tome 38, nº 2 (1997), p. 141-160
<http://www.numdam.org/item?id=CTGDC_1997__38_2_141_0>

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Volume XXXVIII-2 (1997)*

HANDLE-DECOMPOSITIONS OF PL 4-MANIFOLDS

by Maria Rita CASALI and Luca MALAGOLI

Rèsumé. Dans cet article, on étudie les relations entre les décompositions avec anses des 4-variétés PL et l'invariant combinatoire appelé *genre régulier.* En particulier on obtient les caractérisations suivantes des 4-variétés PL M^4 (avec frontière vide ou connexe ∂M^4) vérifiant certaines conditions relatives au genre régulier $\mathcal{G}(M^4)$, au rang de leur groupe fondamental $rk(\pi_1(M^4))$, et au genre régulier de leur frontière $\mathcal{G}(\partial M^4)$:

$$\mathcal{G}(M^4) = rk(\pi_1(M^4)) = m \iff$$

$$M^4 \cong \#_m(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1) \quad \text{ou} \quad M^4 \cong \#_p(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\overset{\sim}{\mathbb{Y}}_{m-p}$$

$$\mathcal{G}(M^4) = \mathcal{G}(\partial M^4) + 1 \quad \iff \quad M^4 \cong (\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\overset{\sim}{\mathbb{Y}}_q$$

où $\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1$ désigne le fibré en 3-sphères sur $\mathbb{S}^1$ orientable ou non, et où $\overset{\sim}{\mathbb{Y}}_s$ désigne le corps avec anses à 4 dimensions orientable ou non de genre $s \geq 0$. Enfin, comme conséquence de ces résultats et de leur généralisation au cas de frontières non-connexes, on complète la classification des 4-variétés PL avec frontière jusqu'au genre 2; plus précisément, on prouve que les seules 4-variétés premières de genre $g \leq 2$ sont $\mathbb{S}^4$, $\mathbb{D}^4$, $\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1$, $\overset{\sim}{\mathbb{Y}}_1$, $\overset{\sim}{\mathbb{Y}}_2$ et $\mathbb{CP}^2$.

1. Introduction.

It is well-known that each closed 4-manifold M^4, admitting a handle-decomposition

$$M^4 = H^{(0)} \cup (H_1^{(1)} \cup \cdots \cup H_{r_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{r_2}^{(2)}) \cup (H_1^{(3)} \cup \cdots \cup H_{r_3}^{(3)}) \cup H^{(4)}$$

(*) Work performed under the auspicies of the G.N.S.A.G.A. of the C.N.R. (National Research Council of Italy) and financially supported by M.U.R.S.T. of Italy (project "Topologia e Geometria"). .

- where $H^{(p)} = \mathbb{D}^p \times \mathbb{D}^{n-p}$ denotes a *handle of index p* (or a *p-handle*) - is completely determined by the handles of index ≤ 2: see [M], or Proposition 1 in the third section. In particular, if M^4 admits a handle-decomposition where no 2-handle appears, then $M^4 \cong \#_{r_1}(\mathbb{S}^3 \underset{(\sim)}{\times} \mathbb{S}^1)$, where $\mathbb{S}^3 \underset{(\sim)}{\times} \mathbb{S}^1$ denotes either the orientable or non-orientable 3-sphere bundle over $\mathbb{S}^1$, according to M^4 being an orientable 4-manifold or not, and the symbol $\cong$ means PL-homeomorphism.

Extending to the boundary case the above characterization is the first aim of the present paper; in fact, the third paragraph is entirely devoted to prove the following statement, where $\overset{(\sim)}{\mathbb{Y}}_s = H^{(0)} \cup (H_1^{(1)} \cup \cdots \cup H_s^{(1)})$ denotes either the orientable or non-orientable 4-dimensional handlebody of genus $s \geq 0$.

Theorem I. *If M^4 (with empty or connected boundary ∂M^4) admits a handle-decomposition where no 2-handle appears, then*

$$M^4 \cong \#_{r_1}(\mathbb{S}^3 \underset{(\sim)}{\times} \mathbb{S}^1) \qquad \textit{if } \partial M^4 = \emptyset$$

$$M^4 \cong \#_{r_3}(\mathbb{S}^3 \underset{(\sim)}{\times} \mathbb{S}^1) \# \overset{(\sim)}{\mathbb{Y}}_{r_1 - r_3} =$$

$$= \begin{cases} \#_{r_3}(\mathbb{S}^3 \times \mathbb{S}^1) \# \mathbb{Y}_{r_1 - r_3} & \textit{if } M^4 \textit{ is orientable with } \partial M^4 \neq \emptyset \\ \#_{r_3}(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1) \# \mathbb{Y}_{r_1 - r_3} & \textit{if } M^4 \textit{ is non-orientable and } \partial M^4 \textit{ is orientable} \\ \#_{r_3}(\mathbb{S}^3 \times \mathbb{S}^1) \# \tilde{\mathbb{Y}}_{r_1 - r_3} & \textit{if both } M^4 \textit{ and } \partial M^4 \textit{ are non-orientable} \end{cases}$$

Further, we study the relationship between handle-decompositions of 4-manifolds (in the particular case considered in Theorem I) and the so called *crystallization theory*, which represents PL n-manifolds by means of pseudosimplicial triangulations admitting exactly $n+1$ vertices.

Within this representation theory (which is reviewed in the second paragraph), a combinatorial invariant - i.e. the *regular genus* - has been defined, which extends to arbitrary dimension the classical notions of genus of a surface and of Heegaard genus of a 3-manifold.

In the present paper we prove that the 4-manifolds M^4 involved in Theorem I are characterized by the equality between their regular genus $\mathcal{G}(M^4)$ and the rank of their fundamental group $rk(\pi_1(M^4))$.

Theorem II. *Let M^4 be a PL 4-manifold with empty or connected boundary ∂M^4. Then:*

$$\mathcal{G}(M^4) = rk(\pi_1(M^4)) = m \quad \Longleftrightarrow$$
$$either \ \ M^4 \cong \#_m(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1) \ \ or \ \ M^4 \cong \#_p(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\# \overset{\sim}{\mathbb{Y}}_{m-p}$$

Moreover, the following condition about regular genus is proved to ensure that the 4-manifold belongs to the class handled in Theorem I.

Theorem III. *Let M^4 be a PL 4-manifold with empty or connected boundary ∂M^4. Then:*

$$\mathcal{G}(M^4) = \mathcal{G}(\partial M^4) + 1 \quad \Longleftrightarrow \quad M^4 \cong (\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\# \overset{\sim}{\mathbb{Y}}_q$$

The proofs of Theorems II and III are exposed in the fourth paragraph, while in the fifth one we generalize all the obtained results to the case of disconnected boundary. Finally, in section six - as a consequence of the whole work - we complete the classification of PL 4-manifolds with (possibly disconnected) boundary up to regular genus two (see Proposition 6 and Tables 1 - 2 - 3).

2. Preliminaries.

In this section we shortly expose the representation theory of PL–manifolds by means of coloured graphs, for which we refer to [FGG], [BM], [Co] and [V]. On the other hand, elementary notions of graph theory may be found in [Ha], while [RS] constitutes an useful reference about piecewise-linear (PL) category.

An $(n+1)$*–coloured graph* (with boundary) is a pair (Γ, γ) where $\Gamma = \big(V(\Gamma), E(\Gamma)\big)$ is a multigraph (for more simplicity we shall call it a *graph*) and $\gamma : E(\Gamma) \longrightarrow \Delta_n = \{0, 1, \ldots, n\}$ is a proper coloration of $E(\Gamma)$ on Δ_n (i.e. $\gamma(e) \neq \gamma(f)$ for every adjacent edges $e, f \in E(\Gamma)$). The coloured graph (Γ, γ) will be often denoted by the only letter Γ of its underlying graph.

For every subset $\beta \subset \Delta_n$, a (possibly disconnected) graph $\Gamma_\beta = \big(V(\Gamma), \gamma^{-1}(\beta)\big)$ is well defined; its connected components are said to be β–*residues* of Γ, or m–residues if m is the cardinality of β. An m–residue is said to be *internal* if it is a regular graph of degree m; otherwise, it is said to be a *boundary* residue. If $\beta = \{i_0, i_1, \ldots, i_h\}$, then we denote by $g_{i_0,i_1,\ldots,i_h}$ (resp. $\dot{g}_{i_0,i_1,\ldots,i_h}$) (resp. $\bar{g}_{i_0,i_1,\ldots,i_h}$) the number of β–residues of Γ (resp. internal β–residues) (resp. boundary β–residues). Later on, we shall write $\hat{\imath}$ instead of $\beta = \Delta_n - \{i\}$.

An $(n+1)$-coloured graph (Γ, γ) is said to be *regular* with respect to the colour $i \in \Delta_n$ if each $\hat{\imath}$–residue is internal. Let G_{n+1} be the class of all $(n+1)$-coloured graphs regular with respect to the color $n \in \Delta_n$; note that, if $(\Gamma, \gamma) \in G_{n+1}$, then every vertex $v \in V(\Gamma)$ must have either degree n (v is called a *boundary vertex)* or degree $n+1$ (v is defined to be an *internal vertex).*

If $(\Gamma, \gamma) \in G_{n+1}$, we call *boundary graph* associated to (Γ, γ) the (regular) n–coloured graph $(\partial\Gamma, {}^{\partial}\gamma)$ constructed as follows:

- $V(\partial\Gamma)$ is the (possibly empty) set of boundary vertices of Γ;
- two vertices $v, w \in V(\partial\Gamma)$ are joined by an edge e of colour ${}^{\partial}\gamma(e) = i \in \Delta_{n-1}$ iff v and w below to the same (boundary) $\{i, n\}$–residue of Γ.

Obviously, $\partial(\partial\Gamma) = \emptyset$; moreover, for each $\beta = \{i_0, i_1, \ldots, i_h\} \subset \Delta_{n-1}$, the number ${}^{\partial}g_\beta$ of β-residues of $\partial\Gamma$ satisfies the relation ${}^{\partial}g_{i_0,i_1,\ldots,i_h} = \bar{g}_{i_0,i_1,\ldots,i_h,n}$.

The starting point of the representation theory of PL-manifolds by means of edge-coloured graphs is the possibility of associating an n–pseudocomplex (see [HW]) $K = K(\Gamma)$ to every graph $(\Gamma, \gamma) \in G_{n+1}$:

- consider an n–simplex $\sigma(v)$ for each element $v \in V(\Gamma)$ and label its $n+1$ vertices by the $n+1$ colours $0, 1, \ldots, n$;
- if $v, w \in V(\Gamma)$ are joined in Γ by an edge of colour i, then identify the $(n-1)$–faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertex coloured by i, so that equally labelled vertices coincide.

The $(n+1)$–coloured graph $(\Gamma, \gamma) \in G_{n+1}$ is said to *represent* the polyhedron $|K(\Gamma)|$ and every homeomorphic space.

It is important to note that, for every $(\Gamma, \gamma) \in G_{n+1}$ and for every $\beta = \{i_0, i_1, \ldots, i_h\} \subset \Delta_n$, each β–residue of Γ uniquely corresponds to an $(n-h-1)$–simplex $\sigma^{n-h-1} \in K(\Gamma)$, whose vertices are coloured by $\Delta_n - \beta$; as a consequence, $K(\Gamma)$ results to be an n–manifold (with boundary) iff the pseudocomplex $K(\Xi)$ is homeomorphic either to the $(n-1)$–sphere $\mathbb{S}^{n-1}$ or to the $(n-1)$–ball $\mathbb{D}^{n-1}$, for each $\hat{\imath}$–residue Ξ of Γ, $i \in \Delta_n$ (see [FGG] for details).

A coloured graph $(\Gamma, \gamma) \in G_{n+1}$ is said to be a *crystallization* of an n–manifold M^n (with empty or connected boundary) if (Γ, γ) represents M^n and $\Gamma_{\hat{\imath}}$ is connected for every $i \in \Delta_n$.

By theorems of [P] and [Ga$_2$], it is known that every n–manifold M^n admits a crystallization (Γ, γ), and that Γ results to be bipartite iff M^n is orientable; moreover, if $\partial M^n \neq \emptyset$ is connected, then $(\partial\Gamma, {}^{\partial}\gamma)$ is a crystallization of ∂M^n ([CG]).

The notion of *regular genus* was introduced in [Ga$_1$] for closed manifolds and extended to the boundary case in [Ga$_3$].

For each coloured graph $(\Gamma, \gamma) \in G_{n+1}$, and for every cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{n-1}, \varepsilon_n = n)$ of Δ_n, it is proved the existence of a 2–cell embedding (see [Ga$_3$]) of Γ onto a suitable surface F_ε, which results orientable iff Γ is bipartite and closed iff $\partial\Gamma = \emptyset$. The regular genus $\varrho_\varepsilon(\Gamma)$ of (Γ, γ) with respect to the permutation ε is defined as the classical genus (resp. half of the genus) of the orientable (resp. non-orientable) surface F_ε; it may be computed by means of the following formula, where $\dot p$ and $\bar p$ respectively denote the number of internal and boundary vertices of Γ:

$$\sum_{i\in\mathbb{Z}_n} \dot g_{\varepsilon_i\varepsilon_{i+1}} + (1-n)\frac{\dot p}{2} + (2-n)\frac{\bar p}{2} + {}^{\partial}g_{\varepsilon_0\varepsilon_{n-1}} = 2 - 2\varrho_\varepsilon(\Gamma)$$

The minimum value $\mathcal{G}(M^n)$ of these $\varrho_\varepsilon(\Gamma)$, computed by changing the crystallization (Γ, γ) of the manifold M^n and the permutation ε, is said to be the *regular genus* of M^n. As far as its properties are concerned, we only remember that it extends to arbitrary dimension the classical notions of genus of a surface and of Heegaard genus of a 3-manifold; moreover, for every n-manifold M^n with $\partial M^n \neq \emptyset$, relation $\mathcal{G}(M^n) \geq \mathcal{G}(\partial M^n)$ holds (see [CP]).

3. 4-manifolds obtained by (partial) boundary identification of handlebodies.

Purpose of the present section is to prove Theorem I; for convenience, we will restate it by making use of the process of (partial) boundary identification of two handlebodies.

If $\overset{|\sim|}{\mathbb{Y}}_p$ and $\overset{|\sim|}{\mathbb{Y}}_q$ $(0 \leq p \leq q)$ are two fixed 4-dimensional handlebodies, then we call *attaching map* every regular embedding φ of type $\varphi : \partial \overset{|\sim|}{\mathbb{Y}}_p \longrightarrow \partial \overset{|\sim|}{\mathbb{Y}}_q$ (in case $p = q$) or $\varphi : \partial \overset{|\sim|}{\mathbb{Y}}_p - int(\mathbb{D}^3) \longrightarrow \partial \overset{|\sim|}{\mathbb{Y}}_q$, $\mathbb{D}^3$ being a 3–ball contained into $\partial \overset{|\sim|}{\mathbb{Y}}_p$. We will denote by $\overset{|\sim|}{\mathbb{Y}}_p \cup_\varphi \overset{|\sim|}{\mathbb{Y}}_q$ the identification space of the two handlebodies via the map φ.

It is now easy to check that Theorem I (as it is stated in the Introduction) is equivalent to the following:

Theorem I′. *Let M^4 be a 4–manifold admitting a decomposition $M^4 = \overset{|\sim|}{\mathbb{Y}}_p \cup_\varphi \overset{|\sim|}{\mathbb{Y}}_q$, by means of suitable $\overset{|\sim|}{\mathbb{Y}}_p$, $\overset{|\sim|}{\mathbb{Y}}_q$ and φ, with $0 \leq p \leq q$.*

(a) If M^4 is closed, then $p = q$ and $M^4 \cong \#_p(\mathbb{S}^3 \underset{|\sim|}{\times} \mathbb{S}^1)$.

(b) If $\partial M^4 \neq \emptyset$, then $M^4 \cong \#_p(\mathbb{S}^3 \underset{|\sim|}{\times} \mathbb{S}^1) \# \overset{|\sim|}{\mathbb{Y}}_{q-p}$.

We shortly recall that in dimension ≤ 4, PL and DIFF categories may be identified (see [KS]). This allows us to translate from DIFF-category into PL-category the following theorem related to the extension of boundary diffeomorphisms of a 4–dimensional handlebody; for its proof, we refer to [M], [Ce], [L] and relative bibliography.

Proposition 1 (Extending theorem). *For each homeomorphism $\varphi : \partial \overset{|\sim|}{\mathbb{Y}}_p \longrightarrow \partial \overset{|\sim|}{\mathbb{Y}}_p$, $p \geq 0$, there exists a unique self-homeomorphism ϕ of $\overset{|\sim|}{\mathbb{Y}}_p$ (up to isotopy) extending φ.* □

It is easy to check that the extending theorem is the key-stone to prove Montesinos's result which states that closed 4-manifolds are determined by the handles of index ≤ 2 of their handle-decomposition (see [M]); moreover, it is quite obvious that statement (a) of Theorem I′ is nothing but a particular case of this statement.

Note that for every fixed $\overset{|\sim|}{\mathbb{Y}}_p$ and $\overset{|\sim|}{\mathbb{Y}}_q$ and whatever φ may be, by extending theorem we can always suppose that the boundary of each orientable (resp. non-orientable) handle of $\overset{|\sim|}{\mathbb{Y}}_p$ is identified (at least partially) with the boundary of a corresponding orientable (resp. non-orientable) handle of $\overset{|\sim|}{\mathbb{Y}}_q$, which will said to be *covered* by φ.

In spite of directly proving Theorem I′ (part (b)), we will prove the following result, which takes into consideration all the cases that may arise.

Lemma 1. *Let M^4 (with $\partial M^4 \neq \emptyset$) be a 4–manifold admitting a decomposition $M^4 = \overset{|\sim|}{\mathbb{Y}}_p \cup_\varphi \overset{|\sim|}{\mathbb{Y}}_q$, by means of suitable $\overset{|\sim|}{\mathbb{Y}}_p$, $\overset{|\sim|}{\mathbb{Y}}_q$ and φ, with $p \leq q$.*

(i) If $\varphi : \partial \overset{|\sim|}{\mathbb{Y}}_p - int(\mathbb{D}^3) \longrightarrow \partial \overset{|\sim|}{\mathbb{Y}}_p$, then $M^4 \cong \#_p(\mathbb{S}^3 \underset{|\sim|}{\times} \mathbb{S}^1) - int(\mathbb{D}^4)$.

(ii) If $\varphi : \partial \mathbb{Y}_p - int(\mathbb{D}^3) \longrightarrow \partial \mathbb{Y}_q$, then $M^4 \cong \#_p(\mathbb{S}^3 \times \mathbb{S}^1) \# \overset{|\sim|}{\mathbb{Y}}_{q-p}$.

(iii) If $\varphi : \partial \mathbb{Y}_p - int(\mathbb{D}^3) \longrightarrow \partial \widetilde{\mathbb{Y}}_q$, then $p < q$ and $M^4 \cong \#_p(\mathbb{S}^3 \times \mathbb{S}^1) \# \widetilde{\mathbb{Y}}_{q-p}$.

(iv) *If* $\varphi : \partial\tilde{\mathbb{Y}}_p - int(\mathbb{D}^3) \longrightarrow \partial\tilde{\mathbb{Y}}_q$ *, and the handles of* $\tilde{\mathbb{Y}}_q$ *not covered by* φ *are all orientable, then* $M^4 \cong \#_p(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\mathbb{Y}_{q-p}$ *.*

(v) *If* $\varphi : \partial\tilde{\mathbb{Y}}_p - int(\mathbb{D}^3) \longrightarrow \partial\tilde{\mathbb{Y}}_q$ *, and at least a non-orientable handle of* $\tilde{\mathbb{Y}}_q$ *is not covered by* φ*, then* $p < q$ *and* $M^4 \cong \#_p(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\tilde{\mathbb{Y}}_{q-p}$ *.*

Proof. Case (i) Since φ is a homeomorphism on the image, $\varphi(\partial \overset{|\sim|}{\mathbb{Y}}_p - int(\mathbb{D}^3)) = \partial \overset{|\sim|}{\mathbb{Y}}_p - int(\mathbb{D}'^3)$ holds, $\mathbb{D}'^3$ being a 3–ball into $\partial \overset{|\sim|}{\mathbb{Y}}_p$. Let $\phi : \partial \overset{|\sim|}{\mathbb{Y}}_p \longrightarrow \partial \overset{|\sim|}{\mathbb{Y}}_p$ be a homeomorphism that extends φ in natural way to the whole boundary of $\overset{|\sim|}{\mathbb{Y}}_p$; since $\mathbb{D}^3 \cup_\varphi \mathbb{D}'^3 \cong \mathbb{S}^3$ is the boundary of a 4–ball $\mathbb{D}^4$, then $\overset{|\sim|}{\mathbb{Y}}_p \cup \varphi \overset{|\sim|}{\mathbb{Y}}_p \cong (\overset{|\sim|}{\mathbb{Y}}_p \cup_\phi \overset{|\sim|}{\mathbb{Y}}_p) - int(\mathbb{D}^4)$. Statement (i) now directly follows from the closed case.

Case (ii) Using the homeomorphism $\mathbb{Y}_q \cong \mathbb{Y}_p\,^{\partial}\#\mathbb{Y}_{q-p}$ (true for each $0 \leq p \leq q$) and the property $X - int(\mathbb{D}^4) = X\#\mathbb{D}^4$ (true for any 4–manifold X), we have: $M^4 = \mathbb{Y}_p \cup_\varphi \mathbb{Y}_q \cong (\mathbb{Y}_p \cup_\varphi \mathbb{Y}_p)^{\partial}\#\mathbb{Y}_{q-p} \cong [\#_p(\mathbb{S}^3 \times \mathbb{S}^1)\#\mathbb{D}^4]^{\partial}\#\mathbb{Y}_{q-p} \cong \#_p(\mathbb{S}^3 \times \mathbb{S}^1)\#\mathbb{Y}_{q-p}$.

Case (iii) Since $\tilde{\mathbb{Y}}_q$ is non-orientable, while all the handles covered by φ are orientable, $p < q$ must obviously hold. If we consider $\tilde{\mathbb{Y}}_q \cong \mathbb{Y}_p\,^{\partial}\#\tilde{\mathbb{Y}}_{q-p}$, where $\mathbb{Y}_p$ represents the handles covered by φ, then $M^4 = \mathbb{Y}_p \cup_\varphi \tilde{\mathbb{Y}}_q \cong (\mathbb{Y}_p \cup_\varphi \mathbb{Y}_p)^{\partial}\#\tilde{\mathbb{Y}}_{q-p} \cong \#_p(\mathbb{S}^3 \times \mathbb{S}^1)\#\tilde{\mathbb{Y}}_{q-p}$ yields; thus, both M^4 and ∂M^4 result to be non-orientable.

Case (iv) Let us consider $\tilde{\mathbb{Y}}_q \cong \tilde{\mathbb{Y}}_p\,^{\partial}\#\mathbb{Y}_{q-p}$, where $\tilde{\mathbb{Y}}_p$ represents the handles of $\tilde{\mathbb{Y}}_q$ covered by φ. Then, $M^4 = \tilde{\mathbb{Y}}_p \cup_\varphi \tilde{\mathbb{Y}}_q \cong (\tilde{\mathbb{Y}}_p \cup_\varphi \tilde{\mathbb{Y}}_p)^{\partial}\#\mathbb{Y}_{q-p} \cong \#_p(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\mathbb{Y}_{q-p}$ is obtained, and M^4 results to be non-orientable with orientable boundary.

Case (v) Similarly to the case (iii), $p < q$ must hold. Moreover, we have $M^4 = \tilde{\mathbb{Y}}_p \cup_\varphi \tilde{\mathbb{Y}}_q \cong (\tilde{\mathbb{Y}}_p \cup_\varphi \tilde{\mathbb{Y}}_p)^{\partial}\#\tilde{\mathbb{Y}}_{q-p} \cong \#_p(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\tilde{\mathbb{Y}}_{q-p}$; thus, both M^4 and ∂M^4 result to be non-orientable. □

Remark 1. It is not difficult to check that the manifolds $\#_p(\mathbb{S}^3 \times \mathbb{S}^1)\#\tilde{\mathbb{Y}}_{q-p}$ and $\#_p(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\tilde{\mathbb{Y}}_{q-p}$ (with $p < q$), determined in cases (iii) and (v) of the previous Lemma, result to be homeomorphic; for, it is sufficient to extend to dimension 4 Theorem 3.17 of [He].

Proof of Theorem I'. The statement is now a direct consequence of Lemma 1. □

4. Main results.

From now, M^4 will denote a connected PL 4–manifold, with empty or connected boundary ∂M^4. Let $(\Gamma, \gamma) \in G_5$ be a crystallization of M^4, $K = K(\Gamma)$ the associated pseudocomplex and $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = 4)$ a cyclic permutation of Δ_4. For each colour $i \in \Delta_4$, we denote by $\varepsilon_{\hat{i}}$ the permutation induced by ε on $\Delta_4 - \{i\}$. In order to simplify the notation, the regular genera $\varrho_\varepsilon(\Gamma)$, $\varrho_{\varepsilon_{\hat 4}}(\partial\Gamma)$ and $\varrho_{\varepsilon_{\hat i}}(\Gamma_{\hat{\varepsilon}_i})$ will be respectively indicated by the symbols ϱ, ${}^\partial\varrho$ and $\varrho_{\hat i}$; moreover, we will often use the only letter i in order to denote the colour ε_i.

If M^4 has empty or connected boundary, then the regular genera mentioned above are related with the number of 2– and 3–residues of Γ, as it is stated in the following formulas (where the indexes are considered in $\mathbb{Z}_5$); they may be easily obtained from [Ca$_1$; Lemma 2], by recalling that ${}^\partial\varrho = \bar{g}_{\varepsilon_0\varepsilon_2 4} - 1 = \bar{g}_{\varepsilon_1\varepsilon_3 4} - 1$ (see [FGG] for details concerning crystallizations of closed 3-manifolds).

$$(1_j) \quad \begin{cases} g_{j-1,j+1} = g_{j-1,j,j+1} + \varrho - \varrho_{\hat j} & \text{if } \partial M^4 = \emptyset \text{ or } 4 \notin \{j-1, j+1\} \\ g_{j-1,j+1} = g_{j-1,j,j+1} + \varrho - \varrho_{\hat j} + \bar{g}_{124} - 1 & \text{if } \partial M^4 \neq \emptyset \text{ and } 4 \in \{j-1, j+1\} \end{cases}$$

$$(2_j) \quad \begin{cases} g_{j,j+2,j+3} = 1 + \varrho - \varrho_{\widehat{j-1}} - \varrho_{\widehat{j+1}} & \text{if } \partial M^4 = \emptyset \text{ or } j \neq 4 \\ g_{124} = \varrho - {}^\partial\varrho - \varrho_{\hat 0} - \varrho_{\hat 3} + \bar{g}_{124} & \text{if } \partial M^4 \neq \emptyset \end{cases}$$

$$(3_j) \quad \begin{cases} \dot{g}_{j,j+2,j+3} = \varrho - {}^\partial\varrho - \varrho_{\widehat{j-1}} - \varrho_{\widehat{j+1}} & \text{if } \partial M^4 \neq \emptyset \text{ and } 4 \notin \{j-1, j+1\} \\ \dot{g}_{j,j+2,j+3} = 1 + \varrho - \varrho_{\widehat{j-1}} - \varrho_{\widehat{j+1}} & \text{if } \partial M^4 = \emptyset \text{ or } 4 \in \{j-1, j+1\} \end{cases}$$

We now introduce a further useful notation. Since M^4 is assumed to be either closed or with connected boundary, $K(\Gamma)$ contains exactly five vertices, which may be labelled by $v_0, \dots, v_4$, so that the vertex v_i corresponds to the $\hat{i}$–residue $\Gamma_{\hat i}$, for every $i \in \Delta_4$. Then, for every subset $\{i_0, \dots, i_h\} \subset \Delta_4$, let $K(i_0, \dots, i_h)$ denote the h–dimensional subcomplex of K generated by the vertices $v_{i_0}, \dots, v_{i_h}$, let $[i_0, \dots, i_h]$ denote the total number of h-simplexes of $K(i_0, \dots, i_h)$, and let $N(i_0, \dots, i_h)$ be a regular neighbourhood of $|K(i_0, \dots, i_h)|$ into $|K|$. Finally, for every cyclic permutation ε and colour i of Δ_4, let $0_{(i)}, 1_{(i)}, 2_{(i)}, 3_{(i)}$ be the elements of $\Delta_4 - \{i\}$ such that $\varepsilon_{\hat{i}} = (0_{(i)}, 1_{(i)}, 2_{(i)}, 3_{(i)})$; note that we may always assume that the regular neighbourhoods $N(i, 1_{(i)}, 3_{(i)})$ and $N(0_{(i)}, 2_{(i)})$ (which are 4-dimensional sub-manifolds of M^4 with connected boundary) give rise to the decomposition $M^4 = N(i, 1_{(i)}, 3_{(i)}) \cup N(0_{(i)}, 2_{(i)})$ (where the two submanifolds intersect only on their (partially identified) boundaries).

In the present section, we will find suitable conditions about the regular genus of Γ, so that the previously described decomposition of M^4 consists of the union of two handlebodies (and so we can classify M^4 by means of Theorem I′).

First, we give a general result about the structure of $K(i, 1_{(i)}, 3_{(i)})$.

Lemma 2. *Let (Γ, γ) be a crystallization of a 4-manifold M^4 (with empty or connected boundary ∂M^4) and let ε be a permutation of Δ_4. For every colour $i \in \Delta_4$, the 2-pseudocomplex $K(i, 1_{(i)}, 3_{(i)})$ contains exactly $[i, 1_{(i)}, 3_{(i)}] = [i, 1_{(i)}] + [i, 3_{(i)}] + \varrho_{\hat{i}} - 1$ triangles.*

Proof. Let $i \in \Delta_4$ be a fixed colour. By direct construction of $K = K(\Gamma)$ from (Γ, γ), it is known that the number of triangles into $K(i, 1_{(i)}, 3_{(i)})$ equals the number of $\{0_{(i)}, 2_{(i)}\}$-residues of Γ, i.e. $[i, 1_{(i)}, 3_{(i)}] = g_{0_{(i)}, 2_{(i)}}$. Moreover, it is not difficult to check that, whichever the colour i may be, there exists a colour $r \in \Delta_4 - \{i\}$ such that $\{0_{(i)}, 2_{(i)}\} = \{r-1, r+1\}$ and $4 \notin \{r-1, r+1\}$.

In particular, the following two cases arise.

- If $i \in \{0, 3, 4\}$ then $(0_{(i)}, 1_{(i)}, 2_{(i)}) = (r-1, r, r+1)$ and $\{i, 1_{(i)}\} = \{s-1, s+1\}$, for a suitable $s \neq 4$; thus, formula (1_r) gives $g_{0_{(i)}2_{(i)}} = g_{0_{(i)}1_{(i)}2_{(i)}} + \varrho - \varrho_{\hat{1}_{(i)}} = [i, 3_{(i)}] + \varrho - \varrho_{\hat{1}_{(i)}}$ and formula (2_s) gives $g_{2_{(i)}3_{(i)}0_{(i)}} = [i, 1_{(i)}] = 1 + \varrho - \varrho_{\hat{1}_{(i)}} - \varrho_{\hat{i}}$.
- If $i \in \{1, 2\}$ then $(2_{(i)}, 3_{(i)}, 0_{(i)}) = (r-1, r, r+1)$ and $\{i, 3_{(i)}\} = \{s-1, s+1\}$, for a suitable $s \neq 4$; thus, formula (1_r) gives $g_{0_{(i)}2_{(i)}} = g_{2_{(i)}3_{(i)}0_{(i)}} + \varrho - \varrho_{\hat{3}_{(i)}} = [i, 1_{(i)}] + \varrho - \varrho_{\hat{3}_{(i)}}$ and formula (2_s) gives $g_{0_{(i)}1_{(i)}2_{(i)}} = [i, 3_{(i)}] = 1 + \varrho - \varrho_{\hat{3}_{(i)}} - \varrho_{\hat{i}}$.

In both cases, an easy computation proves the statement. □

We are now able to state the technical Lemma on which many results of the present paper base themself.

Lemma 3. *Let (Γ, γ) be a crystallization of M^4 and ε a permutation of Δ_4. If there exists a colour $i \in \Delta_4$ such that $\varrho_{\hat{i}} = 0$, then we have:*

(a) if M^4 is closed, then $\quad M^4 \cong \#_\alpha(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)$, where
$\alpha = \varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}} = \varrho - \varrho_{\hat{1}_{(i)}} - \varrho_{\hat{3}_{(i)}}$;

(b) if M^4 has connected boundary, then $\quad M^4 \cong \#_\beta(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1) \# \overset{\sim}{\mathbb{Y}}_\eta$, where
$\beta = \varrho - {}^\partial\varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}}$ *and* $\eta = {}^\partial\varrho + \sum_{n=0}^{3}(-1)^n \varrho_{\hat{n}_{(i)}}$.

Proof. Let M^4 be either closed or with connected boundary, and let $K = K(\Gamma)$ be the pseudocomplex associated to (Γ, γ); we will classify M^4 by making use of the above described decomposition $M^4 = N(i, 1_{(i)}, 3_{(i)}) \cup N(0_{(i)}, 2_{(i)})$, where $i \in \Delta_4$ is such that $\varrho_{\hat{i}} = 0$. At first, we note that Lemma 2 - together with the hypothesis $\varrho_{\hat{i}} = 0$ - ensures that the pseudocomplex $K(i, 1_{(i)}, 3_{(i)})$ contains exactly $[i, 1_{(i)}, 3_{(i)}] = [i, 1_{(i)}] + [i, 3_{(i)}] - 1$ triangles.

Thus, into $K(i, 1_{(i)})$ there are m edges $e_1, \dots, e_m$, $0 \leq m \leq [i, 3_{(i)}] - 1$, which are face of at least two triangles of $K(i, 1_{(i)}, 3_{(i)})$.

If $m = 0$, then we have necessarily $[i, 3_{(i)}] = 1$, and $K(i, 1_{(i)}, 3_{(i)})$ collapses to $K(1_{(i)}, 3_{(i)})$.

If $m \geq 1$, then $K(i, 1_{(i)}, 3_{(i)})$ collapses to $K' = K(i, 3_{(i)}) \cup K(1_{(i)}, 3_{(i)}) \cup \{e_1, \dots, e_m\} \cup \{T_1, \dots, T_l\}$, with $l = m + [i, 3_{(i)}] - 1$; thus, [Ca$_1$; Lemma 5] ensures that K' collapses to the graph $K(1_{(i)}, 3_{(i)})$, too.

In both cases, the number of edges of the 1-dimensional pseudocomplex $K(1_{(i)}, 3_{(i)})$ may be computed by means of formula (2_j), for a suitable choice of the index $j \in \Delta_4$: $[1_{(i)}, 3_{(i)}] = 1 + \varrho - \varrho_{\hat{1}_{(i)}} - \varrho_{\hat{3}_{(i)}}$. This obviously implies that $N(i, 1_{(i)}, 3_{(i)}) \cong \overset{\sim}{\mathbb{Y}}_\alpha$, where $\alpha = \varrho - \varrho_{\hat{1}_{(i)}} - \varrho_{\hat{3}_{(i)}}$.

Now, as far as the structure of the graph $K(0_{(i)}, 2_{(i)})$ is concerned, it is useful to remember the existence of a colour $r \in \Delta_4 - \{i\}$ such that $\{0_{(i)}, 2_{(i)}\} = \{r-1, r+1\}$ and $4 \notin \{r-1, r+1\}$ (see the proof of Lemma 2); further, in order to analize the decomposition $M^4 = N(i, 1_{(i)}, 3_{(i)}) \cup N(0_{(i)}, 2_{(i)})$, the boundary case has to be distinguished from the closed one.

(a) Let us assume that M^4 is closed.

Formula (2_r) yields $[0_{(i)}, 2_{(i)}] = \varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}} + 1$, from which $N(0_{(i)}, 2_{(i)}) \cong \overset{\sim}{\mathbb{Y}}_{\alpha'}$ with $\alpha' = \varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}}$ directly follows. Moreover, $\partial M^4 = \emptyset$ implies $\partial N(i, 1_{(i)}, 3_{(i)}) \cong \partial N(0_{(i)}, 2_{(i)})$; thus, $\alpha = \alpha'$ follows, and by Theorem I$'$(a) the manifold M^4 results to be homeomorphic to $\#_\alpha(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)$.

Note also that in this case we have $\varrho_{\hat{0}_{(i)}} + \varrho_{\hat{2}_{(i)}} = \varrho_{\hat{1}_{(i)}} + \varrho_{\hat{3}_{(i)}}$.

(b) Let us now assume that M^4 has connected boundary.

First of all, note that the existence of an edge of $K(0_{(i)}, 2_{(i)})$ which lies into the boundary of K does not affect the topological structure of $N(0_{(i)}, 2_{(i)})$, since its regular neighbourhood is a collar of a 3–ball $\mathbb{D}^3$ embedded into ∂M^4.

Thus, instead of studing $K(0_{(i)}, 2_{(i)})$, we may consider its subcomplex $\dot{K}(0_{(i)}, 2_{(i)})$, which consists of all edges of $K(0_{(i)}, 2_{(i)})$ which are internal in K and of exactly one boundary edge: by formula (3_r) (where $r \in \Delta_4 - \{i\}$ is such that $\{0_{(i)}, 2_{(i)}\} = \{r-1, r+1\}$ and $4 \notin \{r-1, r+1\}$), $\dot{K}(0_{(i)}, 2_{(i)})$ results to contain exactly $\dot{g}_{i,1_{(i)},3_{(i)}} + 1 = \varrho - {}^{\partial}\varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}} + 1$ edges.

Hence, if $\dot{N}(0_{(i)}, 2_{(i)})$ denotes the regular neighbourhood of $\dot{K}(0_{(i)}, 2_{(i)})$ in K, M^4 admits the decomposition $M^4 \cong N(i, 1_{(i)}, 3_{(i)}) \cup \dot{N}(0_{(i)}, 2_{(i)}) = \overset{(\sim)}{\mathbb{Y}}_\alpha \cup \overset{(\sim)}{\mathbb{Y}}_\beta$, where $\alpha = \varrho - \varrho_{\hat{1}_{(i)}} - \varrho_{\hat{3}_{(i)}}$ and $\beta = \varrho - {}^{\partial}\varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}}$; moreover, since $\dot{K}(0_{(i)}, 2_{(i)})$ contains a boundary edge for K, $\overset{(\sim)}{\mathbb{Y}}_\alpha \cap \overset{(\sim)}{\mathbb{Y}}_\beta = \partial \overset{(\sim)}{\mathbb{Y}}_g \cap \partial \overset{(\sim)}{\mathbb{Y}}_\beta = \partial \overset{(\sim)}{\mathbb{Y}}_\beta - int(\mathbb{D}^3)$ follows, and the attaching map is of type $\varphi : \partial \overset{(\sim)}{\mathbb{Y}}_\beta - int(\mathbb{D}^3) \longrightarrow \partial \overset{(\sim)}{\mathbb{Y}}_\alpha$.

The statement (b) is therefore proved by Theorem I$'$(b); in particular, M^4 is homeomorphic to $\#_\beta(\mathbb{S}^3 \underset{(\sim)}{\times} \mathbb{S}^1) \# \overset{(\sim)}{\mathbb{Y}}_{\alpha-\beta}$, and the difference $\alpha - \beta = {}^{\partial}\varrho + \sum_{n=0}^{3} (-1)^n \varrho_{\hat{n}_{(i)}}$ returns the value η of the statement. $\square$

By a suitable choice of the crystallization (Γ, γ), the indexes α, β and η can be simplified as follows.

Proposition 2. *Let M^4 be a connected PL 4–manifold, with empty or connected boundary ∂M^4. Let (Γ, γ) be a crystallization of M^4 and ε a permutation of Δ_4 such that $\varrho = \varrho_\varepsilon(\Gamma) = \mathcal{G}(M^4)$. If there exists a colour $i \in \Delta_4$ so that $\varrho_{\hat{i}} = 0$, then we have:*

(i) $\varrho_{\hat{j}} = 0$ *for all* $j \in \Delta_4$;

(ii) if $\partial M^4 = \emptyset$, *then* $M^4 \cong \#_\varrho(\mathbb{S}^3 \underset{(\sim)}{\times} \mathbb{S}^1)$;

(iii) if $\partial M^4 \neq \emptyset$, *then* $M^4 \cong \#_{\varrho - {}^{\partial}\varrho}(\mathbb{S}^3 \underset{(\sim)}{\times} \mathbb{S}^1) \# \overset{(\sim)}{\mathbb{Y}}_{{}^{\partial}\varrho}$.

Proof. If M^4 is closed, Lemma 3 (case (a)) ensures that $M^4 \cong \#_\alpha(\mathbb{S}^3 \underset{(\sim)}{\times} \mathbb{S}^1)$, where $\alpha = \varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}} = \varrho - \varrho_{\hat{1}_{(i)}} - \varrho_{\hat{3}_{(i)}}$. Moreover, $\varrho = \mathcal{G}(M^4) = \alpha$

implies that $\varrho_{\hat{n}_{(i)}} = 0$ for every $n \in \Delta_3$; this proves both statement (i) (in case $\partial M^4 = \emptyset$) and statement (ii).

On the other hand, if M^4 has connected boundary, Lemma 3 (case (b)) ensures that $M^4 \cong \#_\beta(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\widetilde{\mathbb{Y}}_\eta$, where $\beta = \varrho - {}^\partial\varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}}$ and $\eta = {}^\partial\varrho + \sum_{n=0}^{3}(-1)^n \varrho_{\hat{n}_{(i)}}$. Moreover, since the regular genus is sub-additive with respect to connected sum, the inequality $\varrho = \mathcal{G}(M^4) \leq \beta + \eta = \varrho - \varrho_{\hat{1}_{(i)}} - \varrho_{\hat{3}_{(i)}}$ holds, from which $\varrho_{\hat{1}_{(i)}} = \varrho_{\hat{3}_{(i)}} = 0$ follows. Finally, ${}^\partial\varrho \geq \mathcal{G}(\partial M^4) = \mathcal{G}(\partial\widetilde{\mathbb{Y}}_\eta) = \mathcal{G}(\#_\eta(\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^1)) = \eta$ implies $\varrho_{\hat{0}_{(i)}} = \varrho_{\hat{2}_{(i)}} = 0$; thus the statements are completely proved. □

We are now able to prove both Theorem II and Theorem III stated in the Introduction.

Proof of Theorem II. It is known that, for every crystallization (Γ, γ) of M and for every choice of $\{i,j\} \in \Delta_4$, a presentation $< X; R >$ for the fundamental group $\pi_1(M)$ of M exists, where X is the set of all $(\Delta_4 - \{i,j\})$-residues of (Γ, γ), but one. Thus, for example, formula (2_1) yields:

$$rk(\pi_1(M^4)) \leq \varrho - \varrho_{\hat{2}} - \varrho_{\hat{4}}$$

It is now easy to check that, if the equality $rk(\pi_1(M^4)) = \varrho$ holds, then (Γ, γ) satisfies the hypothesis of Proposition 2, and hence either $M^4 \cong \#_\varrho(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)$ or $M^4 \cong \#_{\varrho - {}^\partial\varrho}(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\widetilde{\mathbb{Y}}_{{}^\partial\varrho}$ follow. □

Now, instead of simply proving Theorem III, we will prove the following result, that summarizes both Theorem III and the characterization of handlebodies exposed in [Ca_1: Prop. 4].

Theorem III′. *Let M^4 be a PL 4–manifold with connected boundary ∂M^4.*

(a) If $\mathcal{G}(M^4) = \varrho = \mathcal{G}(\partial M^4)$, with $\varrho \geq 0$, then $M^4 \cong \widetilde{\mathbb{Y}}_\varrho$;

(b) if $\mathcal{G}(M^4) = \varrho = \mathcal{G}(\partial M^4) + 1$, with $\varrho \geq 1$, then $M^4 \cong (\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)\#\widetilde{\mathbb{Y}}_{\varrho-1}$.

Proof. Let us fix a crystallization (Γ, γ) of M^4 and a permutation ε of Δ_4 such that $\varrho_\varepsilon(\Gamma) = \mathcal{G}(M^4) = \varrho$. By making use of relations (3_1) and (3_2), and of the inequality $\dot{g}_{rst} \geq 0$ (which holds for every $\{r,s,t\} \subset \Delta_4$), we easily obtain:

(4) $\quad \sum_{i=0}^{3} \varrho_{\hat{i}} \leq 2(\varrho - {}^{\partial}\varrho)$

Now, if $0 \leq \varrho - {}^{\partial}\varrho \leq 1$ is assumed, then a colour $i \in \Delta_3$ must exist such that $\varrho_{\hat{i}} = 0$. From Proposition 2, $\varrho_{\hat{j}} = 0$ for all $j \in \Delta_4$ and $M^4 \cong \#_{\varrho - {}^{\partial}\varrho}(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1) \# \widetilde{\mathbb{Y}}_{\partial_\varrho}$ directly follow; hence, both part (a) (in case $\varrho = {}^{\partial}\varrho$) and part (b) (in case $\varrho = {}^{\partial}\varrho + 1$) result to be proved. $\square$

We conclude the paragraph by noting that the techniques involved in the proof of Theorem III might be useful to solve the following open problem:

Is the classification of PL 4-manifolds with fixed (possibly empty) boundary finite to one with respect to regular genus?

5. The case of disconnected boundary.

In the present section, we will extend the main results of the paper to the case of a PL 4-manifold M^4 with $\mathcal{C} \geq 1$ boundary components; since many proofs essentially involve the same arguments as in the case $\mathcal{C} = 1$, we will give only the statements and the fundamental steps of reasoning.

First of all, if $(\Gamma, \gamma) \in G_5$ is a crystallization of M^4 and $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = 4)$ is a cyclic permutation of Δ_4, we set the following notations:

$$\mathcal{C}_j = \begin{cases} \mathcal{C} & \text{if } j \neq 4 \\ 1 & \text{if } j = 4 \end{cases}$$

${}^{\partial}\varrho = \sum_{h=1}^{\mathcal{C}} \varrho_{\varepsilon_{\hat{4}}}(\Lambda_h)$, $\Lambda_1, \Lambda_2, \ldots, \Lambda_{\mathcal{C}}$ being the $\mathcal{C}$ connected components of $\partial\Gamma$;

$\varrho_{\hat{j}} = \sum_{h=1}^{\mathcal{C}_j} \varrho_{\varepsilon_{\hat{j}}}(H_h^{(j)})$, $H_1^{(j)}, H_2^{(j)}, \ldots, H_{\mathcal{C}_j}^{(j)}$ being the $\mathcal{C}_j$ connected components of $\Gamma_{\hat{\varepsilon}_j}$ (see [Ga$_2$] for details about crystallizations of manifolds with disconnected boundary).

Now, it is easy to check that relations (1_j), (2_j), (3_j) and (4) of section four admit the following generalizations (where, as usual, the letter j often denotes the colour ε_j, and the indexes are considered in $\mathbb{Z}_5$):

$$(1'_j) \quad \begin{cases} g_{j-1,j+1} = g_{j-1,j,j+1} + \varrho - \varrho_{\hat{j}} + \mathcal{C}_j - 1 & \text{if } 4 \notin \{j-1, j+1\} \\ g_{j-1,j+1} = g_{j-1,j,j+1} + \varrho - \varrho_{\hat{j}} + \bar{g}_{124} - 1 & \text{if } 4 \in \{j-1, j+1\} \end{cases}$$

$$(2'_j) \quad \begin{cases} g_{j,j+2,j+3} = \varrho - \varrho_{\widehat{j-1}} - \varrho_{\widehat{j+1}} + \mathcal{C}_{j-1} + \mathcal{C}_{j+1} - 1 & \text{if } j \neq 4 \\ g_{124} = \varrho - {}^{\partial}\varrho - \varrho_{\hat{0}} - \varrho_{\hat{3}} + \bar{g}_{124} + \mathcal{C} - 1 & \end{cases}$$

$$(3'_j) \quad \begin{cases} g_{j,j+2,j+3} = \varrho - {}^{\partial}\varrho - \varrho_{\widehat{j-1}} - \varrho_{\widehat{j+1}} + \mathcal{C} - 1 & \text{if } 4 \notin \{j-1, j+1\} \\ g_{j,j+2,j+3} = \varrho - \varrho_{\widehat{j-1}} - \varrho_{\widehat{j+1}} + \mathcal{C} & \text{if } 4 \in \{j-1, j+1\} \end{cases}$$

$$(4') \quad \sum_{i=0}^{3} \varrho_{\hat{i}} \le 2(\varrho - {}^{\partial}\varrho)$$

Moreover, for every fixed colour $i \in \Delta_4$, M^4 still admits the decomposition $M^4 = N(i, 1_{(i)}, 3_{(i)}) \cup N(0_{(i)}, 2_{(i)})$ (with the same notations used in section four, except from the fact that the h-dimensional subcomplex $K(i_0, \dots, i_h)$ of K is generated by all the vertices labelled by colours of the set $\{i_0, \dots, i_h\} \subset \Delta_4$).

As far as the structure of $K(i, 1_{(i)}, 3_{(i)})$ is concerned, it is not difficult to prove the following generalization of Lemma 2:

Lemma 4. *Let (Γ, γ) be a crystallization of a 4-manifold M^4 (with $\mathcal{C} \ge 1$ boundary components) and let ε be a permutation of Δ_4. For every colour $i \in \Delta_4$, the 2-pseudocomplex $K(i, 1_{(i)}, 3_{(i)})$ contains exactly $[i, 1_{(i)}, 3_{(i)}] = [i, 1_{(i)}] + [i, 3_{(i)}] + \varrho_{\hat{i}} - \mathcal{C}_i$ triangles.* □

In case the existence of a colour $i \in \Delta_4$ such that $\varrho_{\hat{i}} = 0$ is assumed, the structure of $K(i, 1_{(i)}, 3_{(i)})$ may be analized by means of the following Lemma, which is nothing but an extension of [Ca$_1$; Lemma 5].

Lemma 5. *Let H be a two-dimensional labelled pseudocomplex, satisfying the following properties:*

i) H contains $2\mathcal{C}+1$ vertices (with $\mathcal{C} \ge 1$), and exactly $\mathcal{C}$ of them are labelled by colour a, 1 is labelled by colour b, and $\mathcal{C}$ are labelled by colour c;

ii) $1 \le [a,c] \le [a,b] - \mathcal{C}$;

iii) $[a,b,c] = [a,b] + [a,c] - \mathcal{C}$;

iv) the inclusion $j : H(b,c) \to H$ induces an epimorphism $j_* : \pi_1(H(b,c)) \to \pi_1(H)$.

Then, H collapses to $H(b,c)$. □

Lemma 5 ensures that the hypothesis $\varrho_{\hat{i}} = 0$ yields $N(i, 1_{(i)}, 3_{(i)}) \cong \tilde{\mathbb{Y}}_{\varrho - \varrho_{\hat{1}_{(i)}} - \varrho_{\hat{3}_{(i)}}}$.

On the other hand, it is easy to check that $M^4 \cong N(i, 1_{(i)}, 3_{(i)}) \cup \dot{N}(0_{(i)}, 2_{(i)})$, where $\dot{N}(0_{(i)}, 2_{(i)})$ is the regular neighbourhood of the subcomplex of $K(0_{(i)}, 2_{(i)})$, which consists of all edges of $K(0_{(i)}, 2_{(i)})$ which are internal in K and of exactly one boundary edge for each boundary component of K. Thus, M^4 results to be decomposed in the form $\tilde{\mathbb{Y}}_\alpha \cup \tilde{\mathbb{Y}}_\beta$, where

$\alpha = \varrho - \varrho_{\hat{1}_{(i)}} - \varrho_{\hat{3}_{(i)}}$, $\beta = \varrho - {}^{\partial}\varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}}$ and $\widetilde{\mathbb{Y}}_\alpha \cap \widetilde{\mathbb{Y}}_\beta = \partial \widetilde{\mathbb{Y}}_g \cap \partial \widetilde{\mathbb{Y}}_\beta = \partial \widetilde{\mathbb{Y}}_\beta - \cup_{j=1}^{\mathcal{C}}(int(\mathbb{D}_j^3))$, $\mathbb{D}_1^3, \ldots, \mathbb{D}_{\mathcal{C}}^3$ being $\mathcal{C}$ disjoint disks embedded in $\partial \widetilde{\mathbb{Y}}_\beta$.

Finally, a trivial extension of Lemma 1 allows to prove the following generalization of Lemma 3:

Lemma 6. *Let (Γ, γ) be a crystallization of M^4 (with $\mathcal{C} \geq 1$ boundary components) and ε a permutation of Δ_4. If there exists a colour $i \in \Delta_4$ such that $\varrho_{\hat{i}} = 0$, then*

$$M^4 \cong \#_\beta(\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1) \# \widetilde{\mathbb{Y}}_{\eta_1} \# \widetilde{\mathbb{Y}}_{\eta_2} \# \ldots \# \widetilde{\mathbb{Y}}_{\eta_{\mathcal{C}}}$$

where $\beta = \varrho - {}^{\partial}\varrho - \varrho_{\hat{0}_{(i)}} - \varrho_{\hat{2}_{(i)}}$, $\eta_j \geq 0$ for every $j = 1, \ldots, \mathcal{C}$, and $\sum_{j=1}^{\mathcal{C}} \eta_j = {}^{\partial}\varrho + \sum_{n=0}^{3}(-1)^n \varrho_{\hat{n}_{(i)}}$. □

We are now able to state the final results of the section, which are respectively the generalization of Proposition 2, Theorem II and Theorem III′.

Proposition 3. *Let M^4 be a connected PL 4–manifold, with $\mathcal{C} \geq 1$ boundary components. Let (Γ, γ) be a crystallization of M^4 and ε a permutation of Δ_4 such that $\varrho = \varrho_\varepsilon(\Gamma) = \mathcal{G}(M^4)$. If there exists a colour $i \in \Delta_4$ so that $\varrho_{\hat{i}} = 0$, then:*

(i) $\varrho_{\hat{j}} = 0$ for all $j \in \Delta_4$;

(ii) $M^4 \cong \#_{\varrho - {}^{\partial}\varrho}(\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1) \# \widetilde{\mathbb{Y}}_{\eta_1} \# \widetilde{\mathbb{Y}}_{\eta_2} \# \ldots \# \widetilde{\mathbb{Y}}_{\eta_{\mathcal{C}}}$, where $\eta_j \geq 0$ for every $j = 1, \ldots, \mathcal{C}$, and $\sum_{j=1}^{\mathcal{C}} \eta_j = {}^{\partial}\varrho$. □

Proposition 4. *Let M^4 be a connected PL 4–manifold, with $\mathcal{C} \geq 1$ boundary components. Then:*

$$\mathcal{G}(M^4) = rk(\pi_1(M^4)) = m \iff M^4 \cong \#_p(\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1) \# \widetilde{\mathbb{Y}}_{\eta_1} \# \widetilde{\mathbb{Y}}_{\eta_2} \# \ldots \# \widetilde{\mathbb{Y}}_{\eta_{\mathcal{C}}}$$

where $\eta_j \geq 0$ for every $j = 1, \ldots, \mathcal{C}$, and $\sum_{j=1}^{\mathcal{C}} \eta_j = m - p$. □

Proposition 5. *Let M^4 be a connected PL 4–manifold, with $\mathcal{C} \geq 1$ boundary components.*

(a) If $\mathcal{G}(M^4) = \varrho = \mathcal{G}(\partial M^4)$, with $\varrho \geq 0$, then $M^4 \cong \widetilde{\mathbb{Y}}_{\eta_1} \# \widetilde{\mathbb{Y}}_{\eta_2} \# \ldots \# \widetilde{\mathbb{Y}}_{\eta_{\mathcal{C}}}$, where $\eta_j \geq 0$ for every $j = 1, \ldots, \mathcal{C}$, and $\sum_{j=1}^{\mathcal{C}} \eta_j = \varrho$;

(b) *if* $\mathcal{G}(M^4) = \varrho = \mathcal{G}(\partial M^4) + 1$, *with* $\varrho \geq 1$, *then* $M^4 \cong (\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1) \# \widetilde{\mathbb{Y}}_{\eta_1} \# \widetilde{\mathbb{Y}}_{\eta_2} \# \dots \# \widetilde{\mathbb{Y}}_{\eta_{\mathcal{C}}}$, *where* $\eta_j \geq 0$ *for every* $j = 1, \dots, \mathcal{C}$, *and* $\sum_{j=1}^{\mathcal{C}} \eta_j = \varrho - 1$. □

6. Classification of 4–manifolds with low genus.

In this section we will complete the classification of PL 4–manifolds (without assumptions about their boundary) up to regular genus 2. For, we will make use both of our results of sections four and five, and of the following already known results:

I) in arbitrary dimension $n \geq 2$, $\mathcal{G}(M^n) = 0$ is proved to characterize $\mathbb{S}^n$ and $\#_{\mathcal{C}}\mathbb{D}^n$ (see [FG$_1$] and [Ga$_3$]);

II) for every $3 \leq n \leq 5$, if ∂M^n is assumed to be connected, $\mathcal{G}(M^n) = \mathcal{G}(\partial M^n) = \varrho$ is proved to characterize the handlebody $M^n \cong \widetilde{\mathbb{Y}}^n_{\varrho}$ (see [Ca$_1$] and [Ca$_2$]);

III) if $\partial M^4 = \emptyset$ and $\mathcal{G}(M^4) = 1$, then $M^4 \cong \mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1$ (see [Cv$_1$]);

IV) if $\partial M^4 = \emptyset$ and $\mathcal{G}(M^4) = 2$, then either $M^4 \cong \#_2(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)$ or $M^4 \cong \mathbb{CP}^2$ (see [Cv$_2$]).

Let now M^4 be a 4–manifold with (possibly disconnected) boundary; let (Γ, γ) be a crystallization of M^4 and $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = 4)$ a permutation of Δ_4.

For each colour $c \in \Delta_3$ we may construct a new coloured graph $({}^c\Gamma, {}^c\gamma)$ — which is said to be the *c–sewing* of (Γ, γ), according with the notations of [FG$_2$] — by closing all $\{c, 4\}$–boundary residues of Γ by means of an edge of colour 4. As proved in [FG$_2$], $({}^c\Gamma, {}^c\gamma)$ represents the closed pseudomanifold $\widehat{M}$ obtained by capping off each boundary component of M^4 with a cone over it; moreover, if c is consecutive to 4 in ε (i.e. if either $c = \varepsilon_0$ or $c = \varepsilon_3$), then $\varrho = \varrho_\varepsilon(\Gamma)$ equals ${}^c\varrho = \varrho_\varepsilon({}^c\Gamma)$. In a similar way, for each colour $i \in \Delta_4$, one can prove the following relations among the regular genera $\varrho_{\hat{i}} = \varrho_{\varepsilon_{\hat{i}}}(\Gamma_{\hat{i}})$, ${}^\partial\varrho = \varrho_{\varepsilon_{\hat{4}}}(\partial\Gamma)$ and ${}^c\varrho_{\hat{i}} = \varrho_{\varepsilon_{\hat{i}}}({}^c\Gamma_{\hat{i}})$:

$$(5) \quad \begin{cases} {}^c\varrho_{\hat{i}} = \varrho_{\hat{i}} & \forall i \in \Delta_4 - \{c\} \\ {}^c\varrho_{\hat{c}} = \varrho_{\hat{c}} + {}^\partial\varrho & \end{cases}$$

We are now able to classify every PL 4–manifold with genus two.

Proposition 6. *Let M^4 be a connected PL 4–manifold of genus $\mathcal{G}(M^4) = 2$, with $\mathcal{C} \geq 1$ boundary components.*
(a) If $\mathcal{G}(\partial M^4) = 0$, then either $M^4 \cong \mathbb{CP}^2 \#(\#_{\mathcal{C}} \mathbb{D}^4)$ or $M^4 \cong \#_2(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1) \#(\#_{\mathcal{C}} \mathbb{D}^4)$.
(b) If $\mathcal{G}(\partial M^4) = 1$, then $M^4 \cong (\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1) \# \overset{\sim}{\mathbb{Y}}_1 \#(\#_{\mathcal{C}-1} \mathbb{D}^4)$.
(c) If $\mathcal{G}(\partial M^4) = 2$, then either $M^4 \cong \overset{\sim}{\mathbb{Y}}_2 \#(\#_{\mathcal{C}-1} \mathbb{D}^4)$ or $M^4 \cong \overset{\sim}{\mathbb{Y}}_1 \# \overset{\sim}{\mathbb{Y}}_1 \#(\#_{\mathcal{C}-2} \mathbb{D}^4)$.

Proof. Cases (b) and (c) are direct consequences of Proposition 5, parts (b) and (a) respectively.

Let us now examine case (a).
Since $\mathcal{G}(\partial M^4) = 0$, each connected component of ∂M^4 is PL-homeomorphic to $\mathbb{S}^3$ and $M^4 = \widehat{M} \#(\#_{\mathcal{C}} \mathbb{D}^4)$ obviously holds; moreover, [FG$_2$; Prop. 1] ensures that $\mathcal{G}(\widehat{M}) = \mathcal{G}(M^4) = 2$.

Now, let (Γ, γ) be a crystallization of M^4 and ε a permutation of Δ_4 so that $\mathcal{G}(M^4) = 2 = \varrho$. Note that, if $^{\partial}\varrho \neq 0$, relation (4′) and Proposition 3 give $\mathcal{G}(\partial M^4) = \mathcal{G}(\partial(\overset{\sim}{\mathbb{Y}}_{\eta_1} \# \overset{\sim}{\mathbb{Y}}_{\eta_2} \# \ldots \# \overset{\sim}{\mathbb{Y}}_{\eta_{\mathcal{C}}})) = {}^{\partial}\varrho \neq 0$, against the hypothesis; thus, $^{\partial}\varrho = 0$ holds.

Moreover, as far as the crystallization $(^0\Gamma, ^0\gamma)$ of $\widehat{M}$ is concerned, the following two cases may arise (see [Cv$_2$]):

- $^0\varrho_{\hat{i}} = 0$ for all $i \in \Delta_4$, and $\widehat{M} \cong \#_2(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1)$;
- $^0\varrho_{\hat{i}} = 1$ for all $i \in \Delta_4$, and $\widehat{M} \cong \mathbb{CP}^2$.

Hence, since relation (5) gives $^0\varrho_{\hat{i}} = \varrho_{\hat{i}}$ for each $i \in \Delta_4$, the following two possibilities occur:

- $\varrho_{\hat{i}} = 0$ for all $i \in \Delta_4$, and $M^4 \cong \#_2(\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1) \#(\#_{\mathcal{C}} \mathbb{D}^4)$;
- $\varrho_{\hat{i}} = 1$ for all $i \in \Delta_4$, and $M^4 \cong \mathbb{CP}^2 \#(\#_{\mathcal{C}} \mathbb{D}^4)$. □

Proposition 6 completes the classification of PL 4–manifolds (with $\mathcal{C} \geq 0$ boundary components) up to regular genus two; in particular the only prime (with respect to connected sum) 4-manifolds with genus $g \leq 2$ result to be $\mathbb{S}^4$, $\mathbb{D}^4$, $\mathbb{S}^3 \underset{\sim}{\times} \mathbb{S}^1$, $\overset{\sim}{\mathbb{Y}}_1$, $\overset{\sim}{\mathbb{Y}}_2$ and $\mathbb{CP}^2$. The classification is shortly explained in Tables 1 - 2 - 3.

Table 1.

$\mathcal{G}(M^4) = 0$	
$\partial M^4 = \emptyset$	$\mathbb{S}^4$
$\partial M^4 \neq \emptyset$	$\#_c \mathbb{D}^4$

Table 2.

$\mathcal{G}(M^4) = 1$		
	M^4 orientable	M^4 non-orientable
$\partial M^4 = \emptyset$	$\mathbb{S}^3 \times \mathbb{S}^1$	$\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$
$\mathcal{G}(\partial M^4) = 0$	$(\mathbb{S}^3 \times \mathbb{S}^1)\#(\#_c \mathbb{D}^4)$	$(\mathbb{S}^3 \tilde{\times} \mathbb{S}^1)\#(\#_c \mathbb{D}^4)$
$\mathcal{G}(\partial M^4) = 1$	$\mathbb{Y}_1 \#(\#_{c-1} \mathbb{D}^4)$	$\tilde{\mathbb{Y}}_1 \#(\#_{c-1} \mathbb{D}^4)$

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Table 3.

$\mathcal{G}(M^4) = 2$		
	M^4 orientable	M^4 non-orientable
$\partial M^4 = \emptyset$	$\mathbb{CP}^2$ $\#_2 (\mathbb{S}^3 \times \mathbb{S}^1)$	$\#_2 (\mathbb{S}^3 \tilde{\times} \mathbb{S}^1)$
$\mathcal{G}(\partial M^4) = 0$	$\mathbb{CP}^2 \#(\#_c \mathbb{D}^4)$ $\#_2 (\mathbb{S}^3 \times \mathbb{S}^1) \#(\#_c \mathbb{D}^4)$	$\#_2 (\mathbb{S}^3 \tilde{\times} \mathbb{S}^1) \#(\#_c \mathbb{D}^4)$
$\mathcal{G}(\partial M^4) = 1$	$\mathbb{Y}_1 \# (\mathbb{S}^3 \times \mathbb{S}^1) \#(\#_{c-1} \mathbb{D}^4)$	$\mathbb{Y}_1 \# (\mathbb{S}^3 \tilde{\times} \mathbb{S}^1) \#(\#_{c-1} \mathbb{D}^4)$ $\tilde{\mathbb{Y}}_1 \# (\mathbb{S}^3 \tilde{\times} \mathbb{S}^1) \#(\#_{c-1} \mathbb{D}^4)$
$\mathcal{G}(\partial M^4) = 2$	$\mathbb{Y}_2 \#(\#_{c-1} \mathbb{D}^4)$ $\mathbb{Y}_1 \# \mathbb{Y}_1 \#(\#_{c-2} \mathbb{D}^4)$	$\tilde{\mathbb{Y}}_2 \#(\#_{c-1} \mathbb{D}^4)$ $\tilde{\mathbb{Y}}_1 \# \mathbb{Y}_1 \#(\#_{c-2} \mathbb{D}^4)$ $\tilde{\mathbb{Y}}_1 \# \tilde{\mathbb{Y}}_1 \#(\#_{c-2} \mathbb{D}^4)$

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Maria Rita CASALI - Luca MALAGOLI
Dipartimento di Matematica Pura ed Applicata
Università di Modena
Via Campi 213 B
I-41100 MODENA (Italy)
e-mail: CASALI@DIPMAT.UNIMO.IT