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EFFECTIVE TAXONOMIES AND CROSSED TAXONOMIES

by Pierre AGERON

RESUME. Tout endofoncteur pointé d'une catégorie donne naturellement naissance à une taxinomie (ou "catégorie sans identités"). Inversement, toute taxinomie se plonge pleinement dans une taxinomie associée à un endofoncteur pointé. En s'appuyant sur ce cas, on dégage une notion de module croisé de taxinomies, puis on définit la β-accessibilité d'une telle structure à partir de ses objets β-rigides (et non de ses objets β-présentables). L'article se termine par l'étude un peu détaillée d'un exemple, dû pour l'essentiel à Dedekind.

Abstract

Every pointed endofunctor on a category gives rise to a canonical taxonomy (i.e., a "category without identities"). Conversely, every taxonomy has a full embedding into the taxonomy asociated to some pointed endofunctor. Relying on that case, crossed modules of taxonomies are defined, as well as their β-accessibility (from their β-rigid objects and not from their β-presentable ones). Last, a particular model, essentially due to Dedekind, is studied in some detail.

1. Elementary theory of taxonomies

In this section, we insist on ideas and omit proofs : the latter can be found in the unpublished [A].

Consider a pointed endofunctor (F, η) on a category $(\mathbb{C}, \circ)$ (i.e. η is a natural transformation from $id_{\mathbb{C}}$ to F). Consider the graph whose objects are those of $\mathbb{C}$ and whose arrows from A to B are the arrows from $F(A)$ to B in $\mathbb{C}$. Consecutive arrows of that graph compose in a canonical way : if $f : A \longrightarrow B$ and $g : B \longrightarrow C$, let $g \,\square\, f = g \circ \eta_B \circ f : A \longrightarrow C$. Although $\square$ is associative, this construction fails to provide a category because of the lack of identities.

Now define a *taxonomy* as a structure similar to a category except that identities need not exist : the example we have just met will be called the *effective taxonomy* of a pointed endofunctor and will be denoted by $\mathrm{Tax}(F, \eta)$. Say that an object A in a taxonomy is *reflected* (resp. *identified*) if A has an endomorphism (resp. an identity) : morphisms of taxonomies preserve reflected objects, but not identified objects.

Theorem 1. *The functor taking a category to its underlying taxonomy has a left adjoint L and a right adjoint K.*

For every taxonomy $\mathbb{S}$, the cofree category $K(\mathbb{S})$ is Karoubian (i.e. every idempotent splits) : it coincides with the free Karoubian category generated by $\mathbb{S}$. The free category $L(\mathbb{S})$ is "anti-Karoubian" (i.e. every split idempotent is an identity) or, equivalently, "local" (i.e. the class of non-invertible arrows is an ideal).

Any pointed endofunctor gives rise to another interesting taxonomy : define the *abstract taxonomy* $\mathrm{Tax}'(F, \eta)$ as the smallest ideal of $\mathbb{C}$ containing all arrows of the form η_A. Clearly, the abstract taxonomy is a quotient of the effective one, and both have the same reflected objects. Taxonomies associated to pointed endofunctors are generic in the following sense :

Theorem 2. *Every taxonomy has a full embedding into the effective taxonomy of a colimit-preserving pointed endofunctor on a cocomplete category* $\overline{\mathbb{C}}$.

(If needed, we can assume a *well*-pointed endofunctor, i.e. one which satisfies $\eta \circ F = F \circ \eta$.) Effective taxonomies bear a richer structure than merely being a taxonomy. The next notion captures exactly what kind of structure arises :

Definition. Say that $(\mathbb{S}, F, *, \alpha, \delta)$ is a *supertaxonomy* if :

- $\mathbb{S}$ is a graph and F is an endomorphism of $\mathbb{S}$,
- the operation $*$ takes $f : A \longrightarrow F(B)$ and $g : B \longrightarrow C$ to $g * f : A \longrightarrow C$,
- α is a family of arrows $\alpha_A : A \longrightarrow F(A)$ (each object A),
- δ is a family of arrows $\delta_A : A \longrightarrow F(F(A))$ (each object A),
- the following equations hold each time they make sense :

$$h * (g * f) = (h * g) * f, \quad f * \alpha_A = f, \quad \alpha_B * f = f,$$
$$F(g * f) = F(g) * F(f), \quad F(f) * \delta_A = \delta_B * f, \quad \alpha_{F(A)} = F(\alpha_A).$$

Every supertaxonomy has an *underlying taxonomy*, with composition $g \,\square\, f = g * F(f) * \delta_A$. The effective taxonomy of a pointed endofunctor underlies some *effective supertaxonomy*, with $g * f = g \circ f$, $\alpha_A = id_{F(A)}$ and $\delta_A = \eta_{F(A)}$. And finally :

Theorem 3. *Every supertaxonomy is isomorphic to the effective supertaxonomy of a pointed endofunctor.*

As an application of theorems 2 and 3, we can write down natural axioms and rules for a consistent and *constructive* logic where the principle of identity "if A, then A" no longer holds. With one logical constant (0), one modality (?) and one binary connective (*or*), we obtain :

$$\begin{array}{llclcll}
 & A \vdash ?A & & A \vdash ??A & & 0 \vdash A \\
\text{if} & A \vdash B, & \text{then} & ?A \vdash ?B & & \\
\text{if} & A \vdash ?B & \text{and} & B \vdash C, & \text{then} & A \vdash C \\
 & A \vdash ?A \text{ or } B & & B \vdash A \text{ or } ?B & & \\
 & A \vdash ?(A \text{ or } B) & & B \vdash ?(A \text{ or } B) & & \\
\text{if} & A \vdash C & \text{and} & B \vdash C, & \text{then} & A \text{ or } B \vdash C
\end{array}$$

Transitivity of inference is derivable in this system – but reflexivity is not ! Such a logic could reflect the fact that we may no longer hold things that we have held, because the strength of a statement would gradually weaken in case no use is made of it to produce new results. These ideas may apply to computer science where it is important to ensure that some real and useful job is being done.

2. Crossed taxonomies

A *crossed group* (or *crossed module of groups)* is a pair of groups (A, C) with morphisms $\partial : A \longrightarrow C$ and $\phi : C \longrightarrow \mathrm{Aut}(A)$ related by two natural equations. Our selection of axioms for crossed taxonomies will rely on the three following observations in the case where $\mathbb{A}$ is the effective taxonomy of some pointed endofunctor (F, η) on a category $\mathbb{C}$:

1. If $A \xrightarrow[\mathbb{A}]{a} B$ denotes an arrow in $\mathbb{A}$, let $\partial(a) = a \circ \eta_A$. This gives a morphism $\partial : \mathbb{A} \longrightarrow \mathbb{C}$ leaving objects unchanged.

2. If $A \xrightarrow[\mathbb{A}]{a} B$ and $B \xrightarrow[\mathbb{C}]{c} C$, let $c.a = c \circ a : A \xrightarrow[\mathbb{A}]{} C$. Similarly, if $A \xrightarrow[\mathbb{C}]{c} B$ and $B \xrightarrow[\mathbb{A}]{a} C$, define $a.c = a \circ F(c)$: $A \xrightarrow[\mathbb{A}]{} C$. Then the following equations hold :

$$id.a = a \qquad (c' \circ c).a = c'.(c.a) \qquad c.(a' \,\square\, a) = (c.a').a$$
$$a.id = a \qquad a.(c' \circ c) = (a.c').c \qquad (a' \,\square\, a).c = a'.(a.c)$$
$$c.(a.c') = (c.a).c'$$
$$\partial(c.a) = c \circ (\partial a) \qquad \partial a'.a = a' \,\square\, a \qquad \partial(a.c) = (\partial a) \circ c$$

E.g. the last one is derived as follows :

$\partial(a.c) = \partial(a \circ F(c)) = a \circ F(c) \circ \eta_A = a \circ \eta_B \circ c = (\partial a) \circ c.$

These equations imply $\partial(a.\partial a') = \partial(a \,\square\, a')$, but $a.\partial a' = a \,\square\, a'$ need not hold (test Dedekind's model in section 3).

3. Moreover, if $\mathbb{C}$ is a groupoid (but also in other cases, see section 3), we have another operation at our disposal : if $A \xrightarrow[\mathbb{A}]{a} A$ and $A \xrightarrow[\mathbb{C}]{c} B$, let $c[a] = c \circ a \circ F(c)^{-1} : B \xrightarrow[\mathbb{A}]{} B$. Then the following equations hold :

$$id[a] = a \qquad (c' \circ c)[a] = c'[c[a]] \qquad c[a' \,\square\, a] = c[a'] \,\square\, c[a]$$
$$c[a].c = c.a \qquad \partial a'[a] \,\square\, a' = a' \,\square\, a \qquad \partial a'[a \,\square\, a'] = a' \,\square\, a.$$

We can now define abstract crossed taxonomies :

Definition. *A crossed taxonomy consists of a morphism ∂ from a taxonomy $\mathbb{A}$ to a category $\mathbb{C}$, such that the objects of $\mathbb{A}$ are the same as those of $\mathbb{C}$ and are left unchanged by ∂, together with three operations of $\mathbb{C}$ on $\mathbb{A}$ denoted by :*

$$\frac{X \xrightarrow[\mathbb{A}]{a} Y \quad Y \xrightarrow[\mathbb{C}]{c} Z}{X \xrightarrow[\mathbb{A}]{c.a} Z} \qquad \frac{X \xrightarrow[\mathbb{C}]{c} Y \quad Y \xrightarrow[\mathbb{A}]{a} Z}{X \xrightarrow[\mathbb{A}]{a.c} Z}$$

and

$$\frac{X \xrightarrow[\mathbb{A}]{a} X \quad X \xrightarrow[\mathbb{C}]{c} Y}{Y \xrightarrow[\mathbb{A}]{c[a]} Y}$$

satisfying the following equations (whenever they make sense) :

$$id.a = a \qquad (c' \circ c).a = c'.(c.a) \qquad c.(a' \mathbin{\square} a) = (c.a').a$$
$$a.id = a \qquad a.(c' \circ c) = (a.c').c \qquad (a' \mathbin{\square} a).c = a'.(a.c)$$
$$id[a] = a \qquad (c' \circ c)[a] = c'[c[a]] \qquad c[a' \mathbin{\square} a] = c[a'] \mathbin{\square} c[a]$$
$$c.(a.c') = (c.a).c' \qquad c[a].c = c.a$$
$$\partial(c.a) = c \circ (\partial a) \qquad \partial a'.a = a' \mathbin{\square} a \qquad \partial(a.c) = (\partial a) \circ c$$
$$\partial a'[a] \mathbin{\square} a' = a' \mathbin{\square} a.$$

We could define *cocrossed taxonomies* by trading the last equation for its dual (in some sense) : $\partial a'[a \mathbin{\square} a'] = a' \mathbin{\square} a$. The crossed categories in $[P]$ involve an operation similar to our $-[-]$, but in the restrictive context where $\mathbb{A}$ is a skeletal groupoid.

In any crossed taxonomy $(\partial : \mathbb{A} \longrightarrow \mathbb{C}, \ldots)$, the mapping $A \mapsto \mathrm{End}_{\mathbb{A}}(A)$ is the object part of a functor E to the category of semi-groups defined on arrows by $E(c)(a) = c[a]$. Let β be a regular cardinal : let us say that an object A is β-rigid if E is β-continuous at A (i.e. for every small β-filtered category $\mathbb{I}$ and every functor $D : \mathbb{I} \longrightarrow \mathbb{C}$ such that $\mathrm{Colim}\, D \cong A$, then $\mathrm{Colim}(E \circ D) \cong E(A)$). Denote by $\mathrm{Rig}_\beta(\partial)$ the full subcategory of $\mathbb{C}$ consisting of β-rigid objects. The following definition parallels Lair's notion of an accessible (or modelable) category (but here β-presentable objects are replaced by β-rigid ones).

Definition. *A crossed taxonomy $(\partial : \mathbb{A} \longrightarrow \mathbb{C}, \ldots)$ is β-accessible iff :*

1. *$\mathbb{C}$ has all small β-filtered colimits ;*
2. *the category $\mathrm{Rig}_\beta(\partial)$ is equivalent to some small subcategory $\mathbb{C}'$ such that :*
3. *for every object A, the category $\mathbb{C}'/A$ is β-filtered and the canonical functor $\mathbb{C}'/A \longrightarrow \mathbb{C}$ has colimit A.*

The existence of a small full subcategory with property 3. can be seen as the categorical analogue of the topological notion of separability (existence of a countable dense subset). Continuing in this vein, β-accessibility of a crossed taxonomy parallels the requirement that the set of points at which a certain function is continuous be countable and dense – which contradicts Baire's property ! As a matter of fact, it is not a mild requirement : in many cases, β-accessibility is equivalent to the existence of some large cardinal. At an elementary level, this is demonstrated by the example discussed in the next section.

3. Dedekind's model

In this section, $\mathcal{U}$ denotes some set (of sets) such that for every $A \in \mathcal{U}$, one has $A \notin A$ and $A \cup \{A\} \in \mathcal{U}$; $\mathbb{C}$ denotes the small category with objects all sets in $\mathcal{U}$ and with arrows all injections between them. Let (F, η) be the following (not well-)pointed endofunctor on $\mathbb{C}$:

+ if A an object of $\mathbb{C}$, $F(A) = A \cup \{A\}$;
+ if $a : A \longrightarrow B$ is a map of $\mathbb{C}$, $F(a) = a \cup \{(A, B)\}$;
+ $\eta_A : A \longrightarrow F(A)$ is the canonical injection.

In the effective taxonomy $\mathrm{Tax}(F, \eta)$, an arrow from A to B is an injection from $A \cup \{A\}$ into B. The abstract taxonomy $\mathrm{Tax}'(F, \eta)$ is the ideal of $\mathbb{C}$ consisting of all *strict* injections (the class of non-invertible arrows of $\mathbb{C}$ clearly happens to be an ideal, i.e. $\mathbb{C}$ is anti-Karoubian). Given an injective map $c : A \longrightarrow B$, we can think at each arrow $a : A \longrightarrow B$ extending c that may exist in $\mathrm{Tax}(F, \eta)$ as a (constructive) *proof that c is strict.* The reflected objects in $\mathrm{Tax}(F, \eta)$ or $\mathrm{Tax}'(F, \eta)$ are exactly the *infinite sets in the sense of Dedekind* ([D]).

Theorem 4. *The canonical morphism* $\partial : \mathrm{Tax}(F, \eta) \longrightarrow \mathbb{C}$ *underlies a crossed (not cocrossed) taxonomy.*

Proof – If $X \xrightarrow[\mathbb{A}]{a} X$ and $X \xrightarrow[\mathbb{C}]{c} Y$, then define for any $y \in F(Y)$:

$$\begin{array}{lll} c[a](y) = c(a(c^{-1}(y))) & \text{if} & y \in Im(c) \\ c[a](y) = y & \text{if} & y \in Y \setminus Im(c) \\ c[a](y) = c(a(X)) & \text{if} & y = Y \end{array}$$

It is easy to check that $c[a]$ is an injection from $Y \cup \{Y\}$ to Y and that all required equations hold. ∎

Comments. This crossed taxonomy conveys the constructive (i.e. categorical) content of "Dedekind's theorem 68" : in $[D]$, this theorem comes shortly after Dedekind's definition of "infinite" and states that "*every system that contains an infinite part is also infinite*". The proof consists in constructing from any $a : A \xrightarrow[\mathbb{A}]{} A$ (i.e. any proof that A is infinite) and from any injection $c : A \xrightarrow[\mathbb{C}]{} B$ some new arrow $c[a] : B \xrightarrow[\mathbb{A}]{} B$. Dedekind also proved (theorem 72) that a set A is D-infinite iff there is an injection from $\mathbb{N}$ into A. This allows a new trivial proof of theorem 68 which just consists in composing two arrows in $\mathbb{C}$. But Dedekind probably felt that a major intuition about infinity is that bigger than infinite be infinite, so he proved that fact immediately after his definition. However, if one is not willing to accept excluded middle (in $\mathcal{U}$), theorem 68 does not hold any more : to avoid that, some intuitionistic theory of actual infinite could be developped in the framework of free crossed taxonomies. (Dedekind's model is far from free : some relations like $\partial f[\partial f[f]] \,\square\, f = f \,\square\, \partial f[f]$ are related to a combinatorial problem raised by Dehornoy in $[D']$.) Finally, it is easy to check that Dedekind's crossed taxonomy is $\aleph_0$-accessible, the $\aleph_0$-rigid objects being the D-finite sets. If we do not take the axiom of infinity as a primitive, then accessibility of Dedekind's model is in fact equivalent to this axiom.

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