CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 37, nº 4 (1996), p. 279-293

<http://www.numdam.org/item?id=CTGDC_1996__37_4_279_0>

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Volume XXXVII-4 (1996)

WEAKLY HEREDITARY REGULAR CLOSURE OPERATORS

by Temple H. FAY

Résumé. On étend divers résultats de Clementino relatifs à la catégorie des espaces topologiques et aux catégories abéliennes dans le cas des catégories quasi-additives pointées dans lesquelles les produits fibrés de conoyaux sont des conoyaux. En particulier on détermine quand un opérateur de fermeture régulier est faiblement héréditaire. Ces deux types d'opérateurs de fermeture sont centraux dans la théorie des opérateurs de fermeture. Ceci s'applique à la catégorie des groupes.

1 Introduction.

In the theory of categorical closure operators, regular closure operators hold a special place. Indeed, the foundations of the general theory began with Salbany's work [23] in which, essentially, the notion of regular closure operator was defined for the first time. Of course the entire theory, particularly with the category of topological spaces, **TOP**, in mind, has been developed by a number of authors, most notably by Dikranjan and Giuli [8], [9] and Castellini [3], [4].

Weakly hereditary also hold an important place in the theory. They arise from factorization structures for morphisms and are extremely well behaved with regards to categorical compactness (see [20]). Therefore it is of some interest to know when a regular closure operator is weakly hereditary.

This question has been answered recently, in part, by Clementino [7], who showed that for **TOP**, an extremal-epireflective subcategory $\mathcal{F}$ is a disconnectedness (see [1]) if and only if the regular closure operator induced by $\mathcal{F}$ is weakly hereditary. Clementino also shows that for an epireflective subcategory $\mathcal{F}$ of an abelian category (with suitable completeness), $\mathcal{F}$ is a torsion-free subcategory if and only if the regular closure operator induced by $\mathcal{F}$ is weakly hereditary.

Our interest primarily concerns extending Clementino's result to pointed categories more general than abelian ones. In particular, we are interested in the situation for the category of all groups, **GRP**. We extend the ideas of Clementino to cover pointed quasi-additive categories for which the pullback of a cokernel is a cokernel, and clarify the relationship between disconnectedness and the subcategory $\mathcal{F}$ being closed under formation of extensions.

We begin with some general observations concerning regular closure operators. In Section 3, we sharpen some results of Castellini [5] concerning when $\mathcal{F}$-epimorphisms are surjective. The main Section 4 deals with the generalized torsion theories defined by Cassidy, Hébert, and Kelly [2], and explore conditions that assure the regular closure operator induced by a torsion-free class is weakly hereditary.

1 Preliminaries.

Recall that a category $\mathcal{C}$ is called *regular* if it is finitely complete, finitely cocomplete, enjoys the (strong epi, mono) factorization structure for morphisms, and strong epimorphisms are stable under pullbacks. In a regular category, the strong epimorphisms and regular epimorphisms coincide. A pointed category is called *quasi-additive* when a morphism f is a monomorphism whenever $\ker(f) = 0$. Throughout this paper we shall assume $\mathcal{C}$ to be quasi-additive as well as finitely complete and finitely cocomplete with the property that all pullbacks of cokernels are cokernels. These categories enjoy many properties of additive and abelian categories with cokernels in the place of coequalizers (see [2]). In this case, every morphism f factorizes as $f = me$ with m monomorphic and e a cokernel; all strong epimorphisms are cokernels; and composites of cokernels are cokernels. If $f : X \to Y$ is a morphism and $f = me$ is the (cokernel, mono)-factorization of f, then we denote the codomain of e (domain of m) by $f(X)$.

If A is a subobject of an object G, then we write $A \leq G$. A *closure operator* on $\mathcal{C}$ assigns to each subobject A of G, a subobject $c^G(A)$ of G such that for each pair of subobjects A and B of G and each morphism

$f : G \to H$, the following hold:

(i) $A \leq c^G(A)$;

(ii) $A \leq B$ implies $c^G(A) \leq c^G(B)$;

(iii) $f(c^G(A)) \leq c^H(f(A))$ (continuity condition).

A closure operator $c(-)$ is called *idempotent* provided $c^G(c^G(A)) = c^G(A)$ for each subobject A of G; a *weakly hereditary* closure operator satisfies $c^{c^G(A)}(A) = c^G(A)$. The general theory of closure operators has been developed by several authors, see [8], [3] for example.

Let $\mathcal{E}$ and $\mathcal{M}$ be isomorphism closed classes of morphisms. The pair $(\mathcal{E}, \mathcal{M})$ is called a *factorization structure* provided:

(i) for every commutative square $mf = ge$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique morphism d so that $de = f$ and $md = g$ (the $(\mathcal{E}, \mathcal{M})$-diagonalization property);

(ii) every morphism f can be factored as $f = me$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ (the $(\mathcal{E}, \mathcal{M})$-factorization property).

This idea encapsulates and generalizes the usual (surjective, injective) factorization for group homomorphisms.

For a closure operator $c(-)$, we let $\mathcal{E}_c$ be the class of all *c-dense* morphisms (the c-closure of the image equals the codomain) and we let $\mathcal{M}_c$ be the class of all *c-closed embeddings* (monomorphisms with c-closed image). Then the pair $(\mathcal{E}_c, \mathcal{M}_c)$ determines a factorization structure if and only if $c(-)$ is weakly hereditary and idempotent, see [8], [9].

Let $\mathcal{F}$ be any class of objects containing the zero object 0. We call a subobject A of an object G an *$\mathcal{F}$-regular subobject* of G provided there exist an $F \in \mathcal{F}$ and a pair $f, g : G \to F$ so that A is the equalizer of f and g. For categories algebraic over the base category SET in the sense of [19], this simply means $A = \{x \in G \mid f(x) = g(x)\}$.

For an arbitrary subobject A of an object G, we define $c^G_{\mathcal{F}}(A)$ to be the intersection of all $\mathcal{F}$-regular subobjects of G which contain A. Naturally, we say that $c_{\mathcal{F}}(-)$ is the *regular closure operator induced by the class* $\mathcal{F}$. Salbany [23] was the first to introduce such a closure operator for the category of topological spaces; further and more general developments can be found in [4] and [8]. Since $\mathcal{C}$ has coequalizers, if $\widehat{\mathcal{F}}$ denotes the quotient-reflective hull of the class $\mathcal{F}$, then $c_{\widehat{\mathcal{F}}}(-) = c_{\mathcal{F}}(-)$. Thus there is no loss of generality in assuming $\mathcal{F}$ to be closed under products and subobjects when considering a regular closure operator. In this case, $c^G_{\mathcal{F}}(A)$ is $\mathcal{F}$-regular.

Let $\mathcal{F}$ be a quotient reflective class and let θG be the normal subobject of G for which $RG = G/\theta G$ is the reflection of G into $\mathcal{F}$. We let $r_G : G \to RG$ denote the canonical cokernel reflection map.

Proposition 2.1. *If A is an $\mathcal{F}$-regular subobject of G, then $\theta G \leq A$. Moreover, if $\theta G \leq A$, then A is $\mathcal{F}$-regular in G if and only if $A/\theta G$ is $\mathcal{F}$-regular in $G/\theta G$. More generally, for an arbitrary object G and $\mathcal{F}$-regular B in RG, the inverse image $A = r_G^{\leftarrow}(B)$ is $\mathcal{F}$-regular in G.*

The following result is adapted from [5], see also [10].

Lemma 2.2. The Fundamental Lemma for Regular Closure Operators. *If $\mathcal{F}$ is a quotient-reflective subcategory of $\mathcal{C}$ and $A \leq G$, then $r_G^{\leftarrow}(c_F^{RG}(r_g(A))) = c_{\mathcal{F}}^{G}(A)$.*

Proof. Let $Y = c_F^{RG}(r_g(A))$ and let $X = r_G^{\leftarrow}(Y)$. By the previous proposition, X is $\mathcal{F}$-regular in G and since $A \leq X$, we have $c_{\mathcal{F}}^{G}(A) \leq X$. On the other hand, we have $c_{\mathcal{F}}^{G}(A)$ is $\mathcal{F}$-regular, so there exist an $H \in \mathcal{F}$ and $j, k : G \to H$ with $c_{\mathcal{F}}^{G}(A)$ the equalizer of j and k. Since $H \in \mathcal{F}$, there are maps $\widehat{j}, \widehat{k} : RG \to H$ with $\widehat{j} r_G = j$ and $\widehat{k} r_G = k$. Clearly, Y is contained in the equalizer of $\widehat{j}$ and $\widehat{k}$, and thus it follows that $X \leq c_{\mathcal{F}}^{G}(A)$.||

Categorical compactness, first introduced by Manes [21], was developed with respect to an $(\mathcal{E},\mathcal{M})$-factorization structure for morphisms by Herrlich, Salicrup, and Strecker [20]. Monomorphisms belonging to the class $\mathcal{M}$ are called $\mathcal{M}$-closed embeddings and an object G was called *$\mathcal{M}$-compact* provided for every object H and every $\mathcal{M}$-closed embedding $A \to G \times H$, the surjective image $\pi_2(A)$ in H has an $\mathcal{M}$-closed embedding. This definition clearly mimics the well known Kuratowski-Mrówka Theorem which characterizes compact topological spaces. Fay [11] characterizes $\mathcal{M}$-compact modules with respect to the standard factorization structure for morphisms arising from a hereditary torsion theory in terms of relative injectivity. Castellini [3] developed the notion of categorical compactness relative to a regular closure operator. Subsequently, Dikranjan and Giuli, building upon their work on categorical closure operators [8], thoroughly investigated compactness in the module case in [9].

Accordingly, we call an object G *compact with respect to the closure operator* $c(-)$, or more simply *c-compact*, provided for each object H, $\pi_2(A)$

is a c-closed subobject of H whenever A is a c-closed subobject of $G \times H$. If the closure operator $c(-)$ is the regular closure operator induced by the class $\mathcal{F}$, then we call G *$\mathcal{F}$-regular compact* instead of $c_{\mathcal{F}}$-compact. There are two results which significantly ease the testing for $\mathcal{F}$-regular compactness.

Theorem 2.3. [18] *Let G be an arbitrary object. Then the following are equivalent:*
(i) G is $\mathcal{F}$-regular compact;
(ii) $G/\theta G$ is $\mathcal{F}$-regular compact;
(iii) G/C is $\mathcal{F}$-regular compact for every $\mathcal{F}$-regular subobject C of G.

Theorem 2.4. [18] *An object G belonging to $\mathcal{F}$ is $\mathcal{F}$-regular compact if and only if for each object $H \in \mathcal{F}$, $\pi_2 : G \times H \to H$ maps $\mathcal{F}$-regular subobjects onto $\mathcal{F}$-regular subobjects.*

2 $\mathcal{F}$-Epimorphisms.

In this section, we sharpen and extend to a wider categorical setting some results of Castellini [5] concerning the class of $\mathcal{F}$-regular compact objects which belong to the class $\mathcal{F}$, $Comp(\mathcal{F}) \cap \mathcal{F}$, and the question of when $\mathcal{F}$-epimorphisms are surjective. Let $\mathcal{F}$ denote a quotient-reflective subcategory and let $c_{\mathcal{F}}(-)$ and $c_{\mathcal{F}}^{\infty}(-)$ denote the regular closure operator and its weakly hereditary core respectively.

Remark 3.1. *The classes of $c_{\mathcal{F}}^{\infty}$-dense maps and $c_{\mathcal{F}}$-dense maps coincide.*

The following is a sharpening of Theorem 3.2 of [5].

Theorem 3.2. *The following are equivalent:*
(i) The $\mathcal{F}$-epimorphisms are cokernels;
(ii) $c_{\mathcal{F}}^{\infty}(-)$ acts discretely on $\mathcal{F}$;
These conditions hold when every object G belonging to $\mathcal{F}$ is $\mathcal{F}$-regular compact.

Proof. If the $\mathcal{F}$-epimorphisms are cokernels, then for $G \in \mathcal{F}$ and $A \leq G$, the map $A \to c_{\mathcal{F}}^{\infty}(A)$ is an $\mathcal{F}$-epimorphism, hence an isomorphism. Conversely, if $e : A \to G$ is an $\mathcal{F}$-epimorphism, then so is the map $e(A) \to G$. Since $e(A)$ and G belong to $\mathcal{F}$, $c^{\infty}_{\mathcal{F}}(e(A)) = e(A)$ by hypothesis. But $e(A) \to c^{\infty}_{\mathcal{F}}(e(A))$ is an $\mathcal{F}$-epimorphism, so $c_{\mathcal{F}}(e(A)) = G$ and thus $c^{\infty}_{\mathcal{F}}(e(A)) = G$. Hence $e(A) = G$ and e is a cokernel.

If every $G \in \mathcal{F}$ is $\mathcal{F}$-regular compact and $e : A \to B$ is an $\mathcal{F}$-epimorphism with A and B belonging to $\mathcal{F}$, then $e(A) \to B$ is an $\mathcal{F}$-epimorphism, hence is $c_{\mathcal{F}}$-dense. Since A is in $\mathcal{F}$, it is $\mathcal{F}$-regular compact, and hence so is $e(A)$. Since B belongs to $\mathcal{F}$, $e(A)$ is $\mathcal{F}$-regular and thus equals B; i.e., $\mathcal{F}$-epimorphisms are cokernels.||

Example 3.3. *(1) In the category of all groups,* ***GRP****, let $\mathcal{F} = \{G \mid G$ is torsion-free$\}$. Then $c_{\mathcal{F}}(-)$ is weakly hereditary and acts discretely on $\mathcal{F}$.*

(2) In the category of abelian groups, ***AB****, let $\mathcal{F} = \{G \mid G$ is torsion-free$\}$. Then $c_{\mathcal{F}}(-)$ is weakly hereditary but fails to act discretely on $\mathcal{F}$ (c.f. [5]).*

The difference between these two examples is explained by the following partial converse to the above theorem.

Proposition 3.4. *If $c_{\mathcal{F}}(-)$ acts discretely on $\mathcal{F}$, then every object G is $\mathcal{F}$-regular compact.*

Proof. It suffices to assume G belongs to $\mathcal{F}$ and to test for compactness on an arbitrary H also from $\mathcal{F}$. Let A be an $\mathcal{F}$-regular subobject of $G \times H$ and observe that $\pi_2(A) = c^H_{\mathcal{F}}(\pi_2(A))$.||

The following proposition is an immediate extension of Proposition 3.5 of [5].

Proposition 3.5. *If C is additive and $\mathcal{F}$ is quotient-reflective, then the epimorphisms in $Comp(\mathcal{F}) \cap \mathcal{F}$ are cokernels.*

3 When $c_{\mathcal{F}}(-)$ is Weakly Hereditary.

In this section we explore the relationship between weakly hereditary regular closure operators and properties of the class $\mathcal{F}$. In doing so, we extend and amplify some recent results of Clementino [7]. To set the stage for this work, we first establish some notational conventions which will be used throughout the sequel.

Again, we let $\mathcal{F}$ denote a quotient-reflective subcategory $\mathcal{C}$ with reflection $r_G : G \to RG = G/\theta G$, where θG is the smallest normal subobject of G for which the quotient $G/\theta G$ belongs to $\mathcal{F}$. To simplify the notation, for each object G, we set $\tau_{\mathcal{F}}G = c^G_{\mathcal{F}}(0)$.

Proposition 4.1. *$\tau_{\mathcal{F}}(-)$ is a radical with $\tau_{\mathcal{F}}G = \theta G$ for each object G. In particular, $\tau_{\mathcal{F}}G$ is fully invariant in G.*

Proof. The continuity condition for $c_{\mathcal{F}}(-)$ shows that $\tau_{\mathcal{F}}(-)$ is a preradical and is fully invariant. Since $\tau_{\mathcal{F}}G = c^G_{\mathcal{F}}(0)$, it is $\mathcal{F}$-regular, and thus there exist an $F \in \mathcal{F}$ and pair of maps $f, g : G \to F$ with $\tau_{\mathcal{F}}G$ the equalizer of f and g. There are induced maps $\hat{f}, \hat{g} : RG \to F$ so that $\hat{f}r_G = f$ and $\hat{g}r_G = g$. Since $r_G(\theta G) = 0$, it follows that $\hat{f}r_G(\theta G) = \hat{g}r_G(G) = 0$, so $\theta G \leq \tau_{\mathcal{F}}G$. But θG contains 0 and is $\mathcal{F}$-regular. Thus $\tau_{\mathcal{F}}G \leq \theta G$ and equality is obtained. This also shows that $\tau_{\mathcal{F}}(-)$ is a radical.||

Since a regular closure operator is always idempotent, we have the following characterization [8].

Proposition 4.2. *The following are equivalent:*

(i) $c_{\mathcal{F}}(-)$ is weakly hereditary;

(ii) $\mathcal{M}_{c_{\mathcal{F}}}$ is closed under composition;

(iii) $(\mathcal{E}_{c_{\mathcal{F}}}, \mathcal{M}_{c_{\mathcal{F}}})$ is a factorization structure for $\mathcal{C}$.

The next definition is adapted from Clementino [7]. We say the *$\mathcal{F}$-reflections are hereditary* (*with respect to the class* $\{0 \to G \mid G \in \mathcal{C}\}$) provided for each $\mathcal{C}$-object G, the map $P \to 0$ in the pullback diagram:

$$\begin{array}{ccc} P & \to & 0 \\ \downarrow & PB & \downarrow \\ G & \to & RG \end{array}$$

is the $\mathcal{F}$-reflection of P. It is clear that in the above pullback diagram, $P = \theta G \times 0$. Thus P is canonically isomorphic to $\theta G = \tau_{\mathcal{F}}G$. To have $P \to 0$ be the $\mathcal{F}$-reflection is the same as saying $\theta P = P$; *i.e.*, $\tau_{\mathcal{F}}P = P$. This is the same as saying $\tau^2_{\mathcal{F}}G = \tau_{\mathcal{F}}G$.

Following the notation and concepts of [2], we denote by $\mathcal{F}^{\leftarrow}$ the class $\{G \in \mathcal{C} \mid Hom_{\mathcal{C}}(G, F) = 0 \text{ for all } F \in \mathcal{F}\}$. Since this class is always a class of torsion objects for a generalized torsion theory, we denote $\mathcal{F}^{\leftarrow}$ by $\mathcal{T}$. We observe that the class $\mathcal{T}^{\rightarrow} = \{G \in \mathcal{C} \mid Hom_{\mathcal{C}}(T,G) = 0 \text{ for all } T \in \mathcal{T}\}$ contains the class $\mathcal{F}$, is closed under formation of products, subobjects, and extensions, and that the pair $(\mathcal{T},\mathcal{T}^{\rightarrow})$ is a generalized torsion theory. See [2] for more details. Note that it is easy to identify the class $\mathcal{T}$, as it is clear that $\mathcal{T} = \{G \in \mathcal{C} \mid \tau_{\mathcal{F}}G = G\}$. The class $\mathcal{T}$ is closed under formation of coproducts, images, and extensions.

Following [1], in **TOP**, a class of spaces $\mathcal{A}$ is a *disconnectedness* if there is a class of topological spaces $\mathcal{D}$ so that a space X belongs to $\mathcal{A}$ exactly when any continuous function from a space in $\mathcal{D}$ to X is constant. Thus $\mathcal{T}^{\rightarrow}$ could be called the disconnectedness determined by the class $\mathcal{T}$. Clementino [7] showed that for **TOP**, a class $\mathcal{A}$ is a disconnectedness exactly when $\mathcal{A}$ induces a weakly hereditary closure operator.

Since $\tau_{\mathcal{F}}^{\infty}(G) \in \mathcal{T}$, $G \in \mathcal{T}^{\rightarrow}$ implies $\tau_{\mathcal{F}}^{\infty}(G) = 0$. On the other hand, if $T \in \mathcal{T}$ and $f : T \to G$, then $f(T) \subset \tau_{\mathcal{F}}^{\infty}(G)$. Thus $\tau_{\mathcal{F}}^{\infty}(G) = 0$ implies $G \in \mathcal{T}^{\rightarrow}$. This shows that $\mathcal{T}^{\rightarrow} = \{G \mid \tau_{\mathcal{F}}^{\infty}(G) = 0\}$, and consequently we have $\mathcal{F} = \mathcal{T}^{\rightarrow}$ if and only if $\tau_{\mathcal{F}}^2 = \tau_{\mathcal{F}}$.

If $A \leq G$ is $\mathcal{F}$-regular, then by the Fundamental Lemma, we have a pullback diagram:

$$\begin{array}{lcccc} RA & \leftarrow & A & \rightarrow & G \\ \downarrow & & \downarrow & PB & \downarrow \\ B & = & c_{\mathcal{F}}^{RG}(r_G(A)) & \rightarrow & RG \end{array}$$

From $B \in \mathcal{F}$, there is a unique map $\rho_A : RA \to B$ with $\rho_A r_A = r : A \to c_{\mathcal{F}}^{RG}(r_G(A))$. Note that this map ρ_A is necessarily a cokernel since r is a cokernel. It follows that ρ_A is monic if and only if $\tau_{\mathcal{F}}A = \tau_{\mathcal{F}}G$. If $\tau_{\mathcal{F}}^2 = \tau_{\mathcal{F}}$ and A is $\mathcal{F}$-regular in G, then $\tau_{\mathcal{F}}G \leq A$. From $\tau_{\mathcal{F}}$ being a preradical, we have $\tau_{\mathcal{F}}^2 G \leq \tau_{\mathcal{F}}A \leq \tau_{\mathcal{F}}G$. Thus the hypothesis implies that ρ_A is a monomorphism. On the other hand, if each ρ_A is monic, then $\rho_{\tau_{\mathcal{F}}G}$ is monic, which forces $\tau_{\mathcal{F}}^2 G = \tau_{\mathcal{F}}G$.

Finally note that if $c_{\mathcal{F}}(-)$ is weakly hereditary, then

$$\tau_{\mathcal{F}}^2 G = c_{\mathcal{F}}^{\tau_{\mathcal{F}}G}(0) = c_{\mathcal{F}}^{c_{\mathcal{F}}^{G}(0)}(0) = c_{\mathcal{F}}^{G}(0) = \tau_{\mathcal{F}}G.$$

Thus we have proved the following theorem.

Theorem 4.3. *The following are equivalent and imply that $\mathcal{F}$ is closed under extensions:*

(i) $\tau_{\mathcal{F}}^2 = \tau_{\mathcal{F}}$;
(ii) $\mathcal{F}$*-reflections are hereditary;*
(iii) $\mathcal{F} = \mathcal{T}^{\rightarrow}$;
(iv) ρ_A *is monic for every* $\mathcal{F}$*-regular* $A \leq G$.
Moreover, if $c_{\mathcal{F}}(-)$ *is weakly hereditary, then these conditions hold.*

We intend to show that for the category of groups, the converse of this last statement holds. But first, more generally, let A be $\mathcal{F}$-regular in B and let B be $\mathcal{F}$-regular in G. Then we have the commutative diagram:

$$\begin{array}{ccccc} A & \rightarrow & B & \rightarrow & G \\ \downarrow & 2 & \downarrow & 1 & \downarrow \\ r_G(A) & \rightarrow & r_G(B) & \rightarrow & r_G(G) \end{array}$$

It follows that $r_G(A)$ is $\mathcal{F}$-regular in RB and $r_G(B)$ is $\mathcal{F}$-regular in RG. In particular, by the Fundamental Lemma, square 1 is a pullback square. Since B is $\mathcal{F}$-regular in G, $\tau_{\mathcal{F}}B \leq \tau_{\mathcal{F}}G \leq B$. Thus $\tau_{\mathcal{F}}G/\tau_{\mathcal{F}}B \leq B/\tau_{\mathcal{F}}B$. If $\mathcal{F} = \mathcal{T}^{\rightarrow}$, then $\tau_{\mathcal{F}}G/\tau_{\mathcal{F}}B \in \mathcal{T} \cap \mathcal{F} = 0$ and thus $\tau_{\mathcal{F}}G = \tau_{\mathcal{F}}B$. In this case then, $RB = r_G(B) = B/\tau_{\mathcal{F}}G = B/\tau_{\mathcal{F}}B$. Hence it follows that $RA = r_G(A) = A/\tau_{\mathcal{F}}A = A/\tau_{\mathcal{F}}B = A/\tau_{\mathcal{F}}G$. This means that square 2 is a pullback square. Thus the outer rectangle is a pullback square. Hence, by the Fundamental Lemma, A is $\mathcal{F}$-regular in G if and only if RA is $\mathcal{F}$-regular in RG. This argument yields a sharpening of Clementino's Proposition 4.2 as follows:

Theorem 4.4. $c_{\mathcal{F}}(-)$ *is weakly hereditary on* $\mathcal{C}$ *if and only if* $c_{\mathcal{F}}(-)$ *is weakly hereditary on* $\mathcal{F}$ *and* $\mathcal{F} = \mathcal{T}^{\rightarrow}$.

We wish to relate the requirement that $\mathcal{F} = \mathcal{T}^{\rightarrow}$ to $\mathcal{F}$ being closed under formation of extensions. In particular, Clementino's Theorem gives us that when $\mathcal{C}$ is an abelian category, $c_{\mathcal{F}}(-)$ is weakly hereditary precisely when $\mathcal{F}$ is closed under extensions. A similar result hold for the category of all groups, **GRP**, but the verification is somewhat more involved than for the abelian situation.

We say that a monomorphism $A \rightarrow B$ and a cokernel $\alpha : A \rightarrow F$ have a *semiextension* provided there exist an H, with $F \leq H$, and a map $\beta : B \rightarrow H$ so that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & F \\ \downarrow & & \downarrow \\ B & \xrightarrow{\beta} & H \end{array}$$

Note that $\alpha(A) = F \leq \beta(B)$, so that without loss of generality, we may assume β to be a cokernel. We call the above semiextension an ***$\mathcal{F}$-semiextension*** provided $F \in \mathcal{F}$ implies H can be chosen to belong to $\mathcal{F}$. We call a monomorphism $A \to B$ ***$\mathcal{F}$-semiextendible*** provided for every cokernel $\alpha : A \to F$, with $F \in \mathcal{F}$, has an ***$\mathcal{F}$-semiextension.***

Proposition 4.5. *If $\tau_{\mathcal{F}}G \to G$ is $\mathcal{F}$-semiextendible for every G, then $\mathcal{F} = \mathcal{T}^{\to}$.*

Proof. The hypothesis implies $\tau_{\mathcal{F}}G/\tau_{\mathcal{F}}^2G = 0.\|$
We say that the category $\mathcal{C}$ enjoys the *(cokernel, kernel)-Pushout Property* or more simply the *Pushout Property*, if in the following pushout square

$$\begin{array}{ccc} A & \xrightarrow{m} & F \\ e\downarrow & PO & \downarrow f \\ B & \xrightarrow{n} & H \end{array}$$

e a cokernel and m a normal monomorphism imply that n is a monomorphism.

Theorem 4.6. If $\mathcal{C}$ enjoys the Pushout Property, then $\mathcal{F}$ closed under extensions implies for every G, $\tau_{\mathcal{F}}G \to G$ is $\mathcal{F}$-semiextendible.

Proof. Consider the following diagram with indicated pushout square:

$$\begin{array}{ccccc} \tau_{\mathcal{F}}G & \to & G & \to & G/\tau_{\mathcal{F}}G \\ \downarrow & PO & \downarrow & & \| \\ \tau_{\mathcal{F}}G/\tau_{\mathcal{F}}^2G & \to & P & \to & Q \end{array}$$

The image of a normal subobject is normal, and it is easy to see that the quotient Q is isomorphic to $G/\tau_{\mathcal{F}}G$. If $\mathcal{F}$ is closed under extensions, then P belongs to $\mathcal{F}.\|$
The next proposition also shows that closure under extensions implies $\mathcal{F} = \mathcal{T}^{\to}$ under common circumstances.

Proposition 4.7. *If fully invariant implies normal, then $\mathcal{F}$ closed under extensions implies $\mathcal{F} = \mathcal{T}^{\to}$.*

Proof. Under this hypothesis, $\tau_{\mathcal{F}}^2 G$ is normal in G, and thus we have the short exact sequence

$$0 \to \tau_{\mathcal{F}} G/\tau_{\mathcal{F}}^2 G \to G/\tau_{\mathcal{F}}^2 G \to G/\tau_{\mathcal{F}} G \to 0$$

and $\mathcal{F}$ closed under extensions implies $G/\tau_{\mathcal{F}}^2 G$ belongs to $\mathcal{F}$. This in turn implies $\tau = \tau_{\mathcal{F}}^2$.|| **Corollary 4.8.** *In the category of groups, **GRP**, the following are equivalent:*

(i) $c_{\mathcal{F}}(-)$ is weakly hereditary;

(ii) $c_{\mathcal{F}}(-)$ is weakly hereditary on $\mathcal{F}$ and $\mathcal{F}$ is closed under extensions.

We improve this result next by showing that the regular closure operator induced by the class $\mathcal{F}$ closed under subgroups, products, and extensions must act discretely on the class $\mathcal{F}$ and hence the closure under extensions suffices to obtain $c_{\mathcal{F}}(-)$ weakly hereditary in the category of groups.

Theorem 4.9. *In the category of groups, **GRP**, $c_{\mathcal{F}}(-)$ is weakly hereditary if and only if $\mathcal{F}$ is closed under extensions.*

Proof. If $c_{\mathcal{F}}(-)$ is weakly hereditary, then $\mathcal{F} = \mathcal{T}^{\to}$ and $\mathcal{F}$ is closed under extensions by Theorem 4.3. Conversely, we first argue that if $\mathcal{F}$ is non-zero, then it contains all free groups. Note that if $\mathcal{F}$ contains a torsion group, then it contains a cyclic group of prime order, $\mathbb{Z}(p)$. Since $\mathcal{F}$ is closed under extensions, it follows that $\mathbb{Z}(p^2)$ belongs to $\mathcal{F}$ and , by induction, $\mathbb{Z}(p^n)$ for all n belong to $\mathcal{F}$. Since $\mathcal{F}$ is closed under products and subgroups, $\mathcal{F}$ contains the mixed group $\Pi_{n=1}^{\infty}\mathbb{Z}(p^n)$. If $\mathcal{F}$ contains either a mixed group or a torsion-free group, then it contains the infinite cyclic group of integers. Thus all free-abelian groups belong to $\mathcal{F}$. If F is a free group, let $F^{(n)}$ denote the n^{th} term of the derived series for F. Then we have the short exact sequence

$$0 \to F^{(n)}/F^{(n+1)} \to F/F^{(n+1)} \to F/F^{(n)} \to 0.$$

First observe that $F/F^{(1)}$ is free-abelian and since subgroups of free are free, we have $F^{(n)}/F^{(n+1)}$ is free-abelian for each n. Since $\mathcal{F}$ is closed under extensions, by induction, $F/F^{(n+1)}$ belongs to $\mathcal{F}$ for each n. Since the intersection of all the $F^{(n)}$'s is the trivial group, we have F embedded in the product of all the $F/F^{(n)}$'s and thus F belongs to $\mathcal{F}$.

Let G be an arbitrary group belonging to $\mathcal{F}$, and let A be any subgroup of G. We let $G \amalg_A G$ denote the free product of G with itself amalgamating the subgroup A. There are two canonical injections μ_1, $\mu_2 : G \to G \amalg_A G$ with $A = \{x \in G \mid \mu_1(x) = \mu_2(x)\}$. The images $\mu_1(G)$, and $\mu_2(G)$ are called the constituents of $G \amalg_A G$. There is also a canonical homomorphism $\lambda : G \amalg_A G \to G$ enjoying the property that $\lambda\mu_i = 1_G$ for $i = 1, 2$. Each element $w \in G\amalg_A G$ has a normal form $w = x_1 y_1 \cdots x_n y_n$ where the x_i's belong to one constituent and the y_i's belong to the other constituent. It follows that $\lambda(w) = x_1 \bullet y_1 \bullet \cdots \bullet x_n \bullet y_n$ where $\bullet$ denotes the multiplication in the group G. Let K denote the kernel of λ and let $w^{-1}gw \in w^{-1}\overline{G}w \cap K$ where $\overline{G}$ denotes either one of the constituents. Then $\lambda(w^{-1}gw) = w^{-1} \bullet g \bullet w = 1$, which implies $g = 1$. By Kurosh's Subgroup Theorem [22, Corollary 4.9.2], this implies K is free. Thus $G \amalg_A G$ is an extension of an $\mathcal{F}$-group by a free group and therefore belongs to $\mathcal{F}$. This means that A is $\mathcal{F}$-regular and hence $c^G_{\mathcal{F}}(A) = A$. So $\mathcal{F}$ being closed under extensions implies $c_{\mathcal{F}}(-)$ is discrete on $\mathcal{F}$, and hence weakly hereditary on $\mathcal{F}$. The result now follows directly from the above Corollary 4.8.||

Example 4.10. *In the category of groups, **GRP**, if $\mathcal{F}$ is the class of all torsion-free groups, then $c_{\mathcal{F}}(-)$ acts discretely on $\mathcal{F}$ [18], hence is weakly hereditary on **GRP**.*

*If $\mathcal{F}$ is the class of all torsion-free abelian groups, then $c_{\mathcal{F}}(-)$ is not discrete but is weakly hereditary on $\mathcal{F}$, and thus in the category of all abelian groups, **AB**, $c_{\mathcal{F}}(-)$ is weakly hereditary. However, in **GRP**, $\mathcal{F}$ is not closed under extensions and $c_{\mathcal{F}}(-)$ is not weakly hereditary.*

*If $\mathcal{F}$ is the class of all R-groups (groups for which $x^n = y^n$ implies $x = y$), $c_{\mathcal{F}}(-)$ coincides with the isolator [18] on $\mathcal{F}$, and thus is weakly hereditary on $\mathcal{F}$ (the isolator is a weakly hereditary closure operator in **GRP**, see [12]), but $c_{\mathcal{F}}(-)$ is not weakly hereditary on* GRP *as $\mathcal{F}$ fails to be closed under extensions.*

The next observation follows from Proposition 3.3.

Corollary 4.11. *In **GRP**, if $\mathcal{F}$ is closed under extensions, then every group is $\mathcal{F}$-regular compact.*

Finally, we close with:

Proposition 4.12. *If $\mathcal{F}$ is closed under extensions, then $\tau_{\mathcal{F}}^{\infty} = \tau_{\mathcal{F}}^{n}$ if and only if $\tau_{\mathcal{F}}^{\infty} = \tau_{\mathcal{F}}$.*

Proof. Consider the exact sequence

$$0 \to \tau_{\mathcal{F}}^{n-1}G/\tau_{\mathcal{F}}^{n}G \to \tau_{\mathcal{F}}^{n-2}G/\tau_{\mathcal{F}}^{n}G \to \tau_{\mathcal{F}}^{n-2}G/\tau_{\mathcal{F}}^{n-1}G \to 0$$

and note that $\mathcal{F}$ being closed under extensions implies that $\tau_{\mathcal{F}}^{n-2}G/\tau_{\mathcal{F}}^{n}G$ belongs to $\mathcal{F}$. Thus from the exact sequence

$$0 \to \tau_{\mathcal{F}}^{n-2}G/\tau_{\mathcal{F}}^{n}G \to \tau_{\mathcal{F}}^{n-3}G/\tau_{\mathcal{F}}^{n}G \to \tau_{\mathcal{F}}^{n-3}G/\tau_{\mathcal{F}}^{n-2}G \to 0$$

we see that $\tau_{\mathcal{F}}^{n-3}G/\tau_{\mathcal{F}}^{n}G$ belongs to $\mathcal{F}$. Proceeding in this manner a finite number of times, we obtain $\tau_{\mathcal{F}}G/\tau_{\mathcal{F}}^{n}G$ belongs to $\mathcal{F}$. This and the exact sequence

$$0 \to \tau_{\mathcal{F}}G/\tau_{\mathcal{F}}^{n}G \to G/\tau_{\mathcal{F}}^{n}G \to G/\tau_{\mathcal{F}}G \to 0$$

show that $G/\tau_{\mathcal{F}}^{n}G$ belongs to $\mathcal{F}$, and hence $\tau_{\mathcal{F}}G \leq \tau_{\mathcal{F}}^{n}G$.||

4 Acknowledgment.

The author wishes to thank Dikran Dikranjan and Gary L. Walls for helpful conversations. Support from the Technikon Pretoria is greatfully acknowledged.

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