CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 37, nº 3 (1996), p. 179-255

<http://www.numdam.org/item?id=CTGDC_1996__37_3_179_0>

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HELLER'S AXIOMS FOR HOMOTOPY THEORY

by Jerome William HOFFMAN

RESUME. Heller a défini une théorie d'homotopie comme étant une sorte d'hyperfoncteur. Les axiomes qu'il a introduits reflètent certaines des propriétés des limites directes et inverses d'homotopie dans le cas des ensembles simpliciaux. Dans cet article on montre comment enrichir les axiomes usuels relatifs à une catégorie modèle de Quillen pour obtenir des théories d'homotopie dans son sens. On donne divers exemples.

1. Introduction

Recently, Alex Heller has given new formulation of homotopy theory that puts emphasis on the functorial properties of the assignment

$$\mathbf{C} \longrightarrow \mathrm{Ho}(\mathcal{S}^{\mathbf{C}})$$

where $\mathbf{C}$ is an arbitrary small category and $\mathcal{S}$ is the category of "spaces" ie., of simplicial sets, and where $\mathcal{S}^{\mathbf{C}}$ denotes the category of $\mathbf{C}$ - diagrams in $\mathcal{S}$. This is a Quillen model category, with corresponding (homotopy) category of fractions denoted $\Pi(\mathbf{C}) = \mathrm{Ho}(\mathcal{S}^{\mathbf{C}})$. Any functor $F : \mathbf{C} \to \mathbf{D}$ induces

$$\Pi(F) = \mathrm{Ho}(F^*) : \Pi(\mathbf{D}) \longrightarrow \Pi(\mathbf{C})$$

whose left and right adjoints are denoted LF and RF, and are known as the left and right homotopy Kan extensions of F. For the special case where $\mathbf{D} = \mathbf{1}$ these are known as ho(co)lim and were studied by a number of authors, among them Vogt [21], Edwards and Hastings [7] and especially Bousfield and Kan [4]. Following ideas of D.W.Anderson [1], Heller [11] has sought to isolate properties of these adjoints that characterize "homotopy theory". Thus, for Heller, a homotopy theory is an assignment (a "hyperfunctor")

$$\mathbf{C} \longrightarrow \mathrm{T}(\mathbf{C})$$

with properties analogous to $\mathbf{C} \longrightarrow \Pi(\mathbf{C})$. These are codified in a small list of axioms. Actually, there are right and left homotopy theories depending on whether one is more concerned with RF or LF. Some notable features of this approach:

1. Π itself is a (right and left) homotopy theory.
2. If T is a left (resp., right) homotopy theory, then T is tensored (resp. cotensored) over Π.
3. There is a general concept of (co)localization that includes many known examples.
4. There is a theory of algebras.

The proof of 1) is nontrivial. The essential point is the construction of the factorizations that are demanded in a Quillen model category, and for that purpose, Heller has introduced an ingenious generalization of the method by which Steenrod built the classifying space of a topological group . Property 2 asserts roughly that there are external pairings

$$K \otimes X \qquad\qquad \mathrm{Hom}(K, X)$$

which induce "derived functors" Π × T → T . In this sense, the usual homotopy theory of spaces is universal among homotopy theories. His concept of localization is so general that it includes eg. not only the arithmetical localizations at primes, but also the functors that truncate homotopy from above or below. Finally, if H is an algebraic theory in the sense of Lawvere [13], then there is a notion of homotopy H–algebra. These generate a homotopy theory Hoalg(H, T). There is a beginning to the theory of multialgebras, which links up via results of G. Segal [18] to the theory of spectra.

The simplicity of the axioms is achieved in part by ignoring the "spaces" that give rise to a homotopy theory (although in a weak sense, the category T(**1**) generates T by the density theorem of [11]) . However, it seems desirable in view of the possible applications of this theory to other domains of mathematics that one have a way of generating homotopy theories from interesting mathematical objects. In other words, one should put the "spaces" back into the picture. This brings us to the viewpoint put forth by Anderson [1]. In Bousfield and Kan's treatise the problem was posed to study homotopy direct and inverse limits in general model categories. [4, p. 301]. In a sense, this paper addresses that issue. Various authors have proposed axioms that allow one to do homotopy in different contexts. Perhaps the simplest one is that of Baues [2], which is a variation of that of Brown [5]. However, counterexamples show that in themselves they are insufficient to support a theory such as Anderson and Heller have envisaged. The same is true of the more elaborate axioms of Quillen. The main point is that these theories demand the existence of limits over arbitrary small (but possibly infinite) categories whereas the usual axioms are silent about these infinite limits. Heller works exclusively with simplicial sets and utilizes tacitly many properties peculiar to them. The aim of this paper is to show that an extension of familiar axioms allows for the construction of homotopy theories in the sense of Anderson and Heller. The new axioms we give are labeled **LCM**∗, **RCM**∗ , and there are several variants. Inevitably, there is a large overlap with the first third of Heller's book. It is necessary to repeat many of the arguments from [11] in order to see just exactly where the axioms are used. Also, Heller's presentation is often very brief. We present in detail a construction of right homotopy theories, whereas Heller has presented the details for the left homotopy structure associated to simplicial sets.

A word about the length of this article. The goal of this paper is to link the ideas of Heller with other areas of mathematics by providing actual examples of these theories. This paper is therefore intended for the nonexpert as well. We have aimed at presenting complete and rigorous proofs of all our results. Unfortunately, many "facts" , including assertions about so familiar a category as that of simplicial sets, have attained the status of being well - known to experts, without there being any published proofs of said facts. This author was forced to reconstruct and/or rediscover some of these himself. Finally, the question addressed here, that of understanding how the homotopy theories of Heller arise, is not definitively answered. Clearly much remains to be done about the very foundations of this subject. One possible application of this theory that is not covered in this work is to that of sheaves. This is especially important in Algebraic Geometry, where some of the biggest open questions (Hodge conjectures, Beilinson conjectures) ultimately tie into the methods of abstract homotopy theory. A recent work dealing with the homotopy theory of sheaves has been published by Crans [6].

We would like to express thanks to Alex Heller for advice on these questions. Thanks are also due to D. W. Anderson for some communications on his contributions to axiomatic homotopy theory. Some of the diagrams in this paper were set up using Michael Barr's TEXutility.

Note. After this paper was written, A. K. Bousfield sent me a copy of an unpublished manuscript of C. L. Reedy, dating from the late 1970's, in which some of my results (Propositions 3.4 and 3.5) had been proved (in more general form - the special axioms **Path** and **Cyl** are not necessary). This means that our axioms **LCM3c**, **RCM3f**, **LCM5**, and **RCM5** are theorems in every closed model category. In view of the importance of this, we present Reedy's theorems in an appendix to this work. Our proof of his proposition 7.4 is somewhat different from that given in [17]. Also, we have retained our proof of 3.4 because a slight modification of it gives a second proof of 7.5. We have kept these axioms because their use clarifies the logic of the main constructions. Special thanks are due to Bousfield for bringing this to my attention.

Notation. These are all standard except that we use ${}_*J$ to denote Lan_J and J_* to denote Ran_J.

2. Model Structures on $\mathcal{M}^{\mathrm{C}}$

2.1. In any category $\mathcal{C}$ we say that two morphisms $u : A \to X$ and $p : E \to B$ are *transverse* and we write $u \mathrel{\vdash\!\!\vdash\!\!-} p$, if given any map $(f, g) : u \to p$, there exists an arrow $h : X \to E$, such that $p \circ h = g$ and $h \circ u = f$. In other words, the slanted arrow exists in the diagram:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & X \\ {\scriptstyle u}\downarrow & \nearrow{\scriptstyle h} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\ g\ ]{} & Y \end{array}$$

We call such an h, if it exists, a *lifting* of (f, g). If $\mathcal{A}$ and $\mathcal{B}$ are classes of arrows in $\mathcal{C}$ the symbol $\mathcal{A} \mathrel{\vdash\!\!\vdash\!\!-} \mathcal{B}$ has the obvious meaning. We define

$$\begin{aligned} \mathcal{A}^{\vdash\!\vdash\!-} &= \{p : \mathcal{A} \mathrel{\vdash\!\!\vdash\!\!-} p\} \\ {}^{\vdash\!\vdash\!-}\mathcal{B} &= \{u : u \mathrel{\vdash\!\!\vdash\!\!-} \mathcal{B}\} \end{aligned}$$

Note that $\mathcal{A} \mathrel{\vdash\!\!\vdash\!\!-} \mathcal{B} \iff \mathcal{A} \subset {}^{\vdash\!\vdash\!-}\mathcal{B} \iff \mathcal{B} \subset \mathcal{A}^{\vdash\!\vdash\!-}$

Lemma 2.1. 1. *Both* $\mathcal{A}^{\vdash\!\vdash\!-}$ *and* ${}^{\vdash\!\vdash\!-}\mathcal{B}$ *are closed under composition and retraction, and contain all isomorphisms.*

2. $\mathcal{A}^{\vdash\!\vdash\!-}$ *is closed under base - change and products.*
3. ${}^{\vdash\!\vdash\!-}\mathcal{B}$ *is closed under cobase - change and coproducts.*

Proof. These are easy exercises. □

Lemma 2.2. 1. *Let*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & X_n & \longrightarrow \cdots \longrightarrow & X_2 & \xrightarrow{x_2} & X_1 & \xrightarrow{x_1} & X_0 \\ & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_0} \\ \cdots \longrightarrow & Y_n & \longrightarrow \cdots \longrightarrow & Y_2 & \xrightarrow{y_2} & Y_1 & \xrightarrow{y_1} & Y_0 \end{array}$$

be a ladder, and assume that for all n,

(a) $p_n \in \mathcal{A}^{\vdash\!\vdash\!-}$.

(b) *The canonical map* $s_n : X_n \to Y_n \underset{Y_{n-1}}{\times} X_{n-1}$ *is in* $\mathcal{A}^{\vdash\!\vdash\!-}$

Then $p := \lim p_n : X = \lim X_n \to Y = \lim Y_n$ *is in* $\mathcal{A}^{\vdash\!\vdash\!-}$

2. *Let*

$$\begin{array}{ccccccccc} X_0 & \xrightarrow{x_1} & X_1 & \xrightarrow{x_2} & X_2 & \longrightarrow \cdots \longrightarrow & X_n & \longrightarrow \cdots \\ \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u_n} & \\ Y_0 & \xrightarrow{y_1} & Y_1 & \xrightarrow{y_2} & Y_2 & \longrightarrow \cdots \longrightarrow & Y_n & \longrightarrow \cdots \end{array}$$

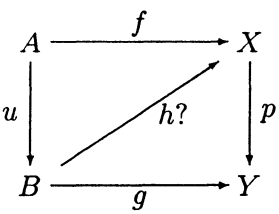
be a ladder, and assume that for all n

(a) $u_n \in {}^{\vdash\!\!+\!\!-}\mathcal{B}$.

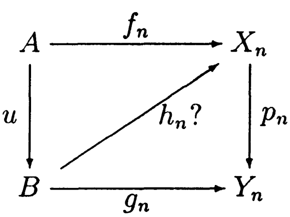
(b) *The canonical map* $t_n \;:\; Y_{n-1} \underset{X_{n-1}}{\vee} X_n \;\to\; X_n$ *is in* ${}^{\vdash\!\!+\!\!-}\mathcal{B}$.

Then $u := \operatorname{colim}\, u_n : X = \operatorname{colim}\, X_n \to Y = \operatorname{colim}\, Y_n$ *is in* ${}^{\vdash\!\!+\!\!-}\mathcal{B}$.

Proof. We prove (a) only since (b) is dual. Let $\xi_n \;:\; X \to X_n$ and $\eta_n : Y \to Y_n$ be the canonical projections. Let



be given with $u \in \mathcal{A}^{\vdash\!\!+\!\!-}$ and with h to be constructed. This gives rise to a family



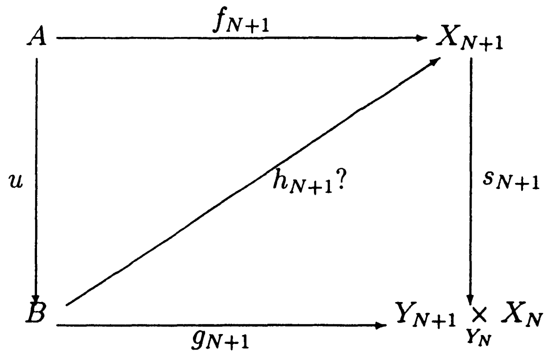
where $f_n := \xi_n \circ f$, $g_n := \eta_n \circ g$ We will inductively construct a family of liftings h_n that are compatible as n varies. h_0 exists because $p_0 \in \mathcal{A}^{\vdash\!\!+\!\!-}$. Suppose that $h_0, ..., h_N$ have been found so that each h_μ is a lifting of $(u,\ p_\mu)$ and such that $x_\mu \circ h_\mu = h_{\mu-1}$ for all $1 \leq \mu \leq N$. Then consider

$$\begin{array}{ccccc} A & \xrightarrow{f_{N+1}} & X_{N+1} & \xrightarrow{x_{N+1}} & X_N \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle p_{N+1}} & & \downarrow{\scriptstyle p_N} \\ B & \xrightarrow{g_{N+1}} & Y_{N+1} & \xrightarrow{y_{N+1}} & Y_N \end{array}$$

Since $p_N \circ h_N = g_N = y_{N+1} \circ g_{N+1}$, we get a unique arrow

$$k_N : B \to Y_{N+1} \underset{Y_N}{\times} X_N$$

with projections $pr_1 \circ k_N = g_{N+1}$, $pr_2 \circ k_N = h_N$. But then h_{N+1} exists as a lifting in



because by hypothesis $s_{N+1} \in \mathcal{A}^{\vdash\vdash}$ and one readily checks the commutativity of the square. The equations $s_{N+1} \circ h_{N+1} = k_N$ and $pr_1 \circ s_{N+1} = p_{N+1}$ show that $p_{N+1} \circ h_{N+1} = g_{N+1}$. That, together with $h_{N+1} \circ u = f_{N+1}$ shows that h_{N+1} is a lifting of $(u,\ p_{N+1})$. Also, $pr_2 \circ s_{N+1} = x_{N+1}$ gives $x_{N+1} \circ h_{N+1} = h_N$ showing that h_{N+1} is compatible with the previous h_μ's

Thus we have built a map from B to the projective system of the X's. We get therefore a unique map $B \to \lim X_n$ with $\xi_n \circ h = h_n$. h is a lifting of (u, p) because both sides in each of the equations to be checked, $p \circ h = g$, $h \circ u = f$ verify the correct universal mapping property for maps to limits, ie. $\eta_n \circ (p \circ h) = \eta_n \circ g$, $\xi_n \circ (h \circ u) = \xi_n \circ f$ □

Corollary 2.3. 1. *In a tower*

$$\cdots \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{x_2} X_1 \xrightarrow{x_1} X_0$$

if each $x_n \in \mathcal{A}^{\vdash\vdash}$ *then the projections*

$$\xi_n : X := \lim X_n \to X_n$$

are all in $\mathcal{A}^{\vdash\vdash}$.

2. *In a tower*

$$X_0 \xrightarrow{x_1} X_1 \xrightarrow{x_2} X_2 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots$$

if each $x_n \in {}^{\vdash\vdash}\mathcal{B}$ *then the injections* $\iota_n : X_n \to X := \operatorname{colim} X_n$ *are all in* ${}^{\vdash\vdash}\mathcal{B}$.

Proof. We do the first one. Without loss of generality, we consider ξ_0. Define the constant projective system $Y_n := X_0$, $y_n := id = p_0$. Then $p_n = x_1 \circ x_2 \circ \ldots \circ x_n$ which is in $\mathcal{A}^{\vdash\vdash}$ because of lemma (2.1.1) . The products $Y_n \underset{Y_{n-1}}{\times} X_{n-1}$ trivially exist and are $= X_{n-1}$. The map $s_n = x_n$ which is in $\mathcal{A}^{\vdash\vdash}$ by hypothesis. Therefore, lemma(2.2.1) applies. The corresponding p is ξ_0 □

Remark. There is an obvious generalization of the above with $\mathbb{N}$ replaced with any well-ordered set (assuming Zorn's lemma).

Lemma 2.4. *Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor. Let $\mathcal{A}_i$, $\mathcal{B}_i$ be classes of morphisms in $\mathcal{C}_i$ for $i = 1, 2$.*

1. *Suppose that G is left-adjoint to F, and that F preserves $\mathcal{B}$, ie.*

$$F(\mathcal{B}_1) \subset \mathcal{B}_2$$

Then G preserves ${}^{\vdash\vdash}\mathcal{B}$, ie

$$G({}^{\vdash\vdash}\mathcal{B}_2) \subset {}^{\vdash\vdash}\mathcal{B}_1$$

2. *Suppose that G is right-adjoint to F, and that F preserves $\mathcal{A}$, ie.*

$$F(\mathcal{A}_1) \subset \mathcal{A}_2$$

Then G preserves $\mathcal{A}^{\vdash\vdash}$, ie

$$G(\mathcal{A}_2^{\vdash\vdash}) \subset \mathcal{A}_1^{\vdash\vdash}$$

Proof. As a) is dual, we do b). Let $(p : X \to Y) \vdash\vdash \mathcal{A}_2$. Consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & G(X) \\ {\scriptstyle u}\downarrow & \nearrow{\scriptstyle h?} & \downarrow{\scriptstyle G(p)} \\ B & \xrightarrow[g]{} & G(Y) \end{array}$$

By adjointness this gives a diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{f^\sharp} & X \\ {\scriptstyle F(u)}\downarrow & \nearrow{\scriptstyle k?} & \downarrow{\scriptstyle p} \\ F(B) & \xrightarrow[g^\sharp]{} & Y \end{array}$$

$$\begin{array}{ccc} F(A) & \xrightarrow{f^\sharp} & X \\ \downarrow{\scriptstyle F(u)} & & \downarrow{\scriptstyle p} \\ F(B) & \xrightarrow{g^\sharp} & Y \end{array}$$

where $\sharp$, $\flat$ are Grothendieck's notation for adjoints. Since

$$F(u) \in \mathcal{A}_1^{\vdash\vdash}$$

a lifting k exists in the above diagram, and we define $h := k^\flat$ to get a lifting in the previous diagram.

$\square$

2.2. Let $\mathcal{M}$ be a functorial closed Quillen model category. Recall that this means that there are three distinguished classes of morphisms in $\mathcal{M}$:

$\mathcal{E}$:	the weak equivalences	$\xrightarrow{\sim}$
Cof :	the cofibrations	$\hookrightarrow$
Fib :	the fibrations	$\twoheadrightarrow$

These are subject to the axioms

CM1 Finite limits and colimits exist in $\mathcal{M}$

CM2 If f and g are maps such that $g \circ f$ is defined, then if any two of $f,\ g,\ g \circ f$ is in $\mathcal{E}$, then so is the third.

CM3 $\mathcal{E}$, Cof, and Fib are closed under retraction.

CM4 Cof $\vdash\!\!\vdash\!\!\!-$ Fib $\bigcap\ \mathcal{E}$, and Cof $\bigcap\ \mathcal{E} \vdash\!\!\vdash\!\!\!-$ Fib

CM5 Any map $f : X \to Y$ admits two functorial factorizations:

(left) $X \xrightarrow{L'f} \widehat{L}f \xrightarrow{L''f} Y$ with $L'f \in \mathrm{Cof} \bigcap \mathcal{E}, L''f \in \mathrm{Fib}$

(right) $X \xrightarrow{R'f} \widehat{R}f \xrightarrow{R''f} Y$ with $R'f \in \mathrm{Cof}, R''f \in \mathrm{Fib} \bigcap \mathcal{E}$

By the *functorality* of the factorizations, we mean that R and L are functors

$$\mathcal{M}^2 \longrightarrow \mathcal{M}^3$$

Concretely this means, eg. that given a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ U & \xrightarrow{g} & V \end{array}$$

there exists $c : \widehat{L}f \to \widehat{L}g$ such that

$$\begin{array}{ccccc} X & \xrightarrow{L'f} & \widehat{L}f & \xrightarrow{L''f} & Y \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\ U & \xrightarrow{L'g} & \widehat{L}g & \xrightarrow{L''g} & V \end{array}$$

commutes.

The elements of Cof $\bigcap\ \mathcal{E}$ (resp. Fib $\bigcap\ \mathcal{E}$) are called *acyclic cofibrations* (resp. *acyclic fibrations*).

Proposition 2.5.

1.
$$\mathrm{Cof} = {}^{\vdash\!\!+\!\!-}(\mathrm{Fib}\bigcap\mathcal{E}),\quad \mathrm{Cof}\bigcap\mathcal{E} = {}^{\vdash\!\!+\!\!-}\,\mathrm{Fib}$$

$$\mathrm{Fib} = (\mathrm{Cof}\bigcap\mathcal{E})^{\vdash\!\!+\!\!-}\ ,\quad \mathrm{Fib}\bigcap\mathcal{E} = \mathrm{Fib}^{\vdash\!\!+\!\!-}$$

2. (a) *Both* Fib *and* Fib $\bigcap\ \mathcal{E}$ *are closed under composition, limits, and contain all isomorphisms.*
   (b) *Both* Cof *and* Cof $\bigcap\ \mathcal{E}$ *are closed under composition, colimits, and contain all isomorphisms.*

Proof. See [15, 16] □

In fact, 1) ⇒ 2) above because of lemma 2.1. From the same lemma we have

Proposition 2.6.

1. *In ladder diagram such as in lemma 2.1.1 in which each* $p_n \in$ Fib *(resp.* Fib $\bigcap\mathcal{E}$*), and for which the* $s_n \in$ Fib*(resp.*Fib $\bigcap\mathcal{E}$*), we have* lim $p_n \in$ Fib *(resp.* Fib $\bigcap\mathcal{E}$*). In particular, in a tower as in corollary 2.3.1 with* $x_n \in$ Fib *(resp.*Fib $\bigcap\mathcal{E}$*), the projections* $\xi_n \in$ Fib *(resp.* Fib $\bigcap\mathcal{E}$*).*
2. *In ladder diagram such as in lemma 2.2.2 in which each* $u_n \in$ Cof *(resp.* Cof $\bigcap\mathcal{E}$*), and for which the* $t_n \in$ Cof *(resp.* Cof $\bigcap\mathcal{E}$*), we have*
   colim $u_n \in$ Cof *(resp.* Cof $\bigcap\mathcal{E}$*). In particular, in a tower as*

in corollary 2.3.2 with $x_n \in \mathrm{Cof}$ *(resp.* $\mathrm{Cof} \bigcap \mathcal{E}$*), the injections* $\iota_n \in \mathrm{Cof}$ *(resp.*$\mathrm{Cof} \bigcap \mathcal{E}$*).*

□

Let ϕ , e denote respectively the initial and final object of the category $\mathcal{M}$, which exist by **CM1**. An object X of $\mathcal{M}$ is called *fibrant* (resp. *cofibrant*) if the unique arrow $X \longrightarrow e$ (resp. $\phi \longrightarrow X$) is a fibration (a cofibration). We let $\mathcal{M}_c$ (resp. $\mathcal{M}_f$, resp. $\mathcal{M}_{cf}$) denote the full subcategory of cofibrant (resp. fibrant, resp. bifibrant = simultaneously fibrant and cofibrant) objects. In any model category, equivalence relations are introduced into $\mathcal{M}_c$ and $\mathcal{M}_f$ called right and left homotopy respectively. These relations induce the same equivalence on the common subcategory $\mathcal{M}_{cf}$, where it is simply called homotopy. We let $\pi^r\mathcal{M}_c$, $\pi^l\mathcal{M}_f, \pi\mathcal{M}_{cf}$ denote the corresponding quotient categories, whose morphisms are the respective homotopy equivalence classes. Let $\mathrm{Ho}(\mathcal{M})$ denote the category of fractions obtained from $\mathcal{M}$ by inverting the weak equivalences. Then

Theorem 2.7. 1. $\mathrm{Ho}(\mathcal{M})$ *exists. The canonical functor*

$$\gamma : \mathcal{M} \longrightarrow \mathrm{Ho}(\mathcal{M})$$

induces equivalences

$$\begin{aligned} \overline{\gamma}_c : \pi^r\mathcal{M}_c &\longrightarrow \mathrm{Ho}(\mathcal{M}) \\ \overline{\gamma}_f : \pi^l\mathcal{M}_f &\longrightarrow \mathrm{Ho}(\mathcal{M}) \\ \overline{\gamma} : \pi\mathcal{M}_{cf} &\longrightarrow \mathrm{Ho}(\mathcal{M}) \end{aligned}$$

where $\overline{\gamma}_c$ *has a fully faithful right adjoint, and hence has a calculus of left fractions, and* $\overline{\gamma}_f$ *has a fully faithful left adjoint, and hence has a calculus of right fractions.*

2. *If A is cofibrant and B is fibrant, then*

$$\pi(\mathcal{M})(A,\ B) \simeq \mathrm{Ho}(\mathcal{M})(\gamma A,\ \gamma B)$$

(When A is cofibrant and B is fibrant, left and right homotopy coincide.)

3. $\gamma(f)$ *is an isomorphism* $\Longleftrightarrow f \in \mathcal{E}$

Proof. See [15, 16] □

Let us remark that many of the familiar model categories, simplicial sets, topological spaces, chain complexes bounded above or below, differential graded algebras, etc. admit structures of functorial Quillen model categories.

2.3. Let $\mathbf{C}$ be a small category and $\mathcal{M}$ be a functorial closed model category. We are going to associate to the category $\mathcal{M}^{\mathbf{C}}$ of functors $\mathbf{C} \longrightarrow \mathcal{M}$ two functorial closed model structures referred to as the *left* and *right* structures. Each of these will require supplementary axioms which will be given below.

Definition 2.1. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{M}^{\mathbf{C}}$. We say that

1. f is a weak equivalence if f_c is a weak equivalence for all $c \in \mathbf{C}_0$. Notation: $f \in \mathcal{E}^{\mathbf{C}}$
2. f is a weak cofibration if f_c is a cofibration for all $c \in \mathbf{C}_0$. Notation: $f \in \mathrm{Cof}^{\mathbf{C}}_w$
3. f is a weak fibration if f_c is a fibration for all $c \in \mathbf{C}_0$. Notation: $f \in \mathrm{Fib}^{\mathbf{C}}_w$.
4. We then define the strong cofibrations, strong fibrations by the equations

$$\mathrm{Fib}^{\mathbf{C}}_s = (\mathrm{Cof}^{\mathbf{C}}_w \cap \mathcal{E}^{\mathbf{C}})^{\vdash\!\dashv}$$

$$\mathrm{Cof}^{\mathbf{C}}_s = {}^{\vdash\!\dashv}(\mathrm{Fib}^{\mathbf{C}}_w \cap \mathcal{E}^{\mathbf{C}})$$

5. We define the left model structure on $\mathcal{M}^{\mathbf{C}}$ as $(\mathcal{E}^{\mathbf{C}}, \mathrm{Cof}^{\mathbf{C}}_s, \mathrm{Fib}^{\mathbf{C}}_w)$ and the right model structure on $\mathcal{M}^{\mathbf{C}}$ as $(\mathcal{E}^{\mathbf{C}}, \mathrm{Cof}^{\mathbf{C}}_w, \mathrm{Fib}^{\mathbf{C}}_s)$.

Notice that if $\mathbf{C}$ is a *discrete* category then $\mathrm{Cof}^{\mathbf{C}}_w = \mathrm{Cof}^{\mathbf{C}}_s$ and $\mathrm{Fib}^{\mathbf{C}}_w = \mathrm{Fib}^{\mathbf{C}}_s$ because of 2.5.

Lemma 2.8. 1. *In the left structure on* $\mathcal{M}^{\mathbf{C}}$, **CM1, CM2, CM3,** *and half of* **CM4**, *namely* $\mathrm{Cof} \vdash\!\dashv\!\!-\!\!- \mathrm{Fib} \bigcap \mathcal{E}$, *holds.*

2. *In the right structure on* $\mathcal{M}^{\mathbf{C}}$, **CM1,CM2, CM3,** *and half of* **CM4**, *namely* $\mathrm{Cof} \bigcap \mathcal{E} \vdash\!\dashv\!\!-\!\!- \mathrm{Fib}$, *holds.*

Proof. Recall that if (co)limits indexed by a small category I with values in any category $\mathcal{M}$ exist, then those I-(co)limits exist in any functor category $\mathcal{M}^{\mathbf{C}}$ and are computed "pointwise", ie.

$$(\lim_I X_i)_c = \lim_I (X_c)_i$$

Therefore, **CM1** holds in $\mathcal{M}^{\mathbf{C}}$ since it holds in $\mathcal{M}$. **CM2** holds because the condition to be a weak equivalence is a pointwise condition. For the same reason, **CM3** is true for $\mathrm{Cof}^{\mathbf{C}}_w$and $\mathrm{Fib}^{\mathbf{C}}_w$, and it is true for $\mathrm{Cof}^{\mathbf{C}}_s$ and $\mathrm{Fib}^{\mathbf{C}}_s$ because of 2.1.1 and the definitions.The half of **CM4** is immediate from those definitions. ☐

The nontrivial part in showing that we have closed model structures defined this way is to build the functorial factorizations

^{L}L and ^{L}R in the left case

^{R}L and ^{R}R in the right case

and to show

$\mathrm{Cof} \bigcap \mathcal{E} \;\vdash\!\!\!\vdash\!\!\!\longrightarrow\; \mathrm{Fib}$ in the left case

$\mathrm{Cof} \;\vdash\!\!\!\vdash\!\!\!\longrightarrow\; \mathrm{Fib} \bigcap \mathcal{E}$ in the right case

Here, ^{L}R denotes the right factorization in the left structure, etc. For this, additional axioms are required.

3. Supplementary Axioms

3.1. These are

LCM1a Arbitrary sums exist in $\mathcal{M}$.
b If $f_i : X_i \to Y_i$ is a family of weak equivalences, with X_i, Y_i cofibrant, then $\coprod f_i$ is a weak equivalence.

RCM1a Arbitrary products exist in $\mathcal{M}$.
b If $f_i : X_i \to Y_i$ is a family of weak equivalences, with X_i, Y_i fibrant, then $\prod f_i$ is a weak equivalence.

LCM2 If $X : \mathbb{N} \to \mathcal{M}$ is a sequence of morphisms and $p : X \to B$ is a colimit cone and if each p_i is an acyclic fibration then $\mathrm{colim}\ p : \mathrm{colim}\ X \to B$ is an acyclic fibration.

LCM2c If $X : \mathbb{N} \to \mathcal{M}$ is a sequence of cofibrations and $p : X \to B$ is a colimit cone and if each p_i is an acyclic fibration then $\mathrm{colim}\ p : \mathrm{colim}\ X \to B$ is an acyclic fibration.

LCM2cc If $X : \mathbb{N} \to \mathcal{M}$ is a sequence of cofibrations and $p : X \to B$ is a colimit cone with each p_i is an acyclic fibration and

if B and the X_i are cofibrant, then
colim p : colim $X \longrightarrow B$ is an acyclic fibration.

RCM2 If $X : \mathbb{N}^{op} \longrightarrow \mathcal{M}$ is a sequence of morphisms
and $u : A \longrightarrow X$ is a limit cone
and if each u_i is an acyclic cofibration then
lim $u : A \longrightarrow$ lim X is an acyclic cofibration.

RCM2f If $X : \mathbb{N}^{op} \longrightarrow \mathcal{M}$ is a sequence of fibrations
and $u : A \longrightarrow X$ is a limit cone
and if each u_i is an acyclic cofibration then
lim $u : A \longrightarrow$ lim X is an acyclic cofibration.

RCM2ff If $X : \mathbb{N}^{op} \longrightarrow \mathcal{M}$ is a sequence of fibrations
and $u : A \longrightarrow X$ is a limit cone
with each u_i is an acyclic cofibration and
if A and the X_i are fibrant, then
lim $u : A \longrightarrow$ lim X is an acyclic cofibration.

LCM3 Let X, Y : $\mathbb{N} \longrightarrow \mathcal{M}$ be sequences of cofibrations
and $f : X \longrightarrow Y$ a morphism
such that each f_i is a weak equivalence.
Then, colim f is a weak equivalence.

LCM3c Let X, Y : $\mathbb{N} \longrightarrow \mathcal{M}$ be sequences of cofibrations
with each X_i, Y_i cofibrant and $f : X \longrightarrow Y$ a morphism
such that each f_i is a weak equivalence.
Then, colim f is a weak equivalence.

RCM3 Let X, Y : $\mathbb{N}^{op} \longrightarrow \mathcal{M}$ be sequences of fibrations
and $f : X \longrightarrow Y$ a morphism
such that each f_i is a weak equivalence.
Then, lim f is a weak equivalence.

RCM3f Let X, Y : $\mathbb{N}^{op} \longrightarrow \mathcal{M}$ be sequences of fibrations
with each X_i, Y_i fibrant and $f : X \longrightarrow Y$ a morphism
such that each f_i is a weak equivalence.
Then, lim f is a weak equivalence.

LCM4 Let

$$\begin{array}{ccc} X & \xrightarrow{f} & U \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ Y & \xrightarrow{g} & V \end{array}$$

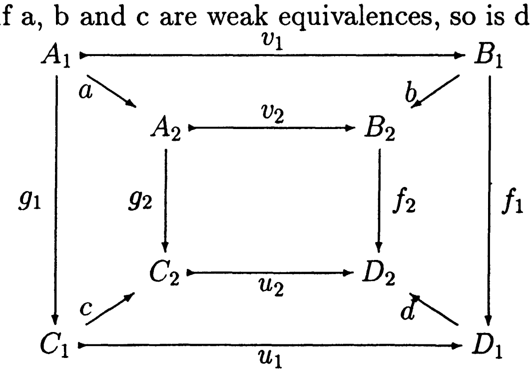
be a co-Cartesian square with u $\in$ Cof and f $\in \mathcal{E}$. Then g $\in$ $\mathcal{E}$.

RCM4 Let

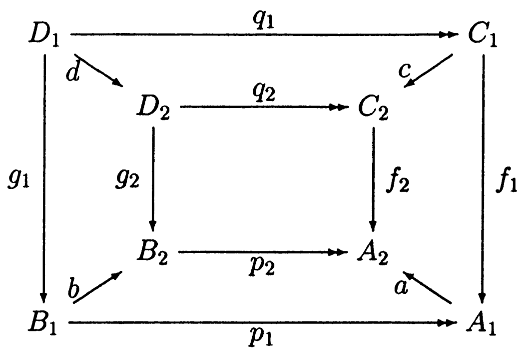
$$\begin{array}{ccc} U & \xrightarrow{g} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{f} & B \end{array}$$

be a Cartesian square with p $\in$ Fib and f $\in \mathcal{E}$. Then g $\in$ $\mathcal{E}$.

LCM5 In the diagram below, suppose that both squares are co-Cartesian, the objects are cofibrant, and all the horizontal arrows are cofibrations.
Then if a, b and c are weak equivalences, so is d.



RCM5 In the diagram below, suppose that both squares are Cartesian, the objects are fibrant, and all the horizontal arrows are fibrations.
Then if a, b and c are weak equivalences, so is d.



3.2. **Remarks.**

1. **CM1** & **LCM1a** $\Longrightarrow$ all colimits exist in $\mathcal{M}$.
   CM1 & **RCM1a** $\Longrightarrow$ all limits exist in $\mathcal{M}$.
2. **LCM2** $\Longrightarrow$ **LCM2c** $\Longrightarrow$ **LCM2cc.**
   RCM2 $\Longrightarrow$ **RCM2f** $\Longrightarrow$ **RCM2ff.**
   LCM3 $\Longrightarrow$ **LCM3c**
   RCM3 $\Longrightarrow$ **RCM3f.**
3. **LCM4** holds if all the objects in question are cofibrant.
   RCM4 holds if all the objects in question are fibrant.
   [2, I. Lemma(1.4), p. 7 and Prop. (2.6), p. 15]

Lemma 3.1.

1. *The classes* $\mathrm{Fib}_w^{\mathbf{C}}$, $\mathrm{Cof}_w^{\mathbf{C}}$, $\mathrm{Fib}_s^{\mathbf{C}}$, $\mathrm{Cof}_s^{\mathbf{C}}$, $\mathrm{Fib}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$, $\mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ *are closed under composition, retraction and contain all isomorphisms.*
2. $\mathrm{Fib}_w^{\mathbf{C}}$, $\mathrm{Fib}_s^{\mathbf{C}}$, $\mathrm{Fib}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ *are closed under all limits.*
3. $\mathrm{Cof}_w^{\mathbf{C}}$, $\mathrm{Cof}_s^{\mathbf{C}}$, $\mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ *are closed under all colimits.*
4. *Assume that* **RCM1a** *is satisfied in* $\mathcal{M}$. *Then*
   (a)
   $$\mathrm{Cof}_s^{\mathbf{C}} \subset \mathrm{Cof}_w^{\mathbf{C}}$$
   .
   (b) $\mathrm{Cof}_s^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ *is closed under composition, retraction, colimits, and contains all isomorphisms.*
5. *Assume that* **LCM1a** *is satisfied in* $\mathcal{M}$. *Then*
   (a)
   $$\mathrm{Fib}_s^{\mathbf{C}} \subset \mathrm{Fib}_w^{\mathbf{C}}$$
   .
   (b) $\mathrm{Fib}_s^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ *is closed under composition, retraction, limits, and contains all isomorphisms.*

Proof. Statements 1), 2), and 3) are trivial consequences of the definitions and lemma 2.1. Part b) of 4) and 5) follow from the respective part a), which implies eg.

$$\mathrm{Fib}_s^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}} \;=\; \mathrm{Fib}_s^{\mathbf{C}} \cap \left(\mathrm{Fib}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}\right)$$

So we prove 4a), 5a) being dual. Let $(u : X \longrightarrow Y)$ be a strong cofibration. We will show that $u_c \mathrel{+\!\!\!+\!\!\!-\!\!\!-} \mathrm{Fib} \bigcap \mathcal{E}$, for all $c \in \mathbf{C}_0$ which by proposition 2.5 shows that u is a weak cofibration. Let

$$(p : E \longrightarrow B)$$

be an acyclic fibration in $\mathcal{M}$. We can identify an object $c \in \mathbf{C}_0$ with a functor $c : \mathbf{1} \longrightarrow \mathbf{C}$ and then $X_c = c^*(X)$ where

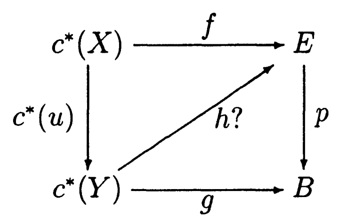
$$c^* : \mathcal{M}^{\mathbf{C}} \longrightarrow \mathcal{M}$$

. The right adjoint to this $c_* := \mathrm{Ran}_c$ is computed as

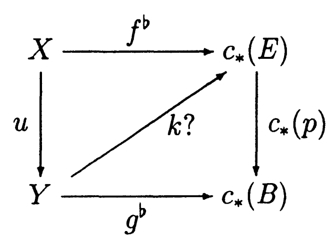
$$\left(c_*(X)\right)_d = \prod_{\gamma \in \mathbf{C}(d,\, c)} X$$

$$\left(c_*(p)\right)_d = \prod_{\gamma \in \mathbf{C}(d,\, c)} p$$

Start with



with h to be constructed. The adjoint of this is



In view of part 1) of this lemma , the above expression for $c_*(p)$ shows that $c_*(p) \in \mathrm{Fib}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$. Therefore an arrow k exists in the above diagram, and we set $h = k^\sharp$, which gives the desired lift.

□

3.3. Let **LCM3cI** stand for the assertion **LCM3c** but with $\mathbb{N}$ replaced with any well-ordered set I. Dually for **RCM3fI**.

Proposition 3.2. *We have the implications*

1. **LCM1a** *&* **LCM3cI** $\Longrightarrow$ **LCM1b.**
2. **RCM1a** *&* **RCM3fI** $\Longrightarrow$ **RCM1b.**

Proof. We do the assertion on the right. Totally order the set of indices I. We let

$$I_a = \{i \in I : i \leq a\}$$

Define

$$\overline{X}_a := \prod_{i \in I_a} X_i, \qquad \overline{Y}_a := \prod_{i \in I_a} Y_i$$

We obtain in this way $\overline{X}$, $\overline{Y}$: $I^{op} \to \mathcal{M}$. Note that each $\overline{X}_a$, $\overline{Y}_a$ is a fibrant object, because fibrations are closed under limits by 2.5. For the same reason, each $\overline{X}_a \to \overline{X}_b$ is a fibration whenever $b \leq a$, because

$$\overline{X}_a = \prod_{b < i \leq a} X_i \times \overline{X}_b$$

Now one can prove by induction that each $f_a : \overline{X}_a \longrightarrow \overline{Y}_a$ is a weak equivalence: it is true for the smallest element $0 \in I$; suppose that it has been shown for $a \in I$. Consider the set

$$T = \{i \in I : a < i\}$$

If T is empty, then $\overline{X}_a := \prod_{i \in I} X_i$, and the theorem is proved. If not, let $a + 1$ denote the smallest member of T. We will show that f_{a+1} is a weak equivalence, completing the induction. But this results from the lemma that follows. □

Lemma 3.3. *In a closed model category $\mathcal{M}$,*

1. *Let $f_i : X_i \longrightarrow Y_i$ be weak equivalences among cofibrant objects, $i = 1, 2$. Then*

$$f_1 \vee f_2 : X_1 \vee X_2 \longrightarrow Y_1 \vee Y_2$$

is a weak equivalence.

2. *Let $f_i : X_i \longrightarrow Y_i$ be weak equivalences among fibrant objects, $i = 1, 2$. Then*

$$f_1 \times f_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$$

is a weak equivalence.

Proof. We do b). In the pull - back,

$$\begin{array}{ccc} X_1 \times X_2 & \longrightarrow & X_1 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle f_1} \\ Y_1 \times X_2 & \longrightarrow & Y_1 \end{array}$$

the horizontal arrows are fibrations and all the objects are fibrant. By remark 3.2.3 above, we conclude that g_1 is a weak equivalence. The same reasoning applied to

$$\begin{array}{ccc} Y_1 \times X_2 & \longrightarrow & X_2 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_2} \\ Y_1 \times Y_2 & \longrightarrow & Y_2 \end{array}$$

shows that g_2 is a weak equivalence. But $f_1 \times f_2 = g_2 \circ g_1$. □

3.4. The supplementary axioms **LCM3**, **LCM5**, **RCM3**, **RCM5** and their variants are of a less elementary nature that the others, and so it is desirable to see if they can be deduced from more basic properties. We will do this, but first let us observe that often, in examples, it is possible to prove these axioms using more direct methods. See section 5. The main new thing to consider is to impose conditions on the *cylinder* or *path* objects in $\mathcal{M}$. Namely,

Cyl There is a functorial cylinder object $A \mapsto A_I$ giving a functorial factorization of the codiagonal

$$\nabla_A : A \vee A \xrightarrow{\sigma_A} A_I \xrightarrow{\pi_A} A$$

with $\sigma_A \in \mathrm{Cof}$ and $\pi_A \in \mathcal{E}$. We assume

(1) If $A \xrightarrow{u} B$ is in $\mathrm{Cof} \cap \mathcal{E}$ then

$$A_I \coprod_{(A \vee A)} (B \vee B) \longrightarrow B_I \text{ is in } \mathrm{Cof} \cap \mathcal{E}.$$

(2) The formation of A_I commutes with colimits.

Path There is a functorial path object $A \mapsto A^I$ giving a functorial factorization of the diagonal

$$\Delta_A : A \xrightarrow{\sigma_A} A^I \xrightarrow{\pi_A} A \times A$$

with $\pi_A \in \text{Fib}$ and $\sigma_A \in \mathcal{E}$. We assume

(1) If $A \xrightarrow{p} B$ is in $\text{Fib} \cap \mathcal{E}$ then

$$A^I \longrightarrow B^I \prod_{(B\times B)} (A \times A) \text{ is in } \text{Fib} \cap \mathcal{E}.$$

(2) The formation of A^I commutes with limits.

3.5. **Remarks.** One important case of this is when $\mathcal{M}$ is a closed *simplicial* model category. In that case, we can take for A_I the external product $A \otimes I$ with a generalized unit interval. Actually, in the arguments that follow we only need $I = \Delta[1]$. The axiom **Cyl** is satisfied *provided* that A is cofibrant : that σ_A is a weak equivalence is [15, II, p.2.7] and that i_A is a cofibration is [15, II, p. 2.3, SM7(b)]. The dual assertion concerning the path axiom requires fibrant objects. In the applications to follow, this will be sufficient. As to the preservation of (co)limits, it follows immediately from the fact that A_I (resp. A^I) is a left (resp. right) adjoint.

Proposition 3.4. 1. *Assume that* **LCM1a** *and* **Cyl** *holds in* $\mathcal{M}_c$. *Then* **LCM3cI** *holds in* $\mathcal{M}$.

2. *Assume that* **RCM1a** *and* **Path** *holds in* $\mathcal{M}_f$. *Then* **RCM3fI** *holds in* $\mathcal{M}$.

Proof. We do the second one. We will suppose that $I = \mathbb{N}$ since the case of an arbitrary well-ordered set is only notationally different (assuming Zorn's lemma). Let then $f : X \longrightarrow Y$ be as in the statement of **RCM3f.** The first step is to refine this to

$$X \xrightarrow{u} Z \xrightarrow{p} Y$$

where $Z : \mathbb{N}^{op} \longrightarrow \mathcal{M}$ is a tower of fibrations. Each u_n is an acyclic cofibration, and each p_n is an acyclic fibration. Since we are assuming that each Y_n is fibrant, it follows that each Z_n is fibrant as well. Also, the canonical maps

$$Z_{n+1} \longrightarrow Z_n \times_{Y_n} Y_{n+1}$$

is an acyclic fibration for every $n \geq 0$. The construction of Z is by induction. First factor f_0:

$$X_0 \xrightarrow{u_0} Z_0 \xrightarrow{p_0} Y_0$$

with u_0 an acyclic cofibration and p_0 an acyclic fibration. Then form the diagram, where the bottom square is a pull-back:

$$\begin{array}{ccc} X_1 & \xrightarrow{x_1} & X_0 \\ \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle u_0} \\ W_1 & \xrightarrow{w_1} & Z_0 \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle p_0} \\ Y_1 & \xrightarrow{y_1} & Y_0 \end{array}$$

Note that v_1 exists, and that q_1 and w_1 are acyclic fibrations, by base-change. Factor v_1:

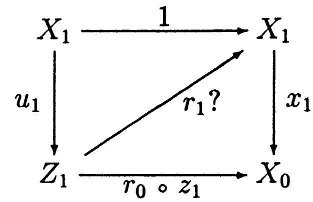
$$X_1 \xrightarrow{u_1} Z_1 \xrightarrow{s_1} W_1$$

with u_1 an acyclic cofibration and s_1 an acyclic fibration. Define $p_1 = q_1 \circ s_1$, which is an acyclic fibration, and $z_1 : Z_1 \longrightarrow Z_0$ as $w_1 \circ s_1$, which is a fibration. Now repeat this procedure with X_0, Y_0, Z_0, and Y_1 replaced by X_1, Y_1, Z_1, and Y_2, and so on.

Clearly, $\lim f = \lim p \circ \lim u$, where the limits exist by the assumption **RCM1a** (see the remarks following the supplementary axioms). Also, $\lim p$ is an acyclic fibration by 2.6. We are reduced to showing that $\lim u$ is a weak equivalence. The strategy of the proof here is that, as is well-known, and an easy exercise in Quillen model categories, an acyclic cofibration among fibrant objects admits a strong deformation retraction (dually, an acyclic fibration among cofibrant objects admits a strong deformation section). In the special case of a simplicial model category, the homotopies in question may be taken to be simplicial homotopies, with path or cylinder object based on the generalized unit interval $\Delta[1]$ itself [15, II, 2.4 Cor. Prop. 4]. We will inductively construct a compatible family of such deformation retraction in the tower, and this will induce a deformation retraction of $\lim u$. This will give what we want. Let then r_0 be a retraction of u_0 and let

$$H_0 : Z_0 \longrightarrow (Z_0)^I$$

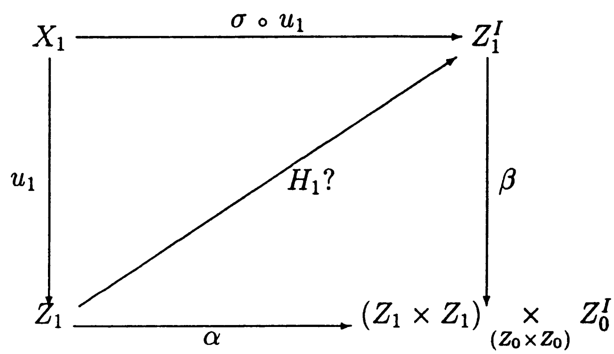
be a homotopy $1 \sim u_0 \circ r_0$, stationary on X_0. We will show how to construct a retraction r_1 of u_1 and a homotopy H_1, $1 \sim u_1 \circ r_1$, stationary on X_1 and such that r_1 and H_1 is compatible with r_0 and H_0 in the obvious sense. The general induction step will be just like this. r_1 is given as a lift in



which exists by our assumptions about u_1 and x_1. This retraction is compatible with r_0. Now construct a map

$$\alpha : \; Z_1 \longrightarrow (Z_1 \times Z_1) \underset{(Z_0 \times Z_0)}{\times} Z_0^I$$

with components $(1, u_1 \circ r_1)$ and $H_0 \circ z_1$, which exists because the two maps equalize on $Z_0 \times Z_0$. We have a diagram



where β has components (p_0, p_1), and z_1^I in the notation of **Path**. One checks that the square commutes. Axiom **Path**, states that β is a fibration, and u_1 is given as an acyclic cofibration, so that a lift H_1 exists. One verifies that H_1 satisfies the conditions.

These all fit together to give

$$\lim\, r : \; \lim\, Z_n \longrightarrow \lim\, X_n$$

which is a retraction of lim u, and

$$\lim\, H : \; \lim\, Z_n \longrightarrow \lim\, (Z_n^I)$$

By axiom **Path**,

$$\lim\, (Z_n^I) \; = \; (\lim\, Z_n)^I$$

is a path-object, and one verifies that lim H is a homotopy $1 \sim \lim\, u \circ \lim\, r$ □

Proposition 3.5. *We have the implications*

1. **Cyl** $\Longrightarrow$ **LCM5**
2. **Path** $\Longrightarrow$ **RCM5**

We do not give the proof of this because a more general result is true (proposition 7.3).

4. The Factorizations

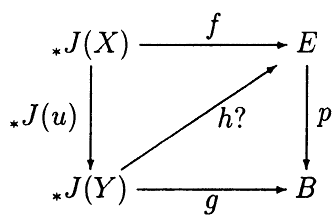
4.1. We verify axiom **CM5** for the left and right model structures on $\mathcal{M}^{\mathbf{C}}$.

Proposition 4.1. *Let $\mathcal{M}$ be a closed Quillen model category. Let $\mathbf{C}$ be a small category and $J : \mathbf{C}_0 \longrightarrow \mathbf{C}$ be the inclusion of the discrete subcategory of objects.*

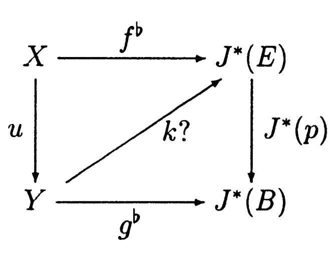
1. *If* **RCM1a** *holds in $\mathcal{M}$, then*
   (a) $J_*(\mathrm{Fib}^{\mathbf{C}_0}) \subset \mathrm{Fib}^{\mathbf{C}}_s$
   (b) $\mathrm{Cof}^{\mathbf{C}}_w \vdash\!\!\dashv\!\!\!- \; J_*(\mathrm{Fib}^{\mathbf{C}_0} \cap \mathcal{E}^{\mathbf{C}_0})$
   (c) $J_*(\mathrm{Fib}^{\mathbf{C}_0} \cap \mathcal{E}^{\mathbf{C}_0}) \subset \mathcal{E}^{\mathbf{C}}$
2. *If* **LCM1a** *holds in $\mathcal{M}$, then*
   (a) ${}_*J(\mathrm{Cof}^{\mathbf{C}_0}) \subset \mathrm{Cof}^{\mathbf{C}}_s$
   (b) ${}_*J(\mathrm{Cof}^{\mathbf{C}_0} \cap \mathcal{E}^{\mathbf{C}_0}) \vdash\!\!\dashv\!\!\!- \; \mathrm{Fib}^{\mathbf{C}}_w$
   (c) ${}_*J(\mathrm{Cof}^{\mathbf{C}_0} \cap \mathcal{E}^{\mathbf{C}_0}) \subset \mathcal{E}^{\mathbf{C}}$

Proof. We prove the second part since the first is dual.

(a) Let $u : X \longrightarrow Y$ be in $\mathrm{Cof}^{\mathbf{C}_0}$. We will prove that a lifting exists in every diagram of the form



where $p \in \mathrm{Fib}^{\mathbf{C}}_w \cap \mathcal{E}^{\mathbf{C}}$. The adjoint of this is



But $J^*(p) \in \mathrm{Fib}^{\mathbf{C}_0} \cap \mathcal{E}^{\mathbf{C}_0}$, and because $\mathbf{C}_0$ is a discrete category, a lifting k exists in the above diagram, because it exists pointwise by **CM4** . Therefore, $h := k^\sharp$ defines a lifting in the previous diagram, and because of the definition of strong cofibrations, this proves the claim.

(b) This is proved by the same argument, but with $p \in \mathrm{Fib}^{\mathbf{C}}_w$.

(c.) This is a pointwise condition to be checked. Let $u \in \mathrm{Cof}^{\mathbf{C}_0} \cap \mathcal{E}^{\mathbf{C}_0}$. Then

$$\left({}_*J(X)\right)_c = \coprod_{\gamma \in \mathbf{C}(d,\ c)} X_d$$

$$\left({}_*J(u)\right)_c = \coprod_{\gamma \in \mathbf{C}(d,\ c)} u_d$$

Each $u_d \in \mathrm{Cof} \cap \mathcal{E}$ and the result that $\left({}_*J(u)\right)_c \in \mathrm{Cof} \cap \mathcal{E}$ follows from 2.5.2(b).

□

Lemma 4.2. *Let $\mathcal{M}$ be a closed model category. Let $J : \mathbf{C}_0 \to \mathbf{C}$ as before be the inclusion of the subcategory of objects of a small category.*

1. *Assume that* **RCM1a** *holds in $\mathcal{M}$ and let $u : J^*(X) \to Y$ be a cofibration in $\mathcal{M}^{\mathbf{C}_0}$. Define X' by co-base extension in*

$$\begin{array}{ccc} {}_*J\,J^*(X) & \xrightarrow{{}_*J(u)} & {}_*J(Y) \\ \Big\downarrow{\scriptstyle \varepsilon_X} & & \Big\downarrow{\scriptstyle v} \\ X & \xrightarrow{w} & X' \end{array}$$

   where η, ε are the unit and counit of the adjunction ${}_*J \dashv J^*$. Then:

$$v^\flat = (J^*v) \circ \eta_Y : Y \to J^*(X')$$

   is a cofibration in $\mathcal{M}^{\mathbf{C}_0}$.

2. *Assume that* **LCM1a** *holds in $\mathcal{M}$ and let $p : Y \to J^*(X)$ be a fibration in $\mathcal{M}^{\mathbf{C}_0}$. Define X' by base extension in*

$$\begin{array}{ccc} X' & \xrightarrow{w} & X \\ \Big\downarrow{\scriptstyle v} & & \Big\downarrow{\scriptstyle \eta_X} \\ J_*J^*(Y) & \xrightarrow{J_*(p)} & J_*J^*(X) \end{array}$$

   where η, $\varepsilon$are the unit and counit of the adjunction $J^* \dashv J_*$. Then:

$$v^\sharp = \varepsilon_Y \circ (J_*v) : J^*(X') \to Y$$

   is a fibration in $\mathcal{M}^{\mathbf{C}_0}$.

Proof. We do 2). We must verify that for all $c \in \mathbf{C}_0$,

$$((\varepsilon_Y) \circ (J^*v))_c \quad = \quad \varepsilon_{Y_c} \circ v_c$$

mapping $(J^*X)_c \quad = \quad X'_c$ to Y_c is a fibration. Observe that

$$(J_*J^*X) \quad = \quad \prod_{\substack{d \in \mathbf{C}_0 \\ \varphi \in \mathbf{C}(c,\, d)}} X_d$$

$$(\eta_X)_c \quad \text{is defined by } pr_\varphi \circ (\eta_X) \quad = \quad X_\varphi$$

$$(\varepsilon_X)_c \quad = \quad pr_{1_c}$$

We have a diagram

$$\begin{array}{ccccc}
X'_c & \xrightarrow{(v^\sharp)_c = \varepsilon_{Y_c} \circ v_c} & Y_c & \xrightarrow{p_c} & X_c \\
\downarrow{\scriptstyle v_c} & & \downarrow{\scriptstyle \theta \times 1} & & \downarrow{\scriptstyle (\eta_X)_c} \\
(\prod_{\varphi \neq 1} Y_d) \times Y_c & \xrightarrow{(\prod p_d) \times 1} & (\prod_{\varphi \neq 1} X_d) \times Y_c & \xrightarrow{1 \times p_c} & (\prod_{\varphi \neq 1} X_d) \times X_c
\end{array}$$

where θ is defined by $pr_\varphi \circ \theta \quad = \quad X_\varphi \circ p_c$ $(\varphi \neq 1)$, and where the products exist by **RCM1a** . We claim that the outer rectangle is the Cartesian square defining X' read at c, and also that the right-hand square is Cartesian as well. Therefore by cancellation, the left-hand square is Cartesian. Now, $\prod p_d \times 1$, being a product of fibrations is a fibration. By base change, $(v^\sharp)_c$ is a fibration as well. (prop. 2.5, and lemma 2.1) The assertion about the outer rectangle amounts to the equation $w_c \quad = \quad p_c \circ v^\sharp \quad = \quad p_c \circ \varepsilon_c \circ v_c$. Using the notation

$$Z(T) \quad = \quad \mathcal{M}(T,\, Z)$$

for the set of T-valued points in Z, the very definition of X' as a fibered product leads to the description

$$\begin{aligned} X'_c \quad = \quad \{((\ldots, y_\varphi, \ldots), x) : y_\varphi \in Y_d(T), x \in X'_c(T), \\ X_\varphi(x) \quad = \quad p_d(y_\varphi) \text{ for } \varphi : c \longrightarrow d\} \end{aligned}$$

The maps v_c, w_c are the first and second projections on the two components of these vectors. The left and right hand side of the proposed equation, applied to a typical vector yields x on the one hand, $(p_c \circ \varepsilon_{Y_c})(\ldots, y_\varphi, \ldots)$ on the other. By the formula for ε given above, this last is $p_c(y_{1_c})$. But the equation $X_\varphi(x) \quad = \quad p_d(y_\varphi)$ applied to $\varphi \quad = \quad 1_c$, gives $x \quad = \quad p_c(y_{1_c})$ which is the equation we want.

The Cartesian nature of the right square is verified as follows. Let U be the fibred product of $(\prod_{\varphi \neq 1} X_d) \times Y_d$ and X_c over $\prod_{\varphi \neq 1} X_d$. As before,

$$U(T) = \{((\ldots, x_\varphi, \ldots ; y), x) : x \in X_c(T),\ x_\varphi \in X_d(T),\ \varphi : c \longrightarrow d,\ y \in Y_d(T) \text{ and } X_\varphi(x) = x_\varphi \text{ for all } \varphi \neq 1, \text{and } p_c(y) = x\}$$

This shows that the map given by

$$y \mapsto ((\ldots, X_\varphi \circ p_c(y), \ldots ; y),\ p_c(y))$$

is a bijection $\zeta : Y_c(T) \simeq U(T)$ which is functorial in T, and for which $pr_1 \circ \zeta = \theta \times 1$ and $pr_2 \circ \zeta = p_c$ proving the claim. □

4.2. Our aim is to construct the factorizations ${}^RL$ and ${}^RR$. The factorizations ${}^LR$ and ${}^LL$ are strictly dual, and in any case are explicitly carried out by Heller in the context of simplicial sets in [11]. By a well - known trick , only ${}^RL$ needs to be given, for we can construct the other one as follows: For let $f : X \longrightarrow Y$ be given in $\mathcal{M}^{\mathbf{C}}$. Let

$$X \xrightarrow{R'f} \widehat{R}f \xrightarrow{R''f} Y$$

be the factorization coming from the functorial right factorization that exists in $\mathcal{M}$ by **CM5** , so that $R'f \in \mathrm{Cof}_w^{\mathbf{C}}$ and $R''f \in \mathrm{Fib}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$. Suppose that a functorial ${}^RL$ has been constructed. Then we use it to factor $R''f$

$$\widehat{R}f \xrightarrow{{}^RL'R''f} {}^R\widehat{L}R''f \xrightarrow{{}^RL''R''f} Y$$

and then we define ${}^RRf$ as follows:

$$\begin{aligned} {}^R\widehat{R}f &:= {}^R\widehat{L}R''f \\ {}^RR'f &:= {}^RL'R''f \circ R'f \\ {}^RR''f &:= {}^RL''R''f \end{aligned}$$

and one sees that ${}^RR'f \in \mathrm{Cof}_w^{\mathbf{C}}$ and ${}^RR''f \in \mathrm{Fib}_s^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ as required. The functoriality of this is evident. Therefore we concentrate on ${}^RL$.

Lemma 4.3. 1. *Assume that axioms* **RCM1a** *and* **LCM4** *hold in $\mathcal{M}$, and assume the following cancellation property is valid:*

$$g \circ f \in \mathrm{Fib} \cap \mathcal{E} \text{ and } f \in \mathcal{E} \Rightarrow g \in \mathrm{Fib} \cap \mathcal{E}$$

For this it is sufficient to assume that Fib = Epi*, the fibrations coincide with the epimorphisms. Then: In order to construct a functorial ${}^LR$ in $\mathcal{M}^{\mathbf{C}}$ it is sufficient to do so for all morphisms between weakly cofibrant objects.*

2. *Assume that axioms* **LCM1a** *and* **RCM4** *hold in $\mathcal{M}$, and assume the following cancellation property is valid:*

$$g \circ f \in \mathrm{Cof} \cap \mathcal{E} \text{ and } g \in \mathcal{E} \Rightarrow f \in \mathrm{Cof} \cap \mathcal{E}$$

For this it is sufficient to assume that Cof = Mono, *the cofibrations coincide with the monomorphisms. Then: In order to construct a functorial* ^{R}L *in* $\mathcal{M}^{\mathbf{C}}$ *it is sufficient to do so for all morphisms between weakly fibrant objects.*

Proof. We do the second one. For any object Z in $\mathcal{M}^{\mathbf{C}}$ we may factor the canonical map $Z \longrightarrow e$ using the functor L in $\mathcal{M}$:

$$Z \xrightarrow{i_Z} \overline{Z} \longrightarrow e$$

where $i_Z \in \mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ and $\overline{Z} \in \mathrm{Fib}_w^{\mathbf{C}}$. Thus, $\overline{Z}$ is weakly fibrant. By definition,

$$\begin{aligned} \overline{Z}_c &= \widehat{L}(Z_c \longrightarrow e) \\ i_{Z_c} &= L'(Z_c \longrightarrow e) \\ \overline{Z}_c \longrightarrow e &= L''(Z_c \longrightarrow e) \end{aligned}$$

Let $f : X \longrightarrow Y$ be any morphism in $\mathcal{M}$. Form the push-out:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle j} \\ \overline{X} & \xrightarrow{g} & T \end{array}$$

and note that $j \in \mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ because of 3.1. Then factor $T \longrightarrow e$ as above to get a commutative diagram :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle k} \\ \overline{X} & \xrightarrow{\overline{f}} & \overline{T} \end{array}$$

with $k \in \mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ because $k = i_T \circ j$. Because the objects $\overline{X}, \overline{T}$ are weakly fibrant, we are assuming the existence of a factorization :

$$\overline{X} \xrightarrow{^{R}L'\overline{f}} {}^{R}\widehat{L\overline{f}} \xrightarrow{^{R}L''\overline{f}} \overline{T}$$

with ${}^R L'\overline{f} \in \mathrm{Cof}^{\mathbf{C}}_w \cap \mathcal{E}^{\mathbf{C}}$, ${}^R L''\overline{f} \in \mathrm{Fib}^{\mathbf{C}}_s$. Form the Cartesian square at right below:

$$\begin{array}{ccccc} X & \xrightarrow{q} & W & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle k} \\ \overline{X} & \xrightarrow{{}^R L'\overline{f}} & {}^R\widehat{L\overline{f}} & \xrightarrow{{}^R L'\overline{f}} & \overline{T} \end{array}$$

The arrow q exists by Cartesian property. By base-change, p is a strong fibration, and since **LCM1a** is in force, p is a weak fibration as well (3.1). Also, by **RCM4** , m is a weak equivalence. But $m \circ q = {}^R L'\overline{f} \circ i_X$ is in $\mathrm{Cof}^{\mathbf{C}}_w \cap \mathcal{E}^{\mathbf{C}}$ and so by the cancellation hypothesis that we are assuming, $q \in \mathrm{Cof}^{\mathbf{C}}_w \cap \mathcal{E}^{\mathbf{C}}$. We get our factorization by setting

$$\begin{aligned} {}^R\widehat{L}f &:= W \\ {}^R L'f &:= q \\ {}^R L''f &:= p \end{aligned}$$

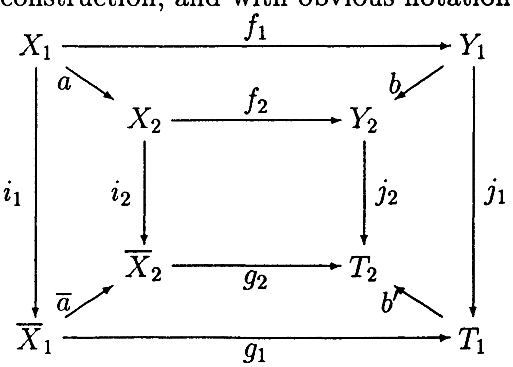
It remains to verify the functoriality of this construction. Given

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ X_2 & \xrightarrow{f_2} & Y_2 \end{array}$$

we need the existence of a map c so that the diagram commutes:

$$\begin{array}{ccccc} X_1 & \xrightarrow{{}^R L'f_1} & {}^R\widehat{L}f_1 & \xrightarrow{{}^R L''f_1} & Y_1 \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\ X_2 & \xrightarrow{{}^R L'f_2} & {}^R\widehat{L}f_2 & \xrightarrow{{}^R L''f_2} & Y_2 \end{array}$$

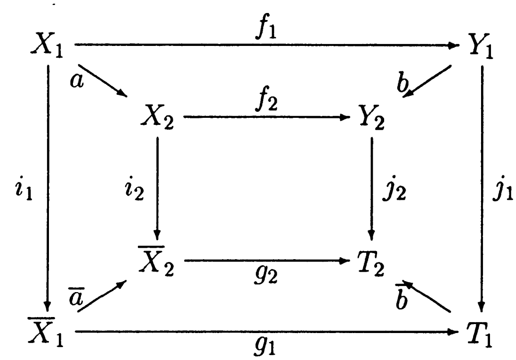
Refering to the above construction, and with obvious notation



$\overline{a}$ exists commuting the left face because $Z \longrightarrow \overline{Z}$ is a functor. We have $(j_2b)f_1 = j_2f_2a = q_2i_2a = (q_2\overline{a})i_1$. Therefore by the co-Cartesian property, b' exists so that the whole cube commutes. We may find $\overline{b}$ in

$$\begin{array}{ccc} T_1 & \xrightarrow{i_{T_1}} & \overline{T}_1 \\ \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle \overline{b}} \\ T_2 & \xrightarrow{i_{T_2}} & \overline{T}_2 \end{array}$$

and get a commutative cube:

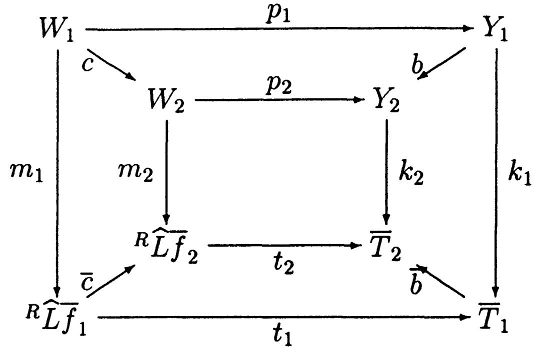


$$k_1 = i_{T_1} \circ j_1, \qquad k_2 = i_{T_2} \circ j_2$$
$$\overline{f}_1 = i_{T_1} \circ g_1, \qquad \overline{f}_2 = i_{T_2} \circ g_2$$

The functoriality of ${}^RL$ among weakly fibrant objects guarantees that $\overline{c}$ exists commuting :

$$\begin{array}{ccccc} \overline{X_1} & \xrightarrow{{}^RL'\overline{f_1}} & {}^R\widehat{L\overline{f_1}} & \xrightarrow{{}^RL''\overline{f_1}} & \overline{T_1} \\ \downarrow{\scriptstyle \overline{a}} & & \downarrow{\scriptstyle \overline{c}} & & \downarrow{\scriptstyle \overline{b}} \\ \overline{X_2} & \xrightarrow{{}^RL'\overline{f_2}} & {}^R\widehat{L\overline{f_2}} & \xrightarrow{{}^RL''\overline{f_2}} & \overline{T_2} \end{array}$$

Then we form the cube , with obvious notation :



where $t_1 = {}^RL''\overline{f}_1$, $t_2 = {}^RL''\overline{f}_2$. We compute:

$$^RL''\overline{f}_2 \circ (\overline{c}m_1) = \overline{b} \circ {}^RL''\overline{f}_1 \circ m_1 = \overline{b}k_1 \circ p_1 = k_2 \circ (bp_1)$$

The Cartesian property then gives a map c commuting the cube. The top face of this cube is the right-hand side of the diagram to be checked. To get the left-hand side, we must show that

$$\begin{array}{ccc} X_1 & \xrightarrow{q_1} & W_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle c} \\ X_2 & \xrightarrow{q_2} & W_2 \end{array}$$

commutes. The morphisms q_1, q_2 exist by the Cartesian property defining W_1, W_2 and are characterized by the equations

$$\begin{aligned} p_1 \circ q_1 &= f_1 \\ p_2 \circ q_2 &= f_2 \\ m_1 \circ q_1 &= {}^RL'\overline{f}_1 \circ i_1 \\ m_2 \circ q_2 &= {}^RL'\overline{f}_2 \circ i_2 \end{aligned}$$

We calculate

$$\begin{aligned} p_2 \circ (cq_1) &= bp_1q_1 \\ &= bf_1 \\ &= f_2a \\ &= p_2 \circ (q_2a) \end{aligned}$$

and

$$\begin{aligned} m_2 \circ (cq_1) &= \bar{c} m_1 q_1 \\ &= \bar{c}^R L' \overline{f}_1 i_1 \\ &= {}^R L' \overline{f}_2 \bar{a} i_1 \\ &= {}^R L' \overline{f}_2 i_2 a \\ &= m_2 \circ (q_2 a) \end{aligned}$$

The above two equations show that $c \circ q_1 = q_2 \circ a$ by the Cartesian property that defines W_2 □

4.3. Let $f : X \to Y$ be a morphism in $\mathcal{M}^{\mathbf{C}}$ and let $J : \mathbf{C}_0 \to \mathbf{C}$ be the inclusion of the discrete category of objects of a small category. We assume **RCM1a** holds in $\mathcal{M}$. We then have the adjunction

$$J^* \dashv J_* \text{ with unit and counit } \eta, \varepsilon$$

We now give the construction of the factorization ${}^R L$. This is done via an inductive procedure. We start from $f = f_0$, $X_0 = Y$. First, we weakly factor f_0 with the aid of the functor L giving the left factorization in $\mathcal{M}$:

$$Lf_0 : X \xrightarrow{L'f_0} \widehat{L}f_0 \xrightarrow{L''f_0} X_0 = Y$$

with $L'f_0 \in \mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$, $L''f_0 \in \mathrm{Fib}_w^{\mathbf{C}}$. We must convert the weak fibration into a strong fibration as requires in the right model structure $(\mathrm{Cof}_w^{\mathbf{C}}, \mathrm{Fib}_s^{\mathbf{C}})$.

We assume that $X_0, \ldots, X_n$ and $f_0, \ldots, f_n$ have been defined. Then we form the diagram

$$\begin{array}{ccccc} \widehat{L}f_n & = & \widehat{L}f_n & = & \widehat{L}f_n \\ \uparrow{\scriptstyle L'f_n} & & \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle L''f_n} \\ X & \xrightarrow{f_{n+1}} & X_{n+1} & \xrightarrow{x_{n+1}} & X_n \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle v_n} & & \downarrow{\scriptstyle \eta_{X_n}} \\ J_*J^*X & \xrightarrow{g_{n+1}} & J_*J^*\widehat{L}f_n & \xrightarrow{z_{n+1}} & J_*J^*X_n \end{array}$$

The square at lower right defines X_{n+1} as the fiber product of the other corners. The arrows v_n, x_{n+1} are the two projections from this product ; $g_{n+1} = J_*J^*L'f_n$, $z_{n+1} = J_*J^*L''f_n$. The arrow u_n exists with $v_n \circ u_n = \eta_{\widehat{L}f_n}$ since the equation $z_{n+1} \circ \eta_{\widehat{L}f_n} = \eta_{X_n} \circ L''f_n$ is true, being a reflection of the naturality of η. Also, f_{n+1} exists, with $x_{n+1} \circ f_{n+1} =$

f_n for the same reason because $z_{n+1} \circ g_{n+1} \circ \eta_X = \eta_{X_n} \circ f_n$ Iterating this we obtain a ladder :

$$\begin{array}{ccccccccc}
\cdots & = & X & = & X & = & X & = & X \\
 & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0 = f} \\
\cdots & \longrightarrow & X_3 & \xrightarrow{x_3} & X_2 & \xrightarrow{x_2} & X_1 & \xrightarrow{x_1} & X_0
\end{array}$$

We define :

$$\begin{aligned}
{}^R\widehat{L}f &:= \lim X_n \\
{}^RL'f &:= \lim f_n \\
{}^RL''f &:= \lim x_n
\end{aligned}$$

This is clearly functorial in f, and we claim that under suitable hypotheses on $\mathcal{M}$ to be made explicit below, this is a left factorization in the right model structure on $\mathcal{M}^{\mathbf{C}}$. ie.

$${}^RL'f \in \mathrm{Cof}^{\mathbf{C}}_w \cap \mathcal{E}^{\mathbf{C}} \text{ and } {}^RL''f \in \mathrm{Fib}^{\mathbf{C}}_s$$

The following dualizes [11, II. Lemma 4.4] :

Proposition 4.4. *Let $\mathcal{M}$ be a functorial closed model category and assume that axioms* **LCM1a** *and* **RCM1a** *hold in $\mathcal{M}$. Then in the above notation :*

1. *All $z_n \in \mathrm{Fib}^{\mathbf{C}}_s$; all $x_n \in \mathrm{Fib}^{\mathbf{C}}_s$.*
2. *${}^RL''f \in \mathrm{Fib}^{\mathbf{C}}_s$.*
3. *If f is in $\mathcal{E}^{\mathbf{C}}$ then*

$${}^RL''f \in \mathcal{E}^{\mathbf{C}} \textit{ and } \mathrm{Cof}^{\mathbf{C}}_w \vdash\!\!\!+\!\!\!\longrightarrow {}^RL''f$$

Proof. (1) By definition of L, $L''f_n \in \mathrm{Fib}^{\mathbf{C}}_w$. Thus, $J^*L''f_n \in \mathrm{Fib}^{\mathbf{C}_0}$, and therefore $z_{n+1} = J_*J^*L''f_n \in \mathrm{Fib}^{\mathbf{C}}_s$ by proposition 4.1, and $x_{n+1} \in \mathrm{Fib}^{\mathbf{C}}_s$ by base change (lemma 3.1).

(2) This follows from (1), the definition of strong fibrations, and corollary 2.3.

(3) If $f = f_0 \in \mathcal{E}^{\mathbf{C}}$ then $L''f_0 \in \mathcal{E}^{\mathbf{C}}$ by **CM2** since $L'f_0 \in \mathcal{E}^{\mathbf{C}}$ by the definition of L and of the weak equivalences in $\mathcal{M}^{\mathbf{C}}$. Therefore, $J^*L''f_0 \in \mathrm{Fib}^{\mathbf{C}_0} \cap \mathcal{E}^{\mathbf{C}_0}$ and proposition 4.1 gives that

$$z_1 = J_*J^*L''f_0 \in \mathrm{Fib}^{\mathbf{C}}_s \cap \mathcal{E}^{\mathbf{C}}$$

and x_1 being deduced from z_1 by base change is also an acyclic strong fibration. Because of the equation $f_0 = x_1 \circ f_1$ and **CM2** we get that $f_1 \in \mathcal{E}^{\mathbf{C}}$ and we may repeat this argument with f_0 replaced by f_1. This continues inductively to give

$$x_n,\ z_n \in \mathrm{Fib}^{\mathbf{C}}_s \cap \mathcal{E}^{\mathbf{C}} \text{ for all } n \geq 0$$

But then by proposition 3.1

$$\xi_0 \;=\; \lim\, x_n \;=\; {}^{R}L''f \;\in\; \mathrm{Fib}^{\mathbf{C}}_{s} \cap \mathcal{E}^{\mathbf{C}}$$

To prove the second part of (3) , the previous argument has shown that $f_n \in \mathcal{E}^{\mathbf{C}}$ so that $L''f_n \in \mathrm{Fib}^{\mathbf{C}}_{w} \cap \mathcal{E}^{\mathbf{C}}$ by definition of L and **CM2** . Hence,

$$J^*L''f_n \;\in\; \mathrm{Fib}^{\mathbf{C}_0} \cap \mathcal{E}^{\mathbf{C}_0}$$

which by proposition 4.1 gives

$$z_{n+1} \;=\; J_*J^*L''f_n \;\in\; \left(\mathrm{Cof}^{\mathbf{C}}_{w}\right)^{++-}$$

Thus,

$$x_{n+1} \;=\; J_*J^*L''f_n \;\in\; \left(\mathrm{Cof}^{\mathbf{C}}_{w}\right)^{++-}$$

by base change (lemma 3.1). Then corollary 2.3 gives

$${}^{R}L'f \;=\; \lim\, x_n \;\in\; \left(\mathrm{Cof}^{\mathbf{C}}_{w}\right)^{++-}$$

□

Proposition 4.5. *Continuing the notation from above, assume that one of the following conditions holds :*

1. **LCM1a** , **RCM1a** , **RCM2f** *hold in $\mathcal{M}$.*
2. **LCM1a** , **RCM1a** , **RCM2ff** *hold in $\mathcal{M}$, and both X and Y are weakly fibrant.*

Then, ${}^{R}L'f \;\in\; \mathrm{Cof}^{\mathbf{C}}_{w} \cap \mathcal{E}^{\mathbf{C}}$

Proof. We apply J^* to the diagram 4.1 and expand it by adding a downstairs :

$$\begin{array}{ccccc}
J^*\widehat{L}f_n & = & J^*\widehat{L}f_n & = & J^*\widehat{L}f_n \\
\uparrow{\scriptstyle J^*L'f_n} & & \downarrow{\scriptstyle J^*u_n} & & \downarrow{\scriptstyle J^*L''f_n} \\
J^*X_n & \xrightarrow{J^*f_{n+1}} & J^*X_{n+1} & \xrightarrow{J^*x_{n+1}} & J^*X_n \\
\downarrow{\scriptstyle J^*\eta} & & \downarrow{\scriptstyle J^*v_n} & & \downarrow{\scriptstyle J^*\eta_{X_n}} \\
J^*J_*J^*X & \xrightarrow{J^*g_{n+1}} & J^*J_*J^*\widehat{L}f_n & \xrightarrow{J^*z_{n+1}} & J^*J_*J^*X_n \\
\downarrow{\scriptstyle \varepsilon_{J^*X}} & & \downarrow{\scriptstyle \varepsilon_{J^*\widehat{L}f_n}} & & \downarrow{\scriptstyle \varepsilon_{J^*X_n}} \\
J^*X & \xrightarrow{J^*L'f_n} & J^*\widehat{L}f_n & \xrightarrow{J^*L''f_n} & J^*X_n
\end{array}$$

Because η, ε are the unit and counit of an adjunction, all three vertical columns compose to the identity. (Recall that $v_n \circ u_n \;=\; \eta_{\widehat{L}f_n}$). Now apply J^* to the diagram 4.1 and interpolate terms from the above diagram to obtain

$$\begin{array}{ccccccccc}
\cdots = & J^*X & = & J^*X & = & J^*X & = & J^*X \\
& \downarrow{\scriptstyle J^*L'f_1} & & \downarrow{\scriptstyle J^*f_1} & & \downarrow{\scriptstyle J^*L'f_0} & & \downarrow{\scriptstyle J^*f_0=J^*f} \\
\cdots \longrightarrow & J^*\widehat{L}f_1 & \xrightarrow{q_1} & J^*X_1 & \xrightarrow{v_0^\sharp} & J^*\widehat{L}f_0 & \xrightarrow{q_0} & J^*X_0 = J^*Y
\end{array}$$

where

$$q_n = J^*L''f_n$$
$$v_n^\sharp = \varepsilon_{J_*\widehat{L}f_n} \circ J^*v_n$$

We check the commutativities. The equation

$$q_n \circ J^*L'f_n \stackrel{?}{=} J^*f_n$$

is true because it is J^* applied to the equation $f_n = L''f_n \circ L'f_n$. The equation

$$v_n^\sharp \circ J^*f_{n+1} \stackrel{?}{=} J^*L'f_n$$

is true because

$$\begin{aligned}
\varepsilon_{J_*\widehat{L}f_n} \circ J^*v_n \circ J^*f_{n+1} &= \varepsilon_{J_*\widehat{L}f_n} \circ J^*\left(v_n \circ f_{n+1}\right) \\
&= \varepsilon_{J_*\widehat{L}f_n} \circ J^*\left(g_{n+1} \circ \eta_X\right) \\
&= \varepsilon_{J_*\widehat{L}f_n} \circ J^*g_{n+1} \circ J^*\eta_X \\
&= J^*L'f_n \circ \varepsilon_{J_*X} \circ J^*\eta_X \\
&= J^*L'f_n
\end{aligned}$$

where we have repeatedly used the identities from diagram 4.1. To check that ${}^RL'f = \lim f_n$ is an acyclic weak fibration we need only verify that this is true pointwise for all c, or equivalently, that

$$J^{*R}L'f = J^*(\lim f_n) = \lim J^*f_n$$

has this property. But in the ladder diagram, the arrows $J^*L'f_n$ are *final* among the vertical arrows, so :

$$J^{*R}L'f = \lim J^*L'f_n$$

Let us observe that :

1. All the horizontal arrows in the ladder above are fibrations.
   Because: For $q_n = J^*L''f_n$ it is true by the definition of L'' and of J^*. For $v_n^\sharp$, we appeal to the definition of v_n in 4.1 and to lemma 4.2.

2. If we assume that Y to start out with is weakly fibrant, then J^*X_n, $J^*\widehat{L}f_n$ are fibrant for all $n \geq 0$.

   Because: it is true for n = 0, since $X_0 = Y$, and q_0 is a fibration by part 1 above . The result follows easily by induction again using part 1 .

Each $(L'f_n)_c$ is a weak equivalence by the definition of L' . Assuming **LCM1a** , **RCM1a** and **RCM2f** , (1) above now suffices to show that

$$\lim\ (L'f_n)_c \in \mathrm{Cof} \cap \mathcal{E}$$

which was to have been shown. Alternatively, assuming that both X and Y are weakly fibrant, and axioms **LCM1a** , **RCM1a** and **RCM2ff** , (1) and (2) show that the exact same conclusion can be drawn. □

In view of lemma 4.3 we have thus proved half of the following, the other half being strictly dual :

Proposition 4.6. *Let $\mathcal{M}$ be a functorial closed model category.*

1. *The factorization ${}^LR$ exists in the left model structure on $\mathcal{M}^{\mathbf{C}}$ given by* $(\mathrm{Cof}_s^{\mathbf{C}}, \mathrm{Fib}_w^{\mathbf{C}})$ *if any of the following sets of axioms hold*
   (a) **LCM1a** , **RCM1a** *and* **LCM2c** .
   (b) **LCM1a** , **RCM1a** , **LCM2cc** , **LCM4** *and the cancellation property*

   $$g \circ f \in \mathrm{Fib} \cap \mathcal{E} \text{ and } f \in \mathcal{E} \Rightarrow g \in \mathrm{Fib} \cap \mathcal{E}$$

   This holds for example if Fib = Epi .
2. *The factorization ${}^RL$ exists in the right model structure on $\mathcal{M}^{\mathbf{C}}$ given by* $(\mathrm{Cof}_w^{\mathbf{C}}, \mathrm{Fib}_s^{\mathbf{C}})$ *if any of the following sets of axioms hold*
   (a) **LCM1a, RCM1a** *and* **RCM2f** .
   (b) **LCM1a** , **RCM1a** , **RCM2f** , **RCM4** *and the cancellation property*

   $$g \circ f \in \mathrm{Cof} \cap \mathcal{E} \text{ and } g \in \mathcal{E} \Rightarrow f \in \mathrm{Cof} \cap \mathcal{E}$$

   This holds for example if Cof = Mono .

Corollary 4.7. 1. *Under the hypotheses of 4.6.1 ,*

$$\mathrm{Cof}_s^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}} \vdash\!\!\!\!\longrightarrow \mathrm{Fib}_w^{\mathbf{C}}$$

2. *Under the hypotheses of 4.6.2 ,*

$$\mathrm{Cof}_w^{\mathbf{C}} \vdash\!\!\!\!\longrightarrow \mathrm{Fib}_s^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$$

Proof. We do the second one. Let $p : E \longrightarrow B$ be in $\mathrm{Fib}_s^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$. We've constructed a factorization

$$p = {}^{R}L''p \circ {}^{R}L'p \quad \text{with} \quad {}^{R}L'p \in \mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}},\ {}^{R}L''p \in \mathrm{Fib}_s^{\mathbf{C}}$$

and ${}^{R}L''p$ is a weak equivalence as well, since p is. We show that p is a retract of ${}^{R}L''p$. We have a lifting in

$$\begin{array}{ccc}
E & \xrightarrow{\quad 1 \quad} & E \\
{\scriptstyle {}^{R}L'p}\downarrow & \nearrow{\scriptstyle k?} & \downarrow{\scriptstyle p} \\
{}^{R}\widehat{L}p & \xrightarrow[{}^{R}L''p]{} & B
\end{array}$$

since $p \in \mathrm{Fib}_s^{\mathbf{C}} = \left(\mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}\right)^{\mathrel{+\!\!+\!-}}$. We get

$$\begin{array}{ccccc}
E & \xrightarrow{{}^{R}L'p} & {}^{R}\widehat{L}p & \xrightarrow{k} & E \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle {}^{R}L''p} & & \downarrow{\scriptstyle p} \\
B & = & B & = & B
\end{array}$$

showing that p is indeed a retract of ${}^{R}L''p$. But proposition 4.4 gives that

$${}^{R}L''p \mathrel{+\!\!+\!-} \mathrm{Cof}_w^{\mathbf{C}}$$

and therefore 2.1 gives

$$p \mathrel{+\!\!+\!-} \mathrm{Cof}_w^{\mathbf{C}}$$

as required. □

Theorem 4.8. *Let $\mathcal{M}$ be a functorial closed model category.*

1. *Suppose that either one of the following sets of axioms is valid in $\mathcal{M}$:*
   (a) **LCM1a** , **RCM1a** *and* **LCM2c** .
   (b) **LCM1a** , **RCM1a** , **LCM2cc** , **LCM4** *and the cancellation property*

   $$g \circ f \in \mathrm{Fib} \cap \mathcal{E} \text{ and } f \in \mathcal{E} \Rightarrow g \in \mathrm{Fib} \cap \mathcal{E}$$

   This holds for example if Fib = Epi .

   Then $\mathcal{M}^{\mathbf{C}}$ is of a functorial closed model category in the left structure given by $\left(\mathrm{Cof}_s^{\mathbf{C}},\ \mathrm{Fib}_w^{\mathbf{C}}\right)$.
2. *Suppose that either one of the following sets of axioms is valid in $\mathcal{M}$:*
   (a) **LCM1a** , **RCM1a** *and* **RCM2f** .

(b) **LCM1a , RCM1a , RCM2ff , RCM4** *and the cancellation property*

$$g \circ f \in \mathrm{Cof} \cap \mathcal{E} \text{ and } g \in \mathcal{E} \Rightarrow f \in \mathrm{Cof} \cap \mathcal{E}$$

This holds for example if Cof = Mono .

Then $\mathcal{M}^{\mathbf{C}}$ *is of a functorial closed model category in the right structure given by* $\left(\mathrm{Cof}_w^{\mathbf{C}},\ \mathrm{Fib}_s^{\mathbf{C}}\right)$.

5. Heller's Axioms

5.1. Let $\mathcal{M}$ be a functorial closed model category satisfying either of the hypotheses of 4.8.1 (resp. 4.8.2) . Then the main result of the previous section was to show that $\mathcal{M}^{\mathbf{C}}$ had the structure of a functorial closed model category, and in particular, the homotopy category

$$T(\mathbf{C}) = \mathrm{Ho}(\mathcal{M}^{\mathbf{C}})$$

exists. We let

$$\gamma^{\mathbf{C}} : \mathcal{M}^{\mathbf{C}} \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathbf{C}})$$

be the canonical localization, which we regard as a *strict* localization. This means (cf. [11, p. 7]) that $\gamma^{\mathbf{C}}$ is a bijection on objects, and we can describe the morphisms as follows. Given X, Y in $\mathcal{M}^{\mathbf{C}}$, we hve a bijection

$$\mathrm{Ho}(\mathcal{M}^{\mathbf{C}})\ (X,\ Y) \simeq \pi(\mathcal{M}^{\mathbf{C}})\ (\overline{X},\ \underline{Y})$$

where $\overline{X}$ is a cofibrant model of X, and where $\underline{Y}$ is a fibrant model of Y, with the right-hand side above denoting the homotopy equivalence classes of morphisms, the right and left homotopy notions coinciding for cofibrant $\overline{X}$ and fibrant $\underline{Y}$. Picture:

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle u} \\ \overline{X} & \xrightarrow{g} & \underline{Y} \\ \downarrow{\scriptstyle p} & & \\ X & & \end{array}$$

Then each f in the homotopy category is represented by a word

$$(\gamma^{\mathbf{C}} u) \circ (\gamma^{\mathbf{C}} g) \circ (\gamma^{\mathbf{C}} p)^{-1}$$

with g uniquely determined up to homotopy. Up to isomorphism, this is independent of the choice of resolutions, $\overline{X}$, $\underline{Y}$, but we can make functorial choices, eg. in the right model structure, as follows :

$$^{R}R(\phi \to X) : \phi \hookrightarrow \overline{X} = {}^{R}\widehat{R}(\phi \to X) \xrightarrow{\sim}\!\!\!\!\rightarrow X$$

$$^{R}L(Y \to e) : Y \overset{\sim}{\hookrightarrow} \underline{Y} = {}^{R}\widehat{L}(Y \to e) \twoheadrightarrow e$$

We can see that $\mathbf{C} \to T(\mathbf{C})$ is a hyperfunctor. This means that given $F : \mathbf{C} \to \mathbf{D}$ there is a functor $TF : T(\mathbf{D}) \to T(\mathbf{C})$ which satisfies several properties relative to composition and natural transformations of functors between small categories (cf. [11, p. 11]). That TF exists results from the fact that

$$\mathcal{M}^{F} = F^{*} : \mathcal{M}^{\mathbf{D}} \to \mathcal{M}^{\mathbf{C}}$$

preserves weak equivalences, which is immediate from the pointwise definition of the weak equivalences in these functor categories, and hence localizes to give a morphism on the fraction categories. The other properties of hyperfunctors are easy to verify. We now turn to the axioms of [11]. We do them explicitly in the right structure, the left structure being dual.

H0. *For any family* $\mathbf{C}_\alpha$ *of small categories, the canonical map*

$$T\left(\coprod_{\alpha} \mathbf{C}_\alpha\right) \longrightarrow \prod_{\alpha} T(\mathbf{C}_\alpha)$$

is an equivalence.

This is essentially clear from the equation

$$\mathcal{M}^{\coprod_{\alpha}\mathbf{C}_\alpha} = \prod_{\alpha} \mathcal{M}^{\mathbf{C}_\alpha}$$

and all the model structures are compatible with this.

H1. *For any* $\mathbf{C}$ *the functor*

$$\mathrm{dgm}_{\mathbf{C}} : T\mathbf{C} \longrightarrow (T\mathbf{1})^{\mathbf{C}}$$

reflects isomorphisms.

That is, $\mathrm{dgm}_{\mathbf{C}}\, f$ is an isomorphism $\Rightarrow$ f is an isomorphism. Recall that $\mathrm{dgm}_{\mathbf{C}}$ is defined on objects as

$$(\mathrm{dgm}_{\mathbf{C}}\ X)_c = X_c$$

or more precisely as

$$(\mathrm{dgm}_{\mathbf{C}}\ \gamma^{\mathbf{C}} X)_c = \gamma(X_c)$$

and if $f : \gamma^{\mathbf{C}} X \to \gamma^{\mathbf{C}} Y$ is a morphism in $T(\mathbf{C})$, represented by the homotopy class of $g : \overline{X} \to \underline{Y}$, in the notations as above, then

$$(\mathrm{dgm}_{\mathbf{C}}\, f)_c = (\gamma u_c) \circ (\gamma g_c) \circ (\gamma p_c)^{-1}$$

If $\mathrm{dgm}_{\mathbf{C}}\, f$ is an isomorphism, then γg_c is an isomorphism for all c, because both γp_c and γu_c are isomorphisms. In a closed model category, the weak equivalences are precisely the maps that become inverted in the homotopy category (cf. theorem 2.7). Therefore, $g_c \in \mathcal{E}$ for all $c \Rightarrow g \in \mathcal{E}^{\mathbf{C}} \Rightarrow \gamma^{\mathbf{C}} g$ is an isomorphism $\Rightarrow$

$$f = (\gamma^{\mathbf{C}} u) \circ (\gamma^{\mathbf{C}} g) \circ (\gamma^{\mathbf{C}} p)^{-1}$$

is an isomorphism also.

H2. *If* $\mathbf{F}$ *is a finite free category, then for all small categories* $\mathbf{C}$

$$\mathrm{dgm}_{\mathbf{F}} \; : \; T(\mathbf{C} \times \mathbf{F}) \longrightarrow (T\mathbf{C})^{\mathbf{F}}$$

is a weak quotient functor.

This means two things :

1. $\mathrm{dgm}_{\mathbf{F}}$ is *full*, ie. for all X, Y

$$\mathrm{Ho}(\mathcal{M}^{\mathbf{C}\times\mathbf{F}})(X,\, Y) \longrightarrow \left(\mathrm{Ho}(\mathcal{M}^{\mathbf{C}})\right)^{\mathbf{F}} (\mathrm{dgm}_{\mathbf{F}} X,\; \mathrm{dgm}_{\mathbf{F}} Y)$$

   is onto.

2. $\mathrm{dgm}_{\mathbf{F}}$ is *replete*, ie. every object of $\left(\mathrm{Ho}(\mathcal{M}^{\mathbf{C}})\right)^{\mathbf{F}}$ is isomorphic with $\mathrm{dgm}_{\mathbf{F}} X$ for some object X in $\mathrm{Ho}(\mathcal{M}^{\mathbf{C}\times\mathbf{F}})$

Since $\pi(\mathcal{M}^{\mathbf{C}\times\mathbf{F}})\ (\overline{X},\ \underline{Y})$ is a quotient of $\mathcal{M}^{\mathbf{C}\times\mathbf{F}}\ (\overline{X},\ \underline{Y})$, (1) will clearly follow from

(1′) For all $X,\ Y$ in $\mathcal{M}^{\mathbf{C}\times\mathbf{F}}$, with X weakly cofibrant and Y strongly fibrant,

$$\mathcal{M}^{\mathbf{C}\times\mathbf{F}}(X,\, Y) \longrightarrow \left(\mathrm{Ho}(\mathcal{M}^{\mathbf{C}})\right)^{\mathbf{F}} (\mathrm{dgm}_{\mathbf{F}} X,\; \mathrm{dgm}_{\mathbf{F}} Y)$$

is onto.

A finite free category is determined by its *graph* Γ whose vertices consist of the the objects of $\mathbf{F}$, and whose edges are the generating morphisms of $\mathbf{F}$. Observe that in a *finite* free category, there is no non-identity morphism from an object to itself, for otherwise its iterates, which must all be distinct, would be infinite in number. We introduce a partial order on $\mathbf{F}_0$ via

$$f \leq g \;\Leftrightarrow\; \mathbf{F}(f,\, g) \neq \phi$$

Notice that $f \leq g$ and $g \leq f \;\Rightarrow\; f = g$ because if $f \neq g$ then the composed arrow $f \longrightarrow g \longrightarrow f$ would either be id_f in which case the category would not be free or it would be id_f, but that's excluded by our previous remarks.

Now given a functor $X : \mathbf{C} \times \mathbf{F} \longrightarrow \mathcal{M}$ we can regard it as a functor $X : \mathbf{F} \longrightarrow \mathcal{M}^{\mathbf{C}}$, and we designate it by the same letter, as this should cause no confusion. Also, we let $\mathcal{M}_1$ denote $\mathcal{M}^{\mathbf{C}}$ so that $\mathcal{M}_1^{\mathbf{F}} = \mathcal{M}^{\mathbf{C}\times\mathbf{F}}$, and we will generally use the subscript 1 to refer to constructs in $\mathcal{M}_1$ so that for example $\mathcal{E}_1$ is the class of weak equivalences in $\mathcal{M}_1$. This understood, we introduce the following condition :

Definition 5.1. An object X in $\mathcal{M}^{\mathbf{C}} = \mathcal{M}_1$ is said to verify (*) if for all $f \in \mathbf{F}_0$ the map

$$\xi \; : \; X_f \;\longrightarrow\; \prod_{\gamma\in\Gamma(f,g)} X_g$$

with components $pr_\varphi \circ \xi \;=\; X_\varphi$ is a strong fibration in $\mathcal{M}^{\mathbf{C}}$

The following dualizes [11, II. Prop. 5.1]

Proposition 5.1. *Let* $X : \mathbf{C} \times \mathbf{F} \;\longrightarrow\; \mathcal{M}$.

1. *If X has property (*) then X is in* $\mathrm{Fib}_s^{\mathbf{C}\times\mathbf{F}}$.
2. *For any Y in $\mathcal{M}^{\mathbf{C}\times\mathbf{F}}$ there exists a morphism $Y \;\longrightarrow\; X$ in* $\mathrm{Cof}_w^{\mathbf{C}\times\mathbf{F}} \cap \mathcal{E}^{\mathbf{C}\times\mathbf{F}}$, *such that X has property (*).*
3. *Therefore, an object of $\mathcal{M}^{\mathbf{C}\times\mathbf{F}}$ has property (*) if and only if it is strongly fibrant.*
4. *For any $W : \mathbf{F} \;\longrightarrow\; \mathrm{Ho}(\mathcal{M}^{\mathbf{C}})$ there exists a strongly fibrant X in $\mathcal{M}^{\mathbf{C}\times\mathbf{F}}$ with* $\mathrm{dgm}_{\mathbf{F}} X \;\simeq\; W$.
5. *If X is weakly cofibrant and Y is strongly fibrant then*

$$\mathcal{M}^{\mathbf{C}\times\mathbf{F}}(X,\; Y) \;\longrightarrow\; \left(\mathrm{Ho}(\mathcal{M}^{\mathbf{C}})\right)^{\mathbf{F}} (\mathrm{dgm}_{\mathbf{F}} X,\; \mathrm{dgm}_{\mathbf{F}} Y)$$

is onto.

Statements (4) and (5) prove axiom **H2** as we've remarked.

Proof. The proofs are all by induction on the number of objects in $\mathbf{F}$. Each statement is seen to be true when $\#\mathbf{F} \;=\; 1$, for in that case $\mathbf{F} \;=\; \mathbf{1}$, and if $*$ is the unique object, then $\Gamma(*,\; *) \;=\; \phi$ and the corresponding product in the definition of property (*) reduces to $\prod_\phi \;=\; e \;=\;$ the terminal object, and to say that X has property (*) merely says that X is a fibrant object in $\mathcal{M}^{\mathbf{C}}$. The remaining statements to be proved follow from general properties of model categories.

We therefore assume that $\#\mathbf{F} > 1$. Let $m \in \mathbf{F}$ be a minimal element for the partial order that was introduced above, and let $\mathbf{F}' \;=\; \mathbf{F}\setminus\{m\}$ be the full subcategory on the remaining objects. If X is a functor with domain $\mathbf{F}$, then X' will denote the corresponding restriction to $\mathbf{F}'$ so that in this notation, $X_f = X'_f$ for $f \in \mathbf{F}'$.

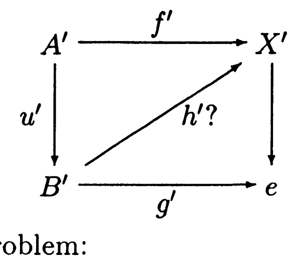
(1) What must be shown is that if X satisfies property (*), then

$$(X \;\longrightarrow\; e) \;\in\; \left(\mathrm{Cof}_w^{\mathbf{C}\times\mathbf{F}} \cap \mathcal{E}^{\mathbf{C}\times\mathbf{F}}\right)^{\leftarrow\!\!\!+\!\!\!-}$$

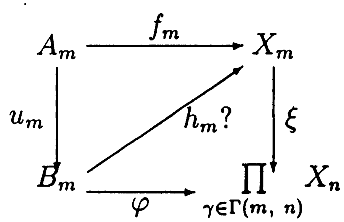
Let $u : A \longrightarrow B$ be an acyclic cofibration and

$$(f,\ g) : u \longrightarrow (X \longrightarrow e).$$

By induction hypothesis, there's a lifting in



Now consider the lifting problem:



where φ is defined by the equation $pr_\varphi = h'_g \circ B_\gamma$ (recall that $\Gamma(m,\ g) \neq \phi \Rightarrow m < g \Rightarrow h'_g$ is defined.) Since (*) is assumed, ξ is in $\mathrm{Fib}_s^{\mathbf{C}}$, and because $i_m \in \mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ there is a lifting h_m. We then define $h : B \longrightarrow X$ by

$$h_g = \begin{cases} h'_g \text{ if } m < g; \\ h_m \text{ otherwise.} \end{cases}$$

We assert that it is a morphism of functors and that it is a lifting.

To see that it is functorial, since any morphism in **F** is uniquely a word in the edges of Γ, it will suffice to consider a $\gamma : p \longrightarrow q$ that is an edge of that graph, and there are three possibilities to consider :

1. $m < p < q$. Then in the equation to be checked

$$X_\gamma \circ h_p = h_q \circ B_\gamma$$

   $h = h'$ and h' was assumed by induction to be a morphism.
2. $m = p = q$. But then, $\gamma = id_m$ as we've already remarked, and the result is trivial.
3. $m = p < q$. In this case, γ is one of the factors appearing in the product in the diagram that defined h_m. Applying pr_γ to the equation there

$$\xi \circ h_m = \varphi$$

gives $X_\gamma \circ h_m$ on the left, by the definition of ξ, and on the right it gives $h'_q \circ B_\gamma = h_q \circ B_\gamma$, by the definition of φ. This is what we want.

That h is a lifting is checked pointwise for all values in $\mathbf{F}$, and it true for all $q > m$ by induction. For $q = m$, it is the equation $h_m \circ i_m = f_m$ in the defining diagram for h_m.

(2) Let Y be given, and suppose inductively that a $u' : Y' \longrightarrow X'$ with $u' \in \mathrm{Cof}^{\mathbf{F}'}_{1,\,w} \cap \mathcal{E}^{\mathbf{F}'}_1$, and X' satisfying (*) relative to $\mathbf{F}'$ has been found. Consider the composite

$$Y_m \xrightarrow{\eta} \prod_{\gamma\in\Gamma(m,\,g)} Y_g \xrightarrow{v} \prod_{\gamma\in\Gamma(m,\,g)} X'_g$$

where $v = \prod_\gamma u'_g$, and η is defined by $pr_\gamma \circ \eta = Y_\gamma$. We may factor this in the model category $\mathcal{M}_1$ as

$$Y_m \xrightarrow{u_m} X_m \xrightarrow{\xi} \prod_{\gamma\in\Gamma(m,\,g)} X'_g$$

defining X_m and $u_m \in \mathrm{Cof}^{\mathbf{C}}_w \cap \mathcal{E}^{\mathbf{C}}$, $\xi \in \mathrm{Fib}^{\mathbf{C}}_s$. We define a functor

$$X : \mathbf{F} \longrightarrow \mathcal{M}_1$$

on objects as

$$X_g = \begin{cases} X'_g \text{ if } m < g; \\ X_m \text{ as defined above, if } m = g. \end{cases}$$

and to define it on morphisms, we need only define it on the generating morphisms in Γ and we may do so arbitrarily because of the freeness of $\mathbf{F}$. If $\gamma : p \longrightarrow q$ is an edge in Γ, and if $m < p \leq q$ then X_γ has already been defined as X'_γ, whereas for $m = p$, γ appears as one of the factors in the product in the diagram defining ξ and we may set $X_\gamma = pr_\gamma \circ \xi$. This is clearly a functor, because of the freeness of $\mathbf{F}$. We define a morphism $u : Y \longrightarrow X$ by the rule

$$u_g = \begin{cases} u'_g \text{ if } m < g; \\ u_m \text{ as defined above, if } m = g. \end{cases}$$

and one must verify that it is indeed a morphism. By the reasoning given in the first part of this proposition, in the equation to be checked :

$$X_\gamma \circ u_p = u_q \circ Y_\gamma$$

we can assume that $\gamma : p \longrightarrow q$ is an edge in the generating graph Γ, and it divides into three cases as before. For $m < p < q$ it holds by induction; for $m = p = q$ it is true because $\gamma = id_m$; for $m = p < q$, γ

is one of the factors in the diagram defining u_m, and applying pr_γ to the equation

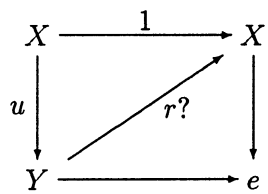
$$\xi \circ u_m = \left(\prod_\delta u_g\right) \circ \eta$$

coming from there gives an equation between $X_\gamma \circ u_m$ on the left side and and

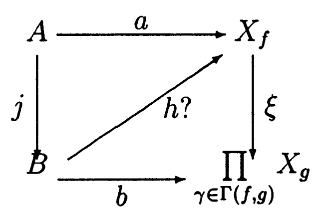
$$pr_\gamma \circ \left(\prod_\delta u_g\right) = u_q \circ pr_\gamma \circ \eta = u_q \circ Y_\gamma$$

on the right.

(1) & (2) $\Rightarrow$ (3) Let Y be a strongly fibrant element of $\mathcal{M}^{\mathbf{C}\times\mathbf{F}}$. Regarded as an element of $\mathcal{M}_1^{\mathbf{F}}$ it is also strongly fibrant. This is because the strong fibrations are defined by a transversality condition relative to the acyclic cofibrations, and those are defined by a pointwise condition, so that (the model structure on $\mathcal{M}_1 = \mathcal{M}^{\mathbf{C}}$ being the right structure) a morphism is an acyclic cofibration whether it is regarded as an element of $\mathcal{M}_1^{\mathbf{F}}$ or $\mathcal{M}^{\mathbf{C}\times\mathbf{F}}$. Let X be a strongly fibrant element of $\mathcal{M}^{\mathbf{C}\times\mathbf{F}}$. Now by (2) we can find a map $u : X \longrightarrow Y$ with u an acyclic cofibration and Y having property (*). But by (1), Y is strongly fibrant as well, and u therefore has a retraction because a lift r exists in



We will verify that X has property(*) by showing the existence of a lift in every diagram



We expand this to

$$\begin{array}{ccccccc}
A & \xrightarrow{a} & X_f & \xrightarrow{u_f} & Y_f & \xrightarrow{r_f} & X_f \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \xi} \\
B & \xrightarrow{b} & \prod_{\gamma\in\Gamma(f,g)} X_g & \xrightarrow{\prod u_g} & \prod_{\gamma\in\Gamma(f,g)} Y_g & \xrightarrow{\prod r_g} & \prod_{\gamma\in\Gamma(f,g)} X_g
\end{array}$$

ξ and η are defined by the formulas $pr_\gamma \circ \xi = X_\gamma$ and $pr_\gamma \circ \eta = Y_\gamma$. It is easy to see the commutativity of the above diagram. The arrow

$$k : B \longrightarrow Y_f$$

exists because, Y having property (*), η is a strong fibration. Define $h = r_f \circ k$. This is the desired lift.

(4) Let $W : \mathbf{F} \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathbf{C}})$ be given. For each $f \in \mathbf{F}$ we have an object W_f of $\mathcal{M}_1$ and for each morphism $\alpha : f \longrightarrow g$ we have a map $W_\alpha : W_f \Rightarrow W_g$ in $\mathrm{Ho}(\mathcal{M}_1)$ where the double arrow indicates that we are thinking of it as a finite composition of actual maps $\longrightarrow$ and of arrows $\longleftarrow$ in $\mathcal{E}_1$. We can find an isomorphism $W \simeq \widetilde{W}$, such that each $\widetilde{W}_f$ is bifibrant, and the W_α will induce $\widetilde{W}_\alpha$, and it is well known (abstract Whitehead theorem cf. 2.7) that $\widetilde{W}_\alpha$ will be represented by an actual map $\longrightarrow$. We can therefore assume from the outset that each W_α is an actual morphism from $\mathcal{M}_1$, well-defined up to homotopy. In the notations of the preceding proof, we suppose inductively that X' has been found with $\mathrm{dgm}_{\mathbf{F}'} X' \simeq W'$. In fact, we assume that

$$u_f : W_f \longrightarrow X'_f$$

with $u_f \in \mathrm{Cof}_1 \cap \mathcal{E}_1$ has been found for each $f \in \mathbf{F}'$ such that

$$\begin{array}{ccc} W_f & \xrightarrow{W_\alpha} & W_g \\ \downarrow{\scriptstyle u_f} & & \downarrow{\scriptstyle u_g} \\ X'_f & \xrightarrow{X'_\alpha} & X'_g \end{array}$$

is homotopy-commutative for all $\alpha : f \longrightarrow g$ in $\mathbf{F}'$. We define X_m (m : minimal element) by factoring

$$W_m \xrightarrow{\omega} \prod_{\gamma \in \Gamma(m,f)} W_f \xrightarrow{v} \prod_{\gamma \in \Gamma(m,f)} X'_f$$

with $v = \prod u'_f$, and ω defined by $pr_\gamma \circ \omega = W_\gamma$ into

$$W_m \xrightarrow{u_m} X_m \xrightarrow{\xi} \prod_{\gamma \in \Gamma(m,f)} X'_f$$

with $u_m \in \mathrm{Cof}_{1,w} \cap \mathcal{E}_1$ and $\xi \in \mathrm{Fib}_{1,s}$. We then define a functor $X : \mathbf{F} \longrightarrow \mathrm{Ho}(\mathcal{M}_1)$ by the rule

$$X_f = \begin{cases} X'_f \text{ if } m < f; \\ X_m \text{ as defined above if } m = f. \end{cases}$$

and we define X_γ to be X'_γ for a morphism in $\mathbf{F}'$ and to be $pr_\gamma \circ \xi$ for $\gamma \in \Gamma(m, f)$ and as Γ freely generates $\mathbf{F}$, this specifies X as a functor. To see that $u : X \longrightarrow W$ defined by

$$u_f = \begin{cases} u'_f \text{ if } m < f; \\ u_m \text{ as defined above if } m = f. \end{cases}$$

is an isomorphism in $\mathrm{Ho}(\mathcal{M}_1)$, it is clear in view of the nature of the maps u_f that it will suffice to prove merely that it is a morphism, or in other words, that the appropriate squares are homotopy commutative. As previously argued, the only really new case to be considered is that of an edge $\gamma : m \longrightarrow f$ in Γ. Applying pr_γ to the defining equation $\omega \circ \prod_\delta u_g = \xi \circ u_m$ gives $u_f \circ pr_\gamma \circ \omega = u_f \circ W_\gamma$ on the left and $pr_\gamma \circ \xi \circ u_m = X_\gamma \circ u_m$ on the right, showing strict commutativity on these edges.

(5) First we note that Y being strongly fibrant, as an element of $\mathcal{M}_1^{\mathbf{F}}$ it is weakly fibrant as well, by lemma 3.1. Observe that the assumption that **LCM1a** holds in $\mathcal{M}$ implies that it holds in the functor category $\mathcal{M}_1 = \mathcal{M}^{\mathbf{C}}$, so that the cited lemma applies. If X is a weakly cofibrant object of $\mathcal{M}_1^{\mathbf{F}}$, by the Whitehead lemma quoted in the previous section, every map $\mathrm{dgm}_{\mathbf{F}'} X \Rightarrow \mathrm{dgm}_{\mathbf{F}'} Y$ is represented by a collection of actual arrows $a_f : X_f \longrightarrow Y_f$ in $\mathcal{M}_1$, for all $f \in \mathbf{F}_0$ such that

$$\begin{array}{ccc} X_f & \xrightarrow{X_\alpha} & X_g \\ \downarrow{\scriptstyle a_f} & & \downarrow{\scriptstyle a_g} \\ Y_f & \xrightarrow{Y_\alpha} & Y_g \end{array}$$

is homotopy-commutative for all $\alpha : f \longrightarrow g$. We must replace the a's by b's such that the corresponding squares are strictly commutative. We inductively assume that this has been done on the subcategory $\mathbf{F}'$, and in fact we will assume that maps $b_f : X_f \longrightarrow Y_f$ and homotopies $a_f \sim b_f$ have been constructed for $f > m$, so that the corresponding squares with the b's are strictly commutative. Consider (m : minimal element)

$$\begin{array}{ccc} X_m & \xrightarrow{a_m} & Y_m \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\ \prod_{\gamma \in \Gamma(m,f)} X_f & \xrightarrow{\beta} & \prod_{\gamma \in \Gamma(m,f)} Y_f \end{array}$$

ξ and η are defined by the formulas $pr_\gamma \circ \xi = X_\gamma$ and $pr_\gamma \circ \eta = Y_\gamma$. $\beta = \prod b_f$ This is homotopy-commutative. Indeed, the γ^{th} projections are the two maps

$$X_m \longrightarrow Y_f$$

given by

$$\begin{aligned} pr_\gamma \circ \eta \circ a_m &= Y_\gamma \circ a_m \sim a_f \circ X_\gamma \sim b_f \circ X_\gamma \\ &= b_f \circ pr_\gamma \circ \xi = pr_\gamma \circ \left(\prod b_g\right) \circ \xi \end{aligned}$$

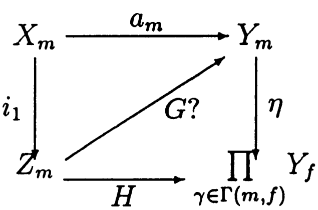
showing the homotopy-commutativity. Let

$$X_m \vee X_m \xrightarrow{i} Z_m \xrightarrow{p} X_m$$

be a cylinder on X_m so that $p \circ i = \nabla_{X_m}$ is a factorization of the folding map. $u \in \mathrm{Cof}_{1,w}$, $p \in \mathcal{E}_1$, and we let H be a homotopy

$$H \circ i_1 = \eta \circ a_m, \; H \circ i_2 = \left(\prod b_f\right) \circ \xi$$

Since X_m is cofibrant, it is well-known ([15, I. Lemma 2, p1.6]) that the insertions $i_1, \; i_2 \; : \; X_m \longrightarrow Z_m$ are acyclic cofibrations, and because we are assuming that Y satisfies (*), η is a strong fibration, and hence, a lift exists in



Define $b_m = G \circ i_2$, which by its very construction is homotopic with a_m. We claim that $b : X \longrightarrow Y$ defined as

$$b_f = \begin{cases} b_f \text{ as previously defined, if } m < f; \\ b_m \text{ as defined above, if } m = f. \end{cases}$$

solves the problem. As before, the commutativities to be checked concerning a given morphism reduce to a consideration of an edge in the generating graph Γ, and there are three cases as before, the only non-trivial one being that of a $\gamma : m \longrightarrow f$. Such a γ appears in the defining diagram for b_m. Then

$$\eta \circ G = H \Rightarrow \eta \circ b_m = \eta \circ G \circ i_2 = H \circ i_2 = \left(\prod b_g\right) \circ \xi$$

and this gives

$$Y_\gamma \circ b_m = pr_\gamma \circ \eta \circ b_m = pr_\gamma \circ \left(\prod b_g\right) \circ \xi = b_f \circ pr_\gamma \circ \xi = b_f \circ X_\gamma$$

This proves **H2.**

□

H3. We do **H3R** since **H3L** is dual and carried out in the context of simplicial sets in [11].

Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be a functor. Then $TF : T(\mathbf{D}) \longrightarrow T(\mathbf{C})$ has an adjoint denoted $R_T F = RF$.

The definition of the adjoint is as follows. Let X be an object of $\mathcal{M}^{\mathbf{C}}$. We factor the canonical map $X \longrightarrow e$ as

$$X \xrightarrow{u} MX := {}^R\widehat{L}(X \to e) \xrightarrow{p} e$$

Where $u = {}^R L'(X \longrightarrow e) \in \mathrm{Cof}_w^{\mathbf{C}} \cap \mathcal{E}^{\mathbf{C}}$ and $p = {}^R L''(X \longrightarrow e) \in \mathrm{Fib}_s^{\mathbf{C}}$. Thus, MX is a strongly fibrant model of X, previously denoted $\underline{X}$. We then define $RF(X) = \mathrm{Ran}_F(MX) = F_*(MX)$ which is clearly a functor $\mathcal{M}^{\mathbf{C}} \longrightarrow \mathcal{M}^{\mathbf{D}}$. The key point is to show that this localizes to the fraction categories $\mathrm{Ho}(\mathcal{M}^{\mathbf{C}}) \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathbf{D}})$ and for this it is necessary to show :

Proposition 5.2. *Assume that* **RCM1** , **RCM3f** , **RCM5** *hold. Then $F_* \circ M$ preserves weak equivalences.*

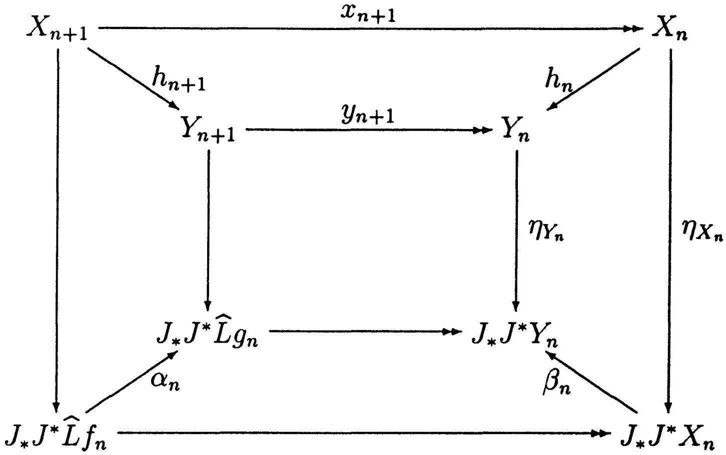
Proof. Let $h : X \longrightarrow Y$ be a weak equivalence in $\mathcal{M}^{\mathbf{C}}$. In what follows we refer to the construction of ${}^R L$ as applied to the morphism $X \longrightarrow e$. Recall that MX is the limit of a tower of strong fibrations:

$$\ldots \longrightarrow X_2 \xrightarrow{x_2} X_1 \xrightarrow{x_1} X_0 = e$$

with compatible maps $f_n : X \longrightarrow X_n$ and a similar tower for MY and maps $g_n : Y \longrightarrow Y_n$. Then Mh is $\lim h_n$ where the h_n are constructed to fit into a diagram

$$\begin{array}{ccccccc}
\ldots \longrightarrow & X_2 & \xrightarrow{x_2} & X_1 & \xrightarrow{x_1} & X_0 = e \\
 & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} \\
\ldots \longrightarrow & Y_2 & \xrightarrow{y_2} & Y_1 & \xrightarrow{y_1} & Y_0 = e
\end{array}$$

Observe that each of the X_n, Y_n is strongly fibrant, since all the horizontal arrows above are strong fibrations. Since we are assuming that **LCM1** is in force, the are all weakly fibrant as well (3.1). The h_n arise inductively from



where the squares are Cartesian, and h_{n+1} exists by Cartesian property. The bottom horizontal arrows are the z_{n+1} that appear in the construction of ${}^RL$, and are strong fibrations by 4.4. The left vertical maps are the v_n that appear there. $\beta_n = J_*J^*h_n$ and $\alpha_n = J_*J^*\widehat{L}(h, h_n)$ where $(h, h_n) : f_n \longrightarrow g_n$. Note that all the objects in the above diagram are weakly fibrant. This can be seen by induction on n. We've noted it above for the X_n, Y_n. But then $J_*J^*X_n$, $J_*J^*Y_n$ are strongly (hence weakly) fibrant because of 4.1. Since the lower horizontal arrows are strong fibrations it follows that $J_*J^*\widehat{L}f_n$, $J_*J^*\widehat{L}g_n$ are strongly (hence weakly) fibrant.

Lemma 5.3. *Under the hypotheses of the proposition,*

$$\alpha_n, \beta_n, h_n \in \mathcal{E}^{\mathbf{C}} \text{ for all } n \geq 0$$

Proof. This is done by induction on n. For $n = 0$, we have $X_0 = Y_0 = e$, $\beta_0 = id.$. We have

$$\begin{array}{ccccc} X & \longrightarrow & \widehat{L}f_0 & \longrightarrow & e \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \widehat{L}(h,h_0)} & & \\ Y & \longrightarrow & \widehat{L}g_0 & \longrightarrow & e \end{array}$$

The left horizontal maps are weak acyclic cofibrations, and the right horizontal ones are strong fibrations. Since **LCM1** is one of our assumptions, these are weak fibrations as well by 3.1. By the hypothesis that $h \in \mathcal{E}^{\mathbf{C}}$ it follows from this diagram and **CM2** that $\widehat{L}(h, h_0) \in \mathcal{E}^{\mathbf{C}}$. Since $\alpha_0 = J_*J^*\widehat{L}(h, h_0)$ the result for $n = 0$ now follows from 5.4 below.

Lemma 5.4. *Under the hypotheses as above,*

1. *Let* $J : \mathbf{C}_0 \longrightarrow \mathbf{C}$ *be the inclusion of the discrete category of objects. Then* $J_* J^*$ *preserves weak equivalences among the weakly fibrant objects.*
2. *Let* $F_0 : \mathbf{C}_0 \longrightarrow \mathbf{D}_0$ *be a functor between discrete small categories. Then* F_{0*} *preserves weak equivalences among the weakly fibrant objects.*

Proof. (1)

$$(J_* J^* X)_c = \prod_{\substack{c' \in \mathbf{C}_0 \\ \gamma \in \mathbf{C}(c,c')}} X_{c'}$$

(2)

$$(F_{0*} X)_d = \prod_{F_0 c = d} X_c$$

In both cases the result follows from the axiom **RCM1b** . □

We now do the induction step n ⇒ n+1. Since we are assuming that α_n, β_n and h_n are weak equivalences, and as we've remarked, all the objects in the big picture-frame diagram are weakly fibrant, the horizontal arrows are strongly, hence weakly, fibrant, we conclude from **RCM 5** that h_{n+1} is also a weak equivalence. But then 5.4 gives that $\beta_{n+1} = J_* J^* h_{n+1}$ is a weak equivalence. The diagram below gives that $\widehat{L}(h, h_{n+1})$ is a weak equivalence:

$$\begin{array}{ccccc} X & \longrightarrow & \widehat{L} f_{n+1} & \longrightarrow & X_{n+1} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \widehat{L}(h,h_{n+1})} & & \downarrow{\scriptstyle h_{n+1}} \\ Y & \longrightarrow & \widehat{L} g_{n+1} & \longrightarrow & Y_{n+1} \end{array}$$

Another appeal to 5.4 gives that $\alpha_{n+1} = J_* J^* \widehat{L}(h, h_{n+1})$ is a weak equivalence, completing the proof of lemma 5.3.

□

Returning to the verification of 5.2, $F : \mathbf{C} \longrightarrow \mathbf{D}$ induces $F_0 : \mathbf{C}_0 \longrightarrow \mathbf{D}_0$ on the discrete categories of objects. Let $J : \mathbf{C}_0 \longrightarrow \mathbf{C}$ and $K : \mathbf{D}_0 \longrightarrow \mathbf{D}$ be the canonical functors. Clearly, $F \circ J = K \circ F_0$ and therefore, $J^* \circ F^* = F_0^* \circ K^*$, which gives $F_* \circ J_* = K_* \circ F_{0,*}$. Lemma 5.4 shows that $K_* \circ F_{0,*}$ preserves weak equivalences since as we've remarked, all the objects are weakly fibrant anyway. Thus it is true as well for $F \circ J$. Now apply F_* to the picture-frame diagram. Because F_* is a right adjoint, it preserves limits, and so the resulting squares are still Cartesian. According to lemma 2.4 F_* preserves $\mathrm{Fib}_s^{\mathbf{C}}$.

Therefore, all the horizontal arrows in this new diagram are strong fibrations. Of course, $F_*(e) = e$ and precisely the same argument that showed that all the objects in the original picture-frame diagram were (strongly, weakly) fibrant, shows that all the objects in this new picture-frame diagram are (strongly, weakly) fibrant. The same argument that showed 5.3 shows that $F_*(\alpha_n)$, $F_*(\beta_n)$, $F_*(h_n)$ are weak equivalences for all n. An appeal to **RCM3f** then shows that

$$RF(h) \;=\; F_*(Mh) \;=\; F_*(\lim h_n) \;=\; \lim F_*(h_n)$$

is a weak equivalence, which was to have been proved. □

Now we claim that this RF is right adjoint to $TF = \mathrm{Ho}(F^*)$, ie. that

$$[F^*X,\, Y]_{\mathbf{C}} \;\simeq\; [X,\, F_*MY]_{\mathbf{D}}$$

functorially in both variables, where

$$[--,\, --]_{\mathbf{C}} \text{ is an abbreviation for } \mathrm{Ho}(\mathcal{M}^{\mathbf{C}})(--,\, --).$$

If $\overline{X} \longrightarrow X$ and $Y \longrightarrow \underline{Y} = MY$ are the canonically defined resolutions given in section (**4.1**), since F^* preserves weak equivalences , the left side is canonically bijective with

$$[F^*\overline{X},\, Y]_{\mathbf{C}} \;\simeq\; [F^*\overline{X},\, MY]_{\mathbf{C}}$$

Also F^* preserves cofibrations (in the right structure on $\mathcal{M}^{\mathbf{C}}$) and so the result we need will follow from

Proposition 5.5. *In the notations above, The adjunction $F^* \dashv F_*$ induces a bijection :*

$$[F^*X,\, Y]_{\mathbf{C}} \;\simeq\; [X,\, F_*Y]_{\mathbf{D}}$$

where X is a weakly cofibrant object of $\mathcal{M}^{\mathbf{D}}$ and Y is a strongly fibrant object of $\mathcal{M}^{\mathbf{C}}$.

Proof. F^* preserves $\mathrm{Cof}_w^{\mathbf{C}}$ (trivially) and F_* preserves $\mathrm{Fib}_s^{\mathbf{C}}$ as we've mentioned (lemma 2.4). Therefore the equality to be proved amounts to a bijection

$$\pi(\mathcal{M}^{\mathbf{C}})(F^*X,\, Y) \;\simeq\; \pi(\mathcal{M}^{\mathbf{D}})(X,\, F_*Y)$$

of homotopy-equivalence classes of morphisms. Right and left homotopy notions coincide here since X, F^*X are cofibrant and Y, F_*Y are fibrant. Because $F^* \dashv F_*$, in Grothendieck's notation $\sharp$, $\flat$ induce reciprocal bijections

$$\mathcal{M}^{\mathbf{C}}(F^*X,\, Y) \;\simeq\; \mathcal{M}^{\mathbf{D}}(X,\, F_*Y)$$

and what must be demonstrated is that this is preserved by the homotopy relation. In fact we show

$$g_1 \sim g_2 \iff g_1^\sharp \sim g_2^\sharp$$

which suffices. ($\Longrightarrow$) Let Z be a cylinder on X, and let

$$H : Z \longrightarrow F_* Y$$

be a homotopy with $H \circ i_1 = g_1$, $H \circ i_2 = g_2$. Since F^* preserves weak equivalences and cofibrations, $F^* Z$ is a cylinder on $F^* X$. Because

$$H^\sharp \circ F^* i_j = (H \circ i_j)^\sharp = g_j^\sharp \text{ for } j = 1,\ 2$$

by the calculus of adjoints, $H^\sharp$ is a homotopy connecting $g_1^\sharp$ and $g_2^\sharp$. ($\Longleftarrow$) We can take any cylinder to give the homotopy, since $F^* X$ is cofibrant and Y is fibrant. So let

$$K : F^* Z \longrightarrow Y$$

be such that $K \circ F^* i_1 = g_1^\sharp$, $K \circ F^* i_2 = g_2^\sharp$. Defining $H = K^\flat$ the calculus of adjoints shows that H is a homotopy connecting g_1 and g_2. □

(H4) We do **H4R**. Here we must assume the existence of coproducts, so that **LCM1** will be in force as well.

Let $P : \mathbf{E} \longrightarrow \mathbf{B}$ be a discrete opfibration. Then

$$TP = \mathrm{Ho}(P^*) : T\mathbf{B} \longrightarrow T\mathbf{E}$$

has a left adjoint $LP = L_T P$ and the square

$$\begin{array}{ccc} T\mathbf{B} & \xrightarrow{TP} & T\mathbf{E} \\ {\scriptstyle \mathrm{dgm}_\mathbf{B}}\downarrow & & \downarrow{\scriptstyle \mathrm{dgm}_\mathbf{E}} \\ (T\mathbf{1})^\mathbf{B} & \xrightarrow{P^*} & (T\mathbf{1})^\mathbf{E} \end{array}$$

has the Beck- Chevalley property.

Recall that this means that in the square deduced by taking adjoints, which is only 2-commutative ,

$$\begin{array}{ccc} T\mathbf{E} & \xrightarrow{LP} & T\mathbf{B} \\ {\scriptstyle \mathrm{dgm}_\mathbf{E}}\downarrow & & \downarrow{\scriptstyle \mathrm{dgm}_\mathbf{B}} \\ (T\mathbf{1})^\mathbf{E} & \xrightarrow{P^*} & (T\mathbf{1})^\mathbf{B} \end{array}$$

the natural transformation φ defined as the composite

$$\begin{array}{ccc} {}_*P \circ \mathrm{dgm}_{\mathbf{E}} & \xrightarrow{1*\eta_T} & {}_*P \circ \mathrm{dgm}_{\mathbf{E}} \circ TP \circ LP \\ \varphi\downarrow & & =\downarrow \\ \mathrm{dgm}_{\mathbf{B}} \circ LP & \xleftarrow{\varepsilon_P * 1} & {}_*P \circ P^* \circ \mathrm{dgm}_{\mathbf{B}} \circ LP \end{array}$$

is an *isomorphism.* Here η_T, ε_T (η, ε) are the unit and counit of the adjunction $LP \dashv TP$ (${}_*P \dashv P^*$).

Recall that a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be a prefibration (resp. preopfibration) if for all objects $b \in \mathcal{B}_0$ the canonical functor

$$I : \mathcal{E}_b \longrightarrow (b \downarrow \mathcal{B}) = b\backslash P$$

has a right adjoint I_*. (resp. the canonical functor

$$I' : \mathcal{E}_b \longrightarrow (\mathcal{B} \downarrow b) = P/b$$

has a left adjoint ${}_*I'$)

If $v : b \longrightarrow Pe$ is an object of $b\backslash P$ we abbreviate

$$I_*(v : b \longrightarrow Pe) := v^*(e)$$

and call it the pull-back or base-change of e along v. Similarly, we set

$${}_*I'(v : Pe \longrightarrow b) := v_*(e)$$

and call it the push-forward or cobase-change of e along v. We say that a prefibration is a fibration if whenever $w : b' \longrightarrow b$ we have transitivity of base-change: $(v \circ w)^* = w^* \circ v^*$ and a preopfibration is an opfibration if $(v \circ w)_* = v_* \circ w_*$ when $w : b \longrightarrow b'$. Finally, we say that an (op)fibration is discrete if each of the category-fibers $\mathcal{E}_b$ is a discrete category.

Lemma 5.6. 1. *Let $P : \mathcal{E} \longrightarrow \mathcal{B}$ be a discrete fibration of categories. Then, for all $b \in \mathcal{B}_0$, the natural functor*

$$I : \mathcal{E}_b \longrightarrow (b \downarrow \mathcal{B}) = b\backslash P$$

is initial.

2. *Let $P : \mathcal{E} \longrightarrow \mathcal{B}$ be a discrete opfibration of categories. Then, for all $b \in \mathcal{B}_0$, the natural functor*

$$I' : \mathcal{E}_b \longrightarrow (\mathcal{B} \downarrow b) = P/b$$

is final.

Proof. We do the case of fibrations. Two things must be checked (see [19, I.9.1.1-9.1.2, pp. 64-65]) :

1. For every object $\xi \in (b\backslash P)_0$ there exists a morphism $\alpha : I\xi' \longrightarrow \xi$ for some $\xi' \in (\mathcal{E}_b)_0$

   If $\xi = (v : b \longrightarrow Pe)$ we define $\xi' = v^*(e)$ and

$$\alpha : I\xi' = II_*\xi \longrightarrow \xi$$

   by the counit of the adjunction $I \dashv I_*$.
2. Given a comutative diagram

$$\begin{array}{ccc} I\xi_3 & \xrightarrow{I\beta_1} & I\xi_1 \\ {\scriptstyle I\beta_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ I\xi_2 & \xrightarrow{\alpha_2} & \xi \end{array}$$

   we must have $\xi_1 = \xi_2$ and $\alpha_1 = \alpha_2$.

   Since we are assuming that $\mathcal{E}_b$ is discrete, we must have $\beta_1 = \beta_2 = id$. Hence, $\xi_1 = \xi_2 = \xi_3$ and $\alpha_1 = \alpha_2$

□

Corollary 5.7. 1. *Let $\mathcal{C}$ be a category with products, and let $P : \mathbf{E} \longrightarrow \mathbf{B}$ be a discrete fibration of small categories. Then*

$$\mathrm{Ran}_P = P_* : \mathcal{C}^{\mathbf{E}} \longrightarrow \mathcal{C}^{\mathbf{B}}$$

exists and is naturally isomorphic with the functor Q given on objects as

$$(QX)_b = \prod_{Pe=b} X_e$$

2. *Let $\mathcal{C}$ be a category with coproducts, and let $P : \mathbf{E} \longrightarrow \mathbf{B}$ be a discrete opfibration of small categories. Then*

$$\mathrm{Lan}_P = {}_*P : \mathcal{C}^{\mathbf{E}} \longrightarrow \mathcal{C}^{\mathbf{B}}$$

exists and is naturally isomorphic with the functor Q given on objects as

$$(QX)_b = \coprod_{Pe=b} X_e$$

Proof. We do the first one, since the second is dual. By Kan's formula,

$$(P_*X)_b = \lim_{b\backslash P} X \circ J_b$$

where $J_b : (b\backslash P) \longrightarrow \mathbf{E}$ is the functor that sends $(v : b \longrightarrow Pe)$ to e. This formula assumes *a priori* that general limits exist in $\mathcal{C}$, but since $I : \mathbf{E}_b \longrightarrow (b\backslash P)$ is initial, we can compute this limit over the discrete category $\mathbf{E}_b$, and this reduces to the formula given, which only requires products. More precisely, (cf. [19, I.9.1.2]) if

$$\theta : X \circ J_b \circ I \longrightarrow \prod_{Pe=b} X_e$$

is the cone defining the limit, then there's a unique

$$\zeta : X \circ J_b \longrightarrow \prod_{Pe=b} X_e$$

such that $\theta = \zeta * I$ and it defines the product on the right as the limit on the left. □

Let $(\mathcal{M}^{\mathbf{E}})_c$ and $(\mathcal{M}^{\mathbf{B}})_c$ be the full subcategories of (weakly) fibrant objects. We know that

$$(\mathcal{M}^{\mathbf{E}})_c \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathbf{E}})$$

is a weak fraction functor (see 2.7) and similarly with $\mathbf{E}$ replaced by $\mathbf{B}$. Both P^* and $_*P$ restrict to these subcategories and asuming the axiom **LCM1** preserve weak equivalences there. This is trivial for P^*. In view of the description of $_*P$ given in corollary 5.7 , this follows from 3.1. Therefore, these induce functors $\mathrm{Ho}(P^*) = TP$ (as we've already noted), and $\mathrm{Ho}(_*P) = LP$. We must check that they are adjoint and that the Beck-Chevalley property holds. On the first point:

Lemma 5.8. *Let*

$$\mathcal{A} \xrightarrow{\lambda} \mathcal{A}[\Sigma^{-1}] = \mathcal{A}'$$

and

$$\mathcal{B} \xrightarrow{\mu} \mathcal{B}[\Xi^{-1}] = \mathcal{B}'$$

be (weak) fraction functors. Given an adjoint pair $F \dashv G$, $\mathcal{A} \leftrightarrows \mathcal{B}$ such that

$$F : \Sigma \longrightarrow \Xi \quad \text{and} \quad G : \Xi \longrightarrow \Sigma$$

the canonically induced functors $\mathbf{F}$,$\mathbf{G}$: $\mathcal{A}[\Sigma^{-1}] \leftrightarrows \mathcal{B}[\Xi^{-1}]$ are adjoint.

Proof. Let $\eta : 1 \longrightarrow G \circ F$ and $\varepsilon : F \circ G \longrightarrow 1$ be the unit and counit. We can see that the lemma is true in the case that λ and μ are *strict* localizations, ie. that λ and μ induce bijections on the objects of their respective categories (see [11]). One can unambiguously define $\mathbf{F}$, $\mathbf{G}$ by the formulae $\mathbf{F}\lambda a := \mu F a$ and $\mathbf{G}\mu b := \lambda G b$ and we can define the unit and counit of $\mathbf{F} \dashv \mathbf{G}$ by $\boldsymbol{\eta} * \lambda = \lambda * \eta$ and $\boldsymbol{\varepsilon} * \mu = \mu * \epsilon$. That this uniquely defines $\boldsymbol{\eta}$ and $\boldsymbol{\varepsilon}$ follows from :

Lemma 5.9. *([8, I. Lemma 1.2]) Let*

$$\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\Phi^{-1}]$$

be a weak fraction functor. Let

$$F_1, F_2 : \mathcal{C}[\Phi^{-1}] \longrightarrow \mathcal{D}$$

be two functors. The assignment $\theta \longrightarrow \theta * \gamma$ *induces a bijection between natural transformations* $F_1 \longrightarrow F_2$ *and* $F_1 \circ \gamma \longrightarrow F_2 \circ \gamma$

□

The above definitions, together with Godement's 5 rules of functorial calculus (see [19, II. 16.1.1]) yield

$$\mu * (F * \eta) = (\mathbf{F} * \boldsymbol{\eta}) * \lambda \quad \text{and} \quad \mu * (\varepsilon * F) = (\boldsymbol{\varepsilon} * \mathbf{F}) * \lambda$$

and

$$\lambda * (\eta * G) = (\boldsymbol{\eta} * \mathbf{G}) * \mu \quad \text{and} \quad \lambda * (G * \varepsilon) = (\mathbf{G} * \boldsymbol{\varepsilon}) * \mu$$

We then compute :

$$\begin{aligned} 1 = \mu * 1 &= \mu * [(\varepsilon * F) \circ (F * \eta)] \\ &= [\mu * (\varepsilon * F)] \circ [\mu * (F * \eta)] \\ &= [(\boldsymbol{\varepsilon} * \mathbf{F}) * \lambda] \circ [(\mathbf{F} * \boldsymbol{\eta}) * \lambda] \\ &= [(\boldsymbol{\varepsilon} * \mathbf{F}) \circ (\mathbf{F} * \boldsymbol{\eta})] * \lambda \end{aligned}$$

Then, (5.9) shows that

$$(\boldsymbol{\varepsilon} * \mathbf{F}) \circ (\mathbf{F} * \boldsymbol{\eta}) = 1$$

A similar argument gives

$$(\mathbf{G} * \boldsymbol{\varepsilon}) \circ (\boldsymbol{\eta} * \mathbf{G}) = 1$$

showing that $\boldsymbol{\eta}$ and $\boldsymbol{\varepsilon}$ are indeed the unit and counit of the adjunction.

Suppose now that λ, μ are merely weak fraction functors. Then according to [11, p. 2], strong fraction functors

$$\lambda_1 : \mathcal{A} \longrightarrow \mathcal{A}'_1 \text{ and } \mu_1 : \mathcal{B} \longrightarrow \mathcal{B}_1$$

exist, inverting Σ and Ξ. By the universal properties of these categories, there are equivalences of categories

$$S, T : \mathcal{A}' \leftrightarrows \mathcal{A}'_1 \text{ and } U, V : \mathcal{B}' \leftrightarrows \mathcal{B}'_1$$

over λ, λ_1, μ, μ_1.Given an adjoint pair

$$\mathbf{F}_1,\ \mathbf{G}_1 : \mathcal{A}'_1 \leftrightarrows \mathcal{B}'_1$$

we create one

$$\mathbf{F},\ \mathbf{G} : \mathcal{A}' \leftrightarrows \mathcal{B}'$$

by the formulas $\mathbf{F} := V \circ \mathbf{F}_1 \circ S$ and $\mathbf{G} := U \circ \mathbf{G}_1 \circ T$. The verifications are left to the reader. □

Finally, we must see the Beck - Chevalley property : that φ is an isomorphism. For $X \in (T\mathbf{E})_0$ we compute $LP(X)$ as $Q(Y)$ where Q is the functor given in 5.7 (2) and where Y is any weakly cofibrant object that resolves X, ie. we have

$$p : Y \longrightarrow X$$

where $p \in \mathrm{Fib}_s^{\mathbf{E}} \cap \mathcal{E}^{\mathbf{E}}$. For instance, we can take $Y = \overline{X}$ as in section 5.1. The counit η_T is described as follows. For each $e \in (\mathbf{E})_0$ the arrow $\gamma(p_e) : \gamma Y_e \to \gamma X_e$ is an isomorphism in $\mathrm{Ho}(\mathcal{M}) = T(\mathbf{1})$. If

$$\iota_e : Y_e \longrightarrow \coprod_{Pe'=Pe} Y_{e'}$$

is the canonical insertion, then one defines $(\eta_T X)_e$ to be the composite

$$\begin{array}{ccccc} (\gamma^{\mathbf{E}}X)_e & \longrightarrow & (P^*QY)_e & = & ([TP \circ LP]\gamma_{\mathbf{E}}X) \\ =\downarrow & & & & =\downarrow \\ \gamma X_e & \xrightarrow{(\gamma p_e)^{-1}} & \gamma Y_e & \xrightarrow{\gamma\iota_e} & \coprod \gamma Y_{e'} \end{array}$$

Recall that the counit $\varepsilon : {}_*P \circ P^* \to 1$ is given by

$$(\varepsilon Z)_b : \coprod_{Pe=b} Z_{Pe} \xrightarrow{\sum id} Z_b$$

Putting it all together, we see that φ_e amounts to the map

$$\gamma X_e \longrightarrow \gamma Y_e$$

given by $(\gamma p_e)^{-1}$ which is an isomorphism.

Theorem 5.10. *Let $\mathcal{M}$ be a functorial closed model category, and suppose that* **LCM1** *and* **RCM1** *hold.*

1. *Assume that one of the following sets of axioms holds:*
   (a) **LCM2c**
   (b) **LCM2cc, LCM4,** *and the cancellation property*

   $$g \circ f \in \mathrm{Fib} \cap \mathcal{E} \text{ and } f \in \mathcal{E} \Rightarrow g \in \mathrm{Fib} \cap \mathcal{E}$$

   This holds for example if Fib = Epi .

   Then

   $$\mathbf{C} \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathbf{C}})$$

   is a left homotopy theory.
2. *Assume that one of the following sets of axioms holds:*
   (a) **RCM2f**
   (b) **RCM2ff, RCM4,** *and the cancellation property*

   $$g \circ f \in \mathrm{Cof} \cap \mathcal{E} \text{ and } g \in \mathcal{E} \Rightarrow f \in \mathrm{Cof} \cap \mathcal{E}$$

   This holds for example if Cof = Mono .

Then

$$\mathbf{C} \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathbf{C}})$$

is a right homotopy theory.

6. Examples

We show a few representative cases, by no means an exhaustive list. We mostly give sketches only, since much of this is well - known. Since each of these is a closed model category, the independent verification of some of these axioms is superfluous. Nonetheless, it is interesting to verify these axioms directly when possible because they often elucidate nontrivial properties in the various cases.

6.1. **Simplicial Sets.** The main facts are stated as Proposition 3.4 in Chapter II of [11]. However, no proofs are given and standard references, eg. [14], [22], do not prove them either. That S is a functorial closed simplicial model category is proved in [15, II, Thm. 3, p. 3.14]. **LCM1** and **RCM1** are easy to prove.

LCM2: A morphism in S is an acyclic fibration if and only if it has the RLP with respect to all inclusions

$$\dot{\Delta}[n] \longrightarrow \Delta[n]$$

and these are small in the sense that

$$\mathrm{Hom}(\dot{\Delta}[n],\ \mathrm{colim}\ Z_n) \;=\; \mathrm{colim}\ \mathrm{Hom}(\dot{\Delta}[n], Z_n)$$

with respect to all filtering colimits. The result follows easily from this.

RCM2: Since cofibrations coincide with monomorphisms, it is easy to see that if each u_i is a cofibration, then lim u_i is a cofibration. The result concerning the weak equivalences then follows from **RCM3**, to be proved below.

LCM3: This follows from proposition 3.4, and the fact that all objects are cofibrant. Alternatively, one can appeal to Bousfield - Kan as follows. Let

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

be a sequence of cofibrations. Then, for any Kan complex Y,

$$\cdots \longrightarrow \mathrm{hom}\ (X_2,\ Y) \longrightarrow \mathrm{hom}\ (X_1,\ Y) \longrightarrow \mathrm{hom}\ (X_0,\ Y)$$

is sequence of fibrations, and all the spaces are fibrant ([14, Thm. 6.9 and Thm. 7.13], hom = function complex). According to [4, XI 4.1(v), p. 299, and XII, Prop. 4.1. p. 334]

$$\lim\ \mathrm{hom}\ (X_i,\ Y) \;\simeq\; \mathrm{holim}\ \mathrm{hom}\ (X_i,\ Y) \;\simeq\; \mathrm{hom}\ (\mathrm{hocolim} X_i,\ Y)$$

are weak equivalences. The first one is evidently isomorphic with hom $(\operatorname{colim} X_i,\ Y)$ so we see that the natural map

$$\xi \ : \ \operatorname{hocolim} X_i \ \longrightarrow \ \operatorname{colim} X_i$$

induces a weak equivalence

$$\hom\ (\operatorname{colim} X_i, Y) \ \longrightarrow \ \hom\ (\operatorname{hocolim} X_i, Y)$$

Taking π_0 of both sides gives a bijection

$$[\operatorname{colim} X_i,\ Y] \ \simeq \ [\operatorname{hocolim} X_i,\ Y]$$

That being true for all Kan complexes Y, we conclude that ξ **is a weak** equivalence ([15, II, Prop. 3, p. 3.15]). Now if $f : X \longrightarrow X'$ is a map of sequences of cofibrations with each f_i a weak equivalence, the homotopy lemma of [4, XII, 4.2, p. 335] shows that hocolim f_i is a weak equivalence. In view of the preceding, this shows that colim f_i is a weak equivalence as well.

RCM3. Proposition 3.4 implies **RCM3f** here. One can also prove this by the reasoning above, which shows that $\lim X_i \ = \ \operatorname{holim} X_i$. Then the homotopy lemma of [4, XI, 5.6, p. 304], shows that holim f_i is a weak equivalence. We can remove the hypothesis that the spaces involved are fibrant by the following argument (shown to me by Heller): Recall the well - known

Lemma 6.1. *Given*

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

where p_1, p_2 are fibrations and g is a weak equivalence. Then, f is a weak equivalence if and only if the induced map on every fiber :

$$F(p_1,\ b) \ = \ p_1^{-1}(b) \ \longrightarrow \ F(p_2,\ g(b)) \ = \ p_2^{-1}(g(b))$$

is a weak equivalence ($b \in B_1$).

Let $f : X \longrightarrow X'$ be as in the statement of **RCM3**. We apply the above to $E_1 = \lim X_i$, $E_2 = \lim X_i'$, $B_1 = X_0$, $B_2 = X_0'$. By 2.6 the projections $\pi_0 : \lim X_i \longrightarrow X_0$ and $\pi_0' : \lim X_i' \longrightarrow X_0'$ are fibrations. Choose any point in E_1 as a base - point. By taking images of this point we can consider all the maps in question as being pointed. The fibers will refer to these points. Then we have

$$F(\pi_0, *) \ = \ \lim F_i, \quad \text{and} \quad F(\pi_0', *) \ = \ \lim F_i'$$

where

$$F_i = \text{fiber of } X_i \longrightarrow X_0 \quad \text{and} \quad F_i' = \text{fiber of } X_i' \longrightarrow X_0'$$

We get in this way a map of towers of fibrations:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & F_3' & \longrightarrow & F_2' & \longrightarrow & F_1' \end{array}$$

in which all the spaces involved are Kan complexes, and the vertical arrows are weak equivalences by the original hypothesis about $X \longrightarrow X'$, and lemma 6.1. Therefore by **RCM3f** we get that $\lim F_i \longrightarrow \lim F_i'$ is a weak equivalence, and again by 6.1 so is $\lim X_i \longrightarrow \lim X_i'$.

We sketch the proof of 6.1. A map $X \longrightarrow Y$ in $\mathcal{S}$ is a weak equivalence if and only if $\mathrm{Ex}^\infty(X) \longrightarrow \mathrm{Ex}^\infty(Y)$ is a weak equivalence ([15, II, Prop. 4, p. 3.19]). Also

$$\mathrm{Ex}^\infty(E_1) \longrightarrow \mathrm{Ex}^\infty(B_1) \text{ is a fibration with fiber } \mathrm{Ex}^\infty(F(p_1, b))$$

([12, Thm. 4.3]). Thus it is sufficient to prove the lemma in the case where all the spaces are Kan complexes. A simple argument then reduces to the case where they are all connected. The result now follows by a 5 - lemma argument comparing the long exact homotopy sequences for $E_1 \longrightarrow B_1$ and $E_2 \longrightarrow B_2$.

RCM 4. This follows from 6.1 because (f, g) induce an isomorphism of fibers $F(q, v) \longrightarrow F(p, f(v))$ for all vertices $v \in V_0$.

LCM 4. This follows from remark 3.2.3 since all objects are cofibrant. We can also derive this from **RCM 4** by the following device. It suffices to show that $\pi_0 \hom(g, K)$ is a bijection for all Kan complexes K ([15, II, Prop.4 (ii), p. 3.19]), which in turn will follow if we show that $\hom(g, K)$ is a weak equivalence. The functor $X \mapsto \hom(X, K)$ carries push - outs to pull - backs and carries cofibrations to fibrations ([15, II, axiom SM7, p. 2.2 and Thm. 3, p. 3.14]). That it preserves weak equivalences, and thus reduces **LCM 4** to **RCM 4** is

Lemma 6.2. *For all Kan complexes K,*

$$X \mapsto \hom(X, K)$$

preserves weak equivalences in $\mathcal{S}$.

Proof. Let $f : X \longrightarrow Y$ be a weak equivalence. We can imbed this in

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \end{array}$$

where $\overline{X}$, $\overline{Y}$ are Kan complexes and u, v are acyclic cofibrations. By axiom SM7 for simplicial closed model categories ([15, II, p. 2.2]), both $\hom(u, K)$ and $\hom(v, K)$ are acyclic fibrations, and thus to check that $\hom(f, K)$ is a weak equivalence it suffices to show that $\hom(\overline{f}, K)$ is a weak equivalence, so that we may assume at the outset that X and Y are Kan complexes. But in $\mathcal{S}$ a weak equivalence among fibrant objects is a homotopy equivalence. Let $g : X \longrightarrow Y$ be a homotopy - inverse, with homotopies

$$M \ : \ X \times I \ \longrightarrow \ X \text{ and } N \ : \ Y \times I \ \longrightarrow \ Y$$

connecting 1, $g \circ f$ and 1, $f \circ g$ respectively. Observe that

$$\hom(X \times I, \ K) \ = \ \hom(X, \ K^I) \ = \ \hom(X, \ K)^I$$

all of these being Kan complexes. That being so, $\hom(X, \ K)^I$ is a path - object for $\hom(X, \ K)$ ([15, II, the proof of Prop. 5]). Therefore, $\hom(M, \ K)$ is a homotopy connecting 1, $\hom(f, \ K) \circ \hom(g, \ K)$. A similar argument shows that $\hom(N, \ K)$ is a homotopy connecting 1, $\hom(g, \ K) \circ \hom(f, \ K)$. Therefore, $\hom(f, \ K)$ is a weak equivalence, as claimed. □

RCM5. This follows from proposition 3.5. Alternatively, one reduces to the case where all the spaces are connected. One puts in base points. The 5 - lemma applied to the homotopy exact sequences shows that (a, b) induce isomorphisms

$$\pi_i(F(p_1, \ *)) \ \simeq \ \pi_i(F(p_2, \ *))$$

But $F(p_1, \ *)) \ = \ F(q_1, \ *))$ and $F(p_2, \ *)) \ = \ F(q_2, \ *))$ by the Cartesian nature of the square. The 5 - lemma applied to the top rows of the diagram gives the isomorphisms

$$\pi_i(D_1, \ *) \ \simeq \ \pi_i(D_2, \ *)$$

LCM5. This follows from proposition 3.5. One can also derive it from **RCM5** by applying $\hom(-, \ K)$, and reasoning as in **LCM4**.

Theorem 6.3.

$$\mathbf{C} \ \mapsto \ \Pi(\mathbf{C}) \ = \ \mathrm{Ho}(\mathcal{S}^{\mathbf{C}})$$

is a left and right homotopy theory.

6.2. **Chain Complexes.** Let R be a ring with unit. There are two cases to consider.

1.) $\mathcal{M} = \mathrm{Ch}_*(R)$, the category of positive chain complexes of (left) R - modules; $\partial : A_{n+1} \longrightarrow A_n$, $A_n = 0$ if $n < 0$. This is a closed model category in which the weak equivalences are the homology isomorphisms (quasi - isomorphisms), the fibrations are epimorphisms in strictly positive degree, and the cofibrations are the monomorphisms

$X \longrightarrow Y$ such that Y_n/X_n is a projective module for all n

2.) $\mathcal{M} = \mathrm{Ch}^*(R)$, the category of positive cochain complexes of (left) R - modules; $d : A^n \longrightarrow A^{n+1}$, $A^n = 0$ if $n < 0$. This is a closed model category in which the weak equivalences are the cohomology isomorphisms (quasi - isomorphisms), the cofibrations are monomorphisms in strictly positive degree, and the fibrations are the epimorphisms

$X \longrightarrow Y$ such that the kernel is an injective module for all n

Note that this is a slight variation of the usual axioms for chain complexes that are bounded below or above . The necessity of this variation has been noted before (see [15, II, p. 6.3]), and was pointed out to me by Heller. One proves that these define closed model categories by the same arguments as in the bounded cases with a careful attention to what happens in dimension 0.

Proposition 6.4.
1.

$$\mathbf{C} \mapsto \mathrm{Ho}(\mathrm{Ch}_*(R)^{\mathbf{C}})$$

is a left homotopy theory.

2.

$$\mathbf{C} \mapsto \mathrm{Ho}(\mathrm{Ch}^*(R)^{\mathbf{C}})$$

is a right homotopy theory.

Proof. In both instances, **LCM1** and **RCM1** are easy to see.
(1) **LCM2**. Each p_i being an epimorphism in strictly positive degrees, so is colim p_i. That colim also preserves weak equivalences is well - known in homological algebra (filtering colim is an exact functor), which also proves **LCM3** and more.
LCM4, **LCM5**. These can be seen by 5 - lemma arguments. For example, in the situation of **LCM4**, we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow[u]{} & Y & \longrightarrow & X/Y & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & U & \xrightarrow[v]{} & V & \longrightarrow & V/U & \longrightarrow & 0 \end{array}$$

in which h is an isomorphism and f is a quasi - isomorphism. Therefore, g is a quasi - isomorphism.
(2) The reasoning here is similar with one exception : lim is no longer an exact functor.
RCM2f. If each u_i is a monomorphism in strictly positive degree then so is lim u_i. That it is also a weak equivalence if the u_i all are follows from:
RCM3, which we prove as follows. In any tower of fibrations

$$\cdots \longrightarrow Z_2 \longrightarrow Z_1 \longrightarrow Z_0$$

we have

$$\mathrm{R}^1 \lim (Z) = 0$$

since all the transition maps are epimorphisms (Mittag - Leffler condition). Also, one knows quite generally that

$$\mathrm{R}^p \lim (Z) = 0 \text{ for all p } \geq 2$$

holds for countable inverse limits. Let $f : X \longrightarrow Y$ be as in the statement of **RCM3**. There are two spectral sequences

$${}'\mathrm{E}_2^{p,q}(X) = \mathrm{R}^p \lim (\mathrm{H}^q(X)) \Longrightarrow \mathbf{R}^{p+q} \lim (X)$$

$${}''\mathrm{E}_2^{p,q}(X) = \mathrm{H}^p(\mathrm{R}^q \lim (X)) \Longrightarrow \mathbf{R}^{p+q} \lim (X)$$

where $\mathbf{R}$ is the hyperderived functor, and we are thinking of lim as a functor between the abelian categories ([9])

$$\mathrm{Ch}^*(R)^I \longrightarrow \mathrm{Ch}^*(R)$$

f induces a map of spectral sequences

$$\mathrm{E}(X) \longrightarrow \mathrm{E}(Y)$$

The above remarks show that the second spectral sequences collapse into isomorphisms:

$$\mathrm{H}^p(\lim (X)) \simeq \mathbf{R}^p \lim (X)$$

$$\mathrm{H}^p(\lim (Y)) \simeq \mathbf{R}^p \lim (Y)$$

On the other hand, the homology isomorphisms f induce isomorphisms of first spectral sequences, and hence by convergence, an isomorphism of their respective abutments, as is well - known. Therefore,

$$\mathrm{H}^p(\lim (X)) \simeq \mathrm{H}^p(\lim (Y))$$

which was to have been shown. □

6.3. Simplicial Objects.

Proposition 6.5. *Let $\mathcal{A}$ be a category closed under limits and colimits having sufficiently many projective objects and possessing a set $\underline{U}$ of small projective generators. Let $s\mathcal{A}$ be the category of simplicial objects over $\mathcal{A}$. Then, $s\mathcal{A}$ has the structure of a simplicial closed model category, and*

$$\mathbf{C} \mapsto \mathrm{Ho}(s\mathcal{A}^{\mathbf{C}})$$

is a left homotopy theory.

Proof. Quillen has shown that $s\mathcal{A}$ has the structure of a simplicial closed model category ([15, II, pp. 4.1 - 4.12]) in which a map f is a weak equivalence (resp. a fibration) if and only if $\underline{\mathrm{Hom}}(P, f)$ is a weak equivalence (resp. a fibration) in $\mathcal{S}$ for all $P \in \underline{U}$. A map is a cofibration if and only if it has the LLP with respect to all acyclic fibrations. We verify supplimentary axioms. **LCM1a** and **RCM1a** are true by hypothesis.

RCM1b. Since

$$\underline{\mathrm{Hom}}(P, \prod X_i) = \prod \underline{\mathrm{Hom}}(P, X_i)$$

our assumption that the X_i are fibrant means that each $\underline{\mathrm{Hom}}(P, X_i)$ is a Kan complex, and ditto for $\underline{\mathrm{Hom}}(P, Y_i)$. The result now follows from **RCM1b** in $\mathcal{S}$ and the description of weak equivalences in $s\mathcal{A}$.

LCM1b. This follows from proposition 3.2, proposition 3.4 and remark 3.5, which also gives a proof of **RCM1b**.

LCM2. By the smallness of $P \in \underline{U}$ we have

$$\underline{\mathrm{Hom}}(P, \operatorname{colim} X_i) = \operatorname{colim} \underline{\mathrm{Hom}}(P, X_i)$$

By our hypotheses, $\underline{\mathrm{Hom}}(P, p_i)$ are all acyclic fibrations in $\mathcal{S}$, where **LCM2** is known to be true.

LCM3c. This follows from proposition 3.4.

LCM5. This follows from proposition 3.5 and remark 3.5. □

6.4. **Commutative DGAs.** Let K be a field of characteristic 0. We let $\mathcal{A}$ be the category of (anti) commutative differential graded algebras over K, and $\mathcal{A}_0$ is the category of augmented commutative DGA's over K. We follow the notations and results of [3].

Proposition 6.6. 1. *$\mathcal{A}$ is a closed model category, and*

$$\mathbf{C} \mapsto \mathrm{Ho}(\mathcal{A}^{\mathbf{C}})$$

is a left homotopy theory.

2. *$\mathcal{A}_0$ is a closed model category, and*

$$\mathbf{C} \mapsto \mathrm{Ho}(\mathcal{A}_0^{\mathbf{C}})$$

is a left homotopy theory.

Proof. We do the unaugmented case. That $\mathcal{A}$ is a closed model category is proved in [3, §4].The weak equivalences are the cohomology isomorphisms, the fibrations are the epimorphisms and the cofibrations are the maps that have the LLP with respect to all the acyclic fibrations. The product is the direct product; the coproduct is the tensor product. Note that infinite products and coproducts exist, the infinite tensor product being the direct limit of the finite ones. Axioms **LCM1a** and **RCM1a** are thus satisfied.

RCM1b. This follows from $\mathrm{H}(\prod X_i) = \prod \mathrm{H}(X_i)$

LCM1b. Since filtering colimit is an exact functor, it is enough to prove that finite tensor product preserves homology isomorphisms, but this follows from Künneth's formula

$$\mathrm{H}(X \otimes Y) \simeq \mathrm{H}(X) \otimes \mathrm{H}(Y)$$

LCM2. Each p_i being an epimorphism, colim p_i clearly is. If each p_i is also a weak equivalence, then so is colim p_i because as was already mentioned, filtering colim is an exact functor. This proves **LCM3** as well.

LCM5. Given squares as in the statement of **LCM5** we apply the functor

$$X \longrightarrow \mathrm{F}(X, Y)$$

where F is the function space construction of [3, §5]. Note that
a) $\mathrm{F}(-, Y)$ carries cofibrations to fibrations; $\mathrm{F}(X, Y)$ is a Kan complex if X is cofibrant. [3, Prop. 5.4].

b) $\mathrm{F}(-, Y)$ carries weak equivalences among cofibrant objects into weak equivalences. [3, Prop. 6.6].

c) $F(-, Y)$ carries push - outs to pull - backs.
To see this last point, note that [3, Lemma 5.2] implies that $F(-, Y)$ applied to a push - out yields a square that has the Cartesian mapping property relative to all finite simplicial sets. But a square with this property truly is Cartesian because every simplicial set is a direct limit of its finite subcomplexes.

Hence applying $F(-, Y)$ to the diagram **LCM5** in $\mathcal{A}$ gives a diagram **RCM5** in $\mathcal{S}$, which in turn shows that

$$F(D_2, Y) \longrightarrow F(D_1, Y)$$

is a weak equivalence of spaces. We get a bijection therefore

$$[D_2, Y] = \pi_0 F(D_2, Y) \rightarrow [D_1, Y] = \pi_0 F(D_1, Y)$$

But the D's being cofibrant (and all objects are fibrant) we have

$$[D_i, Y] = \mathrm{Ho}(\mathcal{A})(D_i, Y)$$

This holds for all Y and so $d : D_1 \longrightarrow D_2$ is invertible in the homotopy category and since $\mathcal{A}$ is a closed model category this shows that d is a weak equivalence as claimed. □

We give a second proof of **LCM5** by a different method. Let us say that a morphism $X \longrightarrow Y$ has the Eilenberg - Moore property (EM) if in every push - out

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ C & \longrightarrow & D = C \otimes_X Y \end{array}$$

the natural map

$$\mathrm{Tor}_X(C, Y) \longrightarrow H(D)$$

is an isomorphism. See [3, 10.4]. Here Tor is the differential torsion product.

Proposition 6.7. *Every cofibration in $\mathcal{A}$ or $\mathcal{A}_0$ is EM.*

Before beginning the proof, let us derive **LCM5** from it. By the fundamental mapping theorem of [10, Cor. 1.8] the homology isomorphisms Ha, Hb, Hc induce an isomorphism

$$\mathrm{Tor}_{A_1}(B_1, C_1) \longrightarrow \mathrm{Tor}_{A_2}(B_2, C_2)$$

and hence an isomorphism Hd since $A_1 \longrightarrow B_1$ and $A_2 \longrightarrow B_2$ are given to be cofibrations.

Proof. We do the unaugmented case. We will first show that the class of EM morphisms are closed under the operations:

a) Composition

b) Cobase change

c) Filtering direct limits

d) Retract

e) Tensor product

Recall that $\mathrm{Tor}_X(C,\ Y)$ is computed as $\mathrm{H}(S \otimes_X Y)$ where $S \longrightarrow C$ is a Künneth resolution as differential graded X - modules (terminology as in [10, pp. 2 - 3]).

a) Consider

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & \longrightarrow & E \end{array}$$

where the top arrows are EM and each square is a push - out. We thus have isomorphisms

$$\mathrm{Tor}_X(C,\ Y) \simeq HD \quad \text{and} \quad \mathrm{Tor}_Y(D,\ Z) \simeq HE$$

Let $S \longrightarrow C$ be an X-module Künneth resolution. Then by the above,

$$S \otimes_X Y \ \longrightarrow\ D$$

is a homology isomorphism of Y- modules. Therefore by the mapping theorem of [10] there is an isomorphism

$$\mathrm{Tor}_Y(S \otimes_X Y,\ Z) \simeq \mathrm{Tor}_Y(D,\ Z) \simeq \mathrm{H}\,E$$

Let $T \longrightarrow Z$ be a Künneth Y-module resolution. Then

$$\begin{aligned} \mathrm{Tor}_Y(S \otimes_X Y,\ Z) &= \mathrm{H}((S \otimes_X Y) \otimes_Y T) \\ &= \mathrm{H}(S \otimes_X T) \\ &= \mathrm{Tor}_X(C,\ T) \\ &= \mathrm{Tor}_X(C,\ Z) \end{aligned}$$

where the last isomorphism follows from the mapping theorem already mentioned. Combining these gives the desired $\mathrm{Tor}_X(C,\ Z) \simeq \mathrm{H}\,E$.

b) The argument here is very similar to the proof of a).

c) This follows from the fact that both homology and tensor product commute with filtering direct limits.

d) This is easy.

e) This follows from a) and b), because a morphism

$$f_1 \otimes f_2 : X_1 \otimes X_2 \longrightarrow Y_1 \otimes Y_2$$

is a composition of the cobase extension of $X_1 \to Y_1$ along $X_1 \to X_1 \otimes X_2$ with the cobase extension of $X_2 \to Y_2$ along $X_2 \to Y_1 \otimes X_2$

One can generate all the cofibrations using the above operations starting from the elementary cofibrations

a) $K \longrightarrow T(m)$

b) $K \longrightarrow S(m)$

c) $S(m+1) \longrightarrow T(m)$

See [3, pp. 20 - 22], whose notation we are following. Therefore it suffices to show that all these elementary ones are EM. For a), b), and those of c) with $m \geq 1$ this follows from [3, Lemma 10.6]. It only remains to verify the EM property for $S(1) \longrightarrow T(0)$. The argument in loc.cit. does not apply because $S(1)$ being not simply connected, one does not have naive convergence of the Eilenberg - Moore spectral sequence. One can show however that $T(0)$, regarded as a differential graded $S(1)$-module is a *distinguished, split* object, and therefore a Künneth resolution of itself. [10, pp. 2 - 6] The required

$$\mathrm{Tor}_{S(1)}(C,\ T(0)) \ \simeq\ \mathrm{H}(C \otimes_{S(1)} T(0))$$

then results essentially from the definition of Tor. In more details: we follow the notations of [10] by defining

$$U \ = \ U_0 \oplus U_{-1} \ = \ S(1) \ = \ K \oplus K.a$$

$$X \ = \ X_0 \oplus X_{-1} \ = \ T(0) \ = \ K[b] \oplus K[b].c$$

where $a^2 = c^2 = 0$ and $db = c$. The natural map $S(1) \to T(0)$ sends a to c. Define

$$\overline{X}_{p,-p} = K.b^p \text{ all other } \overline{X}_{p,q} = 0$$

Then

$$X_0 = \bigoplus_{p \geq 0} \overline{X}_{p,-p} \otimes U_0$$

$$X_{-1} = \bigoplus_{p \geq 0} \overline{X}_{p,-p} \otimes U_{-1}$$

The filtration

$$F_pX = \bigoplus_{m \leq p} \overline{X}_{m,*} \otimes U$$

is by sub differential U - modules. Define

$$X_{p,q} = \bigoplus_{i+j=q} \overline{X}_{p,i} \otimes U_j$$

so that

$$X_{p,-p} = \overline{X}_{p,-p} \otimes U_0 = K.b^p$$
$$X_{p,-p-1} = \overline{X}_{p,-p} \otimes U_{-1} = K.b^p.c$$

In the decomposition of the differential $d = \sum d^r$ we have simply

$$d = d^1 : X_{p,-p} \longrightarrow X_{p-1,-p}$$

Finally, the complex

$$\cdots \longrightarrow E^1_{2,*}X \longrightarrow E^1_{1,*}X \longrightarrow E^1_{0,*}X \longrightarrow HX = K \longrightarrow 0$$

is easily seen to be a resolution. □

Here is a third proof of **LCM5** in case all the objects involved are augmented and connected. By [3, 6.14] we have a long exact homotopy sequence

$$\pi^0(A_1) \longrightarrow \pi^0(B_1) \oplus \pi^0(C_1) \longrightarrow \pi^0(D_1) \longrightarrow \pi^1(A_1) \longrightarrow \cdots$$

and a map to a similar one for A_2, B_2, C_2, D_2. By the Whitehead theorem of [3, 7.10] the isomorphisms Ha, Hb, Hc give rise to homotopy isomorphisms πa, πb, πc. By the 5 - lemma, we obtain isomorphisms

$$\pi^i(D_1) \simeq \pi^i(D_2)$$

But then another appeal to the Whitehead theorem gives the sought - for isomorphism

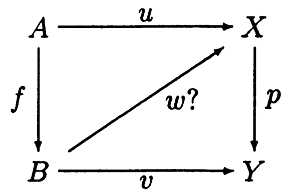
$$\mathrm{H}^i(D_1) \simeq \mathrm{H}^i(D_2)$$

Remark. It might be wondered whether one can prove **LCM5** by appealing to Proposition 3.5. In other words, produce a cylinder object with the required properties. This appears to be impossible for the following reason. Let K be a finite simplicial set. Then the construct $X \otimes AK$ of [3, Lemma 5.2] is not an object $X \otimes K$ in the sense of [15, Axiom SM7, II, p 2.2] but it is an object X^K in that sense. One can ask whether an object $X \otimes K$ exists at all in the category of differential graded algebras, or in other words, whether the functor $X \longrightarrow X^K$ has a left adjoint. This is false because that functor does not preserve inverse limits (tensor product does not commute with products).

7. Appendix : Reedy's Theorems

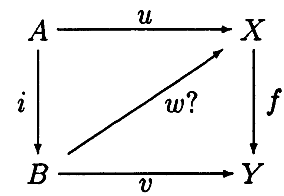
Throughout the following, $\mathcal{M}$ will denote a closed Quillen model category. We begin with a characterization of the weak equivalences among fibrant or cofibrant objects.

Proposition 7.1. 1. *Let A, B be cofibrant objects of $\mathcal{M}$. Then, $f : A \longrightarrow B$ is a weak equivalence if and only if any lifting problem*



where p is a fibration can be solved to the extent that w exists with $p \circ w = v$ and there is a left homotopy $H : u \overset{l}{\sim} w \circ f$ such that $p \circ H$ is stationary.

2. *Let X, Y be fibrant objects of $\mathcal{M}$. Then, $f : X \longrightarrow Y$ is a weak equivalence if and only if any lifting problem*

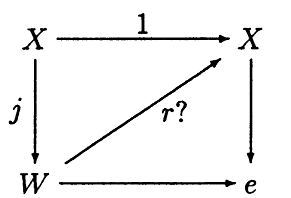


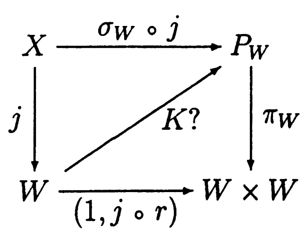
where i is a cofibration can be solved to the extent that w exists with $w \circ i = u$ and there is a right homotopy $H : v \overset{r}{\sim} f \circ w$ such that $H \circ i$ is stationary.

Proof. We do (2). ($\Rightarrow$) We can factor f as

$$X \xrightarrow{j} W \xrightarrow{k} Y$$

with j an acyclic cofibration and k an acyclic fibration. Since X is fibrant, we have a retraction r for j and a homotopy $K : 1 \overset{r}{\sim} j \circ r$ stationary on X, as may be seen by considering the diagrams :





Moreover, it is easily verified that the path object P_W may be chosen to be compatible with a given path object on Y in the sense that we have a diagram:

$$\begin{array}{ccccc} W & \xrightarrow{\sigma_W} & P_W & \xrightarrow{\pi_W} & W \times W \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle k\times k} \\ Y & \xrightarrow{\sigma_Y} & P_Y & \xrightarrow{\pi_Y} & Y \times Y \end{array}$$

Finally, find a lift in

$$\begin{array}{ccc} A & \xrightarrow{j \circ u} & W \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle m?} & \downarrow{\scriptstyle k} \\ B & \xrightarrow[v]{} & Y \end{array}$$

and define $w = r \circ m$ and $H = l \circ K \circ m$, which have the required properties.

($\Leftarrow$) Suppose that f has the stated property. Then f induces a monomorphism of left homotopy classes

$$\pi^l(A,\ X) \longrightarrow \pi^l(A,\ Y)$$

since we can find K in every diagram

$$\begin{array}{ccc} A \vee A & \xrightarrow{(a_1, a_2)} & X \\ {\scriptstyle \sigma_A}\downarrow & \nearrow{\scriptstyle K?} & \downarrow{\scriptstyle f} \\ Z_A & \xrightarrow[H]{} & Y \end{array}$$

with $K \circ \sigma_A = (a_1, a_2)$. f also induces an epimorphism of right homotopy classes

$$\pi^r(A,\ X) \longrightarrow \pi^r(A,\ Y)$$

when A is cofibrant because we can always find v in

$$\begin{array}{ccc} \phi & \longrightarrow & X \\ \downarrow & \nearrow{\scriptstyle v?} & \downarrow{\scriptstyle f} \\ A & \xrightarrow[u]{} & Y \end{array}$$

with $u \overset{r}{\sim} f \circ v$. But when A is cofibrant and X, Y are fibrant these sets of right and left homotopy classes coincide with $\mathrm{Ho}(\mathcal{M})(A,\ X)$, etc. so that we see that f induces a bijection of homotopy classes of morphisms when A is cofibrant, which shows that f becomes invertible in the homotopy category. Since $\mathcal{M}$ is a closed model category, this shows that f is a weak equivalence. □

Proposition 7.2. 1. *In a push-out diagram*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow f & & \downarrow g \\ C & \xrightarrow{j} & D \end{array}$$

in which i is a cofibration, A and C are cofibrant, and f is a weak equivalence, we have that g is a weak equivalence. Therefore, axiom **LCM4** *holds among the cofibrant objects of $\mathcal{M}$.*

2. *In a pull-back diagram*

$$\begin{array}{ccc} D & \xrightarrow{q} & C \\ \downarrow g & & \downarrow f \\ B & \xrightarrow{p} & A \end{array}$$

in which p is a fibration, A and C are fibrant, and f is a weak equivalence, we have that g is a weak equivalence. Therefore, axiom **RCM4** *holds among the fibrant objects of $\mathcal{M}$.*

Proof. We do (2). We are going to apply the criterion of 7.1. Consider a diagram

$$\begin{array}{ccccc} U & \xrightarrow{u} & D & \xrightarrow{q} & C \\ \downarrow i & & \downarrow g & & \downarrow f \\ V & \xrightarrow{v} & B & \xrightarrow{p} & A \end{array}$$

in which i is a cofibration. Our assumptions and that proposition imply that we can find a map $w : V \longrightarrow C$ with $w \circ i = q \circ u$ and a homotopy $H : V \longrightarrow P_A$, $p \circ v \sim f \circ w$ stationary on U. It is easily seen that we may choose a path object on B that has the additional property that there is a natural projection

$$\beta : P_B \longrightarrow (B \times B) \underset{(A \times A)}{\times} P_A$$

which is a fibration. From this is derived a map

$$\beta_1 : P_B \longrightarrow B \underset{A}{\times} P_A$$

where the fiber product is taken relative to the first projection $\pi_1 : P_A \longrightarrow A$, which because A is fibrant is an acyclic fibration, and we can see that β_1 is an acyclic fibration as follows : The map

$$B \underset{A}{\times} P_A \xrightarrow{1\times\pi_1} B \underset{A}{\times} A = B$$

is an acyclic fibration by base - extension. B being fibrant, $\pi_1 : P_B \to B$ is an acyclic fibration, and it factors as

$$P_B \xrightarrow{\beta} (B \times B) \underset{(A\times A)}{\times} P_A \xrightarrow{\tau} B \underset{A}{\times} P_A \xrightarrow{1\times\pi_1} B$$

where τ is the map with components $(pr_1 \circ pr_1, pr_2)$. This shows that $\beta_1 = \tau \circ \beta$ is a weak equivalence. To see that it is a fibration, note the Cartesian square :

$$\begin{array}{ccc} (B \times B) \underset{(A\times A)}{\times} P_A & \xrightarrow{pr_2 \circ pr_1} & B \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle p} \\ B \underset{A}{\times} P_A & \xrightarrow{\pi_2 \circ pr_2} & A \end{array}$$

which shows that τ is a fibration. Define K as a lift in

$$\begin{array}{ccc} U & \xrightarrow{\sigma \circ v \circ i} & P_B \\ \downarrow{\scriptstyle i} & \overset{K?}{\nearrow} & \downarrow{\scriptstyle \beta_1} \\ V & \xrightarrow[(v,H)]{} & B \underset{A}{\times} P_A \end{array}$$

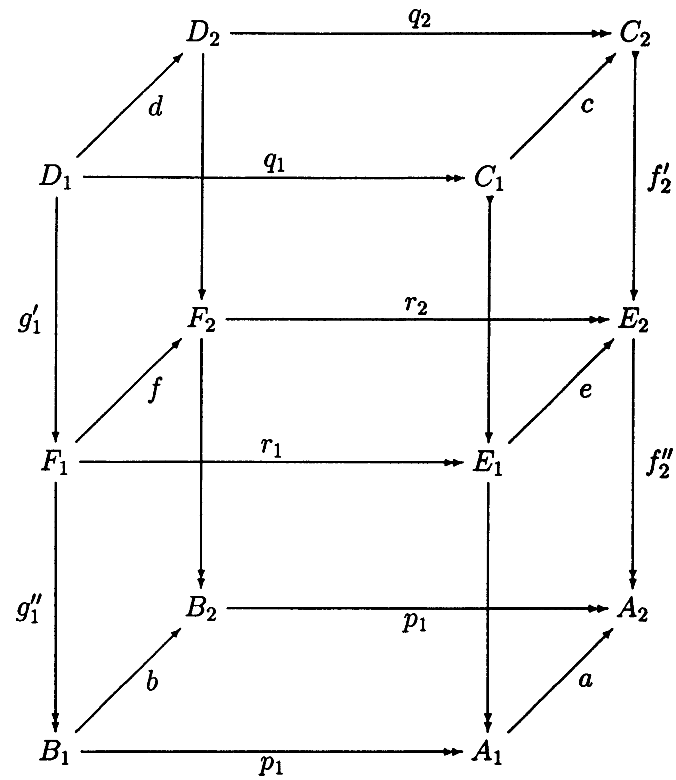
and we define

$$x : V \to D = B \underset{A}{\times} C \ \text{ as } x = (\pi_2 \circ K,\ w)$$

One checks that $x \circ i = u$, that K is a homotopy $v \overset{r}{\sim} g \circ x$, stationary on U, which shows that g is a weak equivalence by 7.1. □

Proposition 7.3. *Axioms* **LCM5** *and* **RCM5** *hold in any closed model category.*

Proof. We do **RCM5**, whose notations we follow. Consider the diagram :



E_2 is defined by factorization $f_2 = f_2'' \circ f_2'$ with f_2'' a fibration and f_2' an acyclic cofibration. E_1 is defined by base - extension along a. An application of 7.2 shows that e is a weak equivalence. The arrow f_1' (not pictured) exists and is therefore a weak equivalence, since c, f_2', and e are. F_1, F_2 are defined by base - extension along p_1, p_2, respectively. The arrows g_1', g_2' exist, and the top front and back rectangles are Cartesian, by cancellation of Cartesian squares. Two applications of 7.2 gives that g_1' and g_2' are weak equivalences. Therefore, d will be a weak equivalence if and only if f is a weak equivalence, so we are reduced to the situation of the lower half of the cube. But now the diagram

$$\begin{array}{ccc} F_1 & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & A_2 \end{array}$$

which is the diagonal of that lower half, is a Cartesian square because it is a composition of the front lower face and the right side face. It is also the composition of the back lower face, which is Cartesian, and the left side face, which is therefore Cartesian, by cancellation of Cartesian squares. As g_1'' and g_2'' are fibrations, one more application of 7.2 shows that f is a weak equivalence.

□

Remark The above argument shows that **RCM5** holds in every fibration category in the sense of [2]. The only change to make is that f_2' is merely a weak equivalence. The point is that 7.2.2 holds in a fibration category. For the same reason, **LCM5** holds in a cofibration category.

The following crucial fact is given without proof in [17].

Proposition 7.4. 1. *Let*

$$A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \cdots$$

be a sequence of cofibrations with each A_n cofibrant. Then :

(a) *We can choose cylinders Z_n on the A_n so that*

$$\begin{array}{ccccc} A_1 \vee A_1 & \xrightarrow{i_1 \vee i_1} & A_2 \vee A_2 & \xrightarrow{i_2 \vee i_2} & \cdots \\ \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_2} & & \\ Z_1 & \xrightarrow{j_1} & Z_2 & \xrightarrow{j_2} & \cdots \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \\ A_1 & \xrightarrow{i_1} & A_2 & \xrightarrow{i_2} & \cdots \end{array}$$

commutes, and

(b) *for all n,*

$$Z_n \underset{(A_n \vee A_n)}{\vee} (A_{n+1} \vee A_{n+1}) \longrightarrow Z_{n+1}$$

is a cofibration.

(c) *Assume that sequential colimits exist in $\mathcal{M}$. For cylinders chosen as in a) and b), $Z = \operatorname{colim} Z_n$ is a cylinder on $A = \operatorname{colim} A_n$.*

2. *Let*

$$\cdots \xrightarrow{p_2} B_2 \xrightarrow{p_1} B_1$$

be a sequence of fibrations with each B_n fibrant. Then :

(a) *We can choose path objects P_n on the B_n so that*

$$\begin{array}{ccccc}
\cdots \xrightarrow{p_2} & B_2 & \xrightarrow{p_1} & B_1 \\
& \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_1} \\
\cdots \xrightarrow{k_2} & P_2 & \xrightarrow{k_1} & P_1 \\
& \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\
\cdots \xrightarrow{p_2 \times p_2} & B_2 \times B_2 & \xrightarrow{p_1 \times p_1} & B_1 \times B_1
\end{array}$$

commutes, and

(b) *for all n,*

$$P_{n+1} \longrightarrow P_n \underset{(B_n \times B_n)}{\times} (B_{n+1} \times B_{n+1})$$

is a fibration.

(c) *Assume that sequential limits exist in $\mathcal{M}$. For path objects chosen as in a) and b), $P = \lim P_n$ is a path object on $B = \lim B_n$.*

Proof. We do (2). The construction of the P's is easily done by induction. To prove c), we must show that $\lim \sigma_n$ is a weak equivalence and that $\lim \pi_n$ is a fibration. The last statement follows immediately from 2.6. The B_n being fibrant, the projections

$$\pi'_n,\ \pi''_n : P_n \longrightarrow B_n \text{ where } \pi'_n = pr_1 \circ \pi_n,\quad \pi''_n = pr_2 \circ \pi_n$$

are acyclic fibrations. Because the composition

$$B \xrightarrow{\lim \sigma_n} P \xrightarrow{\lim \pi'_n} B$$

is the identity, $\lim \sigma_n$ will be a weak equivalence if and only if $\lim \pi'_n$ is one. In fact, we will prove as an application of 2.6 that $\lim \pi'_n$ is an acyclic fibration. The only new point to be checked is that the canonical maps

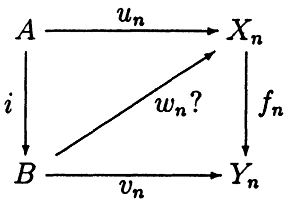
$$P_{n+1} \longrightarrow P_n \underset{B_n}{\times} B_{n+1}$$

(fiber product taken with respect to π'_n) are acyclic fibrations. Since the p_n are fibrations, this point has already been verified in the proof of 7.2 (see the argument showing that β_1 is an acyclic fibration). □

Corollary 7.5. *In any closed model category in which sequential colimits exist,*
LCM3c *holds. In any closed model category in which sequential limits exist,* **RCM3f** *holds.*

Proof. We do the second one.

(first proof, Reedy) We will use the criterion of 7.1. It can be shown that one can choose the w_n in



and the right homotopies $H_n : B \longrightarrow P_n =$ path object on Y_n compatibly as n varies, and so that the conditions of 7.4 are satisfied. Therefore we get maps

$$w : B \longrightarrow \lim X_n \text{ and homotopies } H : B \longrightarrow \lim P_n$$

where the target of the latter really is a path object, satisfying all the conditions of 7.1.

(second proof) The very same argument that was used in proving 3.4 works, but replace the functorial path object used there by a sequence of path objects satisfying the conditions of this proposition.

□

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